UNIT-2

FOURIER SERIES

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Examples of successive approximations to common functions using Fourier series are illustrated above.

In particular, since the superposition principle holds for solutions of a linear homogeneous ordinary differential equation, if such an equation can be solved in the case of a single sinusoid, the solution for an arbitrary function is immediately available by expressing the original function as a Fourier series and then plugging in the solution for each sinusoidal component. In some special cases where the Fourier series can be summed in closed form, this technique can even yield analytic solutions.

Any set of functions that form a complete orthogonal system have a corresponding generalized Fourier series analogous to the Fourier series. For example, using orthogonality of the roots of a Bessel function of the first kind gives a so-called Fourier-Bessel series.

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(m x) \sin(n x)\, d x = \pi \delta_{m n} \tag{1}$$

$$\int_{-\pi}^{\pi} \cos(m x) \cos(n x)\, d x = \pi \delta_{m n} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(m x) \cos(n x)\, d x = 0 \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(m x)\, d x = 0 \tag{4}$$

$$\int_{-\pi}^{\pi} \cos(m x)\, d x = 0 \tag{5}$$

for $m, n \neq 0$, where $\delta_{m n}$ is the Kronecker delta.

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = \cos x$ and $f_2(x) = \sin x$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x), \tag{6}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad (7)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \qquad (8)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx \qquad (9)$$

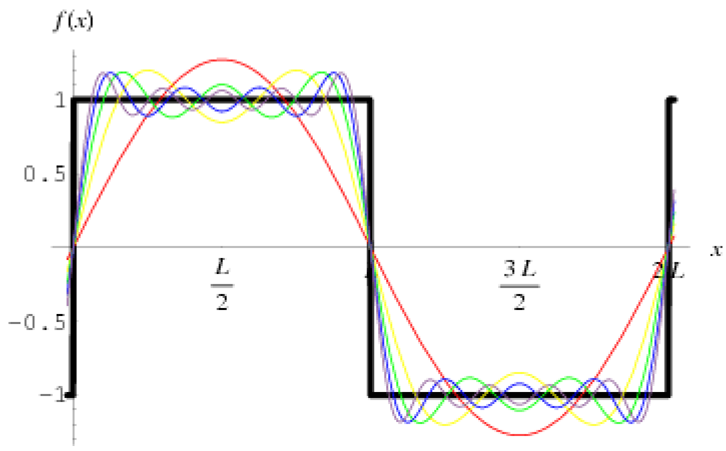
and $n = 1$, 2, 3, ... Note that the coefficient of the constant term a_0 has been written in a special form compared to the general form for a generalized Fourier series in order to preserve symmetry with the definitions of a_n and b_n.

The Fourier cosine coefficient a_n and sine coefficient b_n are implemented in the Wolfram Language as `FourierCosCoefficient`[*expr*, *t*, *n*] and `FourierSinCoefficient`[*expr*, *t*, *n*], respectively.

A Fourier series converges to the function $\bar{f}$ (equal to the original function at points of continuity or to the average of the two limits at points of discontinuity)

$$\bar{f} \equiv \begin{cases} \frac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right] & \text{for } -\pi < x_0 < \pi \\ \frac{1}{2}\left[\lim_{x\to -\pi^+} f(x) + \lim_{x\to \pi_-} f(x)\right] & \text{for } x_0 = -\pi, \pi \end{cases} \qquad (10)$$

if the function satisfies so-called Dirichlet boundary conditions. Dini's test gives a condition for the convergence of Fourier series.



As a result, near points of discontinuity, a "ringing" known as the Gibbs phenomenon, illustrated above, can occur.

For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used to transform the interval of integration from $[-\pi, \pi]$ to $[-L, L]$. Let

$$x \equiv \frac{\pi x'}{L} \quad (11)$$

$$dx = \frac{\pi\, dx'}{L}. \quad (12)$$

Solving for x' gives $x' = Lx/\pi$, and plugging this in gives

$$f(x') = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right). \quad (13)$$

Therefore,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x')\, dx' \quad (14)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx' \quad (15)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x') \sin\left(\frac{n\pi x'}{L}\right) dx'. \quad (16)$$

Similarly, the function is instead defined on the interval $[0, 2L]$, the above equations simply become

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(x')\, dx' \quad (17)$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx' \quad (18)$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x') \sin\left(\frac{n\pi x'}{L}\right) dx'. \quad (19)$$

In fact, for $f(x)$ periodic with period $2L$, *any* interval $(x_0, x_0 + 2L)$ can be used, with the choice being one of convenience or personal preference (Arfken 1985, p. 769).

