UNIT-1
LAPLACE TRANSFORMS

Introduction

In many problems, a function $f(t),\ \ t \in [a,\ b]$ is transformed to another function $F(s)$ through a relation of the type:

$$F(s) = \int_a^b K(t,s) f(t) dt$$

where $K(t,s)$ is a known function. Here, $F(s)$ is called integral transform of $f(t)$. Thus, an integral transform sends a given function $f(t)$ into another function $F(s)$. This transformation of $f(t)$ into $F(s)$ provides a method to tackle a problem more readily. In some cases, it affords solutions to otherwise difficult problems. In view of this, the integral transforms find numerous applications in engineering problems. Laplace transform is a particular case of integral transform (where $f(t)$ is defined on $[0, \infty)$ and $K(s,t) = e^{-st}$). As we will see in the following, application of Laplace transform reduces a linear differential equation with constant coefficients to an algebraic equation, which can be solved by algebraic methods. Thus, it provides a powerful tool to solve differential equations.

It is important to note here that there is some sort of analogy with what we had learnt during the study of logarithms in school. That is, to multiply two numbers, we first calculate their logarithms, add them and then use the table of antilogarithm to get back the original product. In a similar way, we first transform the problem that was posed as a function of $f(t)$ to a problem in $F(s)$, make some calculations and then use the table of inverse Laplace transform to get the solution of the actual problem.

In this chapter, we shall see same properties of Laplace transform and its applications in solving differential equations.

Definitions and Examples

DEFINITION 10.2.1 (Piece-wise Continuous Function)

1. A function $f(t)$ is said to be a piece-wise continuous function on a closed interval $[a,b] \subset \mathbb{R}$, if there exists finite number of points $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ such

that $f(t)$ is continuous in each of the intervals $(t_{i-1},\ t_i)$ for $1 \le i \le N$ and has finite limits as t approaches the end points, see the Figure 10.1.

2. A function $f(t)$ is said to be a piece-wise continuous function for $t \ge 0$, if $f(t)$ is a piece-wise continuous function on every closed interval $[a, b] \subset [0, \infty)$. For example, see Figure 10.1.

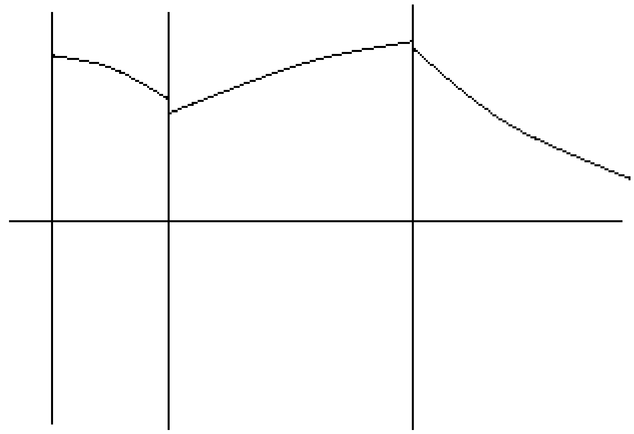

Figure 10.1: Piecewise Continuous Function

DEFINITION 10.2.2 (Laplace Transform) ***Let*** $f : [0, \infty) \longrightarrow \mathbb{R}$ ***and*** $s \in \mathbb{R}$. ***Then*** $F(s),$ ***for*** $s \in \mathbb{R}$ ***is called the LAPLACE TRANSFORM of*** $f(t),$ ***and is defined by***

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t) e^{-st} dt$$

whenever the integral exists.

(Recall that $\int\limits_0^\infty g(t)dt$ ***exists if*** $\lim\limits_{b \longrightarrow \infty} \int\limits_0^b g(t)d(t)$ ***exists and we define*** $\int\limits_0^\infty g(t)dt = \lim\limits_{b \longrightarrow \infty} \int\limits_0^b g(t)d(t)$ ***.)***

Remark 10.2.3

1. Let $f(t)$ be an EXPONENTIALLY BOUNDED function, *i.e.,*

   $$|f(t)| \le Me^{\alpha t} \quad \text{for all} \quad t > 0 \quad \text{and for some real numbers} \quad \alpha \quad \text{and} \quad M \quad \text{with} \quad M > 0.$$

   Then the Laplace transform of f exists.

2. Suppose $F(s)$ exists for some function f. Then by definition, $\lim_{b \longrightarrow \infty} \int_0^b f(t) e^{-st} dt$ exists. Now, one can use the theory of improper integrals to conclude that

$$\lim_{s \longrightarrow \infty} F(s) = 0.$$

Hence, a function $F(s)$ satisfying

$$\lim_{s \longrightarrow \infty} F(s) \text{ does not exist or } \lim_{s \longrightarrow \infty} F(s) \neq 0,$$

cannot be a Laplace transform of a function f.

DEFINITION 10.2.4 (Inverse Laplace Transform) ***Let*** $\mathcal{L}(f(t)) = F(s)$***. That is,*** $F(s)$ ***is the Laplace transform of the function*** $f(t).$ ***Then*** $f(t)$ ***is called the inverse Laplace transform of*** $F(s)$***. In that case, we write*** $f(t) = \mathcal{L}^{-1}(F(s)).$

Properties of Laplace Transform

LEMMA 10.3.1 (Linearity of Laplace Transform)

1. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{L}\big(af(t) + bg(t)\big) &= \int_0^\infty \big(af(t) + bg(t)\big) e^{-st} dt \\ &= a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)). \end{aligned}$$

2.

3. If $F(s) = \mathcal{L}(f(t)),$ and $G(s) = \mathcal{L}(g(t))$, then

$$\mathcal{L}^{-1}\big(aF(s) + bG(s)\big) = af(t) + bg(t).$$

The above lemma is immediate from the definition of Laplace transform and the linearity of the definite integral.

EXAMPLE 10.3.2

1. Find the Laplace transform of $\cosh(at)$.

   Solution: $\cosh(at) = \dfrac{e^{at} + e^{-at}}{2}$. Thus

$$\mathcal{L}(\cosh(at)) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}, \qquad s > |a|.$$

2. Similarly,

$$\mathcal{L}(\sinh(at)) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}, \qquad s > |a|.$$

3. Find the inverse Laplace transform of $\dfrac{1}{s(s+1)}$.

   Solution:

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = 1 - e^{-t}.\end{aligned}$$

4. Thus, the inverse Laplace transform of $\dfrac{1}{s(s+1)}$ is $f(t) = 1 - e^{-t}$.

THEOREM 10.3.3 (Scaling by a) ***Let $f(t)$ be a piecewise continuous function with Laplace transform $F(s)$. Then for*** $a > 0,\ \mathcal{L}(f(at)) = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof. By definition and the substitution $z = at$, we get

$$\mathcal{L}(f(at)) = \int_0^\infty e^{-st} f(at)dt = \frac{1}{a}\int_0^\infty e^{-s\frac{z}{a}} f(z)dz$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}z} f(z)dz = \frac{1}{a}F(\frac{s}{a}).$$

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EXERCISE 10.3.4

1. Find the Laplace transform of

$$t^2 + at + b, \;\; \cos(wt + \theta), \;\; \cos^2 t, \;\; \sinh^2 t;$$

   where a, b, w and θ are arbitrary constants.

