ANNAMACHARYA INSTITUTE OF TECHNOLOGY & SCIENCES:: TIRUPATI
AUTONOMOUS
AK 20 Regulations

Year : I B.Tech – II Sem **Branch of Study: Common to EEE, CE, ME and ECE**

Subject Code	Subject Name	L T P	Credits
Subject Code **20ABS9906**	Subject Name: **Differential Equations and Vector Calculus**	**L** **T** **P** **3** **0** **0**	Credits:3

Course Outcomes:

1.Apply the mathematical concepts of ordinary differential equations of higher order.

2. Solve the differential equations related to various engineering fields .

3. Identify solution methods for partial differential equations that model physical processes .

4. Interpret the physical meaning of different operators such as gradient, curl and divergence .

5. Estimate the work done against a field, circulation and flux using vector calculus

UNIT – I

10 Mark questions:

1. Solve the following differential equations

   a) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=e^{2x}$ b) $(D^3-5D^2+8D-4)y = e^{2x}$ c) $(D+2)(D-1)^2=e^{-2x}+2\sinh x$

   d) $y''-4y'+3y = 4e^{3x}$, $y(0)=-1$, $y'(0)=3$. e) $(D^2-k^2)y = \cosh kx$ f) $(D^2 - 4D + 13)y = e^{2x}$

   g) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=(1-e^x)^2$

2. Solve the following differential equations

   a) $(D^2+D+1)y = \sin 2x$ b) $(D^2-4)y = 2\cos^2 x$ c) $(D^3-1)y = e^x+\sin^2 x+2$

   d) $y''+4y'+4y=3\sin x+4\cos x, y(0)=1, y'(0)=0$

3. Solve the following differential equations

   a) $(D^2+D+1)y = x^3$ b) $(D^3+2D^2+D)y = e^{2x}+x^2+x+\sin 2x$. c) $(D^3-3D^2+4D-2)y = e^x+x+\cos x$

   d) $(D-2)^2 y = 8(e^{2x}+x^2+\sin 2x)$

4. Solve the following differential equations

a) $(D^2-5D+6)y = xe^{4x}$ b) $(D^3-7D^2+14D-8)y = e^x \cos 2x$. c) $(D^2+2D-3)\, y = x^2 e^{-3x}$

d) $(D^2-7D+6)y = (x+1)e^x$ e) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$ f) $y'' - 2y' + 2y = x + e^x \cos x$

g) $\frac{d^2y}{dx^2} - 4y = x \sinh x$

5. Solve the following differential equations

a) $(D^2-2D+1)\, y = x\, e^x \sin x$ b) $(D^2 - 4)y = x \sin \lambda x$ c) $(D^2 - 1)y = x \sin x + (1-x^2)e^x$

d) $(D^2 - 1)y = x \sin 3x + \cos x$

6. Solve by the method of variation of parameters:

a) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ b) $\frac{d^2y}{dx^2} + 4y = \tan 2x$ C) $(D^2+1)y = \sec x$ d) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

e) $(D^2 - 2D)y = e^x \sin x$ f) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ g) $y'' - 2y' + y = e^x \log x$

7. Solve the following simultaneous differential equations

a) $\frac{dx}{dt} + 2y + \sin t = 0,\ \frac{dy}{dt} - 2x - \cos t = 0$ given that x= 0 and y = 1 when t = 0

b) $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

Two mark questions

1. Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 0,\ given\ x(0) = 0, \frac{dx}{dt}(0) = 15$.
2. Solve $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$
3. Solve $(D^3 + D^2 + 4D + 4)y = 0$
4. Solve $\frac{1}{(D-5)^3} e^{5x} \sin x$.
5. Find the complementary function of $(D^2+1)^3\, y = 0$
6. $(D^4 - 4D + 4)y = 0$
7. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$.
8. Find the particular integral of $(D^3 - 5D^2 + 8D - 4)y = e^{2x}$.
9. Find the particular integral of $(D^2 + 4)y = \sin 2x$.
10. Find the particular integral of $(D^2 + D + 1)y = x^3$.
11. Find the complementary function of $(D^3 - 14D + 8)y = 0$.
12. Find the P.I of $(D^2+5D+6)y = e^x$
13. Find the P.I of $(D+2)(D-1)^2y = e^{-2x} + 2\sinh x$
14. Find the P.I of $(D^3+1)y = \cos(2x-1)$
15. Solve the Differential equation $(D^3+1)y = 0$.

16. Find the value of $\frac{1}{D^3}\cos x$, Where $D=\frac{d}{dx}$.

17. Define simultaneous linear differential equation and give an example.

UNIT – 2

10 Mark questions:

1. Solve the following differential equations .

a) $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$

b) $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\log x$

c) $(x^2D^2+xD+1)y=\log x\sin(\log x)$

d) $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$

e) $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$

2. Solve the following differential equations

a) $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+4y=x^2+x+1$

b) $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x$

C) $(2x-1)^2\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}-2y=8x^2-2x+3$

d) $(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=x$

e) $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$

3. A spring for which stiffness k=700 N/m hangs in a vertical position with its upper end is fixed. A mass of 7 kg is attached to the lower end. After coming to rest, the mass is pulled down 0.05 m and released. Find the resulting motion of the mass, neglecting air resistance.

4. A mass weighing 4.9 kg is suspended from a spring. A pull of 10 kg will stretch it to 5 cm. The mass is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the mass from its equilibrium position at time t sec, the maximum velocity and the period of oscillation.

Two mark questions

1 Write the formulae for `A' and `B' in method of variation of parameters

2 Find the value of `A' for $\frac{d^2y}{dx^2}+y=Cosec\ x$ by the method of variation of parameters

3 Find the value of `B' for $(D^2+a^2)y=Tan\ ax$ by the method of variation of parameters

4 Find C F of $(x^2D^2-4xD+6)y=x^2$

5 Transform the D E $(x^2D^2 - xD + 1)y = Log\ x$ into constant coefficient of DE
6 Define Kirchhoff Law's ?
7 Find C F for $\frac{d^2q}{dt^2} + 120\frac{dq}{dt} + 100q = 17\ Sin\ 2t$
8 Write the complete solution of $(5+2x)^2\frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx}8y = 0$
9 Explain Damped oscillations in Mechanical system?
10 Explain Damped oscillations in Electrical system?
11 Form the DE of L-C-R circit.
12 Form the DE of R-L circuit.
13 Explain differential equation of L-C-R circuit
14 Explain Free oscillations(with damping) in Mechanical system?

Unit-3

10Marks Questions

4. Solve the following Equations

(a). $\frac{y^2z}{x}p - xzq = y^2$

(b). $x^2(\mathrm{y-z})p - (z-x)q = z^2(\mathrm{x-y})$

(c). $(\mathrm{mz-ny})\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$

(d). $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

(e). $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

(f). $y^2p + xyq = x(z-2y)$

5. Solve the following equations

(a). $r - 4s + 4t = e^{2x+y}$

(b). $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2\partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

(c).
$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial x^2\partial y} - 6\frac{\partial^3 z}{\partial y^3} = \cos(2\mathrm{x}+\mathrm{y})$

(d). $\left[D^2 - (D')^2\right]z = e^{x-y}\sin(\mathrm{x}+2y)$

(e). $\left[D^2 + 4DD' - 5(D')^2\right]z = \mathrm{in}(2\mathrm{x}+3y)$

2 Marks Questions

1. Solve the following Equations

(a). $p - q = 1$

(b). $\sqrt{p} - \sqrt{q} = 1$

(c). $p(1+q) = \mathrm{qz}$

(d). $\mathrm{q}^2 = z^2\,\mathrm{p}^2(1-p^2)$

(e). $p^2 + q^2 = x + y^2$

(f). $p^2y(1+\mathrm{x}^2) = \mathrm{qx}^2$

(g). $z = px + qy + \sqrt{1+p^2+q^2}$

(h). $p^2y(1+\mathrm{x}^2) = \mathrm{qx}$

2. Solve the following Equations

(a). $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

(d). $D^3 - 3D^2D'2D(D')^2\,z = 0$

(b). $2r + 12s + 9t = 0$

(c). $\frac{\partial 3z}{\partial x^3} - 4\frac{\partial^2 z}{\partial x\partial y} + 4\frac{\partial^2 z}{\partial x\partial y^2} = 0$

3. Solve the following Equations

(a). $(D+3D'+4)^2 = 0\ (D+D'-2)(D+4D'-3)z = 0$

UNIT-IV

10 Mark questions:

1. Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P=(1,2,3) in the direction of the line PQ, where Q=(5,0,4).
2. Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point (2,-1,1) in direction of vector $\bar{i}+2\bar{j}+2\bar{k}$
3. Find the directional derivative of $\phi = x^2 yz + 4xz^2$ at the point (1,-2,1) in the direction of the vector $2\bar{i}-\bar{j}-2\bar{k}$
4. Find the directional derivative of $xyz^2 + xz$ at point (1,1,1) in a direction of the normal to the surface $3xy^2 + y = z$ at (0,1,1)
5. Evaluate the angle between the normal to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3).
6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2+y^2-3$ at the point (2,-1,2).
7. Prove that $\nabla(r^n) = nr^{n-2}\bar{r}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$
8. For a solenoidal vector $\bar{f}$, prove that $\nabla\times(\nabla\times(\nabla\times(\nabla\times\bar{f}))) = curl\ curl\ curl\ curl\ \bar{f} = \nabla^4\bar{f}$.
9. Find $div\,\bar{f}$ where $\bar{f} = r^n\ \bar{r}$. find n if it is sdenoidal
10. Show that $\frac{\bar{r}}{r^3}$ is solenoidal (or) Evaluate $\nabla\frac{\bar{r}}{r^3}$ where $\bar{r} = x\bar{\imath} + y\,\bar{\jmath} + z\,\bar{k}$ and $r = |\bar{r}|$
11. A vector field is given by $\bar{A} = (x^2 + xy^2)\bar{i} + (y^2 + yx^2)\bar{j}$.show that the field is irrotational and find the scalar potential.
12. P.T if $\bar{r}$ is the position vector of any point in space ,then $r^n\,\bar{r}$ irrotational.
13. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.
14. Prove that $div(grad r^n) = \nabla^2(r^n) = n(n+1)r^{n-2}$.
15. Prove that $\nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right] = \frac{-2}{r^3}\bar{r}$
16. Prove that for any vector functions f and g $\nabla\cdot(\bar{f}\times\bar{g}) = \bar{g}\cdot(\nabla\times\bar{f}) - \bar{f}\cdot(\nabla\times\bar{g})$
17. Prove that $\nabla\times(\nabla\times\bar{f}) = curl\ curl\ \bar{f} = \nabla(\nabla.\bar{f}) - \nabla^2\bar{f}$
18. If $\bar{A}$ is a constant vector and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then prove that
$$\nabla\times\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\,\bar{r}}{r^{n+2}}.$$

Two mark questions

1. If $a = x + y + z,\ b = x^2 + y^2 + z^2, c = xy + yz + zx$, prove that $[\text{grad a}, \text{grad b}, \text{grad c}] = 0$
2. Find the directional derivative of the function $2xy + z^2$ at the point (1,-1,3) in the direction of $\bar{i}+2\bar{j}+3\bar{k}$
3. Find a unit normal vector to the surface $xy^3z^2 = 4$ at the point (-1,-1,2)
4. Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).
5. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that a) $\nabla.\bar{r} = 3$ b) $\nabla\times\bar{r} = 0$

6. Find the greatest value of the directional derivative of the function $f = x^2\, y\, z^3$ at the point (2,1,-1) and also find unit normal vector.
7. If $\bar{f} = xy^2\, \bar{\imath} + 2x^2yz\bar{\jmath} - 3yz^2\, \bar{k}$ find $\operatorname{div}\bar{f}$ and Curl $\bar{f}$ at the point (1,-1,1)
8. Find $div\ \bar{f}$ when $\bar{f} = grad\ (x^3 + y^3 + z^3 - 3xyz)$ also find $curl\ \bar{f}$
9. prove that $(\bar{f} \times \nabla)\cdot \bar{r} = o$
10. prove that $(\bar{f} \times \nabla)\times \bar{r} = -2\bar{f}$
11. prove that curl grad F = 0
12. prove that div curl F = 0

UNIT-V

10 Mark questions:

1. Evaluate the integral $\int_C \left[(x^2 + xy)dx + (x^2 + y^2)dy\right]$ where C is the square formed by the lines $x = \pm 1$ and $y = \pm 1$.
2. Find the work done by the force $\overline{F} = (2y+3)\bar{i} + (zx)\bar{j} + (yz - x)\bar{k}$ when it moves a particle from the point (0,0,0) to (2,1,1) along the curve $x = 2t^2, y = t, z = t^3$.
3. Evaluate $\iint_S \overline{F}.\bar{n}\ dS$ where $\overline{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.
4. Verify divergence theorem for $\overline{F} = 4xy\bar{i} - y^2\bar{j} + yz\bar{k}$,where S is the surface of the cube bounded by x = 0,x = 1,y = 0,y =1, z = 0,z = 1.
5. Verify divergence theorem for $\overline{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over the rectangular parallelepiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.
6. Use Divergence theorem to evaluate $\iint_S 4x\bar{i} - 2y^2\,\bar{j} + z^2\,\bar{k}$,where S is the surface bounded by the region $x^2 + y^2 = 4$z = 0 and z = 3.
7. Verify Green's theorem for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the region bounded by $y = \sqrt{x}$ and $y = x^2$.
8. Verify Green's theorem for $\int_C [(xy + y^2)\,dx + x^2\,dy]$ where C is the region bounded by $y = x$ and $y = x^2$.
9. Evaluate by Green's theorem $\oint_C (y - \sin x)dx + \cos x dy$ where C is the triangle enclosed by the lines $y = 0, x = \frac{\pi}{2},\ y = \frac{2}{\pi}x$.
10. Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$.
11. Verify Green's theorem for $\int_C [y^2 dx - x^2 dy]$ where C is the boundary of the triangle whose vertices are (1,0), (0,1), (-1,0).

12. Verify Stoke's theorem for $\overline{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken around a rectangle bounded by the lines $x = \pm a, y = 0$ *and* $y = b$.

13. Verify Stoke's for $\overline{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2 z\bar{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the xy-plane.

14. Use Stoke's theorem evaluate $\oint_C [(x+y)dx + (2x - z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices (2,0,0), (0,3,0), (0,0,6)

Two mark questions

1. If $\overline{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \overline{F}.d\bar{r}$ along the curve C in the xy – plane $y = x^3$ from the point (1,1) to (2,8).
2. If $\overline{F} = (3xy)\bar{i} - (y^2)\bar{j}$, evaluate $\int_C \overline{F}.d\bar{r}$ along the curve C in the xy – plane: $y = 2x^2$ from the point (0,0) to (1,2).
3. Find the work done by the force $\overline{F} = 3x^2\bar{i} + (2zx - y)\bar{j} + z\bar{k}$ along the straight line from (0,0,0) to (2,1,3) along the curve $x = 2t^2, y = t, z = t^3$.
4. Find the work done by the force $\overline{F} = 3x^2\bar{i} + (2zx - y)\bar{j} + z\bar{k}$ along the curve $x^2 = 4y,\ 3x^3 = 8z$ from x = 0 to x = 2.
5. Define work done by force.
6. Define Gauss's Divergence theorem.
7. Define Green's theorem in vectorial form.
8. Define Stoke's theorem.
9. Prove by Stoke's theorem, Curl.grad $\emptyset = \overline{0}$
10. If $\overline{F} = 3y\bar{i} - zx\bar{j} + yz^2\bar{k}$ and S is the surface of the paraboloid 2z = $x^2 + y^2$ bounded by z = 0, evaluate $\iint_S (\nabla \times \overline{F}).\bar{n}\, ds$ using stoke's theorem.

ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES, TIRUPATI
(Autonomous)
Course structure for Four Year Regular B.Tech. Degree Program
(Effective for the batches admitted from 2020-21)
CIVIL ENGINEERING (CE)

Year: I **Semester: II**

Subject Code **20ABS9906**	Subject Name: **Differential Equations and Vector Calculus**	**L T P** **3 0 0**	Credits:3

Course Outcomes:

1.Apply the mathematical concepts of ordinary differential equations of higher order.

2. Solve the differential equations related to various engineering fields .

3. Identify solution methods for partial differential equations that model physical processes .

4. Interpret the physical meaning of different operators such as gradient, curl and divergence .

5. Estimate the work done against a field, circulation and flux using vector calculus .

UNIT I: Linear Differential Equations of Higher Order

Definitions, complete solution, operator D, rules for finding complimentary function, inverse operator, rules for finding particular integral (e^{ax}, sinax (or) cosax, X^k, $e^{ax}v$, x v(x)), method of variation of parameters, simultaneous linear equations with constant coefficients.

UNIT II: Equations Reducible to Linear Differential Equations and Applications

Cauchy's and Legendre's linear equations, Applications to simple pendulum, oscillations of a spring, L-C-R Circuit problems and Mass spring system.

UNIT III: Partial Differential Equations – First order

First order partial differential equations, solutions of first order linear and non-linear PDEs.
Solutions to homogenous and non-homogenous higher order linear partial differential equations.

UNIT IV:Vector differentiation

Scalar and vector point functions, vector operator del, del applies to scalar point functions-Gradient, del applied to vector point functions-Divergence and Curl, vector identities

UNIT V: Vector integration

Line integral-circulation-work done, surface integral-flux, Green's theorem in the plane (without proof), Stoke's theorem (without proof), volume integral, Divergence theorem (without proof) and applications of these theorems.

Text Books :

1. B. S. Grewal, Higher Engineering Mathematics, 44th Edition, Khanna publishers, 2017.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10th Edition, John Wiley & Sons, 2011.

References:

1. Dr.T.K.V.Iyengar, Engineering Mathematics-I,S.Chand publishers
2. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002
3. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics,Laxmi publication,2008
4. B. V. Ramana, Higher Engineering Mathematics, Mc Graw Hill Education.

List of COs	PO no. and keyword	Competency Indicator	Performance Indicator
CO1	PO1:Apply the knowledge of mathematics	1.1	1.1.1
CO2	PO2:Analyse complex engineering problems	2.1	2.1.3
CO3	PO1:Apply the knowledge of mathematics	1.1	1.1.1
CO4	PO1:Apply the knowledge of mathematics	1.1	1.1.1
CO5	PO2:Analyse complex engineering problems	2.1	2.1.3

Higher Order Linear Differential Equations (Constant Coefficients)

CL - 3

Definition :-1. An equation of the form $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = Q(x)$ where $a_n, a_1, \ldots, a_{n-1}, a_0$ and Q are continuous real functions in 'x' defined on an interval I is called a linear differential equation of order 'n' over the interval I.

Ex:- $\frac{d^3y}{dx^3} + 5x^4\frac{d^2y}{dx^2} + 7x^2\frac{dy}{dx} + 8xy = \sin x$ is a linear differential equation of order three.

Definition : 2. A linear differential equation of order 'n' can also be written as $\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(x)\,y = Q(x)$. where $P_1(x), P_2(x), \ldots, P_n(x)$ and $Q(x)$ are written as $P_1, P_2, \ldots, P_n$ and Q.

* If a differential equation is not linear, then it is called a non-linear differential equation. But a differential equation of degree one need not be linear.

Ex:- 1. $\left(\frac{d^2y}{dx^2}\right)^1 + x^2\frac{dy}{dx} + 2xy^2 = \sin x$ is of degree one, but it is not linear because in the third term the co-efficient of y is $2xy$ instead of a function of 'x'.

Ex:2. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=e^x$ is of degree one but not linear because in the second term $\frac{dy}{dx}$ occurs in second degree.

Differential operator: Notation

let the differential operator $\frac{d}{dx}$ be denoted by 'D' and the differential operator $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \cdots, \frac{d^n}{dx^n}$ be denoted respectively by $D^2, D^3, \cdots, D^n$ when applied on a function 'y' of 'x' yield. Thus $Dy=\frac{dy}{dx}$, $D^2y=\frac{d^2y}{dx^2}$, $D^3y=\frac{d^3y}{dx^3}\cdots$ $D^ny=\frac{d^ny}{dx^n}$

Ex:-1. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^x$

$\Rightarrow D^2y+Dy+y=e^x$

$\Rightarrow (D^2+D+1)y=e^x$

Note:- 1. If $Q\neq 0$ for some x in I, then the equation $f(D)y=Q$ is called a linear and non-homogeneous equation

ex:- $\frac{d^3y}{dx^3}+\cos x\frac{d^2y}{dx^2}-2\sin x\frac{dy}{dx}-(\cos x)y=\sin 2x$

$\Rightarrow (D^3+\cos x D^2-2\sin x D-\cos x)y=\sin 2x$.

2. If $Q=0$ for all x in I, then the equation $f(D)y=0$ is called a linear and homogeneous equation.

ex:- $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2x^2y=0$

$\Rightarrow [(1-x^2)D^2-xD+2x^2]y=0$.

General Solution of $f(D)y=0$:-

If $y=y_1(x)$, $y=y_2(x)$ ... $y=y_n(x)$ are 'n' linearly independent solutions of nth order linear equation $f(D)y=0$, then $y=c_1y_1+c_2y_2+\dots+c_ny_n$, where $c_1, c_2 \dots c_n$ are arbitrary constants, is called the general solution of $f(D)y=0$

General Solution of $f(D)y=Q$:-

If $y=y_p(x)$ be a particular solution of $f(D)y=Q$ on an interval I and $y=y_c(x)$ be a solution of $f(D)\cdot y=0$, then $y=y_c+y_p$ is a solution of $f(D)y=Q$.

Complementary function (C.F) and particular Integral (P.I) of $f(D)y=Q$:-

Let $y=y_c+y_p$ be the general solution of $f(D)y=Q$. Then the part 'y_c' of the general solution is called the complementary function (C.F) of $f(D)y=Q$ and the part 'y_p' of the general solution is called the particular integral (P.I) of $f(D)y=Q$.

Auxiliary Equation (A.E) :-

Consider the differential equation $(D^n+P_1D^{n-1}+\dots+P_n)y=Q \Rightarrow f(D)y=Q \longrightarrow (1)$

The algebraic equation $f(m)=0 \Rightarrow m^n+P_1m^{n-1}+\dots+P_n=0$ where $P_1, P_2, \dots, P_n$ are real constants, is called the auxiliary equation (A.E) of $f(D)y=0$.

Since the auxiliary equation $f(m)=0$ is an algebraic equation of degree 'n', it will have 'n' roots. Then three cases arise.

Case.1. $f(m)=0$ may have real and distinct roots.

Case.2. $f(m)=0$ may have real and equal roots.

Case.3. $f(m)=0$ may have real and complex roots.

Case.1. When the Auxiliary Equation has Real and Distinct Roots :-

The Auxiliary Equation of $f(D)y=0$ is

$f(m)=0 \Longrightarrow m^n+P_1m^{n-1}+\cdots+P_n=0$

Let $m_1, m_2, \cdots, m_n$ be 'n' real and distinct roots. We know that if m_1 is a root of the equation $f(m)=0$, then $y=e^{m_1x}$ is a solution of $f(D)y=0$

$\therefore$ The n solutions of $f(D)y=0$ are $y_1=e^{m_1x}$, $y_2=e^{m_2x}$, $y_3=e^{m_3x}$, $\cdots$, $y_n=e^{m_nx}$. Since these 'n' solutions are linearly independent, the general solution of $f(D)y=0$ is

$y=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+\cdots+C_ne^{m_nx}$ where $C_1, C_2 \ldots C_n$ are any real constants.

problems:-

(1). Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Soln:- Given equation in operator form is

$$D^2y - 3Dy + 2y = 0 \qquad \left(\frac{d}{dx} = D,\ \frac{d^2}{dx^2} = D^2,\ \vdots\ \frac{d^n}{dx^n} = D^n \right)$$

$$\Rightarrow (D^2 - 3D + 2)y = 0$$

$$\Rightarrow f(D)y = 0$$

where $f(D) = D^2 - 3D + 2$

The Auxiliary Equation is

$$f(m) = 0$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\therefore m = 1, 2$$

The roots are real and distinct.

So, the complementary function form is

$$y_c = C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\boxed{y_c = c_1 e^{x} + c_2 e^{2x}}$$

and the particular Integral does not Exist because the equation is linear and homogeneous (i.e.)

So that $y_p = 0$

$\therefore$ the Complete Solution

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{2x} + 0.$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x}}$$

(5). Solve $(D^3 + 6D^2 + 11D + 6)y = 0$

Sol:-

Given $f(D) = D^3 + 6D^2 + 11D + 6$

The Auxiliary Equation is

$$f(m) = 0$$

$$\Rightarrow m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m+3)(m+2)(m+1) = 0$$

$$\begin{array}{r|rrrr} -3 & 1 & 6 & 11 & 6 \\ & 0 & -3 & -9 & -6 \\ \hline -2 & 1 & 3 & 2 & \lfloor 0 \\ & 0 & -2 & -2 & \\ \hline & 1 & 1 & \lfloor 0 & \end{array}$$

$\therefore m = -3, -2, -1$ are the real and distinct roots of $f(m) = 0$.

$\therefore$ The complementary function form is

$$y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

and particular Integral does not Exist

$$y_p = 0$$

∴ Hence

The Complete Solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}}$$

(3). Solve the following D.E

(a). $\dfrac{d^2y}{dx^2} - a^2 y = 0$

(b). $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$

(c). $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

(a).
Sol^n:- Given equation in operator form is

$$D^2y - a^2y = 0$$

$$\Rightarrow (D^2 - a^2)y = 0$$

$$f(D)y = 0$$

where $f(D) = D^2 - a^2$

∴ The Auxiliary Eq is

$f(m) = 0 \Rightarrow m^2 - a^2 = 0$

$\Rightarrow (m-a)(m+a) = 0$

$m = a, -a$

So, the C.F is

$y_c = c_1 e^{ax} + c_2 e^{-ax}$.