The coefficients for Fourier series expansions of a few common functions are given in Beyer (1987, pp. 411-412) and Byerly (1959, p. 51). One of the most common functions usually analyzed by this technique is the square wave. The Fourier series for a few common functions are summarized in the table below.

function	$f(x)$	Fourier series
Fourier series--sawtooth wave	$\frac{x}{2L}$	$\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$
Fourier series--square wave	$2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L} - 1\right)\right] - 1$	$\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$
Fourier series--triangle wave	$T(x)$	$\frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{L}\right)$

If a function is even so that $f(x) = f(-x)$, then $f(x)\sin(nx)$ is odd. (This follows since $\sin(nx)$ is odd and an even function times an odd function is an odd function.) Therefore, $b_n = 0$ for all n.

Similarly, if a function is odd so that $f(x) = -f(-x)$, then $f(x)\cos(n x)$ is odd. (This follows since $\cos(n x)$ is even and an even function times an odd function is an odd function.) Therefore, $a_n = 0$ for all n.

The notion of a Fourier series can also be extended to complex coefficients. Consider a real-valued function $f(x)$. Write

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i n x}. \tag{20}$$

Now examine

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) e^{-i m x} \, d x &= \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} A_n e^{i n x} \right) e^{-i m x} \, d x & (21) \\
&= \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi} e^{i (n-m) x} \, d x & (22) \\
&= \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi} \{\cos[(n-m) x] + i \sin[(n-m) x]\} \, d x & (23) \\
&= \sum_{n=-\infty}^{\infty} A_n \, 2\pi \delta_{m n} & (24) \\
&= 2\pi A_m, & (25)
\end{aligned}$$

so

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i n x} \, d x. \tag{26}$$

The coefficients can be expressed in terms of those in the Fourier series

$$\begin{aligned}
A_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos(n x) - i \sin(n x)] \, d x & (27) \\
&= \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos(n x) + i \sin(|n| x)] \, d x & n < 0 \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, d x & n = 0 \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos(n x) - i \sin(n x)] \, d x & n > 0 \end{cases} & (28) \\
&= \begin{cases} \frac{1}{2} (a_n + i b_n) & \text{for } n < 0 \\ \frac{1}{2} a_0 & \text{for } n = 0 \\ \frac{1}{2} (a_n - i b_n) & \text{for } n > 0. \end{cases} & (29)
\end{aligned}$$

For a function periodic in $[-L/2, L/2]$, these become

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i(2\pi n x/L)} \quad (30)$$

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-i(2\pi n x/L)}\, dx. \quad (31)$$

These equations are the basis for the extremely important Fourier transform, which is obtained by transforming A_n from a discrete variable to a continuous one as the length $L \to \infty$.

The complex Fourier coefficient is implemented in the Wolfram Language as `FourierCoefficient`[*expr*, *t*, *n*].

Properties

We say that *f* belongs to if *f* is a 2π-periodic function on **R** which is *k* times differentiable, and its *k*th derivative is continuous.

- If *f* is a 2π-periodic odd function, then $a_n = 0$ for all *n*.
- If *f* is a 2π-periodic even function, then $b_n = 0$ for all *n*.

- If *f* is integrable, , and This result is known as the Riemann–Lebesgue lemma.
- A doubly infinite sequence $\{a_n\}$ in $c_0(\mathbf{Z})$ is the sequence of Fourier coefficients of a function in $L^1([0, 2\pi])$ if and only if it is a convolution of two sequences in . See [12]
- If , then the Fourier coefficients of the derivative *f′* can be expressed in terms of the Fourier coefficients of the function *f*, via the formula .
- If , then . In particular, since tends to zero, we have that tends to zero, which means that the Fourier coefficients converge to zero faster than the *k*th power of *n*.
- Parseval's theorem. If *f* belongs to $L^2([-\pi, \pi])$, then .
- Plancherel's theorem. If are coefficients and then there is a unique function such that for every *n*.
- The first convolution theorem states that if *f* and *g* are in $L^1([-\pi, \pi])$, the Fourier series coefficients of the 2π-periodic convolution of *f* and *g* are given by:

- The second convolution theorem states that the Fourier series coefficients of the product of *f* and *g* are given by the discrete convolution of the and sequences: