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- The Bayes classifier minimizes *risk*.
- If we have complete knowledge of the underlying probability distributions, then Bayes classifier is optimal (for minimizing risk).
- There are other classifiers possible (e.g., nearest neighbour classifier)

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- Another important form for a classifier is

$$h(\mathbf{X}) = 0 \quad \text{if} \quad g(\mathbf{W}, \mathbf{X}) > 0$$

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  g is called Discriminant function.
- Important special case is linear discriminant functions $h(\mathbf{X}) = \text{sgn}(\mathbf{W}^T\mathbf{X})$
- There are different approaches to learn *nonlinear* classifiers.

Bayes Classifier

- In this class we will derive the Bayes classifier for M classes under a general loss function.
- This can actually be looked at as a special case of a more general problem of decision making under uncertainity.

Bayesian Decision Making

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- We want to decide on one of finitely many ‘actions’ based on some observation.
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Bayesian Decision Making

- The task: Decision making under uncertainity
- We want to decide on one of finitely many ‘actions’ based on some observation.
- Our ‘payoff’ or ‘cost’ depends on the *unknown* ‘state of nature’ and the observation gives some (stochastic) information on the ‘state of nature’.
- A Loss function gives ‘costs’ for each decision for every ‘true’ state of nature.
- We want a strategy of decision making that minimizes, e.g., expected loss.

In the context of classifier design

- Observation is the feature vector.
- The 'state of nature' is the 'true' class label of the feature vector.
- We need to decide on a class label based on the observation.

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- Let $h(\mathbf{X}) \in \{\alpha_0, \alpha_1, \cdots, \alpha_{K-1}\}$.
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Bayes Classifier

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  (States of Nature)
- Let $h(\mathbf{X}) \in \{\alpha_0, \alpha_1, \cdots, \alpha_{K-1}\}$.
  The output of classifier would be α_j's.
  (Actions of decision maker)

- In general, we may have $M \neq K$.
  The classifier output need not always be a class label.
- For example, we can have $K = M + 1$ and α_M may denote the decision of ‘rejection’.
- We take $K = M$ unless specified otherwise.

- $L(\alpha_j, C_k)$ – loss when classifier says α_j and ‘true class’ is C_k. We assume that loss function is non-negative.
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- $L(\alpha_j, C_k)$ – loss when classifier says α_j and 'true class' is C_k. We assume that loss function is non-negative.
- Notation makes it easy to understand arguments of loss function.
- As earlier, the risk of a classifier h is

$$R(h) = EL(h(\mathbf{X}), y(\mathbf{X}))$$

- We want the classifier that has the least risk value.

- Given a $\mathbf{X}$, let $R(\alpha_i \mid \mathbf{X})$ denote the expected loss when classifier says α_i and conditioned on $\mathbf{X}$.

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- We saw

$$
\begin{aligned}
R(\alpha_i \mid \mathbf{X}) &= E\left[L(h(\mathbf{X}), y(\mathbf{X})) \mid h(\mathbf{X}) = \alpha_i, \mathbf{X}\right] \\
&= \sum_{j=0}^{M-1} L(\alpha_i, C_j) q_j(\mathbf{X})
\end{aligned}
$$

- In general, we have

$$
R(h(\mathbf{X}) \mid \mathbf{X}) = \sum_{j=0}^{M-1} L(h(\mathbf{X}), C_j) q_j(\mathbf{X})
$$

- Let f denote the density of $\mathbf{X}$. Now risk of any classifier is

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\end{aligned}
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- The optimal classifier:

  for each $\mathbf{X}$, $h(\mathbf{X})$ should minimize $R(h(\mathbf{X}) \mid \mathbf{X})$.

The Bayes Classifier

- Recall $R(h(\mathbf{X}) \mid \mathbf{X}) = \sum_{j=0}^{M-1} L(h(\mathbf{X}), C_j) q_j(\mathbf{X})$

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- The Bayes classifier, h_B for the M-class case is:

  $h_B(\mathbf{X}) = \alpha_i$ if

$$\sum_{j=0}^{M-1} L(\alpha_i, C_j) q_j(\mathbf{X}) \leq \sum_{j=0}^{M-1} L(\alpha_k, C_j) q_j(\mathbf{X}), \ \forall k$$

(Break ties arbitrarily)

The Bayes Classifier

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  (Break ties arbitrarily)

- Thus $R(h_B(\mathbf{X}) \mid \mathbf{X}) \leq R(h(\mathbf{X}) \mid \mathbf{X}), \ \forall h$ and thus Bayes classifier is optimal.

- Take $M = 2$. Now the Bayes classifier is:

  $h_B(\mathbf{X}) = \alpha_0$ if

$$L(\alpha_0, C_0)q_0(\mathbf{X}) + L(\alpha_0, C_1)q_1(\mathbf{X}) \leq$$
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- Same as

$$\frac{q_0(\mathbf{X})}{q_1(\mathbf{X})} \geq \frac{L(\alpha_0, C_1)}{L(\alpha_1, C_0)} \quad \text{if} \quad L(\alpha_0, C_0) = L(\alpha_1, C_1) = 0.$$

- Same as the Bayes classifier we saw earlier.