Hence the C.S is $\boxed{y = y_c + y_p = c_1 e^{ax} + c_2 e^{-ax}}$

Case. 2:- when A.E has some equal roots:-

(i). let $f(m)=0$ have two equal roots $m_1=m_2$ and other distinct roots $m_3, m_4, \ldots, m_n$. Then general solution of $f(D)y=0$ is

$$y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+\cdots+c_ne^{m_nx}$$

(ii). let $f(m)=0$ have three equal roots $m_1=m_2=m_3$ and all other distinct roots $m_4, m_5, \ldots, m_n$. Then general solution of $f(D)y=0$ is

$$y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+\cdots+c_ne^{m_nx}$$

(iii). If $f(m)=0$ has k equal roots $m_1=m_2=\cdots=m_k$ and all other distinct roots $m_{k+1}, m_{k+2}, \ldots$ then general solution is

$$y=(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})e^{m_1x}+c_{k+1}e^{m_{k+1}x}+\cdots+c_ne^{m_nx}.$$

problems:-

(1). Solve $\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=0$

Sol:- Given equation in operator form is

$$D^3y-3Dy+2y=0$$

$$\Rightarrow (D^3-3D+2)y=0.$$

$\Rightarrow f(D)y = 0$

where $f(D) = D^3 - 3D + 2$

The Auxiliary Equation is

$f(m) = 0$

$\Rightarrow m^3 - 3m + 2 = 0$

$\Rightarrow (m^2 + m - 2)(m-1) = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow m-1=0, \quad m^2+m-2=0$

$m=1, \quad m^2+2m-m-2=0$

$m(m+2)-1(m+2)=0$

$(m+2)(m-1)=0$

$m=-2, 1$

$\therefore m=1,1,-2$ are the roots of $f(m)=0$

Here $f(m)=0$ have Two equal roots and all other distinct roots

Then $y=(c_1+c_2x)e^{x}+c_3e^{-2x}$

(2). Solve $(D^4-2D^3-3D^2+4D+4)y=0$

Sol:- Given equation is operator form.

$(D^4-2D^3-3D^2+4D+4)y=0$

$\Rightarrow f(D)y=0$

where $f(D)=D^4-2D^3-3D^2+4D+4$

The Auxiliary Equation is

$$f(m)=0$$

$$\Rightarrow m^4-2m^3-3m^2+4m+4=0$$

$$\Rightarrow (m+1)(m+1)(m^2-4m+4)=0$$

-1	1	-2	-3	4	4
	0	-1	3	0	-4
-1	1	-3	0	4	0
	0	-1	4	-4	
	1	-4	4	0	

$$\Rightarrow m=-1,-1,\ m^2-4m+4=0$$

$$\Rightarrow (m-2)^2=0$$

$$(m-2)(m-2)=0$$

$$m=2,2$$

$\therefore m=-1,-1,2,2$ are roots of $f(m)=0$

Since the roots $-1, 2$ each repeated twice

Hence the general Solution is

$$y=(c_1+c_2x)e^{-x}+(c_3+c_4x)e^{2x} \,/\!/.$$

(2). Solve the following D.E

(i). $(D^4-4D^3+6D^2-4D+1)y=0$

(ii). $\dfrac{d^3y}{dx^3}-3\dfrac{dy}{dx}+2y=0$

(iii). $(D^3-3D^2+3D-1)y=0$

(iv). $(D^4+6D^3+9D^2)y=0$

(v). $(D^3-5D^2+8D-4)y=0.$

(i) Sol^n:-

Given $f(D) = D^4 - 4D^3 + 6D^2 - 4D + 1$

The Auxiliary Equation is

$f(m) = 0$

$\Rightarrow m^4 - 4m^3 + 6m^2 - 4m + 1 = 0$

$\Rightarrow \cancel{(m+1)(m+1)(m^2-4m+4)=0}$

$\Rightarrow \cancel{m+1=0,\ m+1=0},$

$\cancel{m^2-4m+4=0}$

$\Rightarrow \cancel{m=-1,-1},\ \cancel{(m-2)^2=0}$

$\cancel{m=2,2}$

$\Rightarrow m^4 - 4m^3 + 6m^2 - 4m + 1 = 0$

$\Rightarrow (m-1)(m-1)(m-1)(m+1) = 0$

$m = 1, 1, 1, -1$

are root of $f(m) = 0$

Since three roots are equal and other root is distinct

$\therefore$ the C.F is

$$y = (c_1 + c_2 x + c_3 x^2)e^{x} + c_4 e^{-x}$$

1	1	-4	6	-4	1
	0	1	-3	3	-1
1	1	-3	3	-1	0
	0	1	-2	1	
1	1	-2	1	0	
	0	1	-1		
	1	-1	0		

Case.3:- when the A.E has only a pair of Conjugate Complex Roots :-

Let The auxiliary equation $f(m)=0$ are real and Complex. let $a+ib$ (a, b being real and $b\neq 0$) be a Complex roots of $f(m)=0$. Since the co-efficients of $f(m)=0$ are real Constants, the complex roots occur in Conjugate pairs. Hence $a-ib$ is also a root of $f(m)=0$

Let the other real roots of $f(m)=0$ be $m_3, m_4, \dots, m_n$

$\therefore$ The general Solution of $f(D)y=0$ is

$$y = e^{ax}(c_1\cos bx + c_2\sin bx) + c_3e^{m_3x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}.$$

Note:- (i). In case a pair of Conjugate Complex roots $a\pm ib$ of $f(m)=0$ are repeated twice and the remaining roots of $f(m)=0$ are real and distinct, then the general Solution $f(D)y=0$ is

$$y = (c_1+c_2x)e^{ax}\cos bx + (c_3+c_4x)e^{ax}\sin bx + c_5e^{m_5x} + \dots + c_ne^{m_nx}$$

(ii). let $a+\sqrt{b}$ (a, b being real and $b>0$) be an irrational root of $f(m)=0$. Since irrational roots of $f(m)=0$ occur in Conjugate pairs, $a-\sqrt{b}$ is also a root of $f(m)=0$

the general Soln of $f(D)y=0$ is

$$y = e^{ax}\left(c_1 \cosh(x\sqrt{b}) + c_2 \sinh(x\sqrt{b})\right) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

problems:-

(1). Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Sol:- Given equation in operator form is

$$(D^2y + Dy + y) = 0$$

$$\Rightarrow (D^2+D+1)y = 0$$

$$f(D)y = 0$$

where $f(D) = D^2+D+1$

The Auxiliary Equation is $\quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$f(m) = 0 \Rightarrow m^2+m+1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} \quad (i^2=-1)$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$m = \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ are roots of ...

$\therefore$ the General Solution is

$$\boxed{y = e^{-x/2}\left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(x\sqrt{3}/2\right)\right)}$$

(2). Solve $(D^4+8D^2+16)y=0$

Sol^n:- Given equation is

$$(D^4+8D^2+16)y=0$$

$$\Rightarrow f(D)y=0$$

where $f(D) = D^4+8D^2+16$

The auxiliary equation is

$f(m)=0$

$\Rightarrow m^4+8m^2+16=0$

$\Rightarrow (m^2+4)^2=0$

~~$\Rightarrow (m^2+2i)(m^2+2i)^2=0$~~

$\Rightarrow (m^2-(2i)^2)^2=0 \Rightarrow (m^2-(2i)^2)(m^2-(2i)^2)=0$

$\Rightarrow (m-2i)(m+2i)(m-2i)(m+2i)=0$

$\Rightarrow (m-2i)^2(m+2i)^2=0.$

$\Rightarrow m = 2i, 2i, -2i, -2i$ are roots of $f(m)=0$

Since, $2i, -2i$, occur twice each.

$\therefore$ The G.S of $f(D)y=0$ is

$$y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x.$$

(2). Solve $(D^3-14D+8)y=0$

Sol:- Given equation is

$$(D^3-14D+8)y=0$$

$$\Rightarrow f(D)y=0$$

where $f(D)=D^3-14D+8$

$\therefore$ The A.E is

$$f(m)=0 \Rightarrow m^3-14m+8=0$$

$$\Rightarrow (m+4)(m^2-4m+2)=0$$

$$\begin{array}{c|cccc} -4 & 1 & 0 & -14 & 8 \\ & 0 & -4 & +16 & -8 \\ \hline & 1 & -4 & +2 & \boxed{0} \end{array}$$

$$\Rightarrow m+4=0,\ m^2-4m+2=0$$

$$m=-4,\quad m=\frac{+4\pm\sqrt{16-8}}{2}$$

$$=\frac{4\pm\sqrt{8}}{2}=\frac{4\pm2\sqrt{2}}{2}$$

$$m=2\pm\sqrt{2}$$

$\therefore$ $m=-4,\ 2+\sqrt{2},\ 2-\sqrt{2}$ are roots of $f(m)=0$

$\therefore$ The G.S of $f(D)y=0$ is

$$y=c_1e^{-4x}+e^{2x}\left(c_2\cosh(\sqrt{2}x)+c_3\sinh(\sqrt{2}x)\right)$$

(4). Solve the following ODE

(i). $((D-1)^2(D^2+1)^2)y=0$

(ii). $D^3y-8y=0$

(iii). $\frac{d^4y}{dx^4}+4y=0$

(iv). $\frac{d^4y}{dx^4}+18\frac{d^2y}{dx^2}+81y=0$.

Inverse Operator :-

* The operator "D^{-1}" is called the inverse of the differential operator "D". That is, if Q is any function of 'x' defined on an interval I, then $D^{-1}Q$ (or) $\frac{1}{D}Q$ is called the integral of "Q"

Note:- * If Q is any function of x defined on an interval I and α is a constant, then a particular value of $\frac{1}{D-\alpha}Q$ is equal to $e^{\alpha x}\int Q e^{-\alpha x}dx$.

* D is the differential equation operator

$\Rightarrow \frac{1}{D}$ is an integral operator.

problem3:-

(1). Find the particular value of

(a). $\frac{1}{D}x^2$ (b). $\frac{1}{D^2}e^{4x}$ (c). $\frac{1}{D^3}\cos x$

(a).

Sol:- Given $\frac{1}{D}x^2 = \int x^2\,dx = \frac{x^3}{3}$ $\quad \int x^n dx = \frac{x^{n+1}}{n+1}$

(b).

Sol:- Given
$$\frac{1}{D^2}e^{4x} = \frac{1}{D}\left(\frac{1}{D}e^{4x}\right)$$
$$= \frac{1}{D}\left[\int e^{4x}\,dx\right]$$
$$= \frac{1}{D}\left(\frac{e^{4x}}{4}\right) = \int \frac{e^{4x}}{4}\,dx$$
$$= \frac{e^{4x}}{16}$$

(c)

Sol:- Given
$$\frac{1}{D^3}\cos x = \frac{1}{D}\left(\frac{1}{D}\left(\frac{1}{D}\cos x\right)\right)$$
$$= \frac{1}{D}\left(\frac{1}{D}\int \cos x\,dx\right)$$
$$= \frac{1}{D}\left(\frac{1}{D}\sin x\right)$$
$$= \frac{1}{D}\left(\int \sin x\,dx\right)$$
$$= \frac{1}{D}\left(-\cos x\right)$$
$$= -\int \cos x\,dx$$
$$= -\sin x$$

practice problems

(2). Find the particular values of the following

(i). $\frac{1}{D^2} e^{x}$ (ii). $\frac{1}{D^2} x^4$ (iii). $\frac{1}{D^4} \sin x$

(3). Find the particular values of

(a). $\frac{1}{D+1} x$ (b). $\frac{1}{D-2} e^{2x}$ (c). $\frac{1}{D+3} \cos 2x$

(a).
Solution:- Given $\frac{1}{D+1} x = \frac{1}{D-(-1)} x$

$= e^{-x} \int x e^{x} dx$

$= e^{-x} \left\{ x \int e^{x} dx - \int \left(\frac{d}{dx} x \int e^{x} dx \right) \right\}$

$= e^{-x} \left\{ x e^{x} - \int 1 \cdot e^{x} dx \right\}$

$\frac{1}{D+1} x = e^{-x} \left\{ x e^{x} - e^{x} \right\}$

$= e^{-x} e^{x} (x-1)$

$= x-1$

(b).
Sol^n:- Given $\frac{1}{D-2} e^{2x} = e^{2x} \int e^{2x} e^{-2x} dx$

$= e^{2x} \int 1 \cdot dx$

$= e^{2x} \cdot x$

(c).

Solution:- Given

$$\frac{1}{D+3}\cos x = \frac{1}{D-(-3)}\cos x$$

$$= e^{-3x}\int \cos x\, e^{3x}\, dx$$

$$= e^{-3x}\left(\frac{e^{3x}}{1+9}(3\cos x+\sin x)\right)$$

$$\frac{1}{D+3}\cos x = \frac{e^{-3x+3x}}{10}(3\cos x+\sin x)$$

$$= \frac{3\cos x+\sin x}{10}$$ //.

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

practice Problem:-

(4). Find the P.V of (i). $\frac{1}{D-2}e^{3x}$ (ii). $\frac{(x+\sin x)}{D+2}$

(5). Find the particular value of (a). $\frac{1}{(D-2)(D-3)}e^{2x}$

(b). $\frac{1}{(D+1)(D-1)}x$

(a).

Soln:- Given

$$\frac{1}{(D-2)(D-3)}e^{2x} = \frac{1}{(D-2)}\left[\frac{1}{(D-3)}e^{2x}\right]$$

$$= \frac{1}{D-2}\left[e^{3x}\int e^{2x}e^{-3x}\, dx\right]$$

$$= \frac{1}{D-2}\left\{e^{3x}\int e^{-x}\,dx\right\}$$

$$= \frac{1}{D-2}\left\{e^{3x}\frac{e^{-x}}{-1}\right\} \qquad \int e^{ax}dx =$$

$$= \frac{1}{D-2}\left(-e^{2x}\right) \qquad \frac{d}{dx}e^{ax} = a\cdot$$

$$= e^{2x}\int -e^{2x}e^{-2x}\,dx \qquad e^{3x}\cdot e^{-x}$$

$$= -e^{2x}\int 1\,dx \qquad = e^{3x-x}$$

$$\frac{1}{(D-2)(D-3)}e^{2x} = -xe^{2x} \text{ //.} \qquad = e^{2x}$$

(b). Given

Sol^n:-

$$\frac{1}{(D+1)(D-1)}x = \frac{1}{D+1}\left[\frac{1}{D-1}x\right] \qquad \int fg\,dx = f\int g\,dx - \int\left(\frac{df}{dx}\int g\,dx\right)dx$$

$$= \frac{1}{D+1}\left\{e^{x}\int xe^{-x}\,dx\right\}$$

$$= \frac{1}{D+1}\left\{e^{x}\left\{x\int e^{-x}dx - \int\left(\frac{d}{dx}x\int e^{-x}dx\right)dx\right\}\right\}$$

$$= \frac{1}{D+1}\left\{e^{x}\left\{x\frac{e^{-x}}{-1} - \int\frac{e^{-x}}{-1}dx\right\}\right\}$$

$$= \frac{1}{D+1}\left\{e^{x}\left\{-xe^{-x} - e^{-x}\right\}\right\}$$

$$= \frac{1}{D+1}\left\{-e^{x}e^{-x}(x+1)\right\}$$

$$\frac{1}{(D+1)(D-1)}x = \frac{-(x+1)}{D+1} = \frac{1}{D-(-1)}\left(-(x+1)\right).$$

$$= e^{-x}\int -(x+1)e^{x}\,dx$$

$$= -e^{-x}\left\{\int xe^{x}\,dx + \int e^{x}\,dx\right\}$$

$$= -e^{-x}\left[\left(x\int e^{x}dx - \int\left(\frac{d}{dx}x\int e^{x}dx\right)dx\right) + e^{x}\right]$$

$$= -e^{-x}\left[\left(xe^{x} - e^{x}\right) + e^{x}\right]$$

$$= -e^{-x}\left[e^{x}(x-1) + e^{x}\right]$$

$$= -e^{-x}\cdot e^{x}x + e^{-x}e^{x} - e^{-x}e^{x}$$

$$= -x + 1 - 1$$

$$\frac{1}{(D+1)(D-1)}x = -x \quad //.$$

Practice problems :-

(6). Find the P.I of the following

(i). $\frac{1}{(D+3)(D-1)}e^{3x}$ (ii). $\frac{1}{(D+2)(D-2)}x$

(iii). $\frac{1}{(D+3)(D-3)}\cos 3x$.

Method of Finding particular Integral :-

Consider the differential equation $f(D)y = Q$

then the particular integral $y_p = P.I = \frac{1}{f(D)} Q$

Method :-1 P.I of $f(D)y = Q(x)$ when $Q(x) = e^{ax}$ where a is Constant.

working Rule to find value of $\frac{1}{f(D)} e^{ax}$:-

* put $D = a$. If $f(a) \neq 0$, then

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

* If $f(a) = 0$ and no factor of $f(D)$ is repeated

then $y_p = \frac{1}{f(D)} e^{ax} = \frac{x}{f'(a) 1!} e^{ax}$, if $f'(a) \neq 0$

* If $f'(a) = 0$ then, the P.I is

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{x^2/2!}{f''(a)} e^{ax}, \text{ if } f''(a) \neq$$

* Continue this process untill we get the Soluti

Problems :-

(1). Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$

Sol :- Given equation in operator form is

$$D^2y + Dy + y = e^x$$

$(D^2+D+1)y = e^x$

$f(D)y = e^x$

where $f(D) = D^2+D+1$

The A.E is $f(m)=0 \Rightarrow m^2+m+1=0$

$\Rightarrow m = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2}$

$m = \frac{-1\pm i\sqrt{3}}{2}$

$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

The Complementary function is

$y_c = C.F = \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x}$

The particular Integral is

$y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+D+1} e^x$

$y_p = \frac{1}{1+1+1} e^x = \frac{e^x}{3}$

The Complete Solution is $y = y_c + y_p$

$\Rightarrow y = \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)e^{-x/2} + \frac{e^x}{3}$ //

(2). Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$

Sol:- Given equation in operator form is

$D^2y - 3Dy + 2y = e^{2x}$.

$$(D^2-3D+2)y = e^{2x}$$

$$f(D)y = Q(x)$$

where $f(D) = D^2-3D+2$, $Q(x) = e^{2x}$

The A.E is $f(m)=0$

$\Rightarrow m^2-3m+2=0$

$\Rightarrow m=1,2$ are roots of real and dist

∴ the Complementary function is

$$y_c = c_1e^x + c_2e^{2x}$$

The P.I is

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2-3D+2} e^{2x}$$

$$y_p = \frac{e^{2x}}{f(2)} = \frac{1}{4-6+2=0} e^{2x}$$

$$y_p = \frac{x e^{2x}}{f'(2)} = \frac{xe^{2x}}{2D-3}$$

$$= \frac{xe^{2x}}{4-3} = \frac{xe^{2x}}{1}$$

$$y_p = xe^{2x}$$

∴ The complete solution is

$$y = y_c + y_p$$

$$y = c_1e^x + c_2e^{2x} + xe^{2x}$$

(8). Solve the following D.E's

(i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

(ii). $(4D^2 - 4D + 1)y = 100e^{0x}$

(iii). $(D^2 - 5D + 6)y = 4e^{x} + 5$

(iv). $(D^2 + 6D + 9)y = 2e^{-3x}$

(v). $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

(vi). $(D^2 + 4D + 4)y = 18\cosh x$.

(vii). $(D+2)(D-1)^2 y = e^{-2x} + \sinh x$.

(viii). $(D^2 - 3D + 2)y = \cosh x$.

$\left(\cosh x = \frac{e^{x}+e^{-x}}{2} \quad \sinh x = \frac{e^{x}-e^{-x}}{2} \right)$

(viii).

Sol^n:

Given eq in operator form is

$$(D^2 - 3D + 2)y = \cosh x$$

$\Rightarrow f(D)y = Q(x)$

where $f(D) = D^2 - 3D + 2$, $Q(x) = \cosh x$

The A.E is $f(m) = 0 \Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$(m-2)(m-1) = 0 \Rightarrow m = 1, 2$

$\therefore$ The C.F is

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

The P.I is $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{D^2 - 3D + 2} \cosh x$$

$y_p = \frac{1}{D^2-3D+2}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{f(D)}\left(\frac{e^x}{2}+\frac{e^{-x}}{2}\right)$

$= \left(\frac{1}{D^2-3D+2}\left(\frac{e^x}{2}\right)\right) + \left(\frac{1}{D^2-3D+2}\left(\frac{e^{-x}}{2}\right)\right)$

$= \frac{1}{2}\left(\frac{1}{1-3+2}e^x\right) + \left(\frac{1}{2}\frac{e^{-x}}{1+3+2}\right)$

The G.S. is $y = y_c + y_p$

$y_p = \frac{1}{2}\frac{xe^x}{(2D-3)}$

$= \frac{1}{2}\frac{xe^x}{(2-3)}$

$y_p = \frac{xe^x}{-2}$

Working Rule to find P.I. when Q = Sinbx (or) cosbx.

* To find the value $\frac{1}{f(D^2)}$ Sinbx, write $D^2 = -b^2$

Then $P.I = \frac{1}{f(D)}$ Sinbx

$P.I = \frac{1}{f(D^2)}$ Sinbx $= \frac{1}{f(-b^2)}$ Sinbx,

$P.I = \frac{1}{f(-b^2)}$ Sinbx, if $f(-b^2) \neq 0$

Similarly $P.I = \frac{1}{f(D)}$ cosbx $= \frac{1}{f(D^2)}$ cosbx

$P.I = \frac{1}{f(-b^2)}$ cosbx, if $f(-b^2) \neq 0$

* If $f(-b^2) = 0$, then

$P.I = \frac{1}{(D^2+b^2)f(D^2)}$ (Sinbx (or) cosbx)

$P.I = \frac{1}{f(-b^2)} \cdot \frac{1}{D^2+b^2}$ (Sinbx or cosbx)

where $f(-b^2) \neq 0$

Then write,

$\frac{1}{D^2+b^2}$ Sinbx $= \frac{-x}{2b}$ cosbx and

$$\frac{1}{D^2+b^2}\cos bx = \frac{-x}{2b}\sin bx$$

* If $f(D)$ contains odd powers of D, write

$$D^3 = D^2 \cdot D = (-b^2)D, \quad D^5 = D^2 \cdot D^3 = (-b^2) \cdot D^3$$
$$= (-b^2) \cdot D^2 \cdot D$$
$$= (-b^2)(-b^2)D$$

Problems:-

(1). Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x$

Sol:- Given equation in operator form is

$$D^2y - Dy - 2y = \sin 2x$$
$$(D^2 - D - 2)y = \sin 2x$$
$$f(D)y = Q(x)$$

where $f(D) = D^2 - D - 2$, $Q(x) = \sin 2x$

The Auxiliary Equation is

$$f(m) = 0 \Rightarrow m^2 - m - 2 = 0$$
$$\Rightarrow m^2 - 2m + m - 2 = 0$$
$$\Rightarrow m(m-2) + 1(m-2) = 0$$
$$(m-2)(m+1) = 0$$

$$\therefore m = -1, 2$$

The complementary function is

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

The particular Integral is $y_p = \frac{1}{f(D)} Q(x)$

$$y_p = \frac{1}{D^2 - D - 2}\sin 2x$$

$$y_p = \frac{1}{-2^2-D-2}\sin 2x$$

$$= \frac{1}{-4-D-2}\sin 2x$$

$$= \frac{-1}{D+6}\sin 2x.$$

$$y_p = \frac{-1}{D+6}\times\left(\frac{D-6}{D-6}\right)\sin 2x$$

$$= \frac{-(D-6)}{D^2-6^2}\sin 2x$$

$$y_p = \frac{-(D-6)}{D^2-36}\sin 2x$$

$$= \frac{-(D-6)}{-2^2-36}\sin 2x$$

$$= \frac{-D+6}{-40}\sin 2x \qquad \left(D = \frac{d}{dx}\right.$$

$$= \frac{D-6}{40}(\sin 2x) \qquad \Rightarrow \frac{d}{dx}(\sin 2x) = 2\cos 2x$$

$$y_p = \frac{1}{40}(2\cos 2x - 6\sin 2x)$$

∴ The General Solⁿ is

$$y = y_p + y_c$$

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{40}(2\cos 2x - 6\sin 2x)$$

(2). Solve $(D^3+D^2-D-1)y = \cos 2x$

Solⁿ:- Given $f(D) = D^3+D^2-D-1$, $Q(x) = \cos 2x$

The A.E is $f(m) = 0$

$$\Rightarrow m^3 + m^2 - m - 1 = 0$$

$$\Rightarrow (m-1)(m+1)(m+1) = 0$$

$$\Rightarrow m = 1, -1, -1 \text{ are roots of } f(m) = 0 \text{ then}$$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -1 & -1 \\ & 0 & 1 & 2 & \\ \hline -1 & 1 & 2 & 1 & 0 \\ & 0 & -1 & -1 & \\ \hline -1 & 1 & 1 & 0 & \\ & 0 & -1 & & \\ \hline & 1 & 0 & & \end{array}$$

$$y_c = c_1 e^{x} + (c_2 + c_3 x) e^{-x}$$

The P.I is

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^3 + D^2 - D - 1} \cos 2x$$

$$y_p = \frac{1}{-2^2 D + (-2^2) - D - 1} \cos 2x$$

$$= \frac{1}{-4D - 4 - D - 1} \cos 2x$$

$$y_p = \frac{1}{-5D - 5} \cos 2x$$

$$= \frac{1}{-5(D+1)} \cos 2x$$

$$= -\frac{1}{5} \frac{1}{D+1} \times \frac{D-1}{D-1} (\cos 2x)$$

$$= -\frac{1}{5} \frac{(D-1)}{D^2 - 1} \cos 2x$$

$$= -\frac{1}{5} \frac{D-1}{-4-1} \cos 2x$$

$$= \frac{1}{25} (D-1) \cos 2x$$

$$y_p = \frac{1}{25} (-2\sin 2x - \cos 2x)$$

The G.S is

$$y = y_p + y_c$$

$$y = c_1 e^{x} + (c_2 + c_3 x) e^{-x} - \frac{2}{25} \sin 2x - \frac{1}{25} \cos 2x$$

(3). Solve the following D.E's

(i). $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x + \cos 2x$

(ii). $(D^4 + 3D^2 - 4)y = \cos^2 x - \cosh x$ $\qquad \cos^2 x =$

(iii). $(D^2 - 1)y = \cos x$ $\qquad \cosh x = \frac{e^x}{2}$

(iv). $(D^3 + 4D)y = \sin 2x$

(v). $(D^2 - 1)y = \sin x$

(vi). $(D^3 - D^2 + D - 1)y = e^{-x} + \cos x$.