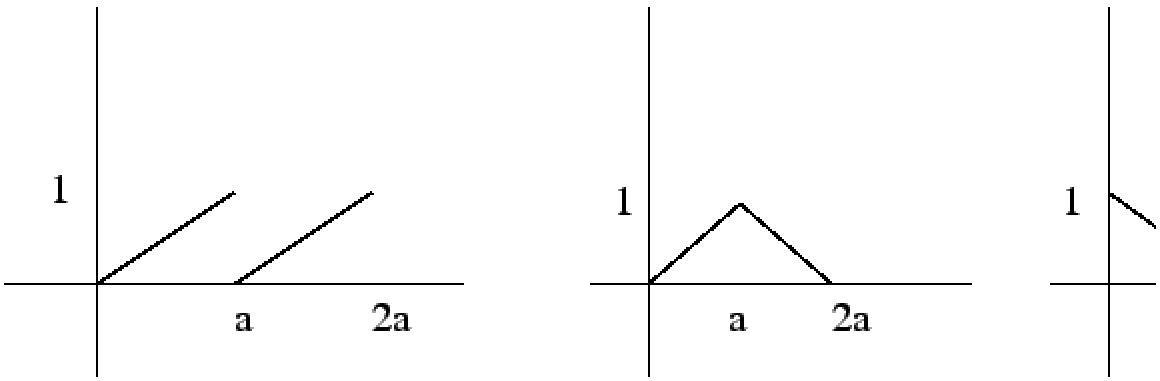

Figure 10.2: $f(t)$

2. Find the Laplace transform of the function $f(\cdot)$ given by the graphs in Figure 10.2.
3. If $\mathcal{L}(f(t)) = \dfrac{1}{s^2+1} + \dfrac{1}{2s+1}$, find $f(t)$.

The next theorem relates the Laplace transform of the function $f'(t)$ with that of $f(t)$.

THEOREM 10.3.5 (Laplace Transform of Differentiable Functions) ***Let $f(t),$ for $t > 0,$ be a differentiable function with the derivative, $f'(t),$ being continuous. Suppose that there exist constants M and T such that $|f(t)| \leq Me^{\alpha t}$ for all $t \geq T.$ If $\mathcal{L}(f(t)) = F(s)$ then***

$$\mathcal{L}(f'(t)) = sF(s) - f(0) \quad \text{for} \quad s > \alpha. \tag{10.3.1}$$

Proof. Note that the condition $|f(t)| \leq Me^{\alpha t}$ for all $t \geq T$ implies that

$$\lim_{b \longrightarrow \infty} f(b)e^{-sb} = 0 \quad \text{for} \quad s > \alpha.$$

So, by definition,

$$\begin{aligned}
\mathcal{L}\big(f'(t)\big) &= \int_0^\infty e^{-st} f'(t)\,dt = \lim_{b \longrightarrow \infty} \int_0^b e^{-st} f'(t)\,dt \\
&= \lim_{b \longrightarrow \infty} f(t)e^{-st}\Big|_0^b - \lim_{b \longrightarrow \infty} \int_0^b f(t)(-s)e^{-st}\,dt \\
&= -f(0) + sF(s).
\end{aligned}$$

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We can extend the above result for n^{th} derivative of a function $f(t)$, if $f'(t), \ldots, f^{(n-1)}(t), f^{(n)}(t)$ exist and $f^{(n)}(t)$ is continuous for $t \geq 0$. In this case, a repeated use of Theorem 10.3.5, gives the following corollary.

COROLLARY 10.3.6 ***Let*** $f(t)$ ***be a function with*** $\mathcal{L}(f(t)) = F(s)$. ***If*** $f'(t), \ldots, f^{(n-1)}(t), f^{(n)}(t)$ ***exist and*** $f^{(n)}(t)$ ***is continuous for*** $t \geq 0,$ ***then***

$$\mathcal{L}\big(f^{(n)}(t)\big) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{10.3.2}$$

In particular, for $n = 2$***, we have***

$$\mathcal{L}\big(f''(t)\big) = s^2 F(s) - s f(0) - f'(0). \tag{10.3.3}$$

COROLLARY 10.3.7 ***Let*** $f'(t)$ ***be a piecewise continuous function for*** $t \geq 0$***. Also, let*** $f(0) = 0$***. Then***

$$\mathcal{L}(f'(t)) = sF(s) \quad \text{or equivalently} \quad \mathcal{L}^{-1}(sF(s)) = f'(t).$$

EXAMPLE 10.3.8

1. Find the inverse Laplace transform of $\frac{s}{s^2+1}$.

Solution: We know that $\mathcal{L}^{-1}(\frac{1}{s^2+1}) = \sin t.$ Then $\sin(0) = 0$ and therefore, $\mathcal{L}^{-1}(\frac{s}{s^2+1}) = \cos t.$

2. Find the Laplace transform of $f(t) = \cos^2(t).$

Solution: Note that $f(0) = 1$ and $f'(t) = -2\cos t \ \sin t = -\sin(2t).$ Also,

$$\mathcal{L}(-\sin(2t)) = \frac{-2}{s^2+4}.$$

Now, using Theorem 10.3.5, we get

$$\mathcal{L}(f(t)) = \frac{1}{s}\left(-\frac{2}{s^2+4}+1\right) = \frac{s^2+2}{s(s^2+4)}.$$