- Take M-class case and consider 0–1 loss function. Then

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&= \sum_{j \neq i} q_j(\mathbf{X}) = 1 - q_i(\mathbf{X})
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- Thus, for M-class case and 0–1 loss function

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- Thus the Bayes classifier is: $h_B(\mathbf{X}) = \alpha_i$ if

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- This is the M-class classifier for 0–1 loss function. Minimizes probability of misclassification.

Bayes Classifier – General Case

- The Bayes classifier that minimizes risk is:

  $h_B(\mathbf{X}) = \alpha_i$ if (Break ties arbitrarily)

$$R(\alpha_i \mid \mathbf{X}) \leq R(\alpha_j \mid \mathbf{X}), \ \forall j.$$

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- Note that this is the most general case. (Even when $L(\alpha_i, C_i) \neq 0$). This is optimal for minimizing risk.

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- Let us consider the two class case and explicitly write down Bayes classifier for some specific class conditional densities.
- For simplicity we write $L(\alpha_i, C_j) = L(i, j)$. We also assume $L(0, 0) = L(1, 1) = 0$.
- For the 2-class case, we decide on C_0 if

$$\frac{q_0(\mathbf{X})}{q_1(\mathbf{X})} = \frac{f_0(\mathbf{X})p_0}{f_1(\mathbf{X})p_1} \geq \frac{L(0,1)}{L(1,0)}$$

Normal class conditional densities

- We start with the simple case of $\mathbf{X} \in \Re$ (hence use X for $\mathbf{X}$) and both class conditional densities normal.

$$f_i(X) = \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left(\frac{-(X - \mu_i)^2}{2\sigma_i^2} \right), \quad i = 0, 1$$

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- Same as

$$\ln(p_0 L(1,0)) + \ln(f_0(X)) > \ln(p_1 L(0,1)) + \ln(f_1(X))$$

- That is, $h_B(X) = 0$ if

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- This is of the form

$$h_B(X) = 0 \quad \text{if} \quad aX^2 + bX + c > 0$$

where a, b, c are some constants.

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- This is of the form

$$h_B(X) = 0 \quad \text{if} \quad aX^2 + bX + c > 0$$

  where a, b, c are some constants.
- Thus the Bayes classifier in this case is a quadratic discriminant function.

some special cases

- The Bayes classifier is: $h_B(X) = 0$ if $\frac{1}{2}X^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) + X\left(\frac{\mu_0}{\sigma_0^2} - \frac{\mu_1}{\sigma_1^2}\right)$ $+\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_0^2}{\sigma_0^2}\right) + \ln\left(\frac{\sigma_1}{\sigma_0}\right) + \ln\left(\frac{p_0 L(1,0)}{p_1 L(0,1)}\right) > 0$

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- Take $\sigma_0 = \sigma_1 = \sigma$, $p_0 = p_1$ and $L(1, 0) = L(0, 1)$.

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- Take $\sigma_0 = \sigma_1 = \sigma$, $p_0 = p_1$ and $L(1,0) = L(0,1)$. Then
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- That is, $X > \frac{\mu_0+\mu_1}{2}$, assuming $\mu_0 > \mu_1$.
- Intuitively the classifier is very clear.

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- Take $\mu_0 = \mu_1 = 0$, $p_0 = p_1$ and $L(1,0) = L(0,1)$. Then
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$$\frac{1}{2}X^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) - \ln\left(\frac{\sigma_0}{\sigma_1}\right) > 0$$
- Assuming $\sigma_0 > \sigma_1$, this is same as
$$X^2 > \frac{\sigma_1^2\sigma_0^2 \ln(\sigma_0/\sigma_1)}{(\sigma_0^2 - \sigma_1^2)}$$

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- The Bayes classifier is: $h_B(X) = 0$ if
$$\frac{1}{2}X^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) + X\left(\frac{\mu_0}{\sigma_0^2} - \frac{\mu_1}{\sigma_1^2}\right)$$
$$+\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_0^2}{\sigma_0^2}\right) + \ln\left(\frac{\sigma_1}{\sigma_0}\right) + \ln\left(\frac{p_0 L(1,0)}{p_1 L(0,1)}\right) > 0$$
- Take $\mu_0 = \mu_1 = 0$, $p_0 = p_1$ and $L(1,0) = L(0,1)$. Then
$$\frac{1}{2}X^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) - \ln\left(\frac{\sigma_0}{\sigma_1}\right) > 0$$
- Assuming $\sigma_0 > \sigma_1$, this is same as
$X^2 > \frac{\sigma_1^2 \sigma_0^2 \ln(\sigma_0/\sigma_1)}{(\sigma_0^2 - \sigma_1^2)}$ (again, intuitively clear).

