Unit -1

Introduction

REVIEW OF DISCRETE TIME SIGNALS AND SYSTEMS

Anything that carries some information can be called as signals. Some examples are ECG, EEG, ac power, seismic, speech, interest rates of a bank, unemployment rate of a country, temperature, pressure etc.

A **signal** is also defined as any physical quantity that varies with one or more independent variables.

A **discrete time signal** is the one which is not defined at intervals between two successive samples of a signal. It is represented as graphical, functional, tabular representation and sequence.

Some of the elementary discrete time signals are unit step, unit impulse, unit ramp, exponential and sinusoidal signals (as you read in signals and systems).

Classification of discrete time signals

Energy and Power signals

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

If the value of E is finite, then the signal x(n) is called energy signal.

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

If the value of the P is finite, then the signal x(n) is called Power signal.

Periodic and Non periodic signals

A discrete time signal is said to be periodic if and only if it satisfies the condition X (N+n) =x (n), otherwise non periodic

Symmetric (even) and Anti-symmetric (odd) signals

The signal is said to be even if x(-n)=x(n)

The signal is said to be odd if x(-n)= - x(n)

Causal and non causal signal

The signal is said to be causal if its value is zero for negative values of 'n'.

Some of the **operations** on discrete time signals are shifting, time reversal, time scaling, signal multiplier, scalar multiplication and signal addition or multiplication.

Discrete time systems

A discrete time signal is a device or algorithm that operates on discrete time signals and produces another discrete time output.

Classification of discrete time systems

Static and dynamic systems

A system is said to be static if its output at present time depend on the input at present time only.

Causal and non causal systems
A system is said to be causal if the response of the system depends on present and past values of the input but not on the future inputs.

Linear and non linear systems
A system is said to be linear if the response of the system to the weighted sum of inputs should be equal to the corresponding weighted sum of outputs of the systems. This principle is called superposition principle.

Time invariant and time variant systems
A system is said to be time invariant if the characteristics of the systems do not change with time.

Stable and unstable systems
A system is said to be stable if bounded input produces bounded output only.

<u>TIME DOMAIN ANALYSIS OF DISCRETE TIME SIGNALS AND SYSTEMS</u>
Representation of an arbitrary sequence
Any signal x(n) can be represented as weighted sum of impulses as given below

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

The response of the system for unit sample input is called impulse response of the system h(n)

$$y(n) = \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n.k)$$

By time invariant property, we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The above equation is called **convolution sum**.
Some of the properties of convolution are commutative law, associative law and distributive law.

Correlation of two sequences
It is basically used to compare two signals. It is the measure of similarity between two signals. Some of the applications are communication systems, radar, sonar etc.
The cross correlation of two sequences x(n) and y(n) is given by

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

One of the important properties of cross correlation is given by

$$r_{xy}(l) = r_{yx}(-l)$$

The auto correlation of the signal x(n) is given by

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

Linear time invariant systems characterized by constant coefficient difference equation

The response of the first order difference equation is given by

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

The first part contain initial condition y(-1) of the system, the second part contains input x(n) of the system.

The response of the system when it is in relaxed state at n=0 or y(-1)=0 is called **zero state response** of the system or **forced response**.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

The output of the system at zero input condition x(n)=0 is called **zero input response** of the system or **natural response**.

The impulse response of the system is given by zero state response of the system

$$y_{zs}(n) = \sum_{k=0}^{n} a^k \delta(n-k)$$

$$= a^n \qquad n \geq 0$$

The total response of the system is equal to sum of natural response and forced responses.

FREQUENCY DOMAIN ANALYSIS OF DISCRETE TIME SIGNALS AND SYSTEMS

A s we have observed from the discussion o f Section 4.1, the Fourier series representation o f a continuous-time periodic signal can consist of an infinite number of frequency components, where the frequency spacing between two successive harmonically related frequencies is 1 / *T p,* and where *Tp* is the fundamental period.

Since the frequency range for continuous-time signals extends infinity on both sides it is possible to have signals that contain an infinite number of frequency components.

In contrast, the frequency range for discrete-time signals is unique over the interval. A discrete-time signal of fundamental period N can consist of frequency components separated by $2n / N$ radians.
Consequently, the Fourier series representation o f the discrete-time periodic signal will contain at most N frequency components. This is the basic difference between the Fourier series representations for continuous-time and discrete-time periodic signals.

4.2.1 The Fourier Series for Discrete-Time Periodic Signals

Suppose that we are given a periodic sequence $x(n)$ with period N, that is, $x(n) = x(n+N)$ for all n. The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{4.2.1}$$

where the $\{c_k\}$ are the coefficients in the series representation.

To derive the expression for the Fourier coefficients, we use the following formula:

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.2}$$

Note the similarity of (4.2.2) with the continuous-time counterpart in (4.1.3). The proof of (4.2.2) follows immediately from the application of the geometric summation formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \tag{4.2.3}$$

The expression for the Fourier coefficients c_k can be obtained by multiplying both sides of (4.2.1) by the exponential $e^{-j2\pi ln/N}$ and summing the product from $n = 0$ to $n = N - 1$. Thus

$$\sum_{n=0}^{N-1} x(n) e^{-j2\pi ln/N} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j2\pi(k-l)n/N} \tag{4.2.4}$$

If we perform the summation over n first, in the right-hand side of (4.2.4), we obtain

$$\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \begin{cases} N, & k-l = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.5}$$

where we have made use of (4.2.2). Therefore, the right-hand side of (4.2.4) reduces to Nc_l and hence

$$c_l = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi ln/N} \qquad l = 0, 1, \ldots, N-1 \tag{4.2.6}$$

Thus we have the desired expression for the Fourier coefficients in terms of the signal $x(n)$.

4.2.3 The Fourier Transform of Discrete-Time Aperiodic Signals

Just as in the case of continuous-time aperiodic energy signals, the frequency analysis of discrete-time aperiodic finite-energy signals involves a Fourier transform of the time-domain signal. Consequently, the development in this section parallels to a large extent, that given in Section 4.1.3.

The Fourier transform of a finite-energy discrete-time signal $x(n)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{4.2.23}$$

Physically, $X(\omega)$ represents the frequency content of the signal $x(n)$. In other words, $X(\omega)$ is a decomposition of $x(n)$ into its frequency components.

We observe two basic differences between the Fourier transform of a discrete-time finite-energy signal and the Fourier transform of a finite-energy analog signal. First, for continuous-time signals, the Fourier transform, and hence the spectrum of the signal, have a frequency range of $(-\infty, \infty)$. In contrast, the frequency range for a discrete-time signal is unique over the frequency interval of $(-\pi, \pi)$ or, equivalently, $(0, 2\pi)$. This property is reflected in the Fourier transform of the signal. Indeed, $X(\omega)$ is periodic with period 2π, that is,

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned} \tag{4.2.24}$$

Hence $X(\omega)$ is periodic with period 2π. But this property is just a consequence of the fact that the frequency range for any discrete-time signal is limited to $(-\pi, \pi)$ or $(0, 2\pi)$, and any frequency outside this interval is equivalent to a frequency within the interval.

The second basic difference is also a consequence of the discrete-time nature of the signal. Since the signal is discrete in time, the Fourier transform of the signal involves a summation of terms instead of an integral, as in the case of continuous-time signals.

Since $X(\omega)$ is a periodic function of the frequency variable ω, it has a Fourier series expansion, provided that the conditions for the existence of the Fourier series, described previously, are satisfied. In fact, from the definition of the Fourier transform $X(\omega)$ of the sequence $x(n)$, given by (4.2.23), we observe that $X(\omega)$ has the form of a Fourier series. The Fourier coefficients in this series expansion are the values of the sequence $x(n)$.

