Unit - I
Special Functions

Introduction:

Special functions are perticular mathematical functions occur in mathematical analysis, physics and in many other applications. In particular, elementary functions are also considered as special functions.

Algebraic functions like polynomial, rational and irrational functions and Transcendental functions like Trigonometric, Inverse Trigonometric, logarithmic, Exponential and Hyperbolic functions taken together constitute the elementary functions.

All functions other than the elementary fuctions are called "Special Functions"

Special Functions include: Beta, Gamma, Bessel, legen and Error functions; chebyshev, Hermite and laguerre polynomials; Sine and Exponential integrals etc.

Many integrals can be evaluated easily by Expre them interms of Beta and Gamma functions.

GAMMA FUNCTION

Def: The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the "Gamma function" and is denoted by $\Gamma(n)$. The integral con only for $n>0$.

Thus, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ where $n>0$

→ Gamma function is also called Eulerian Integral of the Second Kind and an Improper integral of 3^{rd} kind

⇒ The integral $\int_0^\infty e^{-x} x^{n-1} dx$ does not converge if $n \leq 0$

Ex: 1) $\Gamma(n-1) = \int_0^\infty e^{-x} x^{n-1-1} dx$

$= \int_0^\infty e^{-x} x^{n-2} dx$

2) $\Gamma(n-2) = \int_0^\infty e^{-x} x^{n-3} dx$

Properties of Gamma function:

⇒ (1) To show that $\Gamma(1) = 1$

Proof: By the def^n of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \longrightarrow (1)$$

Put $n=1$ in Eq (1), we get

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx$$

$$= \int_0^\infty e^{-x} x^0 dx$$

$$= \int_0^\infty e^{-x} dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty \qquad \left[\because e^{\infty} = \infty, \; e^{-\infty} = 0\right.$$

$$= \frac{e^{-\infty}}{-1} - \frac{e^0}{-1}$$

$$= 0 + \frac{1}{1}$$

$$\Gamma(1) = 1$$

(2)

⇒ (2) To show that $\Gamma(n) = (n-1)\Gamma(n-1)$, where $n>1$

(or) State and prove reduction formula for Gamma function.

<u>proof:</u> By the def^n of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \longrightarrow (1)$$

$$= \left[x^{n-1} \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty (n-1) x^{n-2} \left[\frac{e^{-x}}{-1}\right] dx \quad \left[\because \int uv = uv_1 - \int u' v_1 \, dx\right]$$

$$= 0 + \int_0^\infty (n-1) x^{n-2} e^{-x} dx \quad \left[(\infty)^{n-1} \frac{e^{-\infty}}{-1} - (0)^{n-1} \frac{e^{-0}}{-1}\right]$$

$$= (0-0) = 0$$

$$(\because e^{-\infty} = 0)$$

$$= (n-1) \int e^{-x} x^{n-2} dx$$

$$= (n-1)\Gamma(n-1), \ n>1 \longrightarrow (2) \quad \left(\because \Gamma(n-1) = \int_0^\infty e^{-x} x^{n-2} dx\right)$$

And hence $\Gamma(n+1)$,

Put $n = n+1$ in Eq (2), we get

$$\Gamma(n+1) = (n+1-1)\Gamma(n+1-1)$$

$$\Gamma(n+1) = n\Gamma(n) \longrightarrow (3)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

The relation Eq (3) is called the fundemental ~~recu~~ recurrence relation satisfied by Gamma function.

⇒ If n is a non-negative integer, i.e a positive integer then $\Gamma(n+1) = n!$

<u>proof:</u> By the reduction formula of Gamma function, we have

$\Gamma(n+1) = n\,\Gamma(n)$, by using this

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)(n-2)\,\Gamma(n-2)$$
$$= n(n-1)(n-2)(n-3)\,\Gamma(n-4)$$
$$= n(n-1)(n-2)(n-3)\cdots 3.2.1\,\Gamma(1)$$
$$= n(n-1)(n-2)(n-3)\cdots 3.2.1 \quad [\because \Gamma(1) = 1$$
$$= n!$$

$\therefore \Gamma(n+1) = n!$

Thus $\Gamma(n+1) = n!\,,\ n \in Z^+$

$\Rightarrow \Gamma(n) = (n-1)!$

This shows that the Gamma function can be regarded as a generalization of the elementary factorial function.

Ex: $\Gamma(8) = (8-1)! = 7!$

$\Gamma(7+1) = 7!$

Result - ①: If n is a positive fraction, then

$$\Gamma(n) = (n-1)(n-2)\cdots(n-r)\,\Gamma(n-r) \text{ where } (n-r) > 0$$

Proof: $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$$= (n-1)(n-2)\,\Gamma(n-2)$$
$$= (n-1)(n-2)(n-3)\,\Gamma(n-3)$$

$= (n-1)(n-2)(n-3)(n-4)\Gamma(n-4)$

$\vdots$

$= (n-1)(n-2)(n-3)\ldots\ldots(n-2)\Gamma(n-2)$

This process is continued until the factor remains positive.

Ex: $\Gamma\left(\frac{13}{3}\right) = \left(\frac{13}{3}-1\right)\Gamma\left(\frac{13}{3}-1\right)$

$= \frac{10}{3}\Gamma\left(\frac{10}{3}\right)$

$= \frac{10}{3}\left(\frac{10}{3}-1\right)\Gamma\left(\frac{10}{3}-1\right)$

$= \left(\frac{10}{3}\right)\left(\frac{7}{3}\right)\Gamma\left(\frac{7}{3}\right)$

$= \left(\frac{10}{3}\right)\left(\frac{7}{3}\right)\left(\frac{7}{3}-1\right)\Gamma\left(\frac{7}{3}-1\right)$

$= \left(\frac{10}{3}\right)\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\Gamma\left(\frac{4}{3}\right)$

$= \left(\frac{10}{3}\right)\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}-1\right)\Gamma\left(\frac{4}{3}-1\right)$

$\Gamma\left(\frac{13}{3}\right) = \left(\frac{10}{3}\right)\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)$

$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$

Result-2 If n is a negative fraction, then

$$\Gamma(n) = \frac{\Gamma(n+k+1)}{n(n+1)(n+2)\ldots\ldots(n+k)} \quad \text{where } n \neq 0, -1, -2, \ldots -k$$

Note:

$\Rightarrow$ $\Gamma(n)$ is defined when $n>0$

$\Rightarrow$ $\Gamma(n)$ is defined when "n" is a negative fraction

$\Rightarrow \Gamma(n)$ is not defined when $n=0$ and 'n' is a -ve integer i.e $n = -1, -2, -3 \ldots$

$\Rightarrow$ The complete definition of the Gamma-function for all real values of n except when n is zero or a negative integer is

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad (n>0)$$
$$= \frac{\Gamma(n+1)}{n} \quad (n \neq 0, -1, -2 \ldots)$$

$\Rightarrow$ where K is a least +ve integer for which $(n+k+1)>0$

Other forms of Gamma function:

$\Rightarrow$ To Show $\Gamma(n) = k^n \int_0^\infty e^{-ky} y^{n-1} dy$, $n>0$, $k>0$

Proof: By the defination of gamma fuction, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n>0 \longrightarrow (1)$$

put $x = ky \Rightarrow y = \frac{x}{k}$

$\Rightarrow dx = k\,dy$

Limits: if $x=0 \Rightarrow y=0$

if $x=\infty \Rightarrow y=\infty$

Then Eq (1) $\Rightarrow \Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1} k\,dy$

$$= \int_0^\infty e^{-ky} k^{n-1} y^{n-1} k\,dy$$

$$\boxed{\Gamma(n) = k^n \int_0^\infty e^{-ky} y^{n-1} dy}$$

$$\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma(n)}{k^n} \quad \text{where } n>0,\ k>0$$

② To show . $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}} dy$, $n>0$

and hence prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof: By the def of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \; n>0 \longrightarrow ①$$

Put $x^n = y$ (or) $x = y^{\frac{1}{n}}$

$$n x^{n-1} dx = dy \quad , \; dx = \frac{1}{n} y^{\frac{1}{n}-1}$$

$$x^{n-1} dx = \frac{1}{n} dy \;, \quad dx = \frac{1}{n} y^{\frac{1-n}{n}} dy$$

Limits: if $x = 0 \Rightarrow y = 0$

if $x = \infty \Rightarrow y = \infty$

then Eq① $\Rightarrow = \int_0^\infty e^{-y^{\frac{1}{n}}} \left(y^{\frac{1}{n}}\right)^{n-1} \frac{1}{n} y^{\frac{1-n}{n}} dy$

$$= \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}} y^{\frac{n-1}{n}} y^{-\left(\frac{n-1}{n}\right)} dy$$