- Now let us consider the case of $\mathbf{X} \in \Re^n$ and normal class conditional densities.

$$f_i(\mathbf{X}) = ((2\pi)^n |\Sigma_i|)^{-\frac{1}{2}} \exp(-\frac{1}{2}(\mathbf{X} - \boldsymbol{\mu}_i)^T \Sigma_i^{-1} (\mathbf{X} - \boldsymbol{\mu}_i)), \quad i = 0, 1$$

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- That is,

$$\ln(p_0 L(1,0)) - \frac{1}{2}\ln(|\Sigma_0|) - \frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_0)^T \Sigma_0^{-1} (\mathbf{X}-\boldsymbol{\mu}_0) > \ln(p_1 L(0,1)) - \frac{1}{2}\ln(|\Sigma_1|) - \frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)$$

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- That is

$$\frac{1}{2}\mathbf{X}^T(\Sigma_1^{-1} - \Sigma_0^{-1})\mathbf{X} + \mathbf{X}^T(\Sigma_0^{-1}\boldsymbol{\mu}_o - \Sigma_1^{-1}\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{\mu}_1^T\Sigma_1^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T\Sigma_0^{-1}\boldsymbol{\mu}_0) + \ln\left(\frac{p_0 L(1,0)}{p_1 L(0,1)}\right) + \frac{1}{2}\ln\left(\frac{|\Sigma_1|}{|\Sigma_0|}\right) > 0$$

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$$+\ln\left(\frac{p_0 L(1,0)}{p_1 L(0,1)}\right) + \frac{1}{2}\ln\left(\frac{|\Sigma_1|}{|\Sigma_0|}\right) > 0$$

- Once again, the Bayes classifier is a quadratic discriminant function.

- The Bayes classifier is a discriminant function given by

$$\frac{1}{2}\mathbf{X}^T(\Sigma_1^{-1} - \Sigma_0^{-1})\mathbf{X} + \mathbf{X}^T(\Sigma_0^{-1}\boldsymbol{\mu}_o - \Sigma_1^{-1}\boldsymbol{\mu}_1)$$
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- The Bayes classifier is based on the discriminant function

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- Consider the specisl case $\Sigma_i = \Sigma$.
- Then the quadratic term Vanishes.
- The Bayes classifier now becomes a linear discriminant function.

- In the specisl case $\Sigma_i = \Sigma$, the Bayes classifier is:
- $h_B(\mathbf{X}) = 0$ if $g(\mathbf{X}) > 0$, where

$g(\mathbf{X}) = \mathbf{W}^T\mathbf{X} + w_0$, with

$$\mathbf{W} = \Sigma^{-1}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_1)$$

$$w_0 = \frac{1}{2}(\boldsymbol{\mu}_1^T\Sigma^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0^T\Sigma^{-1}\boldsymbol{\mu}_0) + \ln\left(\frac{p_0 L(1,0)}{p_1 L(0,1)}\right)$$

- This is a linear discriminant function

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- For example, in a 2-class case with 0-1 loss function, given a $\mathbf{X}$, we decide on the class based on whether or not the inequality $p_0 f_0(\mathbf{X}) > p_1 f_1(\mathbf{X})$ is satisfied.
- Given full statistical information, this is the optimal decision.

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- $h(\mathbf{X}) = 0$ if $g(\mathbf{X}) \geq 0$; $h(\mathbf{X}) = 1$ otherwise
- It is not immediately obvious how this is to be extended to the multi-class case.
- The Bayes classifier for the multi-class case is one such generalization.

- Bayes classifier for M-class case is: $h_B(\mathbf{X}) = \alpha_i$ if

$$\sum_{j=0}^{M-1} L(\alpha_i, C_j) q_j(\mathbf{X}) \leq \sum_{j=0}^{M-1} L(\alpha_k, C_j) q_j(\mathbf{X}), \ \forall k$$

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- Consider the case of 0-1 loss function. Then the above is same as $p_i f_i(\mathbf{X}) \geq p_k f_k(\mathbf{X}), \ \forall k$

  or $\ln(p_i f_i(\mathbf{X})) \geq \ln(p_k f_k(\mathbf{X})), \ \forall k$

- Define $g_i(\mathbf{X}) = \ln(f_i(\mathbf{X})) + \ln(p_i)$, $i = 0, 1, \cdots, M-1$.
- Now, the Bayes classifier is:

  Decide on class-i if $g_i(\mathbf{X}) \geq g_j(\mathbf{X}) \; \forall j$

- Define $g_i(\mathbf{X}) = \ln(f_i(\mathbf{X})) + \ln(p_i)$, $i = 0, 1, \cdots, M-1$.
- Now, the Bayes classifier is:

  Decide on class-i if $g_i(\mathbf{X}) \geq g_j(\mathbf{X}) \; \forall j$
- This is the form of discriminant function based classifier for the M-class case.