UNIT-II

* MAGNETOSTATICS

Introduction:

→ An electric field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic field is produced.

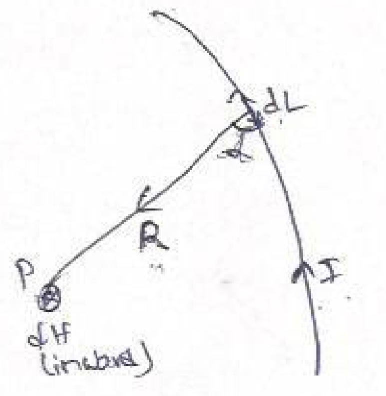
→ A magnetostatic field is produced by a constant current flow (or direct current). This current flow may be due to magnetisation currents as in permanent magnets.

→ Static electric fields characterized by $\vec{E}$ or $\vec{D}$. Similarly Magentic fields characterized by $\vec{H}$ or $\vec{B}$.

where,
- E → Electric field intensity – V/m
- $\vec{D}$ → Electric flux density – C/m²
- $\vec{H}$ → Magnetic field intensity – A/m
- $\vec{B}$ → Magnetic flux density – Web/m² → Tesla

→ As $\vec{E}$ and $\vec{D}$ are related according to $\vec{D} = \varepsilon \vec{E}$,
$\vec{H}$ and $\vec{B}$ are related according to $\vec{B} = \mu \vec{H}$.

Biot - Savart's Law:



Magnetic field dH at P due to current element $Id\vec{L}$

Biot - Savart's law states that the magnetic field intensity dH produced at a point P, by the differential current element $Id\vec{L}$ is propotional to the product $I\,dl$ and the sine of angle α between the element and the line joining P to the element and is inversely propertional to the square of the distance R between p and the element.

i.e.
$$d\vec{H} \propto \frac{I\,dl \sin\alpha}{R^2}$$

$$d\vec{H} = \frac{k\,I\,dl \sin\alpha}{R^2}$$

where k is the constant of proportionality.

In SI units $k = \frac{1}{4\pi}$

$$\therefore\ dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$

But $dl \sin\alpha = d\vec{l} \times \hat{a}_R$

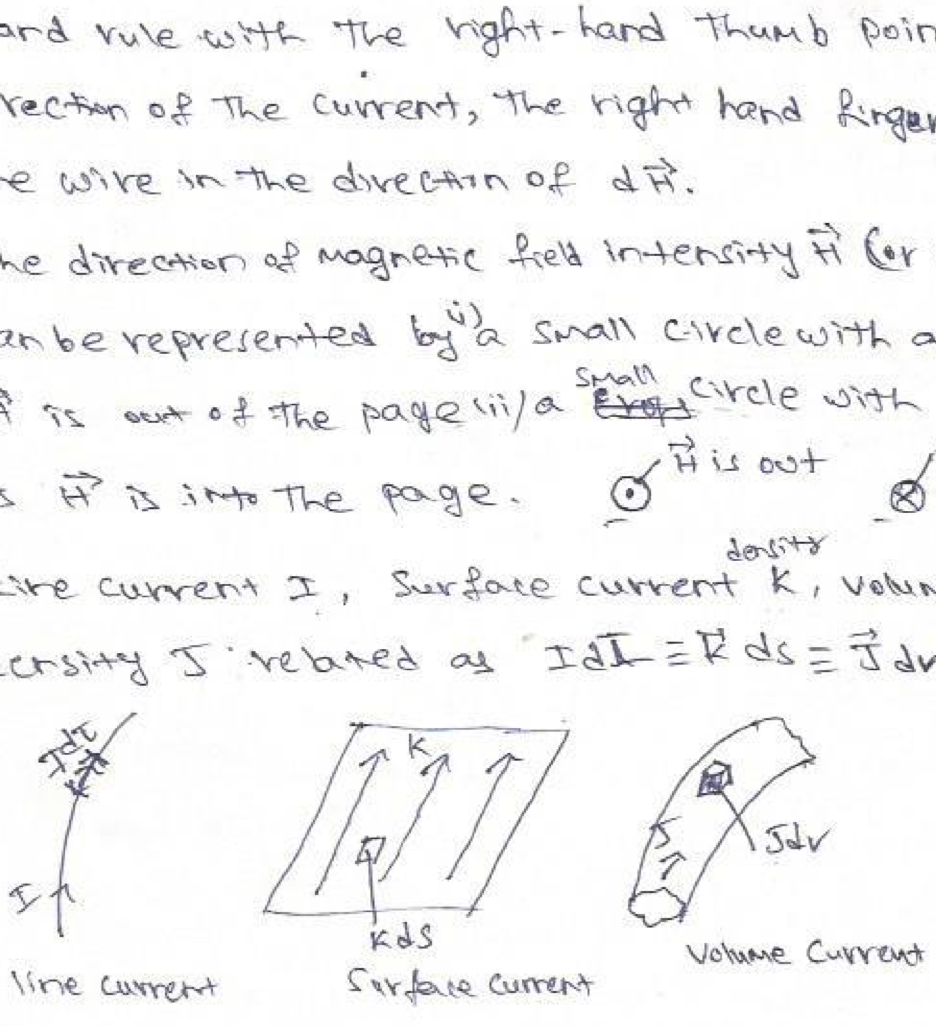
$$dH = \frac{I\,d\vec{l} \times \hat{a}_R}{4\pi R^2}$$

$$\boxed{dH = \frac{I\,d\vec{l} \times R}{4\pi R^3}} \quad \left[\because \hat{a}_R = \frac{\vec{R}}{|R|}\right]$$

→ The direction of $d\vec{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right hand fingers encircling the wire in the direction of $d\vec{H}$.

→ The direction of magnetic field intensity $\vec{H}$ (or current I) can be represented by (i) a small circle with a dot is $\vec{H}$ is out of the page (ii) a small ~~cross~~ circle with a cross is $\vec{H}$ is into the page. $\odot$ $\vec{H}$ is out $\otimes$ $\vec{H}$ is in

→ Line current I, Surface current density K, volume current density J related as $I\,d\vec{l} \equiv \vec{K}\,ds \equiv \vec{J}\,dv$



By using Biot-Savart Law,

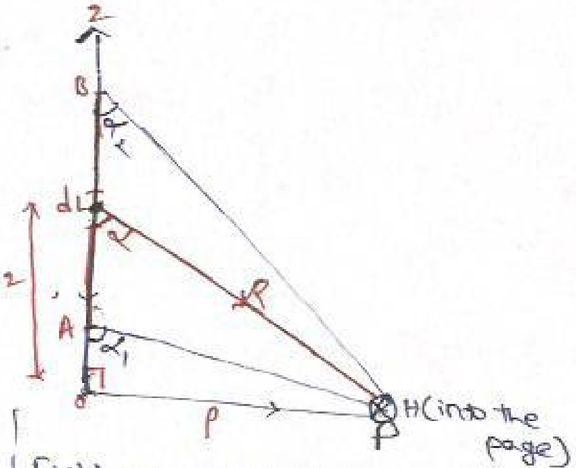
$$\vec{H} = \int_L \frac{I\,d\vec{L} \times \hat{a}_R}{4\pi R^2} \quad \text{(line current)}$$

$$\vec{H} = \int_S \frac{K\,ds \times \hat{a}_R}{4\pi R^2} \quad \text{(Surface current)}$$

$$\vec{H} = \int_V \frac{\vec{J}\,dv \times \hat{a}_R}{4\pi R^2} \quad \text{(Volume current)}$$

Example:-

Consider the field due to a straight current carrying filamentary conductor of finite length AB. Assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles α_2 and α_1 at P, the point at which $\vec{H}$ is to be determined.



Field at point P due to a straight filamentary conductor.

→ The contribution $d\vec{H}$ at P due to an element $d\vec{L}$ at (0, 0, z).

$$d\vec{H} = \frac{I\,d\vec{L} \times \vec{R}}{4\pi R^3}$$

But $d\vec{L} = dz\,a_z$ and $\vec{R} = \rho a_\rho - z a_z$, so,

$$d\vec{L} \times \vec{R} = \rho\,dz\,a_\phi$$

Hence, $$H = \int \frac{I \rho\, dz}{4\pi\,[\rho^2 + z^2]^{3/2}}\, a_\phi$$

→ From the figure, $z = \rho \cot\alpha$, $dz = -\rho \operatorname{cosec}^2\alpha\, d\alpha$,

$$\therefore H = -\frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha\, d\alpha}{[\rho^2 + \rho^2 \cot^2\alpha]^{3/2}}\, a_\phi$$

$$= -\frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha\, d\alpha}{\rho^2\,[1 + \cot^2\alpha]^{3/2}}\, a_\phi$$

$$\vec{H} = \frac{-I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha}{\rho^2 \operatorname{cosec}^3\alpha}\, d\alpha \cdot a_\phi \quad [\because 1+\cot^2\alpha = \operatorname{cosec}^2\alpha]$$

$$= \frac{-I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha \; a_\phi$$

$$= -\frac{I}{4\pi\rho}\left[-\cos\alpha\right]_{\alpha_1}^{\alpha_2} a_\phi$$

$$\boxed{\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] a_\phi}$$

→ $\vec{H}$ is always along the unit vector a_ϕ irrespective of the length of the wire.

→ When the conductor is semiinfinite, so that point A is now at O(0,0,0) while B is at (0, 0, ∞); $\alpha_1 = 90°$, $\alpha_2 = 0°$

Then $\vec{H} = \frac{I}{4\pi\rho} a_\phi$

→ When the conductor is infinite in length, point A is at (0,0,-∞) while B is at (0,0,∞); $\alpha_1 = 180°$, $\alpha_2 = 0°$,

Then $\vec{H} = \frac{I}{2\pi\rho} a_\phi$

→ The unit vector a_ϕ is defined as, $a_\phi = a_\ell \times a_\rho$

where a_ℓ → unit vector along the line current

a_ρ → unit vector along the perpendicular line from the line current to the field point.

Ampere's Circuit Law:-

"Ampere's Circuit law states that the line integral of the tangential component of $\vec{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path". i.e.,

$$\boxed{\oint_L \vec{H} \cdot d\vec{l} = I_{enc}}$$

→ Ampere's law is applied to determine $\vec{H}$ when the current distribution is symmetrical.