(vii). $(D^2 + 16)y = e^{-3x} + \cos 4x$.

(i).

Sol^n:-

Working rule to find $\frac{1}{f(D)} x^k$:-

* Take out common the lowest degree term from $f(D)$.
* Let the remaining factor in the denominator be of the form $(1+f(D))$ (or) $(1-f(D))$.
  Take this factor to numerator as $(1+f(D))^{-1}$ (or) $(1-f(D))^{-1}$
* Now expand $(1 \pm f(D))^{-1}$ in ascending powers of 'D' by the binomial theorem up to D^k and operate upon x^k using each term of the expansion.

Note :- (i). $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

(ii). $(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

(iii). $(1-x)^{-1} = 1+x+x^2+\cdots$

(iv). $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

problems :-

(1). Solve $(D^2-4D+4)y = x^3$

Sol:- Given equation is $(D^2-4D+4)y = x^3$

$$\Rightarrow f(D)y = Q(x)$$

where $f(D) = D^2-4D+4$, $Q = x^3$

The A.E is $f(m)=0$

$$\Rightarrow m^2-4m+4=0$$

$$\Rightarrow (m-2)^2=0$$

$m = 2, 2$ are roots of $f(m)$

$\therefore$ The C.F is

$$y_c = (c_1+c_2x)e^{2x}$$

The P.I is

$$y_p = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-4D+4}x^3$$

$$y_p = \frac{1}{(D-2)^2}x^3$$

$$= \frac{1}{\left(-2\left(1-\frac{D}{2}\right)\right)^2}x^3$$

$(1-x)^{-2} = 1+2x+3x^2+4x^3 \ldots$

$$y_p = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^3$$

$D = \frac{d}{dx}$

$\frac{1}{D} = \int$

$$= \frac{1}{4}\left[\left(1-\frac{D}{2}\right)^{-2}x^3\right]$$

$$= \frac{1}{4}\left[\left(1+2\frac{D}{2}+3\frac{D^2}{4}+4\frac{D^3}{8}+\ldots\right)x^3\right]$$

$$= \frac{1}{4}\left[x^3 + Dx^3 + \frac{3}{4}D^2x^3 + \frac{1}{2}D^3x^3 + \cdots\right]$$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{3}{4_2}(6x) + \frac{1}{8}(6)\right]$$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{9}{2}x + 3\right]$$

$$y_p = \frac{1}{8}\left[2x^3 + 6x^2 + 9x + 6\right]$$

$\therefore$ The general Solution is

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^{2x} + \frac{1}{8}(2x^3 + 6x^2 + 9x + 6)$$

(2). $(D^2 - 3D + 2)y = 2x^2$ Solve this D.Eq

Sol:- Given equation is $(D^2 - 3D + 2)y = 2x^2$

$$\Rightarrow f(D)y = Q(x)$$

where $f(D) = D^2 - 3D + 2$, $Q = 2x^2$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 - 3m + 2 = 0 \Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$\therefore m = 1, 2$ are root of $f(m) = 0$

$\therefore$ The C.F is

$$y_c = c_1 e^x + c_2 e^{2x}$$

The P.I is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$y_p = \frac{1}{(D-1)(D-2)} 2x^2$$

$$= 2\left[\frac{1}{D-2} - \frac{1}{D-1}\right] x^2$$

$$= 2\left[\frac{1}{D-2} x^2 - \frac{1}{D-1} x^2\right]$$

$$y_p = 2\left[\frac{1}{-2(1-\frac{D}{2})} x^2 - \frac{1}{-(1-D)} x^2\right]$$

$$y_p = 2\left[-\frac{1}{2}\left(1-\frac{D}{2}\right)^{-1} x^2 + (1-D)^{-1} x^2\right]$$

$$y_p = -\left(1-\frac{D}{2}\right)^{-1} x^2 + 2(1-D)^{-1} x^2$$

$$y_p = -\left[1+\frac{D}{2}+\frac{D^2}{4}+\cdots\right] x^2 + 2\left[1+D+D^2+\cdots\right] x^2$$

$$= -\left[x^2 + \frac{2x}{2} + \frac{2}{4}\right] + 2\left[x^2+2x+2\right]$$

$$= -x^2 - x - \frac{1}{2} + 2x^2 + 4x + 4$$

$$y_p = x^2 + 3x + \frac{7}{2}$$

$$y_p = \frac{1}{2}\left[2x^2 + 6x + 7\right].$$

∴ The general Solution is

$$y = y_c + y_p$$

$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}(2x^2 + 6x + 7)$$

practice problems :-

(2). Solve the following D.E's

(i). $(D^4-2D^3+D^2)y = x^3$

(ii). $(D^3-D^2-D+1)y = 1+x^2$

(iii). $(D^3+2D^2+D)y = e^{2x}+x^2+x$

(iv). $(D^4+D^2)y = 3x^2$

(v). $(D^4+D^2)y = 3x^2+4\sin x-2\cos x$.

(vi). $(D^2-2D+3)y = \cos x+x^2$

(vi) Sol^n:-

The A.E is $f(m)=0 \Rightarrow m^2-2m+3$

$$m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{2\pm\sqrt{4-12}}{2}$$

$$= \frac{2\pm\sqrt{-8}}{2}$$

$$= \frac{2\pm2\sqrt{2}i}{2}$$

$$m = 1\pm\sqrt{2}i$$

The C.F is

$$y_c = e^{x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$$

The P.I is

$$y_p = \frac{1}{f(D)}$$

$$= \frac{1}{D^2-2D+3}(\cos x+x^2)$$

$$= \frac{1}{D^2-2D+3}\cos x + \frac{1}{D^2-2D+3}x^2$$

$$= \frac{1}{-1^2-2D+3}\cos x + \frac{1}{D^2-2D+3}x^2$$

$$= \frac{1}{-2D+2}\cos x + \frac{1}{3\left(1+\left(\frac{D^2-2D}{3}\right)\right)}x^2$$

$$= \frac{1}{-2(D-1)}\cos 3x + \frac{1}{3}\left(1+\left(\frac{D^2-2D}{3}\right)\right)^{-1} x^3$$

$$= \frac{1}{2}\left[\frac{1}{D-1}\times\frac{D+1}{D+1}\cos 3x\right] + \frac{1}{3}\left\{1-\left(\frac{D^2-2D}{3}\right)+\frac{(D^2-2D)^2}{9}\right\}x^3$$

$$= \frac{1}{2}\left(\frac{D+1}{D^2-1}\cos 3x\right) + \frac{1}{3}\left\{x^3 - \frac{D^2x^3-2Dx^3}{3}\right.$$

$$+ \frac{1}{3}\left\{x^3 - \frac{D^2x^3-2Dx^3}{3} + \frac{D^4+4D^2-4D^3}{9}x^3\right\}$$

To find P.I of $f(D)y = Q$ when $Q = e^{ax}V$ where 'a' is real number and 'V' is a function of 'x' :-

Working Rule to find P.I of $f(D)y = e^{ax}V$:-

* Let the given equation be $f(D)y = e^{ax}V$
* To find P.I $= \frac{1}{f(D)}(e^{ax}.V)$, shift e^{ax} outside and after replacing 'D' by 'D+a', operate V by $\frac{1}{f(D+a)}$
* $\therefore$ P.I $= \frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$

Problems :-

(1). Solve $(D^2-2D+1)y = x^2e^{3x}$

Sol^n:- Given equation is

$$(D^2-2D+1)y = x^2e^{3x}$$

$f(D)y = Q(x)$

where $f(D) = D^2 - 2D + 1$, $Q(x) = x^2 e^{3x}$

The A.E is $f(m) = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$m = 1, 1$ are roots of $f(m) = 0$

$\therefore$ The C.F is

$y_c = (c_1 + c_2 x)e^x$

The particular Integral is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D-1)^2} x^2 e^{3x}$$

$$= e^{3x} \frac{1}{((D+3)-1)^2} x^2$$

$$= e^{3x} \frac{1}{(D+2)^2} x^2$$

$$y_p = \frac{e^{3x}}{4} \frac{1}{(1+\frac{D}{2})^2} x^2$$

$$= \frac{e^{3x}}{4} \left[\left(1+\frac{D}{2}\right)^{-2} x^2 \right]$$

$$y_p = \frac{e^{3x}}{4} \left[\left(1 - 2\frac{D}{2} + 3\frac{D^2}{4} - \cdots\right) x^2 \right]$$

$$= \frac{e^{3x}}{4} \left(x^2 - 2x + \frac{3}{\cancel{4}_2} \cdot 2 \right)$$

$$y_p = \frac{e^{3x}}{4} \left(x^2 - 2x + \frac{3}{2} \right) = \frac{e^{3x}}{8} (2x^2 - 4x + 3)$$

$\therefore$ The G.S is

$$y = y_c + y_p = (c_1 + c_2 x)e^x + \frac{e^{3x}}{8} (2x^2 - 4x + 3)$$

(a). Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

A: Given equation in operator form is

$$D^2y - 6Dy + 13y = 8e^{3x}\sin 2x$$

$$\Rightarrow (D^2 - 6D + 13)y = 8e^{3x}\sin 2x$$

$$\Rightarrow f(D)y = Q(x)$$

where $f(D) = D^2 - 6D + 13$, $Q(x) = 8e^{3x}\sin 2x$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 - 6m + 13 = 0$$

$$m = \frac{+6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2}$$

$m = 3 \pm 2i$ are roots of $f(m) = 0$

$\therefore$ The C.F is

$$y_c = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

The P.I is

$$y_p = \frac{1}{f(D)}Q(x)$$

$$= \frac{1}{D^2 - 6D + 13} 8e^{3x}\sin 2x$$

$$= 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 2x$$

$$= 8e^{3x}\frac{1}{D^2 + 9 + 6D - 6D - 18 + 13}\sin 2x$$

$$y_p = 8e^{3x} \frac{1}{D^2+4} \sin 2x$$

$$= 8e^{3x} \frac{1}{D^2+4} \sin 2x$$

$$= 8e^{3x} \frac{1}{-2^2+4} \sin 2x \quad \left(\frac{1}{D^2+b^2}\sin bx = \frac{x}{2b}\cos bx\right)$$

$$= 8e^{3x} \frac{1}{-4+4} \sin 2x.$$

$$= \cancel{\frac{-4e^{3x}\sin 3x}{9+5}} = 8e^{3x} \frac{x}{2D} \begin{matrix}\cos 2x \\ \cancel{\sin 2x}\end{matrix}$$

$$y_p = \cancel{\frac{-4e^{3x}\sin 3x}{2}} = \frac{8xe^{3x}}{2} \int \begin{matrix}\cos 2x \\ \cancel{\sin 2x}\end{matrix}\, dx$$

$$= 4xe^{3x}\left(\frac{\sin 2x}{2}\right)$$

$$y_p = 2xe^{3x} \sin 2x$$

$\therefore$ The general Solution is

$$y = y_c + y_p$$

$$= e^{3x}(c_1\cos 2x + c_2\sin 2x) - 2xe^{3x}\sin 2x.$$

(3). Solve $(D^2-2D)y = e^x \sin x$.

Sol:- Given equation is

$$(D^2-2D)y = e^x \sin x$$

$$\Rightarrow f(D)y = Q(x). \quad \text{where } f(D) = D^2-2D$$

The A.E is $f(m) = 0$

$$\Rightarrow m^2-2m = 0$$

$$m(m-2) = 0$$

$$m = 0,\ m = 2$$

The C.F. is

$y_c = c_1 e^{0x} + c_2 e^{2x}$

$y_c = c_1 + c_2 e^{2x}$

The particular Integral is

$y_p = \frac{1}{D^2 - 2D} e^x \sin x.$

$= e^x \frac{1}{(D+1)^2 - 2(D+1)} \sin x$

$= e^x \frac{1}{D^2 + 1 + 2D - 2D - 2} \sin x$

$= e^x \frac{1}{D^2 - 1} \sin x$

$= e^x \frac{1}{-1^2 - 1} \sin x$

$= e^x \frac{-1}{2} \sin x$

$y_p = \frac{-e^x \sin x}{2}$

The general Sol^n is

$y = y_c + y_p$

$y = c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2}$

(2). Solve the following D.E's

(i). $(D^2+2)y = x^2e^{3x} + e^x\cos 2x$

(ii). $(D^2-4)y = x\sinh x$

(iii). $(D^2-4D+1)y = e^{2x}\cos^2 x$

(iv). $(5D^2-D-2)y = xe^{-x}$

(v). $(D^2+2D+2)y = xe^x$

(vi). $(D^2-1)y = e^x(1+x^2)$

To find P.I of $f(D)y = Q$ where $Q = xV$ where V is a function of x:

Let $f(D)y = xV$ then

$$P.I = \frac{1}{f(D)}(xV)$$

$$\boxed{P.I = x\frac{1}{f(D)}V - \frac{f'(D)}{(f(D))^2}V}$$

problems:-

(1). Solve $(D^2+4)y = x\sin x$

Sol:

Given equation is $(D^2+4)y = x\sin x$

$$\Rightarrow f(D)y = xV$$

where $f(D) = D^2+4$

The A.E is $f(m) = 0$

$$\Rightarrow m^2+4=0 \Rightarrow m^2-(2i)^2=0$$

$$m^2 = -4 \qquad (m-2i)(m+2i)=0$$

$$m = \pm 2i$$

The C.F is

$$y_c = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

The particular Integral is

$$y_p = \frac{1}{f(D)} xV$$

$$= \frac{1}{D^2+4} x \sin x$$

$$= x \frac{1}{D^2+4} \sin x - \frac{2D}{(D^2+4)^2} \sin x$$

$$y_p = x \frac{(\sin x)}{-1^2+4} - \frac{2D}{(-1^2+4)^2} \sin x$$

$$= \frac{x(\sin x)}{3} - \frac{2D}{9} \sin x$$

$$y_p = \frac{x(\sin x)}{3} - \frac{2}{9} \cos x.$$

$\therefore$ The general Solution is

$$y = y_c + y_p$$

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x \sin x}{3} - \frac{2}{9} \cos x$$

(2). Solve $(D^2+1)y = x^2 \sin 2x$

Sol:- Given equation is $(D^2+1)y = x^2 \sin 2x$

$$\Rightarrow f(D)y = Q(x).$$

where $f(D) = D^2+1$

$\therefore$ The A.E is $f(m) = 0$

$\Rightarrow m^2+1=0 \Rightarrow m^2-(i^2)=0$

$m^2=-1$

$m=\pm i$ are roots of $f(m)=0$

The C.F is

$$y_c=(c_1\cos x+c_2\sin x)e^{0x}$$

$$y_c=(c_1\cos x+c_2\sin x)$$

The particular Integral is

$$y_p=\frac{1}{f(D)}Q(x)$$

$$y_p=\frac{1}{D^2+1}x^2\sin 2x$$

$$=\frac{1}{D^2+1}x(x\sin 2x)$$

$$=\left[x\frac{1}{D^2+1}x\sin 2x-\frac{2D}{(D^2+1)^2}x\sin 2x\right]$$

$$y_p=\left[x\left\{x\frac{1}{D^2+1}\sin 2x-\frac{2D}{(D^2+1)^2}\sin 2x\right\}\right.$$

$$\left.-x\frac{2D}{(D^2+1)^2}\sin 2x+\frac{2(2D)(D^2+1)(2D)}{(D^2+1)^4}\sin\right]$$

$$=x\left\{x\frac{1}{-2^2+1}\sin 2x-\frac{2D}{(-2^2+1)^2}\sin 2x\right\}$$

$$-x\frac{2D}{(-2^2+1)^2}\sin 2x+\frac{8D^2\cancel{(D^2+1)}}{(D^2+1)^3}\sin 2x$$

$$= x\left\{ x\frac{1}{-3}\sin 2x - \frac{2}{9}\,2\cos 2x \right\}$$

$$- x\frac{2D}{9}\sin 2x + \frac{8D^2}{(-2^2+1)^2}\sin 2x$$

$$y_p = \frac{-x^2}{3}\sin 2x - \cancel{\frac{4x}{9}}\;\frac{4x}{9}\cos 2x$$

$$-\frac{2x}{9}(2\cos 2x) + \frac{8}{27}D(2\cos 2x)$$

$$y_p = -\frac{x^2}{3}\sin 2x - \frac{4x}{9}\cos 2x - \frac{4x}{9}\cos 2x$$

$$+\frac{16}{27}(2\sin 2x)$$

$$y_p = -\frac{x^2}{3}\sin 2x - \frac{8x}{9}\cos 2x - \frac{32}{27}\sin 2x$$

The General Solution is

$$y = y_c + y_p$$

$$y = (c_1\cos x + c_2\sin x) - \frac{x^2}{3}\sin 2x$$

$$- \frac{8x}{9}\cos 2x - \frac{32}{27}\sin 2x \quad //$$

(3). Solve the following D.E's

(i). $(D^4+2D^2+1)y = x^2\cos x$

(ii). $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x\sin x$

(iii). $(D^2+4)y = x\cos 2x$

(iv). $(D^2-D)y = x\cos x$. (v). $(D^2-4)y = x\sin x$

(vi). $(D^2+1)y = x^2e^{2x} + x\cos x$

(vii). $(D^2-1)y = x\sin 3x + \cos x$

Ans (ii)

Given eq in operator form is

$$D^2y+3Dy+2y=xe^x\sin x$$

$$(D^2+3D+2)y=xe^x\sin x$$

$$\Rightarrow f(D)y=Q \quad \text{where } f(D)=D^2+3D+2$$

the A.E is $f(m)=0 \Rightarrow m^2+3m+2=0$

$$\Rightarrow m^2+2m+m+2=0 \Rightarrow m(m+2)+1(m+2)=0$$

$$\Rightarrow (m+2)(m+1)=0$$

$$\Rightarrow m=-2,-1$$

$\therefore$ The C.F is

$$y_c=c_1e^{-x}+c_2e^{-2x}$$

The P.I is

$$y_p=\frac{1}{f(D)}Q=\frac{1}{D^2+3D+2}(xe^x\sin x)$$

$$y_p=x\frac{1}{D^2+3D+2}e^x\sin x-\frac{2D+3}{(D^2+3D+2)^2}e^x\sin x$$

$$=x\cdot e^x\frac{1}{(D+1)^2+3(D+1)+2}\sin x-e^x\frac{(2D+3)}{((D+1)^2+3(D+1)+2)^2}\sin x$$

$$=xe^x\frac{1}{D^2+1+2D+3D+3+2}\sin x-e^x\frac{2D+3}{(D^2+1+2D+3D+3+2)^2}\sin x$$

$$=xe^x\frac{1}{(D^2+5D+6)}\sin x-e^x\frac{2D+3}{(D^2+5D+6)^2}\sin x$$

$$=xe^x\frac{1}{-1^2+5D+6}\sin x-e^x\frac{2D+3}{(-1^2+5D+6)^2}\sin x$$

$$=xe^x\frac{1}{5D+5}\sin x-e^x\frac{2D+3}{(5D+5)^2}\sin x$$

$$=\frac{xe^x}{5}\frac{1}{D+1}\sin x-\frac{e^x}{25}\frac{2D+3}{D^2+1+2D}\sin x$$

The method of variation of parameters :-

Given linear differential equation is

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \rightarrow$ (1) where P and Q are functions of 'x' (or) real constants and R is only a function of 'x'. Its homogeneous equation corresponding to eq. (1) is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \rightarrow$ (2).

let $y_c = c_1u + c_2v$ be the general solution of eq.(2) where u, v are functions of x, and c_1, c_2 are real constants hence it is the Comple funct of (1).

Working Rule to find the general Solution of

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ (i) by method of variation of parameters

* In Case the given equation is not in the standard form reduce it to the standard form.
* Find the Solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$. let its Solution be $y_c = c_1u(x) + c_2v(x)$ which is C.F of (i).
* let the P.I of (i) be $y_p = Au + Bv$ where A and B are function of 'x'.

* Find $u\frac{dv}{dx} - v\frac{du}{dx} \Rightarrow uv' - vu'$

* Find 'A' and 'B' by using:

$$A = \int \frac{-vR\,dx}{uv'-vu'}, \quad B = \int \frac{uR\,dx}{uv'-vu'}$$

* The general Solution of eq.(i) is

$$\boxed{y = c_1 u + c_2 v + Au + Bv}$$ //.

Problems:-

(1). Solve $(D^2+a^2)y = \tan ax$ by the method of variat[ion] of parameters.

Sol:-

Given equation is $(D^2+a^2)y = \tan ax$

$$\Rightarrow f(D)y = R(x) \longrightarrow (1)$$

where $f(D) = D^2+a^2$

The A.E is $f(m) = 0$

$$\Rightarrow m^2+a^2 = 0 \Rightarrow m^2-(ai)^2 = 0$$

$$m^2 = -a^2 \quad (m-ai)(m+ai) = 0$$

$$m = \pm ai = a \pm bi$$

The C.F is

$$y_c = e^{0x}(c_1 \cos ax + c_2 \sin ax)$$

$$y_c = c_1 \cos ax + c_2 \sin ax \longrightarrow (2)$$

let the P.I of eq (1) be

$$y_p = Au(x) + Bv(x)$$

$$y_p = A\cos ax + B\sin ax \longrightarrow (1)$$

where A and B are functions of x.