To demonstrate this point, let us evaluate the sequence $x(n)$ from $X(\omega)$. First, we multiply both sides (4.2.23) by $e^{j\omega m}$ and integrate over the interval $(-\pi, \pi)$. Thus we have

$$\int_{-\pi}^{\pi} X(\omega)e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \right] e^{j\omega m} d\omega \tag{4.2.25}$$

The integral on the right-hand side of (4.2.25) can be evaluated if we can interchange the order of summation and integration. This interchange can be made if the series

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

converges uniformly to $X(\omega)$ as $N \to \infty$. Uniform convergence means that, for every ω, $X_N(\omega) \to X(\omega)$, as $N \to \infty$. The convergence of the Fourier transform is discussed in more detail in the following section. For the moment, let us assume that the series converges uniformly, so that we can interchange the order of summation and integration in (4.2.25). Then

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi x(m), & m = n \\ 0, & m \neq n \end{cases} \tag{4.2.26}$$

By combining (4.2.25) and (4.2.26), we obtain the desired result that

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega \tag{4.2.27}$$

If we compare the integral in (4.2.27) with (4.1.9), we note that this is just the expression for the Fourier series coefficient for a function that is periodic with period 2π. The only difference between (4.1.9) and (4.2.27) is the sign on the exponent in the integrand, which is a consequence of our definition of the Fourier transform as given by (4.2.23). Therefore, the Fourier transform of the sequence $x(n)$, defined by (4.2.23), has the form of a Fourier series expansion.

FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM

Before we introduce the DFT, we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus, we establish the relationship between the sampled Fourier transform and the DFT.

5.1.1 Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

We recall that aperiodic finite-energy signals have continuous spectra. Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.1.1}$$

Suppose that we sample $X(\omega)$ *periodically in frequency* at a spacing of $\delta\omega$ radians between successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \leq \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$, as shown in Fig. 5.1. First, we consider the selection of N, the number of samples in the frequency domain.

If we evaluate (5.1.1) at $\omega = 2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.2}$$

The summation in (5.1.2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus

$$\begin{aligned} X\left(\frac{2\pi}{N}k\right) = {} & \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ & + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\ = {} & \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \end{aligned}$$

If we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the result

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-j2\pi kn/N} \tag{5.1.3}$$

for $k = 0, 1, 2, \ldots, N-1$.

The signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \tag{5.1.4}$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier

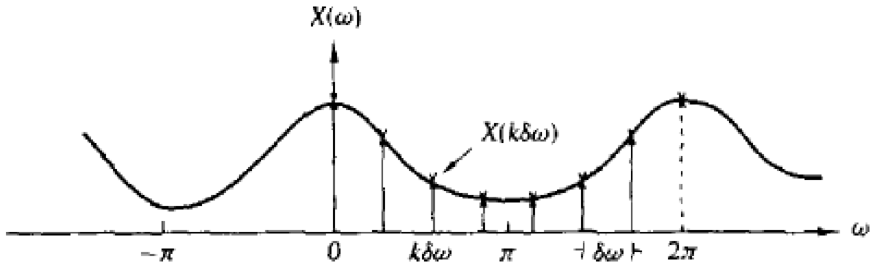

Figure 5.1 Frequency-domain sampling of the Fourier transform.

series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.5}$$

with Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.6}$$

Upon comparing (5.1.3) with (5.1.6), we conclude that

$$c_k = \frac{1}{N}X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \ldots, N-1 \tag{5.1.7}$$

Therefore,

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.8}$$

5.1.2 The Discrete Fourier Transform (DFT)

The development in the preceding section is concerned with the frequency-domain sampling of an aperiodic finite-energy sequence $x(n)$. In general, the equally spaced frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration. Instead, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$, as indicated by the relation in (5.1.4), that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.15}$$

When the sequence $x(n)$ has a finite duration of length $L \leq N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is given as

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases} \tag{5.1.16}$$

Consequently, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, uniquely represent the finite-duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by $N - L$ zeros), the original finite-duration sequence $x(n)$ can be obtained from the frequency samples $\{X(2\pi k/N\}$ by means of the formula (5.1.8).

It is important to note that *zero padding* does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{x(n)\}$. The L equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$ using the reconstruction formula (5.1.13). However, padding the sequence $\{x(n)\}$ with $N - L$ zeros and computing an N-point DFT results in a "better display" of the Fourier transform $X(\omega)$.

In summary, a finite-duration sequence $x(n)$ of length L [i.e., $x(n) = 0$ for $n < 0$ and $n \geq L$] has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \qquad 0 \leq \omega \leq 2\pi \tag{5.1.17}$$

where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \leq n \leq L-1$. When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$, where $N \geq L$, the resultant samples are

$$\begin{aligned} X(k) &\equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \\ X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{5.1.18}$$

where for convenience, the upper index in the sum has been increased from $L - 1$ to $N - 1$ since $x(n) = 0$ for $n \geq L$.

The relation in (5.1.18) is a formula for transforming a sequence $\{x(n)\}$ of length $L \le N$ into a sequence of frequency samples $\{X(k)\}$ of length N. Since the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the relation in (5.1.18) is called the *discrete Fourier transform* (DFT) of $x(n)$. In turn, the relation given by (5.1.10), which allows us to recover the sequence $x(n)$ from the frequency samples

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.19}$$

is called the *inverse DFT* (IDFT). Clearly, when $x(n)$ has length $L < N$, the N-point IDFT yields $x(n) = 0$ for $L \le n \le N-1$. To summarize, the formulas for the DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \tag{5.1.18}$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1 \tag{5.1.19}$$

5.1.3 The DFT as a Linear Transformation

The formulas for the DFT and IDFT given by (5.1.18) and (5.1.19) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.20}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.21}$$

where, by definition,

$$W_N = e^{-j2\pi/N} \tag{5.1.22}$$

which is an Nth root of unity.

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{5.1.24}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. We observe that $\mathbf{W}_N$ is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$ exists, then (5.1.24) can be inverted by premultiplying both sides by $\mathbf{W}_N^{-1}$. Thus we obtain

$$\mathbf{x}_N = \mathbf{W}_N^{-1}\mathbf{X}_N \tag{5.1.25}$$

Relationship to the Fourier series coefficients of a periodic sequence. A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N} \qquad -\infty < n < \infty \tag{5.1.29}$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.30}$$

If we compare (5.1.29) and (5.1.30) with (5.1.18) and (5.1.19), we observe that the formula for the Fourier series coefficients has the form of a DFT. In fact, if we define a sequence $x(n) = x_p(n)$, $0 \le n \le N-1$, the DFT of this sequence is simply

$$X(k) = Nc_k \tag{5.1.31}$$

Furthermore, (5.1.29) has the form of an IDFT. Thus the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

Relationship to the Fourier transform of an aperiodic sequence. We have already shown that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.32}$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.33}$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1.34}$$

bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \leq N$, in which case

$$x(n) = \hat{x}(n) \qquad 0 \leq n \leq N-1 \tag{5.1.35}$$

Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship to the *z*-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5.1.36}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, 0, 1, 2, ..., $N-1$, we obtain

$$\begin{aligned} X(k) &\equiv X(z)|_{z=e^{j2\pi nk/N}} \qquad k = 0, 1, \dots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \end{aligned} \tag{5.1.37}$$

The expression in (5.1.37) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \dots, N-1$, which is the topic treated in Section 5.1.1.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT {X(k)} as follows

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n)z^{-n} \\ X(z) &= \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\right] z^{-n} \\ X(z) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N}z^{-1}\right)^n \\ X(z) &= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi k/N}z^{-1}} \end{aligned} \tag{5.1.38}$$

When evaluated on the unit circle, (5.1.38) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}} \tag{5.1.39}$$

Relationship to the Fourier series coefficients of a continuous-time signal. Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0} \qquad (5.1.40)$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N} \end{aligned} \qquad (5.1.41)$$

It is clear that (5.1.41) is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \qquad (5.1.42)$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN} \qquad (5.1.43)$$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

PROPERTIES OF DFT:

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \circledN x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \circledN y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \circledN X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

LINEAR FILTERING METHODS BASED ON THE DFT

Since the D F T provides a discrete frequency representation o f a finite-duration Sequence in the frequency domain, it is interesting to exp lore its use as a computational tool for linear system analysis and, especially, for linear filtering. We have already established that a system with frequency response *H { w) y* w hen excited with an input signal that has a spectrum possesses an output spectrum.