LEMMA 10.3.9 (Laplace Transform of $tf(t)$) ***Let $f(t)$ be a piecewise continuous function with $\mathcal{L}(f(t)) = F(s).$ If the function $F(s)$ is differentiable, then***

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s).$$

$$\text{Equivalently,} \quad \mathcal{L}^{-1}(-\frac{d}{ds}F(s)) = tf(t).$$

Proof. By definition, $F(s) = \int_0^\infty e^{-st}f(t)dt.$ The result is obtained by differentiating both sides with respect to s. height6pt width 6pt depth 0pt

Suppose we know the Laplace transform of a $f(t)$ and we wish to find the Laplace transform of the function $g(t) = \frac{f(t)}{t}.$ Suppose that $G(s) = \mathcal{L}(g(t))$ exists. Then writing $f(t) = tg(t)$ gives

$$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(tg(t)) = -\frac{d}{ds}G(s).$$

Thus, $G(s) = -\int\limits_a^s F(p)dp$ for some real number a. As $\lim\limits_{s \longrightarrow \infty} G(s) = 0$, we get $G(s) = \int\limits_s^\infty F(p)dp.$

Hence,we have the following corollary.

COROLLARY 10.3.10 ***Let*** $\mathcal{L}(f(t)) = F(s)$ ***and*** $g(t) = \frac{f(t)}{t}.$ ***Then***

$$\mathcal{L}(g(t)) = G(s) = \int_s^\infty F(p)dp.$$

EXAMPLE 10.3.11

1. Find $\mathcal{L}(t\sin(at)).$

   Solution: We know $\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}.$ Hence $\mathcal{L}(t\sin(at)) = \frac{2as}{(s^2+a^2)^2}.$

2. Find the function $f(t)$ such that $F(s) = \frac{4}{(s-1)^3}.$

   Solution: We know $\mathcal{L}(e^t) = \frac{1}{s-1}$ and
   $\frac{4}{(s-1)^3} = 2\frac{d}{ds}\left(-\frac{1}{(s-1)^2}\right) = 2\frac{d^2}{ds^2}\left(\frac{1}{s-1}\right).$

   By lemma 10.3.9, we know that $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s).$ Suppose $\frac{d}{ds}F(s) = G(s).$ Then $g(t) = \mathcal{L}^{-1}G(s) = \mathcal{L}^{-1}\frac{d}{ds}F(s) = -tf(t).$ Therefore,

$$\mathcal{L}^{-1}\left(\frac{d^2}{ds^2}F(s)\right) = \mathcal{L}^{-1}\left(\frac{d}{ds}G(s)\right) = -tg(t) = t^2f(t).$$

   Thus we get $f(t) = 2t^2e^t.$

LEMMA 10.3.12 (Laplace Transform of an Integral) ***If*** $F(s) = \mathcal{L}(f(t))$ ***then***

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.$$

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau.$$

Equivalently,

Proof. By definition,

$$\mathcal{L}(\int_0^t f(\tau)\,d\tau) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,d\tau\right)dt = \int_0^\infty \int_0^t e^{-st} f(\tau)\,d\tau dt.$$

We don't go into the details of the proof of the change in the order of integration. We assume that the order of the integrations can be changed and therefore

$$\int_0^\infty \int_0^t e^{-st} f(\tau)\,d\tau dt = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,dt\,d\tau.$$

Thus,

$$\begin{aligned}
\mathcal{L}(\int_0^t f(\tau)\,d\tau) &= \int_0^\infty \int_0^t e^{-st} f(\tau)\,d\tau dt \\
&= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,dt\,d\tau = \int_0^\infty \int_\tau^\infty e^{-s(t-\tau)-s\tau} f(\tau)\,dt\,d\tau \\
&= \int_0^\infty e^{-s\tau} f(\tau)d\tau\left(\int_\tau^\infty e^{-s(t-\tau)}dt\right) \\
&= \int_0^\infty e^{-s\tau} f(\tau)d\tau\left(\int_0^\infty e^{-sz}dz\right) = F(s)\frac{1}{s}.
\end{aligned}$$

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EXAMPLE 10.3.13

$\mathcal{L}(\int_0^t \sin(az)dz).$

1. Find

   Solution: We know $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}.$ Hence

$$\mathcal{L}(\int_0^t \sin(az)dz) = \frac{1}{s}\cdot\frac{a}{(s^2+a^2)} = \frac{a}{s(s^2+a^2)}.$$

2. Find $\mathcal{L}\left(\int_0^t \tau^2 d\tau\right).$
   Solution: By Lemma 10.3.12

$$\mathcal{L}\left(\int_0^t \tau^2 d\tau\right) = \frac{\mathcal{L}(t^2)}{s} = \frac{1}{s} \cdot \frac{2!}{s^3} = \frac{2}{s^4}.$$

3. Find the function $f(t)$ such that $F(s) = \frac{4}{s(s-1)}.$
   Solution: We know $\mathcal{L}(e^t) = \frac{1}{s-1}.$ So,

$$\mathcal{L}^{-1}\left(\frac{4}{s(s-1)}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s-1}\right) = 4\int_0^t e^\tau d\tau = 4(e^t - 1).$$

LEMMA 10.3.14 (s -Shifting) ***Let*** $\mathcal{L}(f(t)) = F(s).$ ***Then*** $\mathcal{L}(e^{at}f(t)) = F(s-a)$ ***for*** $s > a.$

Proof.

$$\begin{aligned}\mathcal{L}(e^{at}f(t)) &= \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt \\ &= F(s-a) \qquad s > a.\end{aligned}$$

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EXAMPLE 10.3.15

1. Find $\mathcal{L}(e^{at}\sin(bt)).$
   Solution: We know $\mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2}.$ Hence $\mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}.$
2. Find $\mathcal{L}^{-1}\left(\frac{s-5}{(s-5)^2+36}\right).$
   Solution: By s-Shifting, if $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(e^{at}f(t)) = F(s-a)$. Here, $a = 5$ and

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+36}\right)=\mathcal{L}^{-1}\left(\frac{s}{s^2+6^2}\right)=\cos(6t).$$

Hence, $f(t)=e^{5t}\cos(6t).$

Inverse Transforms of Rational Functions

Let $F(s)$ be a rational function of s. We give a few examples to explain the methods for calculating the inverse Laplace transform of $F(s)$.

EXAMPLE 10.3.16

1. DENOMINATOR OF MATHEND000# HAS DISTINCT REAL ROOTS:

$$\text{If}\quad F(s)=\frac{(s+1)(s+3)}{s(s+2)(s+8)}\quad\text{find}\ f(t).$$

Solution: $F(s)=\dfrac{3}{16s}+\dfrac{1}{12(s+2)}+\dfrac{35}{48(s+8)}.$ Thus,

$$f(t)=\frac{3}{16}+\frac{1}{12}e^{-2t}+\frac{35}{48}e^{-8t}.$$

2. DENOMINATOR OF MATHEND000# HAS DISTINCT COMPLEX ROOTS:

$$\text{If}\quad F(s)=\frac{4s+3}{s^2+2s+5}\quad\text{find}\quad f(t).$$

Solution: $F(s)=4\dfrac{s+1}{(s+1)^2+2^2}-\dfrac{1}{2}\cdot\dfrac{2}{(s+1)^2+2^2}.$ Thus,

$f(t) = 4e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t).$

3. DENOMINATOR OF MATHEND000# HAS REPEATED REAL ROOTS:

$$\text{If} \quad F(s) = \frac{3s+4}{(s+1)(s^2+4s+4)} \quad \text{find} \quad f(t).$$

Solution: Here,

$$F(s) = \frac{3s+4}{(s+1)(s^2+4s+4)} = \frac{3s+4}{(s+1)(s+2)^2} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{(s+2)^2}.$$

Solving for a, b and c, we get

$F(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{2}{(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} + 2\frac{d}{ds}\left(-\frac{1}{(s+2)}\right).$ Thus,

$f(t) = e^{-t} - e^{-2t} + 2te^{-2t}.$