Then $u = \cos ax$, $v = \sin ax$, $R = \tan ax$

By method of variation of parameters

$$uv' - vu' = \cos ax\,(a\cos ax) - \sin ax\,(-a\sin ax)$$

$$= a(\cos^2 ax + \sin^2 ax) = a(1)$$

$$uv' - vu' = a$$

Now $$A = \int \frac{-vR}{uv' - vu'}\,dx = -\int \frac{\sin ax \tan ax}{a}\,dx$$

$$= -\frac{1}{a}\int \sin ax \frac{\sin ax}{\cos ax}\,dx$$

$$A = -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$A = -\frac{1}{a}\int \frac{1 - \cos^2 ax}{\cos ax}\,dx$$

$$A = -\frac{1}{a}\left[\int \sec ax - \int \cos ax\,dx\right]$$

$$A = -\frac{1}{a}\left[\frac{1}{a}\log|\sec ax + \tan ax| - \frac{\sin ax}{a}\right]$$

$$A = -\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax \longrightarrow (2)$$

$$B = \int \frac{uR}{uv' - vu'}\,dx$$

$$= \int \frac{\cos ax \tan ax}{a}\,dx = \int \frac{\cos ax \cdot \frac{\sin ax}{\cos ax}}{a}\,dx$$

$$\int \frac{\sin ax}{a}\,dx$$

$$B = \frac{-\cos ax}{a^2} \longrightarrow (4)$$

eq. (3), eq. (4) Substituting in eq (2)

we have

$$y_p = \left(-\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax\right)\cos ax$$

$$+ \left(-\frac{1}{a^2}\cos ax\right)\sin ax$$

$$\boxed{y_p = -\frac{\cos ax}{a^2}\log|\sec ax + \tan ax| + \cancel{\frac{\sin ax \cos ax}{a^2}} - \cancel{\frac{\cos ax \sin ax}{a^2}}}$$

$\therefore$ The General Solution of eq (1) is

$$y = y_c + y_p$$

$$\boxed{y = C_1\cos ax + C_2\sin ax - \frac{1}{a^2}\cos ax \log|\sec ax + \tan ax|}$$

(5). Solve $(D^2+1)y = \operatorname{cosec} x$ by method of variation of parameters

Sol: Given equation is $(D^2+1)y = \operatorname{cosec} x \rightarrow (1)$

where $f(D) = D^2+1$

The A.E is $f(m) = 0$

$\Rightarrow m^2+1 = 0$

$m^2 = -1 = i^2$

$\boxed{m = \pm i}$

The C.F is

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$y_c = c_1 \cos x + c_2 \sin x$

let the P.I of eq (1) is

$y_p = A\cos x + B\sin x \longrightarrow (2)$

where A and B are functions of 'x'

Then $u = \cos x$, $v = \sin x$, $R = \operatorname{cosec} x$

By method of variation of parameters

$uv' - vu' = \cos x(\cos x) - \sin x(-\sin x)$

$= \cos^2 x \sin x - \sin x \cos x \sin^2 x$

$uv' - vu' = 2\sin x \cos x \quad \cos^2 x + \sin^2 x$

$uv' - vu' = 1$

$$A = \int \frac{-vR}{uv' - vu'}\, dx$$

$$= -\int \frac{\sin x \operatorname{cosec} x}{1}\, dx$$

$$= -\int \sin x \cdot \frac{1}{\sin x}\, dx = -\int dx = -x$$

$$\boxed{A = -x} \longrightarrow (3)$$

$$B = \int \frac{uR}{uv' - vu'}\, dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\, dx$$

$$B = \int \frac{\cos x}{\sin x}\, dx = \int \frac{f'(x)}{f(x)}\, dx$$

$$= \log |f(x)|$$

$$B = \log |\sin x| \longrightarrow (4)$$

eq. (3) and eq (4) substituting in eq (2)

we have

$$y_p = -x \cos x + \log|\sin x| \sin x$$

The General Solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 \cos x + c_2 \sin x - x \cos x + \log|\sin x| \sin x}$$

(3). Solve $(D^2-2D)y = e^x \sin x$ by method of variation of parameters

Sol: Given equation is $(D^2-2D)y = e^x \sin x$ → (1)

where $f(D) = D^2-2D$

The A.E is $f(m)=0$

$$\Rightarrow m^2-2m=0$$

$$m(m-2)=0$$

$$\therefore m=0,\ m-2=0$$

$$m=+2$$

$$m=0,\ m=\cancel{\pm\sqrt{2}}\ 2$$

The C.F is

~~$y_c = c_1 e^{0x} + (c_2 \cosh\sqrt{2}x + c_3 \sinh\sqrt{2}x)e^{0x}$~~

~~$y_c = c_1 + c_2\cosh\sqrt{2}x + c_3\sinh\sqrt{2}x$~~

$$y_c = c_1 e^{0x} + c_2 e^{2x}$$

$$y_c = c_1 + c_2 e^{2x}$$

let the P.S of eq (1) be $y_p = A + Be^{2x}$ → (2)

where A, B are functions of 'x'

Then $u(x)=1,\ v(x)=e^{2x},\ R = e^{x}\sin x$

Now, By method of variation of parameters

$$uv'-vu' = 1\cdot 2e^{2x} - e^{2x}(0)$$

$$= 2e^{2x}$$

i) $\int \frac{-vR}{uv'-vu'} dx$

$= \int \frac{-e^{2x} \cdot e^{x} \sin x}{2e^{2x}} dx$

$= \int \frac{-e^{3x} \sin x}{2e^{2x}} dx$

$= -\frac{1}{2} \int e^{3x} \cdot e^{-2x} \sin x \, dx$

$= -\frac{1}{2} \int e^{x} \sin x \, dx \qquad \int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx)$

$= -\frac{1}{2} \left(\frac{e^{x}}{1^2+1^2} (\cos x - \sin x) \right)$

$A = \frac{-e^{x}}{4} (\cos x - \sin x) \longrightarrow (3)$

and $B = \int \frac{uR}{uv'-vu'} dx = \int \frac{1 \cdot e^{x} \sin x}{2e^{2x}} dx = \int e^{x} \cdot e^{-2x} \sin x$

$= \frac{1}{2} \int e^{-x} \sin x \, dx$

$= \frac{1}{2} \left\{ \frac{e^{-x}}{(-1)^2+1^2} (-1 \cos x - \sin x) \right\}$

$= \frac{1}{2} \left\{ \frac{e^{-x}}{2} (-\cos x - \sin x) \right\}$

$B = \frac{-e^{-x}}{4} (\cos x + \sin x) \longrightarrow (4)$

Substituting eq (3), eq (4) in eq (2)

we have

$y_p = \frac{-e^{x}}{4} (\cos x - \sin x) - \frac{e^{-x}}{4} \cdot e^{2x} (\cos x + \sin x)$

$= \frac{-e^{x}}{4} (\cos x - \sin x) - \frac{e^{x}}{4} (\cos x + \sin x)$

$$y_p = \frac{-e^x}{4}\cos x + \cancel{\frac{e^x}{4}\sin x} - \frac{e^x}{4}\cos x - \cancel{\frac{e^x}{4}\sin x}$$

$$y_p = -\frac{\cancel{2}e^x}{\cancel{4}\,2}\cos x$$

$$y_p = \frac{-e^x}{2}\cos x$$

– The General Solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{e^x}{2}\cos x}$$

(4). Solve the following D.E's by method of variation of parameters:

(i). $(D^2 - 2D + 2)y = e^x \tan x$

(ii). $y'' + 3y' + 2y = 12e^x$

(iii). $y'' - 2y' + y = e^x \log x$

(iv). $(D^2 + a^2)y = \sec ax$

(v). $y'' + 2y' + y = x^2 e^{-x}$

(vi). $(D^2 + 1)y = x \cos x$.

(vi). $(D^2 -$ [illegible]$)y = x \sin x$

(vii). $y'' - 2y' + y = e^x \log x$

(viii). $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

(ix). $y'' - y = \frac{2}{1+e^x}$

Simultaneous linear Differential Equations:-

A set of d.eq's that arise from a set of two or more dependent variables and one independent variable is known as a set of Simultaneous linear D.eq's.

We consider Simultaneous linear D.Eq's: $Dy_1 = f_1(y_1, y_2, \dots, y_n, t)$, $Dy_2 = f_2(y_1, y_2, \dots, y_n, t)$ ... $Dy_n = f_n(y_1, y_2, \dots, y_n, t)$ $\longrightarrow (1)$

where $D = \frac{d}{dt}$ and $f_1, f_2, \dots, f_n$ are each function of $y_1, y_2, \dots, y_n, t$ defined on a common set 'A'.

The method of studying this equation is based on the process of elimination. We formulate the equations with one dependent variable and one independent variable by eliminating other dependent variables and then solving the resulting equation's by using previous methods for each of dependent variables.

problems :-

(1). Solve $\frac{dx}{dt}+5x-2y=t$, $\frac{dy}{dt}+2x+y=0$ and the conditions are $x=0,\ y=0$ when $t=0$

Sol^n:- Given $\frac{dx}{dt}+5x-2y=t \longrightarrow (1)$

and $\frac{dy}{dt}+2x+y=0 \longrightarrow (2)$

$\Rightarrow Dx+5x-2y=t \qquad D=\frac{d}{dt}$

$(D+5)x-2y=t \longrightarrow (3)$

and $\Rightarrow Dy+2x+y=0$

$(D+1)y+2x=0$

$2x+(D+1)y=0 \longrightarrow (4)$

Solve eq. (3) and eq. (4), we have

eq. (3) $\times 2 \Rightarrow \cancel{2(D+5)x}-4y=2t$

eq. (4) $\times (D+5) \Rightarrow \cancel{2(D+5)x}+(D+5)(D+1)y=0$

$-4y-(D+1)(D+5)y=2t$

$\Rightarrow -4y-D^2y-5Dy-Dy-5y=2t$

$\Rightarrow (-D^2-6D-9)y=2t$

$\Rightarrow (D^2+6D+9)y=-2t$

$\Rightarrow f(D)y=Q(t)$

where $f(D) = D^2+6D+9$

The A.E is $f(m)=0$

$$\Rightarrow m^2+6m+9=0$$

$$m^2+3m+3m+9=0$$

$$m(m+3)+3(m+3)=0$$

$$(m+3)(m+3)=0$$

$$\Rightarrow m+3=0\ ,\ m+3=0$$

$$m=-3,-3$$

The C.F is

$$y_c=(c_1+c_2 t)e^{-3t}$$

The P.I is

$$y_p=\frac{1}{f(D)}Q(t)$$

$$=\frac{1}{D^2+6D+9}(-2t)$$

$$=-2\left(\frac{1}{9}\,\frac{1}{\left(1+\left(\frac{D^2+6D}{9}\right)\right)}t\right)$$

$$=-\frac{2}{9}\left\{\left(1+\left(\frac{D^2+6D}{9}\right)\right)^{-1}t\right\}$$

$$=-\frac{2}{9}\left\{\left(1-\left(\frac{D^2+6D}{9}\right)\right)t\right\}$$

$$y_p=-\frac{2}{9}\left\{t-\frac{D^2}{9}t-\frac{6}{9}Dt\right\}$$

$$y_p = -\frac{2}{9}\left\{ t + 0 + \frac{1}{3}(1) \right\}$$

$$= -\frac{2}{9}t + \frac{2}{18}\times\frac{2}{3}(1) + \frac{18}{243}$$

$$y_p = -\frac{2}{9}t + \frac{2}{27}\;\frac{1}{27}$$

The General Solution is

$$y = y_c + y_p$$

$$\boxed{y = (c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}} \longrightarrow (5)$$

Substituting y in eq (4)

$$2x + (D+1)\left\{(c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}\right\} = 0$$

$$\Rightarrow 2x + \frac{d}{dt}\left\{c_1 e^{-3t} + c_2 t e^{-3t} - \frac{2}{9}t + \frac{4}{27}\right\}$$
$$+ (c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27} = 0$$

$$\Rightarrow 2x + c_1 e^{-3t}(-3) + c_2\left(t e^{-3t}(-3) + e^{-3t}\right)$$
$$- \frac{2}{9}(1) + 0 + c_1 e^{-3t} + c_2 t e^{-3t} - \frac{2}{9}t + \frac{4}{27} = 0$$

$$\Rightarrow 2x - 3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t} - \frac{2}{9}$$
$$+ c_1 e^{-3t} + c_2 t e^{-3t} - \frac{2}{9}t + \frac{4}{27} = 0$$

$$\Rightarrow 2x - 2c_1 e^{-3t} - 2c_2 t e^{-3t} + c_2 e^{-3t} - \frac{2}{9}t - \frac{2}{9} + \frac{4}{27} = 0$$

$$\Rightarrow x = c_1 e^{-3t} - c_2 t e^{-3t} + \frac{1}{2}c_2 e^{-3t} - \frac{1}{9}t - \frac{1}{9} + \frac{2}{27} = 0$$

$$x = c_1 e^{-3t} - c_2 t e^{-3t} + \frac{1}{2}c_2 e^{-3t} - \frac{1}{9}t - \frac{1}{27} \longrightarrow (6)$$

The Solutions of given System is

$$x = c_1 e^{-3t} - c_2 t e^{-3t} + \frac{1}{2} c_2 e^{-3t} - \frac{t}{9} - \frac{2}{27} \longrightarrow (A)$$

$$y = c_1 e^{-3t} + c_2 t e^{-3t} - \frac{2}{9} t + \frac{2}{27} \longrightarrow (B)$$

Given Conditions $x = 0, y = 0$ when $t = 0$

$$\Rightarrow 0 = c_1 - 0 + \frac{c_2}{2} - 0 - \frac{2}{27}$$

$$0 = c_1 + \frac{c_2}{2} - \frac{2}{27}$$

$$\Rightarrow c_1 + \frac{c_2}{2} = \frac{2}{27}$$

$$2c_1 + c_2 = \frac{4}{27} \longrightarrow (7)$$

and

$$\Rightarrow 0 = c_1 + 0 - 0 + \frac{2}{27}$$

we have $\boxed{c_1 = \frac{-2}{27}}$ Substituting in eq (7)

$$\frac{-4}{27} + c_2 = \frac{4}{27}$$

$$-4 + 27 c_2 = 4$$

$$27 c_2 = 8$$

$$\boxed{c_2 = \frac{8}{27}}$$

Substituting c_1, c_2 in eq (A), (B)

we have,

$$x = \frac{-2}{27} e^{-3t} - \frac{8}{27} t e^{-3t} + \frac{4}{27} e^{-3t} - \frac{t}{9} - \frac{2}{27}$$

$$y = -\frac{2}{27} e^{-3t} + \frac{8}{27} t e^{-3t} - \frac{2}{9} t + \frac{2}{27}$$

//.

(2). Solve the following D.E's

(a). $\frac{dx}{dt} + 2y + \sin t = 0$, $\frac{dy}{dt} - 2x - \cos t = 0$, [illegible]

(b). $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

(c). $\cancel{D^2y}$ $D^2x + 3x - 2y = 0$

$D^2x + D^2y - 3x + 5y = 0$, $x = 0, y = 0$ when $t = 0$

(d). $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 2z$, $\frac{dz}{dt} = 2x$

(a). Sol^n:- Given equation is

$$\frac{dx}{dt} + 2y + \sin t = 0$$

and

$$\frac{dy}{dt} - 2x - \cos t = 0$$

$\Rightarrow Dx + 2y + \sin t = 0 \Rightarrow Dx + 2y = -\sin t \rightarrow (1)$

and

$\Rightarrow Dy - 2x - \cos t = 0 \Rightarrow -2x + Dy = \cos t \rightarrow (2)$

Solving eq (1), eq (2), we have

eq (1) $\times 2 \Rightarrow +2\cancel{Dx} + 4y = -2\sin t$

eq (2) $\times D \Rightarrow -\cancel{2Dx} + D^2y = D\cos t$

$$4y - D^2y = -2\sin t \cdot D\cos t$$

$$-4y - D^2y = -2\sin t + \sin t$$

$$-(D^2+4)y = -\sin t$$

$$(D^2+4)y = \sin t$$

$f(D)y = Q(x)$

where $f(D) = D^2+4$

$\therefore$ The A.E is $f(m) = 0$

$\Rightarrow m^2+4 = 0$

$\Rightarrow m^2 = -4$

$m = \pm 2i$

$\therefore$ The C.F is

$y_c = e^{0t}(c_1 \cos 2t + c_2 \sin 2t)$

$y_c = c_1 \cos 2t + c_2 \sin 2t$

$\therefore$ The P.I is

$y_p = \frac{1}{f(D)} Q$

$= \frac{1}{D^2+4} \sin t$

$= \frac{1}{-1^2+4}$

$y_p = \frac{1}{3} \sin t$

$\therefore$ The general soln is

$y = y_c + y_p$

$y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \sin t \longrightarrow (3)$

Substituting eq (3) in eq (2),

we have

$-2x + D\left(c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \sin t\right) = \cos t$

$\Rightarrow -2x + 2c_1(-\sin 2t) + c_2 2\cos 2t + \frac{1}{3} \cos t = \cos t$

$\Rightarrow -2x - c_1 2\sin t + c_2 2\cos t + \frac{1}{3}\cos t - \cos t = 0$

$\Rightarrow -2x - 2c_1 \sin t + 2c_2 \cos t - \frac{2}{3}\cos t = 0$

$\Rightarrow -2x = 2c_1 \sin t - 2c_2 \cos t + \frac{2}{3}\cos t$

$x = -\frac{\cancel{2}c_1}{\cancel{2}}\sin t + \frac{\cancel{2}c_2}{\cancel{2}}\cos t - \frac{\cancel{2}}{\cancel{2}\times 3}\cos t \rightarrow (1)$

The general solutions of given system of equations

$x = -\frac{c_1}{\cancel{2}}\sin t + \frac{c_2}{\cancel{2}}\cos t - \frac{1}{3}\cos t \rightarrow (A)$

$y = c_1 \cos t + c_2 \sin t + \frac{1}{3}\sin t \rightarrow (B)$

given conditions $x = 0 = y$ when $t = 0$

$\Rightarrow \quad 0 = -\frac{c_1}{\cancel{2}}(0) + \frac{c_2}{\cancel{2}}(1) - \frac{1}{3}(1)$

$\frac{c_2}{\cancel{2}} = \frac{1}{3} = 0$

$\boxed{c_2 = \frac{1}{3}}$

$\Rightarrow \quad 0 = c_1(1) + c_2(0) + \frac{1}{3}(1).$

$\boxed{c_1 = -\frac{1}{3}}$

The values of c_1, c_2 are substituting in (A), (B) equations.

$$\boxed{\begin{aligned} x &= -\frac{1}{3}\sin t + \frac{1}{3}\cos t - \frac{1}{3}\cos t \\ y &= -\frac{1}{3}\cos t + \frac{1}{3}\sin t + \frac{1}{3}\sin t \end{aligned}}$$

(2). Solve $\frac{dx}{dt} = 3x+2y$, $\frac{dy}{dt} + 5x + 3y = 0$

[illegible]

$$\frac{dx}{dt} = 3x+2y$$

and $$\frac{dy}{dt} + 5x + 3y = 0$$

$\frac{d}{dt} = D$

$\Rightarrow Dx = 3x+2y$

$Dx - 3x - 2y = 0 \longrightarrow (1)$

and $(D-3)x - 2y = 0$

$\Rightarrow Dy + 5x + 3y = 0$

$(D+3)y + 5x = 0 \longrightarrow (2)$

$\Rightarrow 5x + (D+3)y = 0$

Solve eq (1), eq (2), we have

eq (1) × 5 $\Rightarrow 5(D-3)x - 10y = 0$

eq. (2) × (D-3) $\Rightarrow 5(D-3)x + (D+3)(D-3)y = 0$

$-10y - (D+3)(D-3)y = 0$

$-10y - (D^2 - 3^2)y = 0$

$-10y - D^2y + 9y = 0$

$-D^2y - y = 0$

$(D^2+1)y = 0$

$\Rightarrow f(D)y = 0$

where $f(D) = D^2+1$

The A.E is $f(m) = 0$

$\Rightarrow m^2+1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$ are roots of $f(m) = 0$

$\therefore$ The C.F is $\quad \alpha \pm \beta i$

$y = y_c = e^{0x}(c_1 \cos t + c_2 \sin t)$

$y_c = e\,(c_1 \cos t + c_2 \sin x)$

$\therefore$ The g.s is

$y = y_c + y_p$

$= (c_1 \cos t + c_2 \sin t) + 0$

$\boxed{y = c_1 \cos t + c_2 \sin t} \longrightarrow (3)$

$\Rightarrow$ (3) is substituting in eq (2) $\left(D = \frac{d}{dt}\right)$

$\Rightarrow 5x + (D+3)(c_1 \cos t + c_2 \sin t) = 0$

$5x + D(c_1 \cos t + c_2 \sin t) + 3(c_1 \cos t + c_2 \sin t) = 0$

$\Rightarrow 5x + c_1(-\sin t) + c_2(+\cos t) + 3c_1 \cos t + 3c_2 \sin t = 0$

$\Rightarrow 5x - c_1 \sin t + c_2 \cos t + 3c_1 \cos t + 3c_2 \sin t = 0$

$5x + c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t) = 0$

$5x = -c_1(3\cos t - \sin t) - c_2(\cos t + 3\sin t)$

(1) $\longleftarrow$ $\boxed{x = \frac{c_1}{5}(3\cos t - \sin t) - \frac{c_2}{5}(\cos t + 3\sin t)}$

Given D.E's soln is

$$\boxed{\begin{aligned} x &= \frac{c_1}{5}(3\cos t - \sin t) - \frac{c_2}{5}(\cos t + 3\sin t) \\ y &= c_1\cos t + c_2\sin t \end{aligned}}$$

//

(3). Solve $\frac{dy}{dx} + y = z + e^{x}$, $\frac{dz}{dx} + z = y + e^{x}$

Sol:- Given D.E is

$$\frac{dy}{dx} + y = z + e^{x}$$

and $$\frac{dz}{dx} + z = y + e^{x}.$$

$$\Rightarrow Dy + y - z - e^{x} = 0$$

$$(D+1)y - z = e^{x} \longrightarrow (1)$$

and $$D\cancel{z} + z - y = e^{x}$$

$$(D+1)z - y = e^{x}$$

$$\Rightarrow -y + (D+1)z = e^{x} \longrightarrow (2)$$

Solve eq (1), (2), we have.

$$\text{eq } (1) \times 1 \Rightarrow (D+1)y - z = e^{x}$$

$$\text{eq } (2) \times (D+1) \Rightarrow -(D+1)y + (D+1)(D+1)z = e^{x}(D+1)$$

$$-z + (D+1)^2 z = e^{x} + (D+1)e^{x}$$

$$-z + (D^2 + 1 + 2D)z = e^{x} + e^{x} + e^{x}$$

$$-\cancel{z}+D^2z+\cancel{z}+2Dz=3e^x$$

$$(D^2+2D)z=3e^x$$

$$D(D+2)z=3e^x$$

$$\Rightarrow f(D)z=\phi(x)$$

where $f(D)=D^2+2D$ (or) $D(D+2)$

The A.E is $f(m)=0$

$$\Rightarrow m^2+2m=0$$

$$m(m+2)=0$$

$$m=0,\ m=-2$$

$\therefore$ The C.F is

$$z_c=\left(c_1e^{0x}+c_2e^{-2x}\right)$$

$$z_c=c_1+c_2e^{-2x}$$

The P.I is

$$z_p=\frac{1}{f(D)}\cdot\phi(x)$$

$$=\frac{1}{D^2+2D}3e^x$$

$$=3\frac{1}{D^2+2D}e^x$$

$$\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$$

$$=3\frac{1}{1^2+2(1)}e^x$$

$$=\cancel{3}\frac{1}{\cancel{3}}e^x$$

$$z_p=e^x$$

the G. S is

$z = z_c + z_p$

$z = (c_1 + c_2 e^{-2x}) + e^x \longrightarrow (2)$

eq (2) sub in eq (1)

$$-y + (D+1)\left[(c_1 + c_2 e^{-2x}) + e^x\right] = 0$$

$$-y + D\{c_1 + c_2 e^{-2x} + e^x\} + c_1 + c_2 e^{-2x} + e^x$$

$$-y + 0 + c_2 e^{-2x}(-2) + e^x + c_1 + c_2 e^{-2x} + e^x$$

$$-y = -2c_2 e^{-2x} + c_2 e^{-2x} + e^x + e^x + c_1$$

$$\boxed{y = c_2 e^{-2x} - 2e^x - c_1}$$

∴ Finally the Sol^n of given D E is

$$\boxed{\begin{aligned} y &= c_2 e^{-2x} - c_1 - 2e^x \\ z &= (c_1 + c_2 e^{-2x}) + e^x \end{aligned}}$$

(4). Solve following D. E's

(i). $\frac{dy}{dx} = y, \frac{dz}{dx} = 2y + z$

(ii). $\frac{dx}{dt} = 5x + 4y, \frac{dy}{dt} = -x + y$

(4) (i). Given Differential equation is operating form is

$\Rightarrow Dy - y = 0$

$\Rightarrow (D-1)y = 0 \Rightarrow (D-1)y + 0z = 0 \longrightarrow (1)$

and $Dz = 2y + z$

$\Rightarrow Dz - 2y - z = 0$

$\Rightarrow (D-1)z - 2y = 0$

$\Rightarrow -2y + (D-1)z = 0 \longrightarrow (2)$

Solve eq (1) and eq (2),

we get

eq (1) $\times 2 \Rightarrow 2(D-1)y + 0z = 0$

eq (2) $\times (D-1) \Rightarrow -2(D-1)y + (D-1)(D-1)z = 0$

$(D-1)(D-1)z = 0$

$\Rightarrow f(D)z = 0$

where $f(D) = (D-1)(D-1)$

The A. E is $f(m) = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\Rightarrow m = 1, 1$

The C. F is

$z_c = (c_1 + c_2 x)e^x$

The P. I is

$z_p = 0$

$\therefore$ The general Sol^n is

$$z = z_c + z_p = (c_1 + c_2 x) e^x \quad \text{//} \longrightarrow (3)$$

eq (3) is substituting in eq (2)

we have

$$-2y + (D-1)\{(c_1 + c_2 x) e^x\} = 0$$

$$\Rightarrow -2y + D\{(c_1 + c_2 x) e^x\} - (c_1 + c_2 x) e^x = 0$$

$$\Rightarrow -2y + \frac{d}{dx}(c_1 e^x + c_2 x e^x) - c_1 e^x - c_2 x e^x = 0$$

$$-2y + c_1 e^x + c_2 (x e^x + e^x \cdot 1) - c_1 e^x - c_2 x e^x = 0$$

$$-2y + \cancel{c_1 e^x} + \cancel{c_2 x e^x} + c_2 e^x - \cancel{c_1 e^x} - \cancel{c_2 x e^x} = 0$$

$$\Rightarrow -2y + c_2 e^x = 0$$

$$\Rightarrow -2y = -c_2 e^x$$

$$\boxed{y = \frac{c_2}{2} e^x}$$

$\therefore$ The Sol^n of given Simultaneous D.E is

$$\boxed{\begin{aligned} y &= \frac{c_2}{2} e^x \\ z &= (c_1 + c_2 x) e^x \end{aligned}}$$

//.

Unit- 2

(25-05-2021)

Equations Reduces to linear Differential Equations and Applications:

Cauchy's (Euler) Linear Differential (or) Homogeneous Linear Eq'n:

An equation of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = Q(x)$, where $P_1, P_2, \ldots, P_n$ are real constants and $Q(x)$ is a function of 'x' is called a homogeneous linear equation (or) Euler-cauchy's linear equation of order 'n'.

The equation in the operator form is

$$(x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_n) y = Q(x) \longrightarrow (1), \text{ where } \frac{d}{dx} \equiv D.$$

Cauchy's linear differential equation can be transformed into a linear equation with constant co-efficients by the change of independent variable with the substitution

$$x = e^z \text{ (or) } z = \log x, \quad x > 0 \longrightarrow (2)$$

let us denote $\frac{d}{dx} \equiv D$ and $\frac{d}{dz} = \theta$

also, we denote $\frac{d^2}{dz^2} = \theta^2, \frac{d^3}{dz^3} = \theta^3, \ldots, \frac{d^n}{dz^n} = \theta^n$

Then, eq (1) $x \frac{dy}{dx} = xDy = \theta y$

$$x^2 \frac{d^2 y}{dx^2} = x^2 D^2 y = \theta(\theta - 1) y$$

$$x^3 \frac{d^3 y}{dx^3} = x^3 D^3 y = \theta(\theta - 1)(\theta - 2) y$$

and so on

$$x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} = x^{n-1}D^{n-1}y$$

$$= \left[\theta(\theta-1)(\theta-2)\ldots(\theta-(n-2))\right]y$$

$$x^n\frac{d^ny}{dx^n} = x^nD^ny = \left[\theta(\theta-1)(\theta-2)\ldots(\theta-(n-1))\right]y$$

Substituting the above values in eq. (1),

we get

$$\{\theta(\theta-1)\ldots(\theta-n+1)\}y + \{P_1\theta(\theta-1)\ldots(\theta-n+2)\}y$$

$$+\ldots+P_ny = Q(e^z)$$

$$\Rightarrow \boxed{f(\theta)y = Z}$$

where $f(\theta) = \theta(\theta-1)\ldots(\theta-n+1) + (P_1\theta(\theta-1)\ldots(\theta-n+2))$

$+\ldots+P_n$

$Z = Q(e^z)$

This is a linear differential equation with constant coefficients.