The output sequence *y(n)* is determined from its spectrum via the inverse Fourier transform. Computationally, the problem with this frequency domain approach is

that are functions o f the continuous variable. As a consequence, the computations cannot be done on a digital computer, since the computer can only store and perform computations on quantities at discrete frequencies.
On the other hand, the DFT does lend itself to computation on a digital computer. In the discussion that follows, we describe how the DFT can be used to perform linear filtering in the frequency domain. In particular, we present a computational procedure that serves as an alternative to time-domain convolution.
In fact, the frequency-domain approach based on the DFT, is computationally m ore efficient than time-domain convolution due to the existence of efficient algorithms for computing the DFT . These algorithms, which are described in Chapter 6, are collectively called fast Fourier transform (FFT) algorithms.

5.3.1 Use of the DFT in Linear Filtering

In the preceding section it was demonstrated that the product of two DFTs is equivalent to the circular convolution of the corresponding time-domain sequences. Unfortunately, circular convolution is of no use to us if our objective is to determine the output of a linear filter to a given input sequence. In this case we seek a frequency-domain methodology equivalent to linear convolution.

Suppose that we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M. Without loss of generality, let

$$x(n) = 0, \qquad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \qquad n < 0 \text{ and } n \geq M$$

where $h(n)$ is the impulse response of the FIR filter.

The output sequence $y(n)$ of the FIR filter can be expressed in the time domain as the convolution of $x(n)$ and $h(n)$, that is

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{5.3.1}$$

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L + M - 1$.

The frequency-domain equivalent to (5.3.1) is

$$Y(\omega) = X(\omega)H(\omega) \tag{5.3.2}$$

If the sequence $y(n)$ is to be represented uniquely in the frequency domain by samples of its spectrum $Y(\omega)$ at a set of discrete frequencies, the number of distinct samples must equal or exceed $L+M-1$. Therefore, a DFT of size $N \geq L+M-1$, is required to represent $\{y(n)\}$ in the frequency domain.

Now if

$$\begin{aligned} Y(k) &\equiv Y(\omega)|_{\omega=2\pi k/N} \qquad & k = 0, 1, \ldots, N-1 \\ &= X(\omega)H(\omega)|_{\omega=2\pi k/N} \qquad & k = 0, 1, \ldots, N-1 \end{aligned}$$

then

$$Y(k) = X(k)H(k) \qquad k = 0, 1, \ldots, N-1 \tag{5.3.3}$$

where $\{X(k)\}$ and $\{H(k)\}$ are the N-point DFTs of the corresponding sequences $x(n)$ and $h(n)$, respectively. Since the sequences $x(n)$ and $h(n)$ have a duration less than N, we simply pad these sequences with zeros to increase their length to N. This increase in the size of the sequences does not alter their spectra $X(\omega)$ and $H(\omega)$, which are continuous spectra, since the sequences are aperiodic. However, by sampling their spectra at N equally spaced points in frequency (computing the N-point DFTs), we have increased the number of samples that represent these sequences in the frequency domain beyond the minimum number (L or M, respectively).

Since the $N = L + M - 1$-point DFT of the output sequence $y(n)$ is sufficient to represent $y(n)$ in the frequency domain, it follows that the multiplication of the N-point DFTs $X(k)$ and $H(k)$, according to (5.3.3), followed by the computation of the N-point IDFT, must yield the sequence $\{y(n)\}$. In turn, this implies that the N-point circular convolution of $x(n)$ with $h(n)$ must be equivalent to the linear convolution of $x(n)$ with $h(n)$. In other words, by increasing the length of the sequences $x(n)$ and $h(n)$ to N points (by appending zeros), and then circularly convolving the resulting sequences, we obtain the same result as would have been obtained with linear convolution. Thus with zero padding, the DFT can be used to perform linear filtering.

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