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}} dy \longrightarrow ②$$

$$\int_0^\infty e^{-y^{\frac{1}{n}}} dy = n\Gamma(n)$$

Hence to, prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

put $n = \frac{1}{2}$ in Eq②, then

$$\Gamma(\tfrac{1}{2}) = \frac{1}{(\frac{1}{2})}\int_0^\infty e^{-y^{1/\frac{1}{2}}} dy$$

$$= 2\int_0^\infty e^{-y^2} dy$$

$$= 2\left(\frac{1}{2}\sqrt{\pi}\right) \quad \left[\because \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}\right]$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

⇒ To show $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy,\ n>0$

Proof: By the def^n of gamma fuction, we have

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx,\ n>0 \longrightarrow (1)$$

put $e^{-x} = y \Rightarrow -x = \log y$

$$x = \log y^{-1}$$

$$x = \log\frac{1}{y}$$

$$dx = \frac{1}{\left(\frac{1}{y}\right)}\left(-\frac{1}{y^2}\right) dy$$

$$= y\left(-\frac{1}{y^2}\right) dy$$

$$dx = \frac{-1}{y} dy$$

Limits: if $x = 0 \Rightarrow y = 1$

if $x = \infty \Rightarrow y = 0$

$$\text{Eq}(1) \Rightarrow \Gamma(n) = \int_1^0 y\left(\log\frac{1}{y}\right)^{n-1}\left(\frac{-1}{y}\right) dy$$

$$\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy,\ n>0$$

⇒ To show $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

Complete definition of gamma function:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \quad n>0$$

$$= \frac{\Gamma(n+1)}{n}, \quad n \neq 0, -1, -2, \ldots$$

Note: (1) Gamma function is defined for $n>0$

2) Gamma function is defined for negative fraction

3) Gamma function is undefined for $n=0$ & $n=$ -ve integers

Problems:

⇒ S.T $\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$ where $n>0$, $k>0$

Sol: Consider L.H.S $= \int_0^\infty e^{-kx} x^{n-1}\,dx$

put $kx = t \Rightarrow x = \frac{t}{k}$

$dx = \frac{1}{k}dt$

limits: $kx = t$

if $x=0 \Rightarrow t=0$

if $x=\infty \Rightarrow t=\infty$

∴ t limits are 0 to ∞

$$\therefore \int_0^\infty e^{-kx} x^{n-1}\,dx = \int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{n-1}\frac{1}{k}\,dt$$

$$= \frac{1}{k^n}\int_0^\infty e^{-t} t^{n-1}\,dt \quad \left(\because \int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma(n)\right)$$

$$= \frac{1}{k^n}\Gamma(n)$$

(2) S.T $\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\Gamma(n)$ where $n>0$

Sol: Consider L.H.S $= \int_0^\infty e^{-x^{\frac{1}{n}}}\,dx$

put $x^{\frac{1}{n}} = t$

$x = t^n$

$dx = n t^{n-1}\,dt$

Limits: $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore$ t limits are 0 to ∞

$$\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = \int_0^\infty e^{-t}\, n\, t^{n-1}\,dt$$

$$= n\int_0^\infty e^{-t} t^{n-1}\,dt$$

$$\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\,\Gamma(n) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx\right.$$

(3) Evaluate $\int_0^\infty e^{-4x} x^{\frac{3}{2}}\,dx$

Sol: Given that $\int_0^\infty e^{-4x} x^{\frac{3}{2}}\,dx$

put $4x = t \Rightarrow x = \frac{t}{4}$

$dx = \frac{1}{4}\,dt$

Limits: $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty e^{-4x} x^{\frac{3}{2}}\,dx = \int_0^\infty e^{-t}\left(\frac{t}{4}\right)^{\frac{3}{2}} \frac{1}{4}\,dt$$

$$= \frac{1}{4^{\frac{3}{2}}.4}\int_0^\infty e^{-t}\, t^{\frac{3}{2}}\,dt$$

$$= \frac{1}{4^{\frac{5}{2}}}\int_0^\infty e^{-t}\, t^{\frac{5}{2}-1}\,dt$$

$$= \frac{1}{(2^2)^{\frac{5}{2}}} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{2^5} \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{32} \cdot \frac{3}{4} \sqrt{\pi}$$

$$= \frac{3}{128} \sqrt{\pi}$$

H.W

⇒ (4) Evaluate $\int_0^\infty e^{-2x} x^6 \, dx = \frac{1}{128} \Gamma(7) = \frac{1}{128} \cdot 720 = \frac{45}{8}$

Hint: put $2x = t$

H.W

⇒ (5) Evaluate $\int_0^\infty t^{-\frac{3}{2}} (1 - e^{-t}) \, dt$

Sol: $\int_0^\infty t^{-\frac{3}{2}} (1 - e^{-t}) dt = \int_0^\infty t^{-\frac{3}{2}} dt - \int_0^\infty e^{-t} t^{-\frac{3}{2}} dt$

$$= \left[(1 - e^{-t}) \left(\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right) - \int (0 + e^{-t}) \left[\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right] dt \right]_0^\infty$$

$$= (0 - 0) - \int_0^\infty e^{-t} \left(\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} \right) dt$$

$$= 2 \int_0^\infty e^{-t} \, t^{\frac{1}{2} - 1} \, dt$$

$$= 2 \Gamma\left(\frac{1}{2}\right)$$

$$= 2\sqrt{\pi}$$

(6) Evaluate $\int_0^\infty a^{-bx^2} dx$ put $a^{-bx^2} = e^{-t}$

Sol: $\int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{\log a^{-bx^2}} dx$ $\log(a^{-bx^2}) = \log(e^{-t})$

$= \int_0^\infty e^{-bx^2 \log a} dx$ $+bx^2 \log a = +t$

$x^2 = \dfrac{t}{b \log a}$

Put $bx^2 \log a = t$

$x^2 = \dfrac{t}{b\log a} \Rightarrow x = \sqrt{\dfrac{t}{b \log a}}$

$dx = \dfrac{1}{\sqrt{b \log a}} \dfrac{1}{2\sqrt{t}} dt$ $x = \sqrt{\dfrac{t}{b\log a}}$

$= \dfrac{1}{2\sqrt{b \log a}} t^{-\frac{1}{2}} dt$ $dx = \dfrac{1}{\sqrt{b\log a}} \dfrac{1}{2\sqrt{t}} dt$

Limits: $t = bx^2 \log a$

if $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-bx^2 \log a} dx = \int_0^\infty e^{-t} \dfrac{1}{2\sqrt{b \log a}} t^{-\frac{1}{2}} dt$

$= \dfrac{1}{2\sqrt{b \log a}} \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt$

$= \dfrac{1}{2\sqrt{b \log a}} \Gamma\left(\frac{1}{2}\right)$

$= \dfrac{1}{2\sqrt{b \log a}} \sqrt{\pi}$

(7) Evaluate $\int_0^\infty 3^{-4x^2} dx$ $\Rightarrow$ put $3^{-4x^2} = e^{-t}$

(8) Express the integral $\int_0^\infty \dfrac{x^c}{c^x} dx$ where $c > 1$ in terms of gamma function put $c^{-x} = e^{-t}$

$x \log c = +t$

$x = \dfrac{t}{\log c}$

Sol: Given that

$$\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty x^c c^{-x} dx$$

$$= \int_0^\infty x^c e^{\log c^{-x}} dx = \int_0^\infty x^c e^{-x\log c} dx$$

put $x \log c = t$

$$x = \frac{t}{\log c} \Rightarrow dx = \frac{1}{\log c} dt$$

Limits: if $x=0 \Rightarrow t=0$

if $x=\infty \Rightarrow t=\infty$

$$\therefore \int_0^\infty x^c c^{-x} dx = \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t} \frac{dt}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t}\, t^{(c+1)-1} dt$$

$$= \frac{1}{(\log c)^{c+1}} \Gamma(c+1)$$

(9) S.T $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma(n)$, $n>0$

Sol: put $\log \frac{1}{x} = t$

$$\frac{1}{x} = e^t \Rightarrow x = e^{-t}$$

$$dx = -e^{-t} dt$$

Limits: if $x=0 \Rightarrow t = \log\left(\frac{1}{0}\right) = \infty$

if $x=1 \Rightarrow t = \log\left(\frac{1}{1}\right) = \log 1 = 0$

$$\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \int_\infty^0 t^{n-1}(-e^{-t}) dt$$

$$= -\int_0^\infty t^{n-1}(-e^{-t}) dt$$