The differential operator $f(\theta)$ and the inverse operator $\frac{1}{f(\theta)}$ obey the properties of $f(D)$ and $\frac{1}{f(D)}$.

Hence $f(\theta)y = Z$ can be solved by the methods discussed already in this previous chapter.

problem :- 1 Solve $3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$

Sol^n:- Given Equation in operator form is

$$(3x^2D^2 + xD + 1)y = x \longrightarrow (1)$$

which is Homogeneous linear equation

put $x = e^z \Rightarrow z = \log x$, $x > 0$

let $\theta = \frac{d}{dz}$ then

$$xD = \theta, \quad x^2D^2 = \theta(\theta - 1) \longrightarrow (2)$$

eq. (2), substituting in eq. (1)

we have

$$(3\theta(\theta-1) + \theta + 1)y = e^z$$

$$(3\theta^2 - 3\theta + \theta + 1)y = e^z$$

$$(3\theta^2 - 2\theta + 1)y = e^z$$

$$\Rightarrow f(\theta)y = Q(e^z) = Q(z)$$

where $f(\theta) = 3\theta^2 - 2\theta + 1$ Then

the A.E is $f(m) = 0$

$$\Rightarrow 3m^2 - 2m + 1 = 0$$

$$\Rightarrow m = \frac{+2 \pm \sqrt{4 - 12}}{2 \times 3}$$

$-1 = i^2$

$$= \frac{2 \pm \sqrt{-8}}{6}$$

$$= \frac{2 \pm 2\sqrt{2}i}{6}$$

$$m = \frac{1 \pm \sqrt{2}i}{3} = \frac{1}{3} \pm \frac{\sqrt{2}}{3}i$$

$$= a \pm bi$$

The C.F is

$$y_c = e^{z/3}\left(c_1 \cos\frac{\sqrt{2}}{3}z + c_2 \sin\frac{\sqrt{2}}{3}z\right)$$

The P.I is

$$y_p = \frac{1}{f(\theta)} Q(e^z)$$

$$= \frac{1}{3\theta^2 - 2\theta + 1} e^z$$

$$= \frac{1}{3(1)^2 - 2(1) + 1} e^z$$

$$= \frac{1}{4-2} e^z$$

$$\boxed{y_p = e^z/2}$$

The General Solution is

$$y = y_c + y_p$$

$$\boxed{y = e^{z/3}\left(c_1 \cos\frac{\sqrt{2}}{3}z + c_2 \sin\frac{\sqrt{2}}{3}z\right) + e^z/2}$$

Here $z = \log x$

$$\Rightarrow y = e^{\frac{\log x}{3}}\left(c_1 \cos\frac{\sqrt{2}}{3}\log x + c_2 \sin\frac{\sqrt{2}}{3}\log x\right) + \frac{e^{\log x}}{2}$$

$$\left(\because e^{\frac{1}{3}\log x} = e^{\log x^{1/3}} = x^{1/3} \text{ and } e^z = e^{\log x} = x\right)$$

$$(e^{\log_e a} = a).\ (e^{\log a} = a)$$

$$\Rightarrow \boxed{y = x^{1/3}\left(c_1 \cos\frac{\sqrt{2}}{3}\log x + c_2 \sin\frac{\sqrt{2}}{3}\log x\right) + \frac{x}{2}}$$

(2). Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol^n:- Given equation in the operator form is

$x^2D^2y - xDy + y = \log x$

$\rightarrow (x^2D^2 - xD + 1)y = \log x \longrightarrow (1)$

$\Rightarrow$ let $x = e^z \Rightarrow z = \log x$

and take $xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$, $\frac{d}{dx} = D$

from eq (1), we have $\qquad \frac{d}{dz} = \theta$

$(\theta(\theta-1) - \theta + 1)y = z$

$\Rightarrow (\theta^2 - \theta - \theta + 1)y = z$

$(\theta^2 - 2\theta + 1)y = z$

$(\theta-1)^2 y = z$

$\Rightarrow f(\theta)y = Q(z)$

where $f(\theta) = (\theta-1)^2$

The A.E is $f(m) = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow (m-1)(m-1) = 0$

$m = 1, 1$

$\therefore$ The C.F is

$y_c = (c_1 + c_2 z)e^z$

The P.I. is

$y_p = \frac{1}{f(\theta)} Q(z) = \frac{1}{(\theta-1)^2} z$

$$y_p = \frac{1}{(\theta-1)^2} z = \frac{1}{(-1(1-\theta))^2}$$

$$= \frac{1}{(-1)^2(1-\theta)^2} z \qquad (1-x)^{-2} = 1+2x+3x^2 \ldots$$

$$= (1-\theta)^{-2} z$$

$$= \left(1+2\theta+3\theta^2+\cdots\right) z$$

$$= \left(z + 2\frac{d}{dz}(z) + 3\frac{d^2}{dz^2}(z)\right)$$

$$= z + 2(1) + 0$$

$$y_p = z+2$$

The general Soln is $\qquad e^{\log_e a} = a$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 z)e^z + z + 2$$

Here $z = \log x$

$$y = (c_1 + c_2 \log x)e^{\log x} + \log x + 2$$

$$\boxed{y = (c_1 + c_2 \log x)x + \log x + 2}$$

(3). Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Sol^n: Given equation in operator form is

$$x^3D^3y + 2x^2D^2y + 2y = 10\left(x + \frac{1}{x}\right)$$

$$\left(x^3D^3 + 2x^2D^2 + 2\right)y = 10\left(x + \frac{1}{x}\right) \longrightarrow (1)$$

let $x = e^z \Longrightarrow z = \log x$, $x > 0$

and $\frac{d}{dx} = D$, $\frac{d}{dz} = \theta$

Then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$

from eq (1), we have

$$\left(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2\right)y = 10\left(e^z + \frac{1}{e^z}\right)$$

$$\left(\theta(\theta^2 - 2\theta - \theta + 2) + 2\theta^2 - 2\theta + 2\right)y = 10\left(e^z + e^{-z}\right)$$

$$\left(\theta^3 - \cancel{2\theta^2} - \theta^2 + \cancel{2\theta} + \cancel{2\theta^2} - \cancel{2\theta} + 2\right)y = 10e^z + 10e^{-z}$$

$$\left(\theta^3 - \theta^2 - \cancel{2\theta} + 2\right)y = 10e^z + 10e^{-z}$$

$$\Longrightarrow f(\theta)y = Q(e^z) = Q(z)$$

where $f(\theta) = \theta^3 - \theta^2 - \cancel{2\theta} + 2$

The A. E is $f(m) = 0$

$$\Longrightarrow m^3 - m^2 - \cancel{2m} + 2 = 0$$

$$\Longrightarrow m^3 - m^2 + 2 = 0$$

$$\Longrightarrow (m+1)(m^2 - 2m + 2) = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & 0 & -1 & +2 & -2 \\ \hline & 1 & -2 & +2 & 0 \end{array}$$

$m = -1$, $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

$m = 1 \pm i, -1$

the C.F is

$$\boxed{y_c = c_1 e^{-z} + (c_2 \cos z + c_3 \sin z) e^{z}}$$

the P.I is

$$y_p = \frac{1}{f(\theta)} Q(e^z) \qquad \frac{d}{d\theta}(\theta^3 - \theta^2 + 2) = 3\theta^2 - 2\theta$$

$$= \frac{1}{\theta^3 - \theta^2 + 2}\left(10e^{z} + 10e^{-z}\right)$$

$$y_p = \frac{1}{\theta^3 - \theta^2 + 2} 10e^{z} + \frac{1}{\theta^3 - \theta^2 + 2} 10e^{-z}$$

$$= \frac{10e^{z}}{(1)^3 - (1)^2 + 2} + \frac{10e^{-z}}{(-1)^3 - (-1)^2 + 2}$$

$$= \frac{10e^{z}}{2} + \frac{10e^{-z}}{-1-1+2=0} \qquad f(a)=0$$

$$= \frac{10e^{z}}{2} + \frac{10z/1!}{3\theta^2 - 2\theta} e^{-z}$$

$$= \frac{10e^{z}}{2} + \frac{10ze^{-z}}{3(-1)^2 - 2(-1)}$$

$$= 5e^{z} + \frac{10ze^{-z}}{3+2}$$

$$y_p = 5e^{z} + \frac{10z}{5} e^{-z}$$

$$\boxed{y_p = 5e^{z} + 2ze^{-z}}$$

the general Solution is

$$y = y_c + y_p$$

$$= c_1 e^{-z} + (c_2 \cos z + c_3 \sin z) e^{z} + 5e^{z} + 2ze^{-z}$$

Here $x = e^z \rightarrow z = \log x$

$$\therefore y = c_1 e^{-\log x} + \left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) e^{\log x} + 5e^{\log x} + 2z e^{-\log x}$$

$$= c_1 e^{\log x^{-1}} + \left(c_2 \cos(\log x) + c_3 \sin(\log x)\right)(x) + 5x + 2 e^{\log x^{-1}} (\log x)$$

$$= c_1 x^{-1} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) + 5x + 2(\log x)\, x^{-1}$$

$e^{n \log a} = e^{\log a^n}$

$$\boxed{y = \frac{c_1}{x} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) + 5x + \frac{2\log x}{x}}$$

4). Solve $(x^2D^2 + 2xD + 12)y = x^3 \log x$

5). Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1+x)^2$

6). Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$

7). Solve $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$

8). Solve (i). $(x^2D^2 + 4xD - 16)y = (\log x)^2$

(ii). $(x^2D^2 + 3xD + 1)y = \frac{1}{(1-x)^2}$

2) Given eqn is $(x^2D^2 + 2xD - 12)y = x^3 \log x$ ——— (1)

Take $x = e^z$ or $z = \log x$, $x > 0$

[illegible] $\frac{dz}{dx} = \frac{1}{x}$

Then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

The given form substituting in eq(1)

we have

$$(\theta(\theta-1) + 2\theta - 12)y = (e^z)^3 \cdot z$$

$$\Rightarrow (\theta^2 - \theta + 2\theta - 12)y = e^{3z} \cdot z$$

$$\Rightarrow (\theta^2 + \theta - 12)y = z \cdot e^z$$

$$\Rightarrow f(\theta)y = \phi(z)$$

where $f(\theta) = \theta^2 + \theta - 12$, $\phi(z) = z \cdot e^z$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 + m - 12 = 0$$

$$m^2 + 4m - 3m - 12 = 0$$

$$m(m+4) - 3(m+4) = 0$$

$$(m+4)(m-3) = 0$$

$$\Rightarrow m+4 = 0,\ m-3 = 0$$

$$m = -4,\ m = +3$$

$$\therefore m = -4, 3$$

The C.F is

$y_c = c_1 e^{-4z} + c_2 e^{3z}$

The particular Integral is

$$y_p = \frac{1}{f(\theta)} \phi(z)$$

$$= \frac{1}{\theta^2 + \theta - 12} z \cdot e^z$$

$$= e^z \frac{1}{(\theta+1)^2 + (\theta+1) - 12} z$$

$$= e^z \frac{1}{\theta^2 + 1 + 2\theta + \theta + 1 - 12} z$$

$$= e^z \frac{1}{\theta^2 + 3\theta - 10} z \qquad \left(\frac{1}{f(\theta)} x^k\right) = (1 \pm f(\theta))$$

$$= \frac{e^z}{-10} \left\{ \frac{1}{(1 - (\theta^2 + 3\theta))} z \right\}$$

$$y_p = \frac{-e^z}{10} \left(1 - (\theta^2 + 3\theta)\right)^{-1} z$$

$$= \frac{-e^z}{10} \left\{ 1 + (\theta^2 + 3\theta) \right\} z$$

$$= \frac{-e^z}{10} \left\{ z + \theta^2 z + 3\theta z \right\}$$

$$= \frac{-e^z}{10} \left\{ z + \frac{d^2}{dz^2} z + 3 \frac{d}{dz} z \right\}$$

$$= \frac{-e^z}{10} \left\{ z + 0 + 3(1) \right\}$$

$$y_p = -\frac{e^z z}{10} - \frac{3e^z}{10}$$

The general sol^n is

$y = y_c + y_p$

$$y = c_1 e^{-4z} + c_2 e^{3z} - \frac{z e^{z}}{10} - \frac{3e^{z}}{10}$$

Here $z = \log x$

$$y = c_1 e^{-4\log x} + c_2 e^{3\log x} - \frac{(\log x)e^{\log x}}{10} - \frac{3e^{\log x}}{10}$$

$$\left(e^{n\log a} = e^{\log a^n}\right)$$

$$= c_1 e^{\log x^{-4}} + c_2 e^{\log x^3} - \frac{(\log x)e^{\log x}}{10} - \frac{3e^{\log x}}{10}$$

$$\left(e^{\log a} = a\right)$$

$$= c_1 x^{-4} + c_2 x^3 - \frac{(\log x)x}{10} - \frac{3x}{10}$$

$$\boxed{y = \frac{c_1}{x^4} + c_2 x^3 - \frac{x\log x}{10} - \frac{3x}{10}}$$

Legendre's Linear Equation :-

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = Q(x)$$

where $P_1, P_2, \dots, P_n$ are real constants and $Q(x)$ is a function of 'x' is called Legendre's linear equation.

This can be solved by the substitution

$a+bx = e^z \Rightarrow z = \log(a+bx)$ and $\theta = \frac{d}{dz}$, $D = \frac{d}{dx}$

problems :-

(1). Solve $(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2+x+1$

Sol :- Given equation in operator form is

$$(x+1)^2 D^2 y - 3(x+1)Dy + 4y = x^2+x+1$$

$$\Rightarrow \left[(x+1)^2 D^2 - 3(x+1)D + 4\right] y = x^2+x+1$$

This is a Legendre's diff eqn

let $x+1 = u$ so that $x = u-1 \Rightarrow \frac{du}{dx} = 1$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot 1$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}$$

The given eq becomes

$$u^2 \frac{d^2y}{du^2} - 3u \frac{dy}{du} + 4y = (u-1)^2 + (u-1) + 1$$

$$\Rightarrow u^2 \frac{d^2y}{du^2} - 3u \frac{dy}{du} + 4y = u^2 + 1 - 2u + u - 1 + 1$$

$\Rightarrow u^2 \frac{d^2y}{du^2} - 3u\frac{dy}{du} + 4y = u^2 - u + 1 \longrightarrow (2)$

Let $u = e^z \Rightarrow z = \log u$

Denote $\frac{d}{dz} = \theta$, then

$u^2D^2 = \theta(\theta-1)$, $uD = \theta$

from eq (2)

$$(\theta(\theta-1) - 3\theta + 4)y = u^2 - u + 1$$

$$(\theta^2 - \theta - 3\theta + 4)y = (e^z)^2 - e^z + 1$$

$$(\theta^2 - 4\theta + 4)y = e^{2z} - e^z + 1$$

$$\Rightarrow f(\theta)\,dy = Q(e^z) = Q(z)$$

where $f(\theta) = \theta^2 - 4\theta + 4$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

The C.F is

$$y_c = (c_1 + c_2 z)e^{2z}$$

The P.I is

$$y_p = \frac{1}{f(\theta)} Q(e^z)$$

$$= \frac{1}{\theta^2 - 4\theta + 4}\left(e^{2z} - e^z + 1\right)$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 4} e^{2z} - \frac{1}{\theta^2 - 4\theta + 4} e^z + \frac{1}{\theta^2 - 4\theta^{\ldots}}$$

$$y_p = \frac{1}{D^2-4D+4}e^{2z} + \frac{1}{D^2-4D+4}e^{z} + \frac{1}{D^2-4D+4}e^{0z}$$

$$= \frac{1}{2^2-4(2)+4}e^{2z} + \frac{1}{1^2-4(1)+4}e^{z} + \frac{1}{0^2-4(0)+4}e^{0z}$$

$$= \frac{z e^{2z}}{2D-4} + \frac{1}{1}e^{z} + \frac{1}{4}e^{0z}$$

$$y_p = \frac{z e^{2z}}{2(2)-4=0} + e^{z} + \frac{1}{4}$$

$$y_p = \frac{z^2 e^{2z}}{2\cdot 2!} + e^{z} + \frac{1}{4}$$

$$y_p = \frac{z^2}{4}e^{2z} + e^{z} + \frac{1}{4} \,//.$$

$\therefore$ The general solution is

$$y = y_c + y_p$$

$$= (c_1 + c_2 z)e^{2z} + \frac{z^2}{4}e^{2z} + e^{z} + \frac{1}{4}$$

$$= (c_1 + c_2 \log u)e^{2\log u} + \frac{(\log u)^2}{4}e^{2\log u} + e^{\log u} + \frac{1}{4}$$

$$y = (c_1 + c_2 \log u)e^{\log u^2} + \frac{(\log u)^2}{4}e^{\log u^2} + e^{\log u} + \frac{1}{4}$$

$$= (c_1 + c_2 \log u)u^2 + \frac{(\log u)^2}{4}u^2 + u + \frac{1}{4}$$

$$\boxed{y = (c_1 + c_2 \log(1+x))(1+x)^2 + \frac{(\log(1+x))^2}{4}(1+x)^2 + (1+x) + \frac{1}{4}}$$

(2). Solve the D.E $(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=x$

Sol:- Given equation in operator form is

$$(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=x \longrightarrow (1)$$

put $2x-1=t \Rightarrow x=\frac{t+1}{2} \Rightarrow 1=\frac{1}{2}\frac{dt}{dx}+\frac{1}{2}$

$$2=\frac{dt}{dx}$$

Now $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot 2$

$$\frac{dy}{dx}=2\frac{dy}{dt}$$

ILly $\frac{d^2y}{dx^2}=2^2\frac{d^2y}{dt^2}$

$$\frac{d^3y}{dx^3}=2^3\frac{d^3y}{dt^3}$$

Substitute these terms in eq (1)

we get $t^3 2^3\frac{d^3y}{dt^3}+t\,2\frac{dy}{dt}-2y=x$

$$\Rightarrow 8t^3\frac{d^3y}{dt^3}+2t\frac{dy}{dt}-2y=\left(\frac{t+1}{2}\right)\longrightarrow (2)$$

let $z=\log t \Rightarrow e^z=t$, $\frac{d}{dz}=\theta$

Then $t\frac{d}{dt}=\theta$, $t^2\frac{d^2}{dt^2}=\theta(\theta-1)$, $t^3\frac{d^3}{dt^3}=\theta(\theta-1)(\theta-2)$

from eq (2), we have

$$\Rightarrow\left(8\theta(\theta-1)(\theta-2)+2\theta-2\right)y=\frac{e^z+1}{2}$$

$$\Rightarrow\left(8\theta(\theta^2-2\theta-\theta+2)+2\theta-2\right)y=\frac{e^z+1}{2}$$

$(8\theta^3-16\theta^2-8\theta^2+16\theta+2\theta-2)y = \frac{e^z+1}{2}$

$\Rightarrow (8\theta^3-24\theta^2+18\theta-2)y = \frac{e^z+1}{2}$

$\Rightarrow f(\theta)y = Q(e^z) = Q(z)$

where $f(\theta) = 8\theta^3-24\theta^2+18\theta-2$

The A.E is $f(m)=0$

$\Rightarrow 8m^3-24m^2+18m-2=0$

$$\begin{array}{c|cccc} 1 & 8 & -24 & 18 & -2 \\ & 0 & 8 & -16 & 2 \\ \hline & 8 & -16 & 2 & \boxed{0} \end{array}$$

$\Rightarrow (m-1)(8m^2-16m+2)=0$

$m-1=0$, $\quad m = \frac{16 \pm \sqrt{256-64}}{16}$

$m=1$,

$= \frac{16 \pm \sqrt{192}}{16} = \frac{16 \pm \sqrt{24 \times 8}}{16}$

$= \frac{16 \pm 4\sqrt{12}}{16} = 1 \pm \sqrt{12}/4$

$m = 1 \pm \sqrt{3}/2$

$\therefore m = 1, 1 \pm \frac{\sqrt{3}}{2}$

The C.F is

$y_c = c_1e^z + e^z\left(c_2\cosh\frac{\sqrt{3}}{2}z + c_3\sinh\frac{\sqrt{3}}{2}z\right)$

The P.I is

$y_p = \frac{1}{8\theta^3-24\theta^2+18\theta-2}\left(\frac{e^z+1}{2}\right)$

$= \frac{1}{2}\left(\frac{1}{8\theta^3-24\theta^2+18\theta-2}e^z\right) + \frac{1}{2}\frac{e^{0z}}{8\theta^3-24\theta^2+18\theta-2}$

$= \frac{1}{2}\frac{e^z}{8-24+18-2=0} + \frac{1}{2}\frac{1}{0-0+0-2}$

$= \frac{1}{2}\frac{ze^z}{24\theta^2-48\theta+18} + \left(-\frac{1}{4}\right)$

$= \frac{ze^z}{2(24-48+18)} - \frac{1}{4}$

$$= \frac{ze^{z}}{2(6)} - \frac{1}{4}$$

$$y_p = \frac{ze^{z}}{12} - \frac{1}{4}$$

$\therefore$ the general solⁿ is

$$y = y_c + y_p$$

$$y = c_1 e^{z} + e^{z}\left(c_2 \cosh \frac{\sqrt{3}}{2}z + c_3 \sinh \frac{\sqrt{3}}{2}z\right)$$

$$\quad - \frac{ze^{z}}{12} - \frac{1}{4}$$

Here $z = \log t$, and $t = 2x-1$

$$y = c_1 e^{\log t} + e^{\log t}\left(c_2 \cosh \frac{\sqrt{3}}{2}(\log t) + c_3 \sinh \frac{\sqrt{3}}{2}\log t\right)$$

$$\quad - \frac{\log t \, e^{\log t}}{12} - \frac{1}{4}$$

(3). Solve $\left[(1+x)^2 D^2 + (1+x) D + 1\right] y = 4\cos\left(\log(x+1)\right)$

(4). $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$ Solve

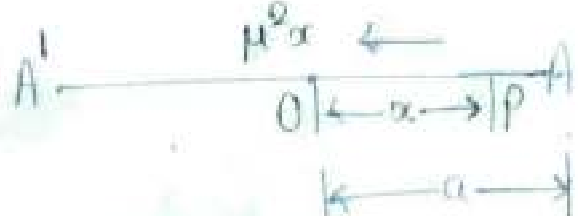
(5). Solve $\left((1+2x)^2 D^2 - 6(1+2x) D + 16\right) y = 8(1+2x)^2$

Applications of linear Differential equations :-

The linear differential equations with Constant coefficients play an important role in the study of electrical and mechanical oscillatory System.

Simple Harmonic motion :-

when the acceleration of a particle is proportional to ~~the~~ its displacement from a fixed point and is always directed towards it, then the motion is said to be "Simple Harmonic".



If the displacement of the particle at any time "t", from fixed point "O" is "x",

$a \propto \cancel{\mu^2} x \Rightarrow a = -\mu^2 x.$ $\qquad a \propto s$

$a = -ks$

then

$\frac{d^2x}{dt^2} = -\mu^2 x$ $\qquad a = \frac{dv}{dt}$

$\Rightarrow \frac{d^2x}{dt^2} + \mu^2 x = 0$ $\qquad v = \frac{dx}{dt}$

$\Rightarrow D^2x + \mu^2 x = 0$ $\qquad \therefore D = \frac{d}{dt}$

$(D^2 + \mu^2)x = 0$ $\qquad a = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$a = \frac{d^2x}{dt^2}$

$\Rightarrow f(D)x = 0$ where $f(D) = D^2 + \mu^2$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 + \mu^2 = 0$$

$$m^2 = -\mu^2$$

$$m^2 = \mu^2 i^2 \Rightarrow m^2 - (\mu i)^2 = 0$$

$$\boxed{m = \pm \mu i}$$

$\therefore$ The Solution is

$$\boxed{x = c_1 \cos \mu t + c_2 \sin \mu t}$$

$\therefore$ its velocity at $P = \frac{dx}{dt} = V$

$$\Rightarrow V = \frac{d}{dt}(c_1 \cos \mu t + c_2 \sin \mu t)$$

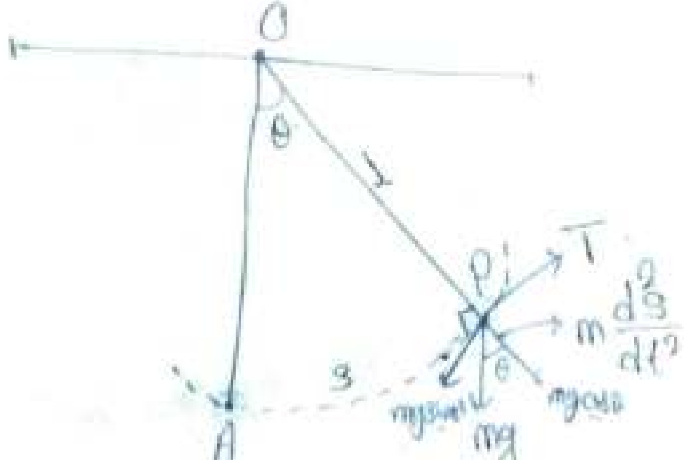
$$= -\mu c_1 \sin \mu t + c_2 \cos \mu t\,(\mu)$$

$$V = \mu(-c_1 \sin \mu t + c_2 \cos \mu t)$$

Simple pendulum:

A heavy particle attached by a light string to a fixed point and oscillating under gravity constitutes a "Simple pendulum".

let 'O' be the fixed point, 'l' be the length of the string and 'A' be the position of the bob initially. If 'p' be the position of the bob at any time 't', such that arc $AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.