By applying stoke's theorem,

$$I_{enc} = \oint_L \bar{H} \cdot d\bar{L} = \int_S (\nabla \times \vec{H}) \cdot ds$$

But,

$$I_{enc} = \int_S \vec{J} \cdot ds$$

By comparing above two equations, we can write,

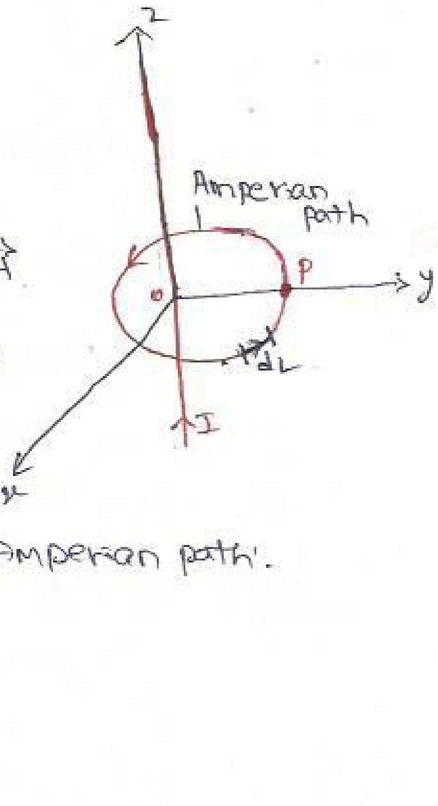
$$\boxed{\nabla \times \vec{H} = \vec{J}}$$

* This is the third Maxwell's equation, it is Ampere's Law in differential (or point) form.

Applications of Ampere's Law:

Ampere's Circuit law is used to determine $\vec{H}$ for symmetrical current distribution.

(i) Infinite Line Current:-



Consider an infinitely long filamentary current I along the z-axis. To determine $\vec{H}$ at an observation point 'P', allow a closed path pass through P. This path, on which Ampere's law is to be applied, is known as an 'Amperian path'.

→ The path encloses the whole current I,

$$I = \oint \bar{H} \cdot d\bar{L} \qquad \vec{H} \to H_\phi a_\phi$$
$$d\vec{L} \to \rho d\phi a_\phi$$

$$I = \oint H_\phi a_\phi \cdot \rho d\phi a_\phi$$
$$= H_\phi \int_{\phi=0}^{2\pi} \rho d\phi$$
$$I = H_\phi 2\pi \rho$$

$$H_\phi = \frac{I}{2\pi\rho}$$

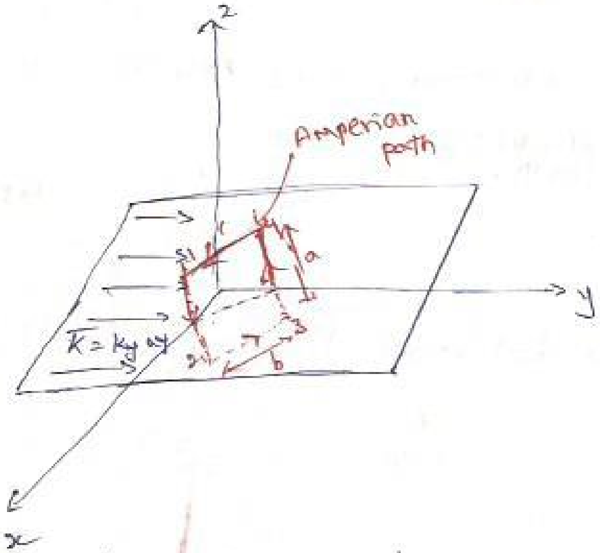
In general, this equation can be written as

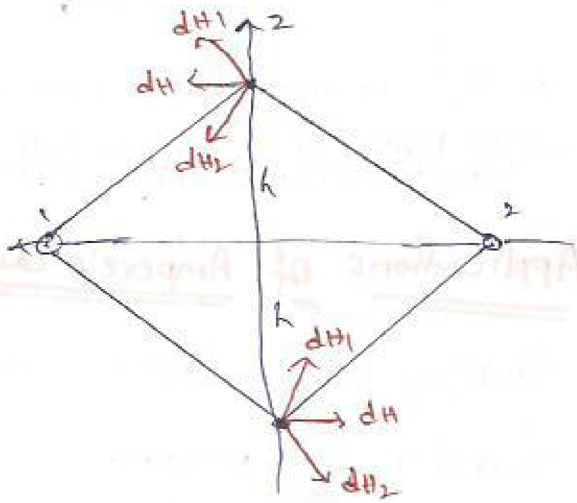
$$\vec{H} = H_\phi a_\phi$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} a_\phi} \rightarrow \text{For infinite line current.}$$

(ii) Infinite sheet of current:

→ Consider an infinite current sheet in the $z=0$ plane. If the sheet has a uniform current density $\vec{K} = K_y a_y$ A/m.





→ Applying Ampere's law to the rectangular closed path 1-2-3-4-1 gives

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = K_y b$$

→ $\vec{H}$ on one side of the sheet is the negative of that on the other side. Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of $\vec{H}$ for a pair are the same for the infinite current sheets, i.e.

$$\vec{H} = \begin{cases} H_0 a_x, & z > 0 \\ -H_0 a_x, & z < 0 \end{cases}$$

where H_0 is to be determined.

→ Evaluating the line integral of $\vec{H}$ along the closed path 1-2-3-4-1 gives,

$$\oint \vec{H} \cdot d\vec{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \vec{H} \cdot d\vec{l}$$

$$= 0(-a) + (-H_0)(-b) + 0(a) + H_0(b) = 2H_0 b$$

$\oint \vec{H} \cdot d\vec{l} = 2H_0 b = K_y b$

$\therefore H_0 = \frac{1}{2} K_y$

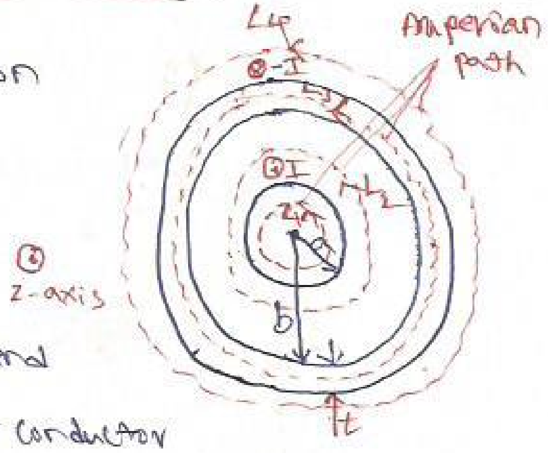
$$\therefore H = \begin{cases} \frac{1}{2} K_y a_x, & z > 0 \\ -\frac{1}{2} K_y a_x, & z < 0 \end{cases}$$

In general, for an infinite sheet of current density $\vec{K}$ A/m

$$\boxed{H = \frac{1}{2} \vec{K} \times \hat{a}_n}$$

where a_n is a unit normal vector directed from the current sheet to the point of interest.

(iii) Infinitely Long Coaxial Transmission Line:



→ Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis.

→ The inner conductor has radius a and carries current I while the outer conductor has inner radius 'b' and thickness 't' and carries return current $-I$.

→ To determine $\vec{H}$ every where assume that the current is uniformly distributed in both conductors.

→ Since the current distribution is symmetrical, we apply Ampere's law along the Amperean path for each of four possible regions: $0 \le \rho \le a$, $a \le \rho \le b$, $b \le \rho \le b+t$, and $\rho \ge b+t$.

→ For region $0 \le \rho \le a$, Apply ampere's law to path L_1:

$$\oint_{L_1} \vec{H} \cdot d\vec{l} = I_{enc} = \int J \cdot ds$$

Since the current is uniformly distributed over the cross section, $\vec{J} = \frac{I}{\pi a^2} a_z$, $ds = \rho \, d\phi \, d\rho \, a_z$

$$\therefore I_{enc} = \int_S J \cdot dS$$

$$= \frac{I}{\pi a^2} \int\int \rho \, d\phi \, d\rho$$

$$= \frac{I}{\pi a^2} (2\pi) \frac{\rho^2}{2}$$

$$I_{enc} = \frac{I \rho^2}{a^2}$$

But $\oint_{L_1} \vec{H} \cdot d\vec{L} = H\phi \int dl$ $\left[\because \int dL = \int_0^{2\pi} \rho d\phi \, a_\phi \right]$

$$= H\phi \, 2\pi \rho \qquad = \rho \, [\phi]_0^{2\pi}$$

$$= \frac{I \cdot \rho^2}{a^2} = I_{enc} \qquad \int dL = 2\pi\rho$$

$$\therefore H\phi = \frac{I\rho}{2\pi a^2}$$

→ For region $a \le \rho \le b$, apply ampere's Law to path L_2,

$$\oint_{L_2} \vec{H} \cdot d\vec{L} = I_{enc} = I$$

$$H\phi \, 2\pi \rho = I$$

$$H\phi = \frac{I}{2\pi\rho}.$$

Since the whole current I is enclosed by L_2.

→ For region $b \le \rho \le b+t$,

$$\oint \vec{H} \cdot d\vec{L} = H\phi \cdot 2\pi\rho = I_{enc}.$$

Where $I_{enc} = I + \int \vec{J} \cdot d\vec{s}$

Where $\vec{J}$ is curren density of the outer conductor and is along $-a_z$ direction.

$$\therefore J = -\frac{I}{\pi\left[(b+t)^2 - b^2\right]} a_z$$

$$I_{enc} = I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

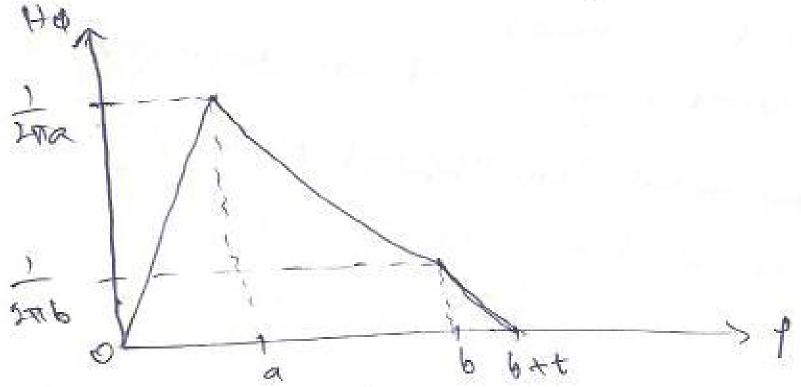
→ For region $\rho \geq b+t$, for path L_4

$$\oint_{L_4} \vec{H} \cdot d\vec{L} = I - I = 0$$

$\vec{H} = H_\phi a_\phi$

$$\underline{H_\phi = 0}$$

$$\vec{H} = \begin{cases} \frac{I\rho}{2\pi a^2} a_\phi & 0 \leq \rho \leq a \\ \frac{I}{2\pi\rho} a_\phi & a \leq \rho \leq b \\ \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] a_\phi & b \leq \rho \leq b+t \\ 0 & \rho \geq b+t \end{cases}$$



Magnetic flux Density (B):-

Maxwell's Equation

→ The magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ according to,

$$\boxed{\vec{B} = \mu_0 \vec{H}}$$

where μ_0 is a constant as the permeability of free space. The constant is in henry/meter (H/m) and has the value of

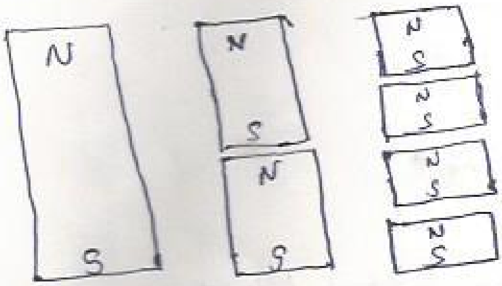
$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ H/m}}$$

The magnetic flux through a Surface S is given by

$$\boxed{\psi = \int_S \vec{B} \cdot d\vec{S}}$$

where the magnetic flux ψ is in webers (Wb) and the magnetic flux density $(\vec{B})$ is in weber/square meter (Wb/m^2) or Tesla.

Ex:-



→ Successive division of a bar magnet results in pieces with north and South poles, magnetic poles cannot be isolated.

→ An isolated magnetic charge does not exist. Thus the total flux through a closed surface in a magnetic field must be zero; i.e.

$$\boxed{\oint_S B \cdot ds = 0}$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields. The magnetostatic field is not conservative, magnetic flux is conserved.

→ By applying the divergence theorem,

$$\oint_S B \cdot ds = \int_v \nabla \cdot B \, dv = 0.$$

or, $\boxed{\nabla \cdot B = 0}$

This equation is the fourth Maxwell's equation.

Maxwell's Equation For Static EM Fields:

Differential or Point form	Integral form	Remarks
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{s} = \int_v \rho_v dv$	Gauss's law
$\nabla \cdot \bar{H} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$	Nonexistance of magnetic monopole
$\nabla \times \bar{E} = 0$	$\oint_L \bar{E} \cdot d\bar{l} = 0$	Conservativeness of electrostatic fields
$\nabla \times \bar{H} = J$	$\oint_L \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{s}$	Ampere's law

Magnetic Scalar and Vector potentials:-

→ In electrostatics, it is seen that there exists a scalar potential V, which is related to the electric field intensity $\bar{E}$ as $\bar{E} = -\nabla V$

→ In case of magnetic fields there are two types of potentials which can be defined:

(i) The scalar magnetic potential denoted as V_m

(ii) The vector magnetic potential denoted as $\vec{A}$

To define scalar and vector magnetic potentials, let us use two vector identities which are, $\nabla \times \nabla V = 0$; $V \to$ scalar

$\nabla \cdot (\nabla \times \bar{A}) = 0$; $\bar{A} \to$ vector

Scalar Magnetic potential:

→ If V_m is the scalar magnetic potential then

$$\nabla \times \nabla V_m = 0$$

The scalar magnetic potential is related to the magnetic field intensity $\bar{H}$ as, $\bar{H} = -\nabla V_m$

$$\therefore \nabla \times (-\bar{H}) = 0 \qquad \text{i.e } \nabla \times \bar{H} = 0$$

But $\nabla \times \bar{H} = J$ $\qquad$ i.e $\bar{J} = 0$

→ Scalar magnetic potential V_m can be defined for source free region where $\bar{J}$ (current density) is zero.

$$\bar{H} = -\nabla V_m \quad \text{only for } \bar{J} = 0.$$

→ Magnetic scalar potential can be expressed interms of $\bar{H}$ as,

$$V_{m,a,b} = -\int_b^a \bar{H} \cdot d\bar{L} \ldots \text{ Specified path}$$

Laplace's Equation for scalar magnetic potential:

Magnetic flux in a closed path is zero.

$$\psi = 0$$

$$\oint \bar{B} \cdot d\bar{s} = 0$$

By using Divergence theorem,

$$\oint \bar{B} \cdot d\bar{s} = \int_v (\nabla \cdot \bar{B})\, dv = 0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot (\mu_0 \bar{H}) = 0 \quad [\because \bar{B} = \mu_0 \bar{H}]$$

where $\mu_0 \neq 0$,

$$\nabla \cdot \bar{H} = 0$$

$$\nabla \cdot (-\nabla V_m) = 0$$

$$\boxed{\nabla^2 V_m = 0} \quad \text{for } \bar{J} = 0$$

This is Laplace's equation for scalar magnetic potential.

Vector Magnetic potential:

The vector magnetic potential is denoted as $\bar{A}$ and measured in Wb/m.

$$\therefore \nabla \cdot (\nabla \times \bar{A}) = 0 \qquad \bar{A} \rightarrow \text{Vector magnetic potential}$$

We know that $\nabla \cdot \bar{B} = 0$ $\qquad \bar{B} \rightarrow$ magnetic flux density.

$$\therefore \boxed{\bar{B} = \nabla \times \bar{A}}$$

Curl of vector magnetic potential is The flux density

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \times \frac{\bar{B}}{\mu_0} = \bar{J}$$

$$\nabla \times \bar{B} = \mu_0 \bar{J}$$

$$\nabla \times \nabla \times \bar{A} = \mu_0 \bar{J}$$

Using vector identity to $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$$J = \frac{1}{\mu_0} \left[\nabla(\nabla \cdot A) - \nabla^2 \bar{A}\right]$$

For defining $\vec{A}$ the current density need not be zero.

Poisson's Equation for Magnetic field :-

W.K.T for a vector magnetic field $\vec{A}$,

$$\nabla \times \vec{A} = \bar{B}$$

But to completely define $\bar{A}$ its divergence must be known.

Assume $\nabla \cdot \bar{A} = 0$.

Then $\bar{J} = \frac{1}{\mu_0} \left[-\nabla^2 \bar{A}\right]$

$$\boxed{\nabla^2 \bar{A} = -\mu_0 \bar{J}}$$

This is the possion's equation for magneto static fields

$\bar{A}$ due to Differential Current Element:

→ Consider The differential element dL carring current I. Then according to Biot-Savart law The vector magnetic potential $\bar{A}$ at a distance R from the differential current element is given by,

$$\bar{A} = \oint \frac{\mu_0 I d\bar{L}}{4\pi R} \text{ wb/m for line current}$$

$$\bar{A} = \oint_S \frac{\mu_0 \bar{K} dS}{4\pi R} \text{ wb/m for surface current}$$

$$\bar{A} = \oint_V \frac{\mu_0 \bar{J} dv}{4\pi R} \text{ wb/m for volume current.}$$

Forces due to Magnetic field:

→ There are at least three ways in which force due to magnetic field can be experienced. The force can be,

(a) due to a moving charged particle in a $\bar{B}$ field,

(b) on a current element in an external $\bar{B}$ field,

(c) Between two current elements.

A) Force on a Charged Particle:-

→ The electric force F_e on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity $\vec{E}$ by

$$\bar{F}_e = Q\bar{E} \longrightarrow (1)$$

if Q is positive, $\bar{F}_e$ and $\bar{E}$ have the same direction.

→ A magnetic field can exert force only on a moving charge. The magnetic force F_m experienced by a charge Q moving with a velocity $\bar{u}$ in a magnetic field $\bar{B}$ is,

$$\bar{F}_m = Q\bar{u} \times \bar{B} \longrightarrow (2)$$

F_m is perpendicular to both $\bar{u}$ and $\bar{B}$

→ F_e is independent of the velocity of the charge and can perform work on the charge, F_m depends on the charge velocity and is normal to it. F_m cannot perform work.

→ For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\bar{F} = \bar{F}_e + \bar{F}_m$$

$$= Q\bar{E} + Q(\bar{u} \times \bar{B})$$

$$\boxed{\bar{F} = Q(\bar{E} + \bar{u} \times \bar{B})} \longrightarrow (3)$$

This is known as the Lorentz force equation.

→ If the mass of the charged particle moving in $\bar{E}$ and $\bar{B}$ fields is m, Newton's second law of motion,

$$\bar{F} = m\frac{d\bar{u}}{dt} = Q(\bar{E} + \bar{u} \times \bar{B})$$

This eq. is used in determining the motion of charged particles in $\bar{E}$ and $\bar{B}$ fields.

Force on a charged particle

State of particle	E-field	B field	E & B - field
Stationary	$Q\bar{E}$	–	$Q\bar{E}$
Moving	$Q\bar{E}$	$Q\bar{u}\times\bar{B}$	$Q(\bar{E}+\bar{u}\times\bar{B})$

(B) Force on a current element:

→ To determine the force on a current element $I d\bar{l}$ of a current-carrying conductor due to the magnetic field $\bar{B}$.

the conduction current is,

$$\bar{J} = \rho_v \bar{u}$$

W.K.T relationship between current elements:

$$I d\bar{l} = \bar{K}\, ds = \bar{J}\, dv$$

$$\therefore\ I d\bar{l} = \rho_v \bar{u}\, dv$$

$$\boxed{I d\bar{l} = dQ\bar{u}}$$

An elemental charge dQ moving with velocity $\bar{u}$ is equivalent to a conduction current element $I\, d\bar{l}$.

$$\bar{F} = Q\bar{u}\times\bar{B}$$

$$d\bar{F} = dQ\,\bar{u}\times\bar{B}$$

$$d\bar{F} = I d\bar{l}\times\bar{B}$$

If the current is through a closed path L or circuit,

The force on the circuit is, $\boxed{\bar{F} = \oint_L I d\bar{l}\times\bar{B}}$

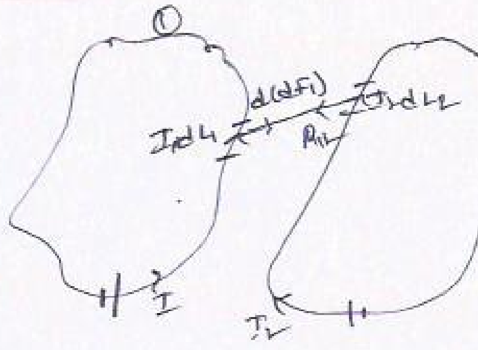
Similarly for surface current element or a volume current element,

$$d\bar{F} = \bar{K}\, ds\times\bar{B} \quad \text{or} \quad d\bar{F} = \bar{J}\, dv\times\bar{B}$$

$$\bar{F} = \int_S \bar{K}\, ds\times\bar{B} \quad \text{or} \quad \bar{F} = \int_v \bar{J}\, dv\times\bar{B}$$

The magnetic field $\bar{B}$ is defined as the force per unit current element.

(c) Force Between Two Current Elements:



Force between two current loops.

→ Let us consider the force b/w two elements $I_1 d\bar{L}_1$ and $I_2 d\bar{L}_2$. According to Biot-Savart's law, both current elements produce magnetic fields.

→ Now find the force $d(dF_1)$ on element $I_1 d\bar{L}_1$ due to the field dB_2 produced by element $I_2 d\bar{L}_2$,

$$d(d\bar{F}_1) = I_1 d\bar{L}_1 \times dB_2$$

But from Biot-Savart's law,

$$d\bar{B}_2 = \frac{\mu_0 I_2 d\bar{L}_2 \times a_{R_{21}}}{4\pi R_{21}^2}$$

Hence

$$d(dF_1) = \frac{\mu_0 I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times a_{R_{21}})}{4\pi R_{21}^2}$$

The total Force F_1 on current loop 1 due to current loop 2

$$F_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \hat{a}_{R_{21}})}{R_{21}^2}$$

The force F_2 on loop 2 due to the magnetic field B_1 from loop 1 is obtained as $F_2 = -F_1$.

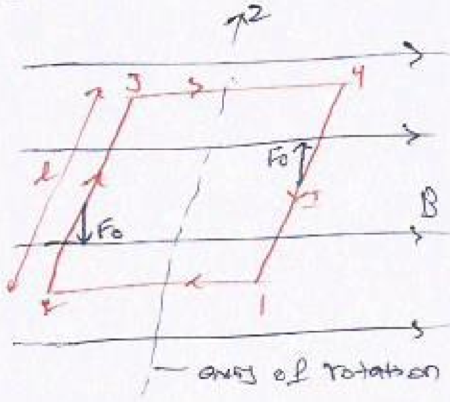
Magnetic Torque and Moment :-

→ The torque $\bar{T}$ (or mechanical moment of force) on the loop is the vector product of the $\bar{F}$ and the moment arm $\bar{r}$.

i.e. $\bar{T} = \bar{r} \times \bar{F}$

units are N-M.

→ Let us apply this to rectangular loop of length l and width w placed in a uniform magnetic field $\bar{B}$.



(9)

The $d\bar{l}$ is parallel $\bar{B}$ along sides 12 and 34 of the loop and no force is exerted on those sides.

$$\bar{F} = I\int_{2}^{3} d\bar{l}\times\bar{B} + I\int_{4}^{1} d\bar{l}\times\bar{B}$$

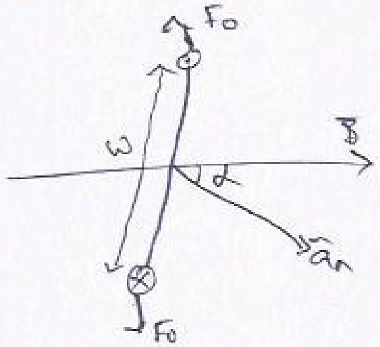
$$= I\int_{0}^{l} dz\,\hat{a}_z\times\bar{B} + I\int_{l}^{0} dz\,\hat{a}_z\times\bar{B}$$

$$= F_0 - F_0$$

$$\bar{F} = 0$$

where $|F_0| = IBl$ because $\bar{B}$ is uniform.

Thus, no force is exerted on the loop as a whole.



→ If the normal to the plane of the makes angle α with $\bar{B}$, the torque on the loop is,

$$|\bar{T}| = |\bar{F_0}|\,w\sin\alpha$$

$$T = BIlw\sin\alpha$$

$lw = S$, the area of the loop. Hence,

$$T = BIS\sin\alpha$$

→ Let us define a new quantity,

$$\boxed{m = IS a_n}$$

is called magnetic dipole moment (in A/m^2) of the loop. a_n is a unit normal vector to the plane of the loop.

→ The magnetic dipole moment is the product of current and area of the loop; its direction is normal to the loop.

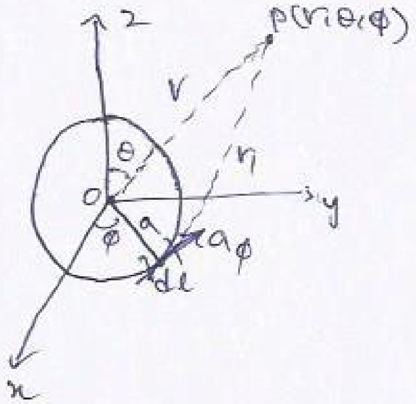
$\therefore$ Torque $\boxed{\bar{T} = \bar{m}\times\bar{B}}$

Magnetic Dipole:

A bar magnet or a small filamentary current loop is usually referred to as a ~~message~~ magnetic Dipole.

Let us determine the magnetic field $\bar{B}$ at an observation point $P(r,\theta,\phi)$ due to a circular loop carrying current I.

The magnetic vector potential at P is



$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\bar{l}}{r_1}$$

Let the differential length at $(a,\pi/2,\phi)$

$$d\bar{l} = a \sin\phi \, d\phi \, \hat{a}_\phi$$

Then $$\vec{A} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a \sin\phi}{r_1} d\phi \, \hat{a}_\phi$$

From the figure,

$$r_1^2 = r^2 + a^2 - 2ar \sin\theta \sin\phi_1$$

$$r_1^2 = r^2 \left(1 + \frac{a^2}{r^2} - \frac{2a}{r} \sin\theta \sin\phi_1\right)$$

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a^2}{r^2} - \frac{2a}{r} \sin\theta \sin\phi_1\right)^{-1/2}$$

Since $r >> a$, using series expansion

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a}{r} \sin\theta \sin\phi_1 + \text{higher oder terms}\right)$$

Neglecting higher order terms,

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a}{r} \sin\theta \sin\phi_1\right)$$

$$\therefore \vec{A} = \frac{\mu_0 I a \hat{a}_\phi}{4\pi r} \int_0^{2\pi} \left(1 + \frac{a}{r} \sin\theta \sin\phi\right) \sin\phi \, d\phi$$

$$= \frac{\mu_0 I a \hat{a}_\phi}{4\pi r} \left[\int_0^{2\pi} \sin\phi \, d\phi + \frac{a \sin\theta}{r} \int_0^{2\pi} \sin^2\phi \, d\phi\right]$$

$$= \frac{\mu_0 I a}{4\pi r} \hat{a}_\phi \left(0 + \frac{a \sin\theta}{r}\left(\frac{2\pi}{2}\right) + 1\right)$$

$$\vec{A} = \frac{\mu_0 I \pi a^2 \sin\theta \, \hat{a}_\phi}{4\pi r^2}$$

or

$$\boxed{A = \frac{\mu_0 m \times a_r}{4\pi r^2}}$$ where $m = I\pi r^2 \hat{a}_r$

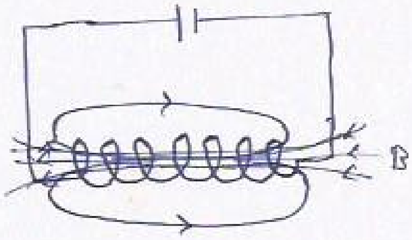
$\because \hat{a}_z \times \hat{a}_r = \sin\theta \, a_\phi$

The magnetic flux density $\vec{B} = \nabla \times \vec{A}$

$$\boxed{B = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \, \hat{a}_r + \sin\theta \, a_\theta)}$$

Inductors and Inductances:

A circuit carrying current I produces a magnetic field $\bar{B}$ which causes a flux $\psi = \int \bar{B} \cdot d\bar{s}$ to pass through each turn of the circuit.



If the circuit has N identical turns the flux linkage, λ is

$$\lambda = N\psi$$

→ If the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it.

$$\lambda \propto I$$

$$\lambda = LI \qquad L = \frac{\lambda}{I}$$

Where L is a constant of proportionality called the inductance of the circuit.

→ A circuit or a part of a circuit that has inductance is called an inductor.

Inductance of an inductor is the ratio of the magnetic flux linkage λ to the current I through the inductor;

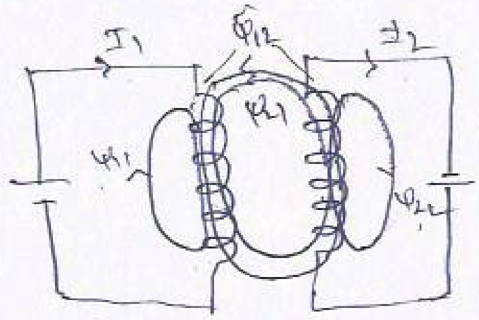
$$\boxed{L = \frac{\lambda}{I} = \frac{N\psi}{I}}$$ Units henry(H), wb/A

This inductance commonly referred as self-inductance since the linkages are produced by the inductor itself.

→ The magnetic energy stored in an inductor is,

$$W_m = \frac{1}{2} L I^2.$$

$$L = \frac{2W_m}{I^2}$$



→ Now, Let us consider two circuits carrying current I_1 and I_2. Four components of fluxes ψ_{11}, ψ_{12}, ψ_{21}, ψ_{22} are produced.

The flux ψ_{12}, is the flux passing through circuit 1, due to current I_2 in circuit 2, If B_2 in the field due to I_2 and S_1 is the area of circuit 1, Then

$$\psi_{12} = \int_{S_2} B_2 . ds$$

→ The mutual inductance M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1 \psi_{12}$ on circuit 1 to current I_2,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2}$$

Similarly $M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1}$

If the surrounding medium is linear $M_{12} = M_{21}$

→ Self inductance of circuit 1 & 2 are,

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \psi_1}{I_1} \quad \& \quad L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \psi_2}{I_2}$$

where $\psi_1 = \psi_{11} + \psi_{12}$ & $\psi_2 = \psi_{21} + \psi_{22}$

The total energy in the magnetic field is,

$$W_m = W_1 + W_2 + W_3 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2 //$$

Magnetic Energy:

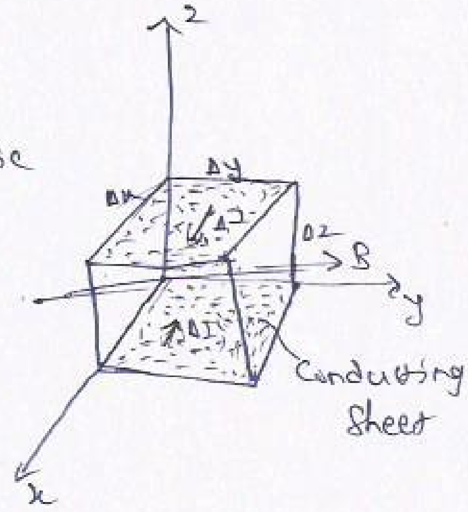
→ The potential energy in an electrostatic field,

$$W_E = \frac{1}{2}\int \bar{D}.\bar{E}\, dv = \frac{1}{2}\int \varepsilon E^2 dv \quad [\because \bar{D} = \varepsilon \bar{E}]$$

→ Similarly in magnetic fields the energy can be expressed as

$$W_m = \frac{1}{2} L I^2$$

The energy is stored in the magnetic field B of the inductor.



→ Consider defferential volume in a magnetic field, Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI.

Each volume has an inductance

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{\mu H\, \Delta x\, \Delta z}{\Delta I}$$

where $\Delta I = H \Delta y$.

then
$$\Delta W_m = \frac{1}{2} \Delta L\, \Delta I^2$$
$$= \frac{1}{2} \mu H^2 \Delta x\, \Delta y\, \Delta z$$
$$\Delta W_m = \frac{1}{2} \mu H^2 \Delta v$$

The magneto static energy density w_m (in J/m³) is defined as

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2$$

hence,
$$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\bar{B}.\bar{H} = \frac{B^2}{2\mu}$$

Thus the energy in a magnetic field in a linear medium is,

$$W_m = \int w_m dv$$

$$\boxed{W_m = \frac{1}{2}\int \bar{B}.\bar{H} dv = \frac{1}{2}\int \mu H^2 dv}$$