∴ the equation of motion along PT is

$F = ma$

$F \propto ma$

$= m\frac{dv}{dt}$

$= m\frac{d}{dt}\left(\frac{ds}{dt}\right)$

$\Rightarrow m\frac{d^2s}{dt^2}$

mg sinθ

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$\Rightarrow \frac{d^2s}{dt^2} = -g\sin\theta$$

$$\Rightarrow \frac{d^2(l\theta)}{dt^2} = -g\sin\theta$$

$$\Rightarrow l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$= \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\left\{\theta - \frac{\theta^3}{3!} + \dots\right\}$$

$= -\frac{g}{l}\theta$ to a first approximation

$$\Rightarrow \frac{d^2\theta}{dt^2} + g/l\,\theta = 0 \qquad D = \frac{d}{dt}$$

$$\left(D^2\theta + \frac{g}{l}\theta\right) = 0$$

$$\left(D^2 + \frac{g}{l}\right)\theta = 0$$

$$\Longrightarrow f(D)\theta = 0$$

where $f(D) = D^2 + \frac{g}{l}$

The A.E is $f(m) = 0$

$$\Longrightarrow m^2 + \frac{g}{l} = 0$$

$$m^2 = -\frac{g}{l}$$

$$m^2 = \frac{g}{l}i^2$$

$$m = \pm\sqrt{g/l}\,i$$

∴ The Solution is

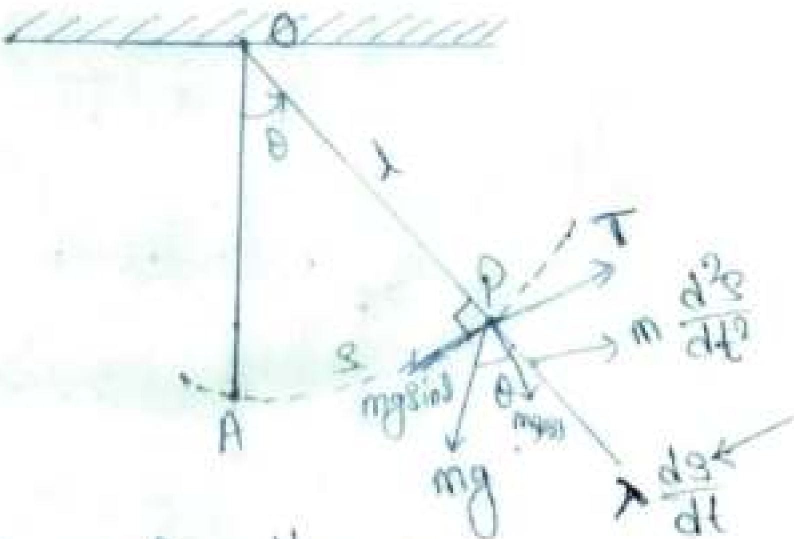
$$\theta = c_1\cos\sqrt{g/l}\,t + c_2\sin\sqrt{g/l}\,t$$

Thus, the motion of the bob is simple harmonic and the time of an oscillation is

$2\pi\sqrt{g/l}$ $\qquad$ $m\pi\sqrt{g/l}$

Example:- A simple pendulum of length 'l' is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of motion.

Solⁿ: let the position of the bob (of mass m), at any time 't' be 'p' and 'O' be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and arc $AP = s = l\theta$. (See in figure below).



the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta - \lambda\frac{ds}{dt}$$

where λ is constant

$F \propto V$
$F = -\lambda V$
$F = -\lambda\frac{ds}{dt}$

$$\Rightarrow m\frac{d^2s}{dt^2} + mg\sin\theta + \lambda\frac{ds}{dt} = 0$$

$$\Rightarrow \frac{d^2s}{dt^2} + g\sin\theta + \frac{\lambda}{m}\frac{ds}{dt} = 0$$

$$\Rightarrow \frac{d^2 s}{dt^2} + \frac{\lambda}{m}\frac{ds}{dt} + g\sin\theta = 0$$

Since $s = l\theta$ and replacing $\sin\theta$ by θ since it is small and writing $\frac{\lambda}{lm} = 2k$

we got

$$\frac{d^2 l\theta}{dt^2} + \frac{\lambda}{m}\frac{d(l\theta)}{dt} + g\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{\lambda}{ml}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{l}\theta = 0 \longrightarrow (1)$$

which is required differential equation

eq. (1) can be written operating form

$$(D^2 + 2kD + \omega)\theta = 0$$

where $\omega = g/l$

$D = \frac{d}{dt}$

$$\Rightarrow f(D)\theta = 0$$

where $f(D) = D^2 + 2kD + \omega$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 + 2km + \omega = 0$$

$$m = \frac{-2k \pm \sqrt{4k^2 - 4\omega}}{2\cdot 1}$$

$$= \frac{-2k \pm 2\sqrt{k^2 - \omega}}{2}$$

$$m = -k \pm \sqrt{k^2 - \omega}$$

$$m = -k \pm \sqrt{-(w-k^2)}$$

$$= -k \pm \sqrt{(w-k^2)i^2}$$

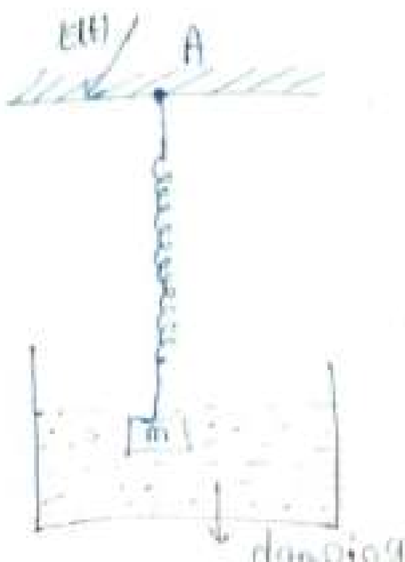
$$m = -k \pm i\sqrt{w-k^2};$$

$\therefore$ the solution of eq (1) is

$$\theta = e^{-kt}\left(c_1 \cos(\sqrt{w-k^2})t + c_2 \sin(\sqrt{w-k^2})t\right)$$

Spring mass system (or). oscillating spring-mass system :-

let a mass (m) be suspended at one of the ends of a simple spring whose other end is fixed. let k be the spring constant (restoring force constant). let 'r' be the damping constant (or damping force) and $x(t)$ be the displacement of the mass. let $E(t)$ be the external force (or applied force).



$$f_R \propto x \Rightarrow f_R = -kx$$

$$f_D \propto \frac{dx}{dt} \Rightarrow f_D = -r\frac{dx}{dt}$$

$$f_F \propto E(t) \Rightarrow f_F = E(t)$$

By Newton's Second law of motion

we have $F = ma$

$$-kx - r\frac{dx}{dt} + F(t) = m\frac{d^2x}{dt^2}$$

$$\boxed{m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F(t)} \longrightarrow (1)$$

Case (i): Free Oscillations ($F(t) = 0$):-

(i). Equation of the motion of the mass ~~with~~ and ~~damping~~ restoring force with out damping force.

$$m\frac{d^2x}{dt^2} + kx = 0 \longrightarrow (2)$$

(ii). Equation of the motion of the mass with restoring force and damping force

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \longrightarrow (3)$$

Case (ii): Force Oscillations ($F(t)$ is present):-

(i). Equation of the motion of the mass with out damping

$$m\frac{d^2x}{dt^2} + kx = F(t) \longrightarrow (4)$$

(ii). Equation of the motion of the mass with damping force

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F(t) \longrightarrow (5)$$

problem:-1. Consider a spring mass system with $m = 0.5$, $k = 17$, $r = 3$. let $x(t)$ be the displacement of the mass at any time 't'. write the equation of the mass with an external force $F(t) = 20t$. If $x(0) = 3$, $x'(0) = 1$, find $x(t)$.

Sol:- Given that

mass $(m) = 0.5$

$k = 17$ (spring constant)

$r = 3$ (damping constant)

$F(t) = 20t$ (External force)

The equation of motion of the mass is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F(t)$$

$$0.5\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 17x = 20t$$

$$\Rightarrow \frac{1}{2}\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 17x = 20t$$

$$\Rightarrow \frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 34x = 40t \longrightarrow (1)$$

also given that

$x(0)=3,\ x'(0)=1$

eq (1) can be write in operator form is

$(D^2+6D+34)x=40t \qquad \therefore D=\frac{d}{dt}$

$\rightarrow f(D)x=Q(t)$

where $f(D)=D^2+6D+34$

The A.E is $f(m)=0$

$$m^2+6m+34=0$$

$$m=\frac{-6\pm\sqrt{36-136}}{2}$$

$$=\frac{-6\pm\sqrt{-100}}{2}$$

$$=\frac{-6\pm 10i}{2}$$

$$m=-3\pm 5i$$

$\therefore$ The C.F is

$x_c=e^{3t}(c_1\cos 5t+c_2\sin 5t)$

The particular Integral is

$x_p=\frac{1}{f(D)}\cdot Q(t)$

$\frac{1}{D^2+6D+34}\cdot 40t$

$x_p=40\cdot\frac{1}{D^2+6D+34}\cdot t$

$$x_p = 40 \cdot \frac{1}{34} \cdot \frac{1}{1+\left(\frac{D^2+6D}{34}\right)} \cdot t$$

$$= \frac{40}{34} \cdot \left(1+\left(\frac{D^2+6D}{34}\right)\right)^{-1} t$$

$$= \frac{40}{34} \left\{1-\left(\frac{D^2+6D}{34}\right)\right\} t \qquad D^2 t$$

$$= \frac{40}{34} \left\{t - \frac{D^2}{34}t - \frac{6}{34}Dt\right\}$$

$$= \frac{40}{34} \left\{t - 0 - \frac{6}{34}\right\}$$

$$x_p = \frac{40t}{34} - \frac{\cancel{40}^{20}}{\cancel{34}_{17}} \cdot \frac{\cancel{6}^{3}}{\cancel{34}_{17}}$$

$$x_p = \frac{40t}{34} - \frac{60}{289}$$

The general Solution is

$$x(t) = x_c + x_p$$

$$x(t) = e^{3t}\left(c_1 \cos 5t + c_2 \sin 5t\right) + \frac{40t}{34} - \frac{60}{289} \longrightarrow (2)$$

Since $x(0) = 3$, $x'(0) = 1$

$x(0) = 3$

$$\Rightarrow x(0) = e^0\left(c_1 \cos(0) + c_2 \sin(0)\right) + 0 - \frac{60}{289}$$

$$3 = c_1 - \frac{60}{289}$$

$$c_1 = 3 + \frac{60}{289} = \frac{867+60}{289}$$

$$\boxed{c_1 = \frac{927}{289}}$$

$x'(0) = 1$

$$\Rightarrow x'(t) = e^{-3t}\cdot(-3)\left(c_1 \sin 5t \cdot 5 + 5\cos 5t\right) + \frac{40}{34}$$

$$x'(t) = -3e^{-3t}\left(-5c_1 \sin 5t + 5c_2 \cos 5t\right) + \frac{40}{34}$$

$$\Rightarrow x'(0) = -3\left(0 + 5c_2\right) + \frac{40}{34}$$

$$1 = -15c_2 + \frac{40}{34}$$

$$-15c_2 = 1 - \frac{40}{34}$$

$$= -\frac{6}{34}$$

$$c_2 = \frac{\cancel{6}}{\cancel{450}} = \frac{\cancel{3}}{\cancel{210}\ 255}$$

$$\boxed{c_2 = \frac{2}{85}}$$

Substituting c_1 and c_2 in eq (23) we get Complete Solⁿ is

$$\boxed{x(t) = e^{-3t}\left(\frac{927}{289}\cos 5t + \frac{2}{85}\sin 5t\right) + \frac{40}{30}t - \frac{66}{289}}$$

(2). A Spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. if the Spring is stretched to a length of 0.7 m and then released with initial velocity zero. Find the position of the mass at any time.

$x'(0) = 1$

$\therefore x'(t) = e^{-3t}(-3)\left(c_1 \sin 5t + c_2 \cos 5t\right) + \frac{40}{34}$

$x'(t) = e^{-3t}\left(-5c_1 \sin 5t + 5c_2 \cos 5t\right) + \frac{40}{34}$

$\therefore x'(0) = -3\left(c_1 + 5c_2\right) + \frac{40}{34}$

$1 = -15c_2 + \frac{40}{34}$

$-15c_2 = 1 - \frac{40}{34}$

$= -\frac{6}{34}$

$c_2 = \frac{6}{510} = \frac{3}{255}$

$\boxed{c_2 = \frac{2}{85}}$

Substituting c_1 and c_2 in eq (2)

we get Complete Soln is

$$\boxed{x(t) = e^{-3t}\left(\frac{927}{289}\cos 5t + \frac{2}{85}\sin 5t\right) + \frac{40}{30}t - \frac{66}{289}}$$

(2). A Spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it Stretched to a length of 0.7 m. if the Spring is stretched to a length of 0.7 m and then released with initial velocity zero. Find the position of the mass at any time.

Sol:- Given that

$m = 2$ kg (mass)

$F = 25.6$ N (Force)

$x = 0.7 - 0.5$

$x = 0.2$ (length) and Condition velocity is zero

Now By hooke's law $\Rightarrow x(0) = 0.2,\ x'(0) = 0$

The force req to stretch the spring

is $F = Kx$

$25.6 = K \cdot 0.2$

$K = 128$

The equation of the motion of the mass with out damping and External force is

$$m\frac{d^2x}{dt^2} + Kx = 0$$

$$2\frac{d^2x}{dt^2} + 128x = 0$$

$$\frac{d^2x}{dt^2} + 64x = 0 \longrightarrow (1)$$

The eq (1) Can be write in operator form

is $D^2x + 64x = 0$

$(D^2 + 64)x = 0$

$f(D)x = 0$

where $f(D) = D^2 + 64$

the A.E is $f(m) = 0$

$$m^2 + 64 = 0$$

$$m^2 = -64$$

$$m = \pm 8i$$

$\therefore$ The solution of eq (1) is

$$x(t) = C_1 \cos 8t + C_2 \sin 8t \quad \text{--- (2)}$$

Since $x(0) = 0.2$

eq (2) $\Rightarrow x(0) = C_1 \cos(0) + C_2 \sin(0)$

$$0.2 = C_1 + 0$$

$$\boxed{C_1 = 0.2}$$

and $x'(0) = 0$

eq. (2) $\Rightarrow x'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t$

$$x'(0) = 0 + 8C_2$$

$$0 = 8C_2$$

$$\boxed{C_2 = 0}$$

Substituting C_1, C_2 values in eq (2)

we get

$$x(t) = (0.2) \cos 8t + 0$$

$$= (0.2) \cos 8t$$

$$\boxed{x(t) = \cos 8t}$$

Given that

$m = 2\,kg$ (mass)

$F = 25.6\,N$ (force)

$x = 0.7 - 0.5$

$x = 0.2$ (length)

and also given

Initial velocity is zero

$x(t) = x(0) = 0.2$

$x'(0) = 0$

By hook's law

The force is stretched the spring

$F \alpha x$

is $F = -kx$

$25.6 = -k \cdot 0.2 \Rightarrow k = \frac{-25.6}{0.2}$

$k = -128$

The equation of the motion of the mass

without damping force and external force

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$2 \frac{d^2x}{dt^2} + (-128)x = 0$$

$$\frac{d^2x}{dt^2} - 64x = 0$$

$$\frac{d^2x}{dt^2} (D^2 - 64)x = 0$$

$$\Rightarrow f(D)x = 0$$

[illegible]

The general sol^n is

$$x(t) = c_1 e^{-8t} + c_2 e^{8t} \longrightarrow (1)$$

given conditions

$$x(0) = 0.2, \quad x'(0) = 0$$

$\Rightarrow x(0) = 0.2$

$$\Rightarrow x(0) = c_1 e^0 + c_2 e^0$$

$$0.2 = c_1 + c_2 \longrightarrow (2)$$

$\Rightarrow x'(0) = 0$

$$x'(t) = -8c_1 e^{-8t} + 8c_2 e^{8t}$$

$$x'(0) = -8c_1 + 8c_2$$

$$0 = -8c_1 + 8c_2 \longrightarrow (3)$$

Solving eq, (2), (3), we have

eq (2) $\times 8 \Rightarrow 1.6 = \cancel{8c_1} + 8c_2$

eq (3) $\quad 0 = -\cancel{8c_1} + 8c_2$

$$1.6 = 16 c_2 \Rightarrow c_2 = \frac{1.6}{16} = 0.1$$

eq (2) $\Rightarrow c_1 = 0.2 - 0.1 = 0.1$

$$\boxed{c_1 = 0.1}$$

from eq (1)

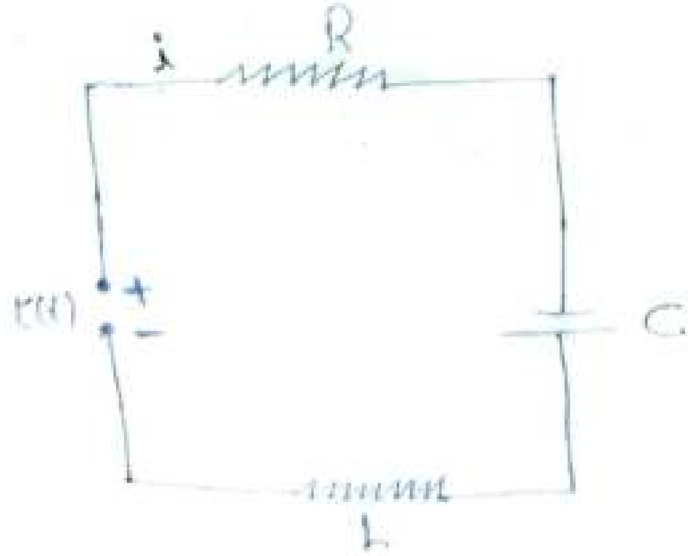
$$x(t) = (0.1)e^{-8t} + (0.1)e^{8t}$$

Kirchoff's Law of voltage :-

In any electric circuit the sum of voltage drops is equal to the E.M.F (electro motive force) (i.e $E(t)$)

L-C-R Circuit :- (or) Inductance - capacitor - Resistor Circuit :-

Consider, a simple electrical circuit as shown figure which contains of a resistor (R) in Ω, a capacitor (C) in farad, an inductance L in henry and electro motive force (E.M.F), $E(t)$ in volts.



The current "i" is ampeare and the charge q in columbs

By Kirchhoff's law,

we have

$$L\frac{di}{dt} + Ri + \frac{q}{c} = E(t) \quad \text{---} (4)$$

but $i = \frac{dq}{dt}$ Rate of charge with respect to time.

from eqn (4)

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t) \quad \text{---} (5)$$

Assume q_0 be the initial charge

and i_0 be the initial Current.

(i.e $q = q_0$, $i = i_0$ at $t = 0$)

$$\left\{ \begin{array}{l} \text{i.e } q(0) = q_0 \\ q'(0) = i_0 \end{array} \right\} \quad \begin{array}{l} \text{i.e } q(t) = q_0 \\ q'(t) = i_0 \end{array}$$

problem:-1. An LCR Circuit is Connect in Series has $R = 180\,\Omega$, $C = \frac{1}{280}$ F, $L = 20$ H, and an applied voltage $E(t) = 10 \sin t$. Assume no initial charge on the capacitor but an initial Current is 1 amp. Find the Subsequent charge and Current in the Circuit.

Sol:

Given that

$R = 180\,\Omega$

$C = \frac{1}{280}\,F$

$L = 20\,H$

$E(t) = 10\sin t$

By Kirchoff's law, we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

$$20\frac{d^2q}{dt^2} + 180\frac{dq}{dt} + 280q = 10\sin t$$

$$\frac{d^2q}{dt^2} + 9\frac{dq}{dt} + 14q = \frac{1}{2}\sin t \longrightarrow (1)$$

En eq (1) Can be written in operator form

$$D^2q + 9Dq + 14q = \frac{1}{2}\sin t$$

$$(D^2 + 9D + 14)q = \frac{1}{2}\sin t$$

$$\Rightarrow f(D)q = Q(t)$$

where $f(D) = D^2 + 9D + 14$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 + 9m + 14 = 0$$

$$\Rightarrow m^2 + 7m + 2m + 14 = 0$$

$$m(m+7) + 2(m+7) = 0$$

$$(m+7)(m+2) = 0$$

$$m = -2, -7$$

The C.F is

$$q_c = c_1e^{-7t} + c_2e^{-2t}$$

//.

The P.I is

$$y_p = \frac{1}{D^2+9D+14} \cdot \frac{1}{2} \sin t$$

$$= \frac{1}{2} \frac{1}{D^2+9D+14} \sin t$$

$$y_p = \frac{1}{2} \frac{1}{-1^2+9D+14} \sin t$$

$$= \frac{1}{2} \frac{1}{9D+13} \sin t$$

$$= \frac{1}{2} \cdot \frac{1}{9} \frac{1}{D+13/9} \sin t$$

$$= \frac{1}{18} \cdot \left\{ \frac{1}{D+13/9} \sin t \right\}$$

$$y_p = \frac{1}{18} \left\{ e^{-13/9 t} \int e^{13/9 t} \sin t \right\}$$

$$= \frac{1}{18} \left\{ e^{-13/9 t} \left[\frac{e^{13/9 t}}{\left(\frac{13}{9}\right)^2 + 1^2} \left(\frac{13}{9} \sin t - \cos t \right) \right] \right\}$$

$$= \frac{1}{18} \left\{ \frac{1}{\frac{169}{81}+1} \left(\frac{13}{9} \sin t - \cos t \right) \right\}$$

$$= \frac{1}{\cancel{18}_2} \left\{ \frac{\cancel{81}\, 9}{169+81} \left(\frac{13}{9} \sin t - \cos t \right) \right\}$$

$$y_p = \frac{9}{2 \times 250} \left(\frac{13}{9} \sin t - \cos t \right)$$

$$y_p = \frac{9}{500} \left(\frac{13}{9} \sin t - \cos t \right)$$

The general soln is

$$y(t) = y_c + y_p$$

$$\boxed{y(t) = c_1 e^{-7t} + c_2 e^{-2t} + \frac{9}{500} \left(\frac{13}{9} \sin t - \cos t \right)}$$

Unit - 3 09-06-2021

Partial Differential Equation

Definition: A differential equation which involves the partial derivatives of dependent variable w.r.t two or more than two independent variables is called "partial differential Equation".

Ex:- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2 \Rightarrow p+q = z^2 \Rightarrow p+q-z^2=0$
$\Rightarrow F(z,p,q)=0$

$\frac{\partial^2 z}{\partial x^2} + 2\sin x \frac{\partial z}{\partial y} = xz$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$ are called first and Second order partial derivatives.

Partial Differential Equations of First order :-

The general form of a first order partial differential equation is $F(x,y,z,p,q)=0$. →(1)
where x, y are the two independent variables, z is the dependent variable and $p = \frac{\partial z}{\partial x} = z_x$, $q = \frac{\partial z}{\partial y} = z_y$.

Complete Solution :-

Consider the first order P.D.E $F(x,y,z,p,q)=0$ →(2) then $z = F(x,y,a,b) = 0$ is known as Complete Solution.

General Solution:-

Case (i): [illegible]

[illegible]

[illegible]

[illegible]

Auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then $\phi(u,v) = 0$ (or) $u = \phi(v)$ (or) $v = \phi(u)$

be a general soln of eq (1) where ϕ is arbitrary function of u, v.

Case (ii):- let $F(x,y,z,p,q) = 0$ be a non-linear P.D.E of first order $\longrightarrow$ (A)

let $z = f(x,y,a,b)$ be the Complete solution of eq (A) $\longrightarrow$ (2) (or) $\psi(x,y,z,a,b) = 0$ be the complete solution of eq (A)

put $b = \phi(a)$ where ϕ is arbitrary function of a and in eq (2)

$$\psi(x,y,z,a,\phi(a)) = 0 \longrightarrow (3)$$

Then the elimination of a from $\psi(x,y,z,a,\phi(a)) = 0$ and $\frac{\partial \psi}{\partial a} = 0$ gives general Solution of eq (A)

Singular Solution :-

Consider the point [illegible]

Let $\psi(x,y,z,a,b)=0$ be the Complete [illegible] then singular Solution is given by eliminating a,b from $\phi(x,y,z,a,b)=0$, $\frac{\partial\phi}{\partial a}=0$, $\frac{\partial\phi}{\partial b}=0$

General Solution of Quasi linear P.D.E of first order:

Consider the linear partial differential equation $P(x,y,z)p+Q(x,y,z)q=R(x,y,z) \longrightarrow (1)$

The lagrange's Auxilary Equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \longrightarrow (2)$$

let $u(x,y,z)=c_1$, $v(x,y,z)=c_2$ be two independent Solutions of eq (2) Then

$$\phi(u,v)=0 \text{ (or) } u=\phi(v), \text{ (or) } v=\phi(u)$$

be the general Solution of eq (1) where ϕ is arbitrary function.

Type (i):- Take two Different pairs of fractions in eq (2) and we get $u(x,y,z)=c_1$, $v(x,y,z)=c_2$

problems:

(1). Solve $yzp + zxq = xy$

Sol:- Given

$yzp + zxq = xy \longrightarrow (1)$

eq (1) Compare to $Pp + Qq = R$

Then $P = yz,\ Q = zx,\ R = xy$

The lagrange's Auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \longrightarrow (2)$$

Take first and second in eq (2)

we have $\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x}$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow x\,dx = y\,dy$$

Integrating on b.s

we get $\int x\,dx = \int y\,dy$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1^2}{2}$$

$$\Rightarrow \boxed{x^2 - y^2 = c_1^2}$$

$u = c_1$

and Take Second and Third in eq (2)

we have $\frac{dy}{\cancel{z}x} = \frac{dz}{xy}$

$\Rightarrow y\,dy = z\,dz$

Integrating on b.s

we get

$\int y\,dy = \int z\,dz +$

$\frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$

$\boxed{y^2 - z^2 = c_2}$

$v = c_2$

The general solution is

$\phi(u,v) = 0$ (or) $u = \phi(v)$ (or) $v = \phi(u)$

$\Rightarrow \phi(x^2-y^2, y^2-z^2) = 0$. (or) $x^2-y^2 = \phi(y^2-z^2)$

(c). Solve the following Linear partial Differential Equation.

(a). $xp+yq = 3z$

(b). $\sqrt{x}p+\sqrt{y}q = \sqrt{z}$

(c). $y^2p-xyq = x(z-2y)$

(d). $py-qx = yz^2$

(e). $\frac{y^2z}{x}p+xzq = y^2$

Sol.

(e). Given L.P.D.E is

$\frac{y^2z}{x}p+xzq = y^2$

$\Rightarrow y^2z\,p+x^2zq = xy^2 \longrightarrow (1)$

eq (1) is Compared with $Pp+Qq = R$

Then $P = y^2z$, $Q = x^2z$, $R = xy^2$

the Lagrange's A.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2} \longrightarrow (2)$$

take first and second pairs of eq (2)

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}}$$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2}$$

$$x^2 dx = y^2 dy$$

Integrating on b.s

$$\int x^2 dx = \int y^2 dy \qquad \therefore \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$$

$$\frac{x^3}{3} - \frac{y^3}{3} = \frac{c_1}{3}$$

$$\boxed{x^3 - y^3 = c_1}$$

$$\boxed{u = c_1}$$

and also take ~~Second~~ first and Third pairs of eq (2)

we get $\frac{dy}{x^2z} =$

$$\frac{dx}{\cancel{y^2}z} = \frac{dz}{x\cancel{y^2}}$$

$$\frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx = z\,dz$$

Int on b.s

$$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$\boxed{x^2 - z^2 = c_2}$$

$$\boxed{v = c_2}$$

$\therefore$ The G.S is

$$\phi(u, v) = 0$$

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

Type (ii):- One solution is obtained by taking two suitable functions in Lagrange auxiliary equation and the second solution is obtained by making use of 1st solution.

Problems:-

1. Solve $p - q = \log(x+y)$

Sol:- Given L.P.D.Eqn

$$p - q = \log(x+y) \longrightarrow (1)$$

Comparing eq (1) into $Pp + Qq = R$

we get $P = 1$, $Q = -1$, $R = \log(x+y)$

The Lagrange's Auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \longrightarrow (2)$$

Taken from eq (2), we have

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\Rightarrow dx = -dy$$

Integrating on b.s

we have $\int dx = -\int dy$

$$x = -y + c_1$$

$$\boxed{x + y = c_1} \quad \text{i.e } \boxed{u = c_1}$$

and also $\frac{-dy}{1} = \frac{dz}{\log(x+y)}$ $\quad (\because x+y = c_1)$

$$-dy = \frac{dz}{\log c_1}$$

Integrating w.r.t.

we get

$$-\int dy = \int \frac{1}{\log c_1}\, dz$$

$$-y = \frac{1}{\log c_1} \int dz$$

$$-y \log c_1 = z + c_2 \qquad (\because c_1 = x+y)$$

$$-y \log(x+y) - z = c_2$$

$$\boxed{y \log(x+y) + z = c_2} \quad \boxed{\text{i.e. } v = c_2}$$

$\therefore$ The general solution is

$$\phi(u, v) = 0$$

$$\phi(x+y,\ y\log(x+y) + z) = 0$$

(5). Solve the following 1. P.D equations

(i). $py - qx = yz^2 e^{-x^2-y^2}$

(ii). $(y^2+z^2)p - xyq = -zx$

(iii). $(z^2 - yz - y^2)p + (xy + zx)q = xy - zx$

Type (ii):- Consider $Pp+Qq=R$ then lagrange's auxilary equation is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=$$

choose multipliers P_1, Q_1, R_1

so that, $P_1P+Q_1Q+R_1R=0$ then

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{P_1dx+Q_1dy+R_1dz}{0}$$

So, we have take

$$P_1dx+Q_1dy+R_1dz=0$$

which can be integrate on b.s

problems:-

(1). Find general Solution of $(z-y)p+(x-z)q=y-x$

Sol:- Given Linear P.D.E is

$$(z-y)p+(x-z)q=y-x \longrightarrow (1)$$

Comparing eq (1) into $Pp+Qq=R$

we have $P=z-y,\ Q=x-z,\ R=y-x$

The lagrange's auxilary equation is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x} \longrightarrow (2)$$

choose multipliers $P_1=1,\ Q_1=1,\ R_1=1$

then $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{P_1dx+Q_1dy+R_1dz}{0}$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{dx+dy+dz}{z-y+x-z+y-x}$$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{dx+dy+dz}{0}$$

so that,

$$dx+dy+dz=0$$

Integrating on b.s

we get $\int dx + \int dy + \int dz = 0$

$$\Rightarrow \boxed{x+y+z=c_1}$$

$$u=c_1$$

and also

Take multipliers $P_1 = x$, $Q_1 = y$, $R_1 = z$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{x\,dx+y\,dy+z\,dz}{xz-xy+xy-yz+yz-xz}$$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

So that, $x\,dx+y\,dy+z\,dz=0$

Integrating on b.s

we get $\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2/2$$

$$\boxed{x^2+y^2+z^2=c_2}$$

$$v=c_2$$

The general solution is

$$\phi(u,v)=0$$

$$\phi(x+y+z,\ x^2+y^2+z^2)=0$$

Find general solutions of following linear P.D.E

(i). $(y-zx)p+(x+yz)q=x^2+y^2$

(ii). $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

(iii). $(mz-ny)\dfrac{\partial z}{\partial x}+(nx-lz)\dfrac{\partial z}{\partial y}=ly-mx$

Type (iv):- Multipliers may be chosen (more than once) such that the numerator $P_1dx+Q_1dy+R_1dz$ is an exact differential of the denominator $P_1P+Q_1Q+R_1R$. Now combine eq $\dfrac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$ with a fraction of eq (1) to get an integral.

Problem:-1 Solve $x^2p+y^2q=(x+y)z$

Given $x^2p+y^2q=(x+y)z \longrightarrow (1)$

equation (1) substituting in $Pp+Qq=R$

we have $P=x^2,\ Q=y^2,\ R=(x+y)z$

The auxiliary equation of linear P.D.E.

is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \longrightarrow (2)$$

take first and second terms of eq (2) i.e

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating on b.s

$$\int \frac{1}{x^2}\,dx = \int \frac{1}{y^2}\,dy$$

$$\int x^{-2}\,dx = \int y^{-2}\,dy \qquad \left(\int x^n\,dx = \frac{x^{n+1}}{n+1}\right)$$

$$\frac{x^{-2+1}}{-2+1} = \frac{y^{-2+1}}{-2+1}$$

$$-\frac{1}{x} = -\frac{1}{y} + \frac{1}{c}$$

$$-\frac{1}{x} + \frac{1}{y} = \frac{1}{c} = c_1 \Rightarrow \frac{-y+x}{xy} = c_1$$

$$y - x = xy\,c_1$$

$$\boxed{\frac{y-x}{xy} = c_1} \Rightarrow u = c_1$$

and take $P_1 = \frac{1}{x}$, $Q_1 = -\frac{1}{y}$, $R = -1/z$

then $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{\frac{1}{x}x^2 - \frac{1}{y}y^2 - [illegible]}$

So that

$$-\frac{1}{x}\,dx - \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

Integrating on b.s

$$-\int \frac{1}{x}\,dx - \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = 0$$

$$-\log x - \log y + \log z = c_2$$

$$-\log xy + \log z = c_2 \qquad \log x + \log y - \log z = \log c_2$$

$$\log (xy)^{-1} + \log z = c_2 \qquad \log xy - \log z = \log c_2$$

$$\log \frac{1}{xy} + \log z = c_2 \qquad \log \frac{xy}{z} = \log c_2$$

$$\Rightarrow \boxed{\frac{xy}{z} = c_2} \Rightarrow v = c_2$$

$\therefore$ The general Solution is

$$\phi(u, v) = 0$$

$$\Rightarrow \phi\left(\frac{y-x}{xy}, \frac{xy}{z}\right) = 0$$

(or)

$$u = \phi(v) \Rightarrow u = \phi\left(\frac{xy}{z}\right)$$

Solve $(y+z)p + (z+x)q = x+y$

Given P.D.E:

$$(y+z)p + (z+x)q = x+y \longrightarrow (1)$$

Comparing eq (1) into $Pp + Qq = R$

we have $P = y+z,\ Q = z+x,\ R = x+y$

Here the Lagrange's P, Q, R

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{--- (2)}$$

Taking (2), we have $P_1 = 1,\ Q_1 = -1,\ R_1 = 0$

$$\frac{dx - dy + 0\,dz}{y+z-z-x} = \frac{0\,dx + dy - dz}{0+z+x-x-y} \quad P_1 = 0, Q_1 = 1, R_1 = -1$$

$$\frac{dx - dy - 0}{y+\cancel{z}-\cancel{z}-x} = \frac{0 + dy - dz}{\cancel{z}+\cancel{x}-\cancel{x}-y}$$

$$\frac{dx - dy}{y-x} = \frac{dy - dz}{z-y}$$

$$\Rightarrow \frac{d(x-y)}{y-x} = \frac{d(y-z)}{z-y}$$

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

Int on b.s

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) - \log(y-z) = \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

$\boxed{u = c_1}$

also, we take $P_1 = 1, Q_1 = 0, R_1 = -1$

and $P_1 = 1, Q_1 = -1, R_1 = 0$

$$\frac{dx - dz}{y + z - x - y} = \frac{dx - dy}{y + \cancel{z} - \cancel{z} - x}$$

$$\frac{dx - dz}{z - x} = \frac{dx - dy}{y - x}$$

$$\Rightarrow \frac{d(x-z)}{-(x-z)} = \frac{d(x-y)}{-(x-y)}$$

Int on b.s

$$\int \frac{d(x-z)}{x-z} = \int \frac{d(x-y)}{x-y}$$

$$\log(x-z) = \log(x-y) + \log c_2$$

$$\Rightarrow \log(x-z) - \log(x-y) = \log c_2$$

$$\log\left(\frac{x-z}{x-y}\right) = \log c_2$$

$$\boxed{\frac{x-z}{x-y} = c_2 \qquad v = c_2}$$

The general soln is

$$\phi(u,v) = 0 \Rightarrow \phi\left(\frac{x-y}{y-z}, \frac{x-z}{x-y}\right) = 0$$

Non-linear Equation of the first order:

The equations in which p and q occur other than in the first degree are called non-linear partial differential equations of the first order. The complete solution of such an equation contains only two arbitrary constants (i.e equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.

Here we have to discuss four standard forms of these equations

Form - I :- Equation of the type $f(p,q)=0$ (i.e Equation containing p and q only):-

Let the complete solution be $z = ax+by+c \rightarrow (1)$ of $f(p,q)=0$.

Hence $\frac{\partial z}{\partial x} = a$, $\frac{\partial z}{\partial y} = b$

i.e $p = a$ & $q = b$

on substituting these values in $f(p,q)=0$ we get $f(a,b)=0$

form this find the value of 'b' in terms of 'a' and substitute the value of 'b' in eq (1), that will be the required solution.

problems:-

1. Solve $p^2+q=1$

Sol:- Given non-linear equation is

$$p^2+q=1$$

$$\Rightarrow p^2+q-1=0 \longrightarrow (A)$$

$$\Rightarrow f(p,q)=0 \longrightarrow (1)$$

So, the complete solution of eq (1) is

$$z=ax+by+c \longrightarrow (2)$$

Here $a=p$, $b=q$

from eq (A), we have

$$a^2+b-1=0$$

$$b=1-a^2$$

Substituting 'b' value in eq (2)

we get $\boxed{z=ax+(1-a^2)y+c}$

which is required complete solution

2. Solve $pq=1$.

Sol:- Given non-linear P.D.E is

$$pq=1$$

$$\Rightarrow pq-1=0 \longrightarrow (1)$$

$$\Rightarrow f(p,q)=0$$

So, the Complete Solution of eq (1)

is $z = ax + by + c$ ---> (2)

Here $a = p$, $b = q$

from eq (1), we have

$ab - 1 = 0$

$\Rightarrow ab = 1$

$\Rightarrow b = 1/a$

Substituting 'b' value in eq (2),

we get

$$z = ax + (1/a)y + c$$

which is required Complete Solution of given non linear P.D.E of first order.

(3). Solve following Non linear P.D.E. to find complete Solution

(i). $\sqrt{p} + \sqrt{q} = 1$

(ii). $p + q = pq$

(iii). $p = e^q$

(iv). $p^2 + q^2 = m^2$

(v). $p^2 - q^2 = 1$

(vi). $2p^2 + 6p + 9q + 4 = 0$

(vii). $p + q = 1$.

Form - II :- Equation of the type $f(z,p,q)=0$:-

In this form, the equations are not containing 'x' & 'y'.

Thus to solve $f(z,p,q)=0$

* Assume $u=x+ay$ and substitute $p=\frac{dz}{du}$, $q=a.\frac{dz}{du}$ in the given equation.
* on substituting the above values $f(z,p,q)=0$ becomes $f\left(z,\frac{dz}{du},a.\frac{dz}{du}\right)=0$, which is an ordinary differential equation of first order.
* Solve the resulting ordinary differential equation in 'z' and 'u' and finally replace $u=x+ay$, we get required solution.

Problems :-

1). Solve $p(1+q)=qz$

Sol:- Given equation is

$$p(1+q)=qz \longrightarrow (1)$$

let $u=x+ay$ and $p=\frac{dz}{du}$, $q=a.\frac{dz}{du}$
$=a.p$

Substituting above values in eq (1)

we get,

$$\cancel{\frac{dz}{du}}\left(1+a.\frac{dz}{du}\right)=z.a.\cancel{\frac{dz}{du}}$$

$$1+a\frac{dz}{du}=za$$

$$a\frac{dz}{du}=za-1$$

$$\frac{a}{az-1}dz=du$$

Integrating on b.s

$$\int\frac{a}{az-1}dz=\int du.$$

$$\log(az-1)=u+c$$

$$u=\log(az-1)+c \qquad \because u=x+ay$$

$$\Rightarrow x+ay=\log(az-1)+c$$

which is required Complete Solution

(2). Find the Complete Solution of following non-linear P.D.E

(i). $p^2z^2+q^2=1$

(ii). $q^2=z^2p^2(1-p^2)$

(iii). $p(1+q^2)=q(z-a)$

(iv). $p^2+q^2=z$

(v). $z^2=1+p^2+q^2$

Form - IV: Equation of the type $z = px + qy + f(p, q)$ (i.e Clairaut's equation):-

Its Complete Solution is $z = ax + by + f(a, b)$ which is obtained by writing 'a' for p and 'b' for q in the given equation.

Problems :-

1). Solve $z = px + qy + \sqrt{1+p^2+q^2}$

Sol:- Given equation is

$$z = px + qy + \sqrt{1+p^2+q^2} \longrightarrow (1)$$

as (1) Comparing in the form $z = px + qy + f(p, q)$

i.e C.S

where $f(p, q) = \sqrt{1+p^2+q^2}$

Hence put $a = p$, $q = b \Rightarrow f(a, b) = \sqrt{1+a^2+b^2}$

we have. $z = ax + by + f(a, b)$

$$z = ax + by + \sqrt{1+a^2+b^2}$$

which is Complete Solution of given clairaut's equation.

(2). Solve $z = px + qy + \log(pq)$

Sol:- Given equation is

$$z = px + qy + \log(pq) \longrightarrow (1)$$

$$\Rightarrow z - px - qy - \log(pq) = 0$$

$$\Rightarrow f(x, y, z, p, q) = 0$$

Now, the complete solution of eq (1) is

$$z = ax + by + f(a, b) \longrightarrow (2)$$

Comparing eq (1) into $z = px + qy + f(p, q)$

where $f(p, q) = \log pq$

put $a = p$, $b = q$

we have $f(a, b) = \log(ab)$

from eq (2), we get $z = ax + by + f(a, b)$

The required complete solution is

$$z = ax + by + \log(ab)$$

(3). Find the complete sol^n in the following eqtns

p, q, t

(i). $(p+q)(z - px - qy) = 1$

(ii). $z = px + qy + p^2 q^2$

(iii). $z = px + qy + \sin(p+q)$

(iv). $z = px + qy + \sin(p+q)$

Form-III: The equation of the type $f(x,p) = F(y,q)$

(i.e. the equations in which z is absent and the terms containing 'x' and 'p' can be separated from those containing 'y' and 'q').

As a trial solution assume that

$$f(x,p) = F(y,q) = a \text{ (say)} \longrightarrow (1)$$

Then $f(x,p) = a$, $F(y,q) = a$

Solving for 'p' and 'q' we get

$$p = \phi_1(x), \quad q = \phi_2(y)$$

Since

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy$$

$$dz = \phi_1(x)dx + \phi_2(y)dy \longrightarrow (2)$$

Integrating on b.s

$$\int dz = \int \phi_1(x)\,dx + \int \phi_2(y)\,dy$$

$$z = \int \phi_1(x)\,dx + \int \phi_2(y)\,dy + b$$

which is the desired Complete Solution containing two Constants 'a' and 'b'.

Problems :-

(1). Solve $pz = xy$

Sol:-

Given equation

$pz = xy$

$\Rightarrow \frac{p}{x} = \frac{y}{z} = a$ (say) $\longrightarrow (1)$

$\Rightarrow f(x,p) = F(y,z) = a$

then $\frac{p}{x} = a$, $\frac{y}{z} = a$

$\Rightarrow p = ax$, $z = \frac{y}{a}$

$\Rightarrow p = \phi_1(x)$, $z = \phi_2(y)$

Since $dz = \phi_1(x)\,dx + \phi_2(y)\,dy$

$dz = ax\,dx + \frac{y}{a}\,dy$

Integrating on b.s

$\int dz = a\int x\,dx + \frac{1}{a}\int y\,dy + b$

$z = a\frac{x^2}{2} + \frac{1}{a}\frac{y^2}{2} + b$

$$\boxed{z = \frac{ax^2}{2} + \frac{y^2}{2a} + b}$$

which is required Complete Solution of given $f(x,y,p,z) = 0$ //.

Q1. Solve $p^2 - q^2 = x - y$

Given equation

$p^2 - q^2 = x - y$

$\Rightarrow p^2 - x = q^2 - y = a$ (say) — (1)

$\Rightarrow f(x,p) = f(y,q) = a$

Now $p^2 - x = a$, $q^2 - y = a$

$p^2 = x + a$, $q^2 = y + a$

$p = \sqrt{x+a}$, $q = \sqrt{y+a}$

$\Rightarrow p = \phi_1(x)$, $q = \phi_2(y)$

Since $dz = \phi_1(x)\,dx + \phi_2(y)\,dy$

$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$

Integrate on b.s

$\int dz = \int \sqrt{x+a}\,dx + \int \sqrt{y+a}\,dy$

$z = \int (x+a)^{1/2}\,dx + \int (y+a)^{1/2}\,dy$

$z = \dfrac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \dfrac{(y+a)^{1/2+1}}{\frac{1}{2}+1} + b$

$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$

$$\boxed{\frac{3z}{2} = (x+a)^{3/2} + (y+a)^{3/2} + \frac{3b}{2}}$$ //

(5). Solve the following non-linear P.D.E and find complete Solution?

(i). $p^2+q^2=x^2+y^2$

(ii). $\sqrt{p}+\sqrt{q}=x+y$

(iii). $p+q=\sin x+\sin y$

[illegible].

Charpit's Method :-

We now explain a general method for finding the complete integral for a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y, z, p, q) = 0 \longrightarrow (1)$$

Since z depends on 'x' and 'y', we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = p\,dx + q\,dy \longrightarrow (2)$$

The Lagrange's Auxiliary Equation of eq (1)

is
$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

We choose two suitable fraction, which gives a relation containing 'p' or 'q' or both 'p' and 'q'

This will give $g(x, y, z, p, q, a) = 0 \longrightarrow (3)$

where 'a' is arbitrary constant.

Solve eq (1), (3), we get the p and q

Substitute p, q in dz (2)

we get $dz = \phi_1(x,y,z)\,dx + \phi_2(x,y,z)\,dy \longrightarrow (4)$

By solving eq (4)

we get Complete Solution of eq (1)

which is of the form

$F(x,y,z,a,b)=0.$

problems :-

(1). Find complete Solution of $zpq = p+q$

Soln:- Given equation

$zpq = p+q$

$\Rightarrow zpq - p - q = 0 \longrightarrow (1)$

$\Rightarrow f(x,y,z,p,q)=0$

$f_p = zq-1,\ f_x = 0$

$f_q = zp-1,\ f_y = 0$

$f_z = pq$

Now, the A.E of charpits eq is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-qf_q - pf_p} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-zq+1} = \frac{dy}{-zp+1} = \frac{dz}{-zpq+q-pqz+p} = \frac{dp}{p^2q} = \frac{dq}{q^2p}$$

Take $\frac{dp}{p^2\cancel{q}} = \frac{dq}{q^2p}$

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating on b.s

$$\int \frac{1}{p}\, dp = \int \frac{1}{q}\, dq$$

$$\log p = \log q + \log a$$

$$\log p - \log q = \log a$$

$$\log\left(\frac{p}{q}\right) = \log a$$

$$\frac{p}{q} = a$$

$$\boxed{p = qa} \quad (\Rightarrow p = \phi_1(x,y,z,a))$$

Substitute 'p' in eq ① $\quad p = \cancel{a} \cdot \frac{1+a}{\cancel{a}z} = \frac{1+a}{z}$

$$q^2az - qa - q = 0$$

$$q^2az - q(1+a) = 0.$$

$$\cancel{q}(1+a) = q^{\cancel{2}}az$$

$$\boxed{q = \frac{1+a}{az}} \quad (\Rightarrow q = \phi_2(x,y,z)$$

Since $\quad dz = \phi_1(x,y,z_{a})\,dx + \phi_2(x,y,z_{a})\,dy$

$$dz = (qa)\,dx + \left(\frac{1+a}{az}\right)dy$$

Integrating on b.s

$$\int dz = \int qa\, dx + \int \frac{1+a}{az}\, dy$$

$$z = a\int \frac{1+a}{az}$$

$$dz = \left(\frac{1+a}{z}\right)\cancel{x}\, dx + \left(\frac{1+a}{az}\right) dy$$

$$z\, dz = \cancel{a}(1+a)\, dx + \frac{1+a}{a} dy$$

Integrating on b.s

$$\int z\, dz = \int a(1+a)\, dx + \int \frac{1+a}{a}\, dy$$

$$\frac{z^2}{2} = a(1+a)x + \left(\frac{1+a}{a}\right)y + b$$

$F(x, y, z, a, b) = 0$

which is Complete Soln

(2). Solve $(p^2+q^2)y - qz = 0$

(3). Solve $2z + p^2 + qy + 2y^2 = 0$

(4). Solve the following equation and find the complete Solution

(i). $z = p^2x + q^2x$

(ii). $1 + p^2 = qz$

(iii). $z^2 = pqxy$

(iv). $q + xp = p^2$

(v). $pxy + pz + qy = yz$

(vi). z

Homogeneous Linear Equation with Constant Co-efficients:-

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \longrightarrow (1)$$

in which a's are Constants, is called a homogeneous linear P.D.E of the n^{th} order with Constant co-efficients. It is called homogeneous because all terms contain derivatives of the Same order.

on writing, $\frac{\partial^r}{\partial x^r} = D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$

Then eq (1) becomes $\left(\because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}\right)$

$$\left(D^n + a_1 D^{n-1} D'^{r} + D' + \cdots + a_n D'^n\right) z = F(x,y)$$

(or).

briefly $f(D, D') z = F(x,y)$

As in the case of Ordinary linear equations with Constant co-efficients the complete solution of eq (1) consists of two parts, namely:

The complementary function and the particular integral.

The complementary function is the complete solution of the equation $f(D,D')z=0$ which must contain 'n' arbitrary functions.

The particular integral is the particular solution of eq. ③.

Rules for finding the Complementary function:-

Consider the equation

$$\frac{\partial^2 z}{\partial x^2} + a_1\frac{\partial^2 z}{\partial x \partial y} + a_2\frac{\partial^2 z}{\partial y^2} = 0 \longrightarrow ①$$

which is in operator form is $\frac{\partial}{\partial x}=D$

$$D^2 z + a_1 DD'z + a_2 D'^2 z = 0 \quad \frac{\partial}{\partial y}=D'$$

$$(D^2 + a_1 DD' + a_2 D'^2)z = 0 \longrightarrow ②$$

$$\Rightarrow f(D,D')z = 0.$$

the eq ② is called the A.E is $f(m,1)=0$

let its root be $m=\frac{D}{D'}$

From eq ①

$$\left(\left(\frac{D}{D'}\right)^2 + a_1\frac{D}{D'} + a_2\right)z = 0.$$

$$\Rightarrow (m^2 + a_1 m + a_2)z = 0$$

$f(m)z = 0$ $\qquad f(m,1) = 0_{11}$

case(i):- If the roots be real and distinct then eq (2) is equivalent to

$$(D - m_1 D')(D - m_2 D')z = 0$$

Here the complete solution of eq (1) is

$$\boxed{z = f(y + m_1 x) + \phi(y + m_2 x)}$$

$z = f_1(y+m_1 x) + f_2(y+m_2 x)$ (or) $z = \phi_1(y+m_1 x) + \phi_2(y+m_2 x)$

case(ii):- If the roots be equal $(m_1 = m_2 = m_3)$ then the complete solution of eq (1) is

$$\boxed{z = f(y + m_1 x) + x\phi(y + m_1 x)}$$

$z = \phi_1(y+m_1 x) + x\phi_2(y+m_1 x) + x^2\phi_3(y+m_1 x)$

problems:-

(1). Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

sol^n:- Given equation

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0 \longrightarrow (1)$$

eq (1) can be written in operator form we get

$$2D^2 z + 5DD'z + 2D'^2 z = 0$$

$$\Rightarrow (2D^2 + 5DD' + 2D'^2)z = 0 \longrightarrow (2)$$

$$\Rightarrow f(D, D')z = 0$$

$\therefore$ The A.E of eq (2) is ~~f(m) = 0~~ $f(m,1)=0$

we have. $\therefore f\left(\frac{D}{D'}\right)=0$

$$\left(2\frac{D^2}{D'^2}+5\frac{DD'}{D'^2}+2\right)z=0$$

$$\left(2\left(\frac{D}{D'}\right)^2+5\left(\frac{D}{D'}\right)+2\right)z=0.$$

$$\Rightarrow (2m^2+5m+2)z=0$$

$$\Rightarrow f(m,1)z=0.$$

$\therefore$ The A.E is

$$\Rightarrow f(m,1)=0.$$

$$2m^2+5m+2=0$$

$$m=\frac{-5\pm\sqrt{25-16}}{4}$$

$$=\frac{-5\pm\sqrt{9}}{4}$$

$$=\frac{-5\pm3}{4}=\frac{-2}{4},\ -8/4,$$

$$m=-1/2,\ -2$$

$\therefore$ The Complete Soln of eq (1) is

$$z=f(y+m_1x)+\phi(y+m_2x)$$

$$\boxed{z=f\left(y-\frac{1}{2}x\right)+\phi(y-2x)}$$

Solve. $4r+12s+9t=0$

Case (iii) :- If m_1, m_2 are Complex Conjugate pair Say $m_1 = a+ib$, $m_2 = a-ib$ and $m_3, m_4, m_5 \dots m_n$ are real and distinct then the Complementary function of a Complete Solution is

$$z = \phi_1(y+m_1x) +$$

$$z = \left[\phi_1(y+ax+ibx) + \phi_1(y+ax-ibx)\right]$$
$$+ i\left[\phi_2(y+ax+ibx) - \phi_2(y+ax-ibx)\right]$$
$$+ \phi_3(y+m_3x) + \phi_4(y+m_4x) + \dots + \phi_n(y+m_nx)$$

problems :-

(3) Find the Complete Solution of following PDE

(i). $(D_x^2 - D_xD_y - 6D_y^2)z = 0$

(ii). $(D_x^3 - 3D_x^2D_y + 2D_y^2D_x)z = 0$

(iii). $(D^2 + 6DD' + 9(D')^2)z = 0$

(iv). $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$

(v) $(D^3 - 4D^2D' + 3(D')^3)z = 0$

(vi) $(D_x^4 - D_y^4)z = 0$

(3) (i). $(D_x^2 - D_x D_y - 6D_y^2)z = 0$

Given equation is $\quad D_x = \frac{\partial}{\partial x}$

$(D_x^2 - D_x D_y - 6D_y^2)z = 0 \longrightarrow (1) \quad D_y = \frac{\partial}{\partial y}$

$\Rightarrow f(D_x, D_y)z = 0$

(or)

$\Rightarrow f(D, D')z = 0$

where $f(D, D') = D^2 - DD' - 6D'^2$

$\Rightarrow \quad f(D/D', 1) = \frac{D^2}{D'^2} - \frac{D}{D'} - 6(1)$

The A.E is $f(m, 1) = 0$.

$m^2 - m - 6 = 0$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m-3)(m+2) = 0$

$\Rightarrow m = 3, -2$.

∴ The Complementary function is

$\boxed{Z_C = Z = f(y + (-2)x) + \phi(y + 3x)}$

(or)

$\boxed{Z = \phi_1(y - 2x) + \phi_2(y + 3x)}$

(v)

Given equation is

$$\left(D^3 - 4D^2D' + 3D'^3\right)z = 0 \longrightarrow (1)$$

$\longrightarrow f(D, D')z = 0$

where $f(D, D') = D^3 - 4D^2D' + 3D'^3$

$$f(D/D', 1) = \left(\frac{D}{D'}\right)^3 - 4\frac{D^2}{D'^2} + 3$$

The A.E is $f(m, 1) = 0$. $\left(m = \frac{D}{D'}\right)$

$$m^3 - 4m^2 + 3 = 0.$$

$(m-1)(m^2 - 3m - 3) = 0$

$$\begin{array}{c|cccc} 1 & 1 & -4 & 0 & 3 \\ & 0 & 1 & -3 & -3 \\ \hline & 1 & -3 & -3 & \underline{|0} \\ & 0 & 3 & & \\ \hline & 1 & 0 & & \end{array}$$

$m - 1 = 0$, $m^2 - 3m - 3 = 0$

$m = 1$, $m = \dfrac{3 \pm \sqrt{9+12}}{2}$

$= \dfrac{3 \pm \sqrt{21}}{2}$

$m = 1$, $m = \dfrac{3}{2} \pm \dfrac{\sqrt{21}}{2}$

Rules for finding the particular Integral

Consider the equation

$$(D^2 + a_1 DD' + a_2 D'^2) z = F(x,y)$$

$$\Rightarrow f(D,D') = F(x,y)$$

Then $P.I = \frac{1}{f(D,D')} F(x,y) = z_p$

Case (i) :- when $F(x,y) = e^{ax+by}$

Then the P.I is

$$z_p = P.I = \frac{1}{f(D,D')} e^{ax+by}$$

$$z_p = \frac{1}{f(a,b)} e^{ax+by}$$

Suppose $f(a,b) = 0$. Then

$$P.I = \frac{x}{\frac{\partial}{\partial D_x} f(a,b)} \cdot e^{ax+by} \qquad f' = \frac{\partial}{\partial D_x} f$$

Suppose $f'(a,b) = 0$. Then

$$P.I = \frac{x^2}{\frac{\partial}{\partial D_x} f'(a,b)} e^{ax+by}$$

problems :-

(1). Solve $r - 4s + 4t = e^{2x+y}$

Given equation is

$D^2z - 4DD'z + 4D'^2z = e^{2x+y}$

$\Rightarrow (D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

$\Rightarrow f(D,D')z = F(x,y)$

where $f(D,D') = D^2 - 4DD' + 4D'^2 = \left(\frac{D}{D'}\right)^2 - 4\frac{D}{D'} + 4$

$F(x,y) = e^{2x+y}$

$\therefore$ The A.E is $f(m,1) = 0$

$m^2 - 4m + 4 = 0$

$\cancel{(m-2m)^2 = 0}$

$m^2 - 2m - 2m + 4 = 0$

$m(m-2) - 2(m-2) = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

$\therefore$ The C.F is

$z_c = f_1(y+2x) + x f_2(y+2x)$

and The P.I is

$z_p = P.I = \frac{1}{f(D,D')} F(x,y)$

$= \frac{1}{(D-2D')^2} e^{2x+y}$

$$Z_p = \frac{1}{(2-2(1))^2 = 0} e^{2x+y}$$

$$Z_p = \frac{x/1!}{\frac{\partial}{\partial D}(D^2 - 4DD' + 4D'^2)} e^{2x+y}$$

$\frac{\partial}{\partial D}(D(y)) = x$

$$= \frac{x}{2D - 4D' + 0} e^{2x+y}$$

$= y \frac{\partial}{\partial D}$

$= y \cdot 1$

$$Z_p = \frac{x}{2(2) - 4(1) = 0} e^{2x+y}$$

$\frac{\partial}{\partial D}(y^2) = 0$

$$Z_p = \frac{x^2/\cancel{2!}}{\frac{\partial}{\partial D}(2D - 4D')} e^{2x+y}$$

$\frac{\partial}{\partial D} D = 1$

$$Z_p = \frac{x^2}{\cancel{1}} \left(\frac{1}{2-0} \right) e^{2x+y}$$

$$\boxed{Z_p = \frac{x^2}{2} e^{2x+y}}$$

$\therefore$ The Complete Solution is

$$Z = Z_c + Z_p$$

$$\boxed{Z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}}$$

2). Solve $(D^2 - 2DD' + D'^2)Z = e^{x+y}$

(3). Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^2 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$

(4). Solve $4r + 12s + 9t = e^{3x-2y}$

(5). Solve $(D_x^2 + 5D_x D_y + 6D_y^2)z = e^{x-y}$

(6). Solve $r - 4s + 4t = e^{2x+y}$

(7). Solve $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^2 z}{\partial x^2 \partial y} + 5\dfrac{\partial^3 z}{\partial x \partial y^2} - 2\dfrac{\partial^3 z}{\partial y^3} = e^{2x+y}$

(8). Solve $(D^2 - 2DD' + D'^2)z = e^{x+y}$

(9). Solve $(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = e^{3x+y}$

Case (ii):- when $F(x,y) = \sin(ax+by)$ (or) $\cos(ax+by)$

Then the P.I is of the form

$$z_p = \frac{1}{f(D,D')} F(x,y)$$

Here $F(x,y) = \sin(ax+by)$ (or) $\cos(ax+by)$

$$\therefore z_p = \frac{1}{f(D,D')} \sin(ax+by)$$

$$= \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

Replace $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$

$$\therefore P.I = z_p = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$$

problems :-

(1). Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

Sol:- Given PDE in operator form (or) Symbolic form is

$$(D^2 + 2DD' + D'^2) z = \sin(2x+3y)$$

$$\Rightarrow f(D,D') z = F(x,y)$$

where $f(D,D') = D^2 + 2DD' + D'^2$

$$F(x,y) = \sin(2x+3y)$$

The A.E of $f(m,1) = 0$

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0.$$

$(m+1)(m+1)=0$

$m=-1,-1$ roots are equal

The C.F is

$$z_c = \phi_1(y+(-1)x) + x\phi_2(y+(-1)x)$$

$$z_c = \phi_1(y-x) + x\phi_2(y-x)$$

Now, the P.I is

$$z_p = \frac{1}{f(D,D')} F(x,y) \qquad D^2=-a^2,\quad D'^2=-b^2$$

$$= \frac{1}{D^2+2DD'+D'^2} \sin(2x+3y) \qquad DD'=-a.b$$

Replace $D^2=-2^2$, $DD'=-2(3)$, $D'^2=-3^2$

$$\Rightarrow z_p = \frac{1}{-4-12-9} \sin(2x+3y)$$

$$z_p = -\frac{\sin(2x+3y)}{25}$$

∴ The Complete Solution is

$$z = z_c + z_p$$

$$\boxed{z = \phi_1(y-x) + x\phi_2(y-x) - \frac{1}{25}\sin(2x+3y)}$$

(2). Solve $(D+1)(D+D'-1)z = \sin(x+2y)$

(3). Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$ $\quad 2\sin a\cos b$ $\quad \sin(a+b) + \sin(a-b)$

(4). Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos 2x \cdot \cos 2y$ $\quad 2\cos a\cos b$ $\quad \cos(a+b) + \cos(a-b)$

(5). Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

(6). Solve $(D_x^2 + D_x D_y - 6D_y^2)z = \cos(2x-y)$

(7). Solve $(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$

Case (iii):- when $f(x,y) = x^m \cdot y^n$:-

The partial integral is of the form

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

Here $f(x,y) = x^m \cdot y^n$, where m, n are positive constants

$$P.I = z_p = \frac{1}{f(D,D')} x^m y^n$$

$$= [f(D,D')]^{-1} x^m \cdot y^n$$

* If $n < m$, then $[f(D,D')]^{-1}$ expanded in powers of $\frac{D'}{D}$

* If $m < n$, then $[f(D,D')]^{-1}$ expanded in powers of $\frac{D}{D'}$

problems :-

(1). Solve $(D^2 + D'^2)z = x^2y^2$

Sol:

Given PDE is $(D^2+D'^2)z = x^2y^2$

$$\Rightarrow f(D,D')z = f(x,y)$$

where $f(D,D') = D^2 + D'^2$

$F(x,y) = x^2 y^2$

The A.E is $f(m,1)=0$

$\Rightarrow m^2+1=0$

$m^2=-1$

$m^2=i^2$

$\boxed{m=\pm i}$ $\alpha=0, b=1$ $a\pm ib$ $a+ib$ $a-ib$

The Complementary function is

$Z_C = \left[f_1(y+ix) + f_1(y-ix)\right]$

$+ i\left[f_2(y+ix) - f_2(y-ix)\right]$

The P.I is

$Z_P = \frac{1}{f(D,D')} F(x,y)$

$= \frac{1}{D^2+D'^2} x^2y^2$ $(m=n)$

$= \frac{1}{D^2}\left[\frac{x^2y^2}{\left(1+\frac{D'^2}{D^2}\right)}\right]$ $(1+x)^{-1} = 1-x+x^2+\cdots$

$Z_P = \frac{1}{D^2}\left\{\left(1+\frac{D'^2}{D^2}\right)^{-1} x^2y^2\right\}$

$= \frac{1}{D^2}\left\{\cancel{x^2y^2}\ 1 - \frac{D'^2}{D^2} + \frac{D'^4}{D^4} - \cdots\right\} x^2y^2$

$= \frac{1}{D^2}\left\{x^2y^2 - \frac{1}{D^2}\frac{\partial^2}{\partial y^2} x^2y^2\right\}$ $\frac{\partial^2}{\partial y^2} y^2 = 2y = 2$

$$Z_p = \frac{1}{D^2}\left\{x^2y^2 + \frac{1}{D^2}\,2x^2\right\}$$

$$= \frac{1}{D^2}\left\{x^2y^2 + \frac{12x^3}{D\,3}\right\}$$

$$= \frac{1}{D^2}\left\{x^2y^2 + \frac{x^4}{3\cdot 4}\right\} \qquad D = \frac{\partial}{\partial x}$$

$$= \frac{1}{D^2}\left\{x^2y^2 + \frac{x^4}{6}\right\} \qquad D' = \frac{\partial}{\partial y}$$

$$= y^2\frac{1}{D^2}x^2 + \frac{1}{6}\frac{1}{D^2}x^4$$

$$= y^2\frac{1}{D}\frac{x^3}{3} - \frac{1}{6}\frac{1}{D}\frac{x^5}{5}$$

$$Z_p = y^2\frac{x^4}{12} - \frac{1}{6}\frac{x^6}{30}$$

$$\boxed{Z_p = \frac{x^4y^2}{12} - \frac{x^6}{180}}$$

∴ The Complete Solution is given by

$$Z = Z_c + Z_p$$

$$Z = \left[\phi_1(y+ix) + \phi_1(y-ix)\right] + i\left[\phi_2(y+ix) + \phi_2(y-ix)\right] + \frac{x^4y^2}{12} - \frac{x^6}{180}$$

(2). Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

Sol:-

Given PDE in operator form (or)

Symbolic form is

$$D^3 z - 2D^2 D' z = 2e^{2x} + 3x^2 y$$

$$(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$$

$$\Longrightarrow f(D, D')z = F(x,y)$$

where $f(D, D') = D^3 - 2D^2 D'$

$F(x,y) = 2e^{2x} + 3x^2 y$

~~The Complementary function is~~

~~$Z_c =$~~ The A.E is $f(m,1) = 0$

$$m^3 - 2m^2 = 0$$

$$m^2(m-2) = 0$$

$$m = 0, 0, 2$$

The C.F is

$$Z_c = \phi_1(y+0x) + x\phi_2(y+0x) + \phi_3(y+2x)$$

$$Z_c = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$$

The P.I is

$$Z_p = \frac{1}{f(D,D')} F(x,y)$$

$$= \frac{1}{D^3 - 2D^2 D'} (2e^{2x} + 3x^2 y)$$

$$Z_p = \frac{1}{D^3 - 2D^2D'}\left(2e^{2x} + 3x^2y\right)$$

$$Z_p = \frac{1}{D^3 - 2D^2D'}2e^{2x} + \frac{1}{D^3 - 2D^2D'}3x^2y$$

$$= \frac{1}{D(2) - D}2e^{2x} + \frac{1}{2D^2D'\left(\frac{D^3}{2D'D^2} - 1\right)}3x^2y$$

$$= \frac{1}{\cancel{2}D}\cancel{2}e^{2x} - \frac{1}{2D^2D'\left(1 - \frac{D^3}{2D'D^2}\right)}3x^2y$$

$$Z_p = \int e^{2x}\,dx - \frac{1}{2D^2D'}\left(1 - \frac{D^3}{2D'D^2}\right)^{-1}3x^2y$$

$$= \frac{e^{2x}}{2} - \frac{1}{2D^2D'}\left[1 + \frac{D^3}{2D'D^2}\right]3x^2y$$

$$= \frac{e^{2x}}{2} - \frac{1}{2D^2D'}\left\{3x^2y + \frac{1}{2D'D^2}0\right\}$$

$$= \frac{e^{2x}}{2} - \frac{1}{2D^2}3x^2\frac{y^2}{2}$$

$$= \frac{e^{2x}}{2} - \frac{3}{4}\frac{1}{D}y^2\int x^2\,dx$$

$$= \frac{e^{2x}}{2} - \frac{3}{4}y^2\frac{1}{D}\frac{x^3}{3}$$

$$= \frac{e^{2x}}{2} - \frac{\cancel{3}y^2x^4}{4\times 12^4}$$

$$Z_p = \frac{e^{2x}}{2} - \frac{3x^4y^2}{16}$$

$\therefore$ The complete Soln is

$$z = z_c + z_p$$

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) + \frac{e^{2x}}{2} - \frac{5x^4y^2}{16}$$

(3) Solve the following P.D.E and find C.F

(i). $(D^2 - 3DD' + 2D'^2)z = 24xy$

(ii). $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial y^2} = x^3y^3$

(iii) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x+2y)$

(iv). $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Case (vi) :- when $f(x,y) = e^{ax+by} v(x,y)$:-

The particular integral is of the form

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

Here $f(x,y) = e^{ax+by} v(x,y)$

where $v(x,y)$ is function of x, y

$$P.I = \frac{1}{f(D,D')} e^{ax+by} v(x,y)$$

$$Z_p = P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)} v(x,y)$$

Problems :-

(1). Solve $(D^2 - D'^2)z = e^{x-y} \sin(x+2y)$

Sol^n:-

Given PDE is

$$(D^2 - D'^2)z = e^{x-y} \sin(x+2y)$$

$$\Rightarrow f(D,D')z = e^{ax+by} v(x,y)$$

where $f(D,D') = D^2 - D'^2$

$$f(x,y) = \sin(x+2y) \cdot e^{ax+by}$$

The A.E is $f(m,1) = 0$

$$\Rightarrow (m^2 - 1) = 0$$

$$m^2 = 1$$

$$\boxed{m = \pm 1}$$

The C.F is

$$\boxed{Z_c = \phi_1(y-x) + \phi_2(y+x)}$$

The P.I is

$$Z_p = P.I = \frac{1}{f(D,D')}\left(e^{ax+by}\, V(x,y)\right) f(x,y)$$

$$= \frac{1}{D^2-D'^2} e^{x-y} \sin(x+2y)$$

$$= e^{x-y} \frac{1}{(D+1)^2-(D'-1)^2} \sin(x+2y)$$

$$Z_p = e^{x-y} \frac{1}{D^2+1+2D-D'^2-1+2D'} \sin(x+2y)$$

$$= e^{x-y} \frac{1}{D^2-D'^2+2D+2D'} \sin(x+2y)$$

$D^2 = -a^2$
$D'^2 = -b^2$
$DD' = -ab$

$$Z_p = e^{x-y} \frac{1}{-1^2+\cancel{2}^2+2D+2D'} \sin(x+2y)$$

$$Z_p = e^{x-y} \frac{-1}{2D+2D'+3} \sin(x+2y)$$

$$\cancel{Z_p = \frac{1}{3} e^{x-y} \sin(x+2y)}$$

$$Z_p = e^{x-y} \frac{1}{(2D+2D')+3} \sin(x+2y)$$

$$Z_p = e^{x-y} \frac{(2D+2D')-3}{(2D+2D')^2-3^2} \sin(x+2y)$$

$$Z_p = e^{x-y} \frac{2D+2D'-3}{4D^2+4D'^2+8DD'-9} \sin(x+2y)$$

$$Z_p = e^{x-y}\,\frac{(2D+2D'-3)}{4D^2+4D'^2+8DD'-9}\sin(x+2y)$$

$$Z_p = e^{x-y}\,\frac{2D\sin(x+2y)+2D'\sin(x+2y)-3\sin(x+2y)}{4D^2+4D'^2+8DD'-9}$$

$$Z_p = e^{x-y}\left\{\frac{2\cos(x+2y)+4\cos(x+2y)-3\sin(x+2y)}{4D^2+4D'^2+8DD'-9}\right\}$$

$$4(-1)^2+4(2^2)+8(-1)(2)-9$$

$$= e^{x-y}\left\{\frac{6\cos(x+2y)-3\sin(x+2y)}{4D^2+4D'^2+8DD'-9}\right\}$$

$$= e^{x-y}\left\{\frac{6\cos(x+2y)-3\sin(x+2y)}{4(-1^2)+4(-2^2)+8(-1)(+2)-9}\right\}$$

$$= e^{x-y}\left\{\frac{6\cos(x+2y)-3\sin(x+2y)}{-4-16-16-9}\right\}$$

$$Z_p = \frac{e^{x-y}}{-45}\left\{6\cos(x+2y)-3\sin(x+2y)\right\}$$

$$Z_p = \frac{-e^{x-y}}{45}\left\{6\cos(x+2y)-3\sin(x+2y)\right\}$$

The Complete Sol^n is

$$Z = Z_c + Z_p$$

$$Z = \phi_1(y-x)+\phi_2(y+x)-\frac{e^{x-y}}{45}\left\{6\cos(x+2y)-3\sin(x+2y)\right\}$$

(2). Solve $(D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3) z = e^x \cos 2y$

Sol:- Given PDE is

$(D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3) z = e^x \cos 2y$ $\longrightarrow (1)$

~~The A.E is $f(m,1)=0$~~

$(m^3 - m^2 - m - 1) z = e^{ax+by}$ ~~$\cos$~~ $f(x,y)$ ~~$\cos(2y)$~~

$\Rightarrow f(D_x, D_y) z = e^{ax+by} f(x,y) = f(x,y)$

where $f(D_x, D_y) = D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3$

$f(x,y) = \cos 2y\, e^x.$

The A.E is $f(m,1)=0$

$\Longrightarrow m^3 + m^2 - m - 1 = 0$

$(m+1)(m+1)(m-1) = 0$

$\Rightarrow m = -1, -1, +1$

$$\begin{array}{r|rrrr} +1 & 1 & +1 & -1 & -1 \\ & 0 & +1 & 2 & 1 \\ \hline -1 & 1 & 2 & +1 & 0 \\ & 0 & -1 & -1 & \\ \hline -1 & 1 & 1 & 0 & \end{array}$$

The C.F is

$z_c = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x) + 0$

The P.I is

$z_p = \dfrac{1}{f(D_x, D_y)} e^{ax+by} f(x,y)$

$$Z_p = \frac{1}{D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3} e^{ax+by} \cos(by+ax)$$

$$Z_p = e^x \frac{1}{(D+1)^3 + (D+1)^2(D') - (D+1)(D'^2) - D'^3} \cos 2y$$

$$Z_p = e^x \frac{\cos 2y}{D^3 + 3D^2 + 3D + 1 + (D^2 + 2D + 1)D' - DD'^2 - D'^2 - D'^3}$$

$$= e^x \frac{\cos 2y}{D^3 + 3D^2 + 3D + 1 + D^2D' + 2DD' + D' - DD'^2 - D'^2 - D'^3}$$

$D^2 = -a^2$

$D'^2 = -b^2$

$DD' = -ab$

$$= e^x \frac{\cos 2y}{D(0) + 0 + 3D + 1 + D'(0) + (0) + D' - D(-2^2) - (-2^2) - D'(-2^2)}$$

$$= e^x \frac{\cos 2y}{3D + 1 + D' + 4 + 4D' + 4D}$$

$$= e^x \frac{\cos 2y}{3D + 5D' + 5 + 4D}$$

$$z_p = e^x \frac{1}{(7D+5D')^2+25} \times \frac{(7D+5D')+5}{(7D+5D')+5} \cos 2y$$

$$= e^x \frac{7D\cos 2y + 5D'\cos 2y - 5\cos 2y}{(7D+5D')^2 - 25}$$

$$= e^x \frac{0 + 5(-2\sin 2y) - 5\cos 2y}{49D^2 + 25D'^2 + 70DD' - 25}$$

$$z_p = e^x \frac{-10\sin 2y - 5\cos 2y}{9D^2 + 25D'^2 + 30DD' - 25}$$

$$= e^x \left\{ \frac{-10\sin 2y - 5\cos 2y}{9(-0^2) + 25(-2^2) + 30(0)(2) - 25} \right\}$$

$$= e^x \left\{ \frac{-10\sin 2y - 5\cos 2y}{-0 - 100 - 0 - 25} \right\}$$

$$\boxed{z_p = \frac{+e^x}{+125}\{10\sin 2y + 5\cos 2y\}}$$

$\therefore$ The Complete Solution is

$$z = z_C + z_p \qquad \boxed{z = \phi_1(y-x) + x\phi_2(y+x) + x^2\phi_3(y+x) + \frac{e^x}{25}(2\sin 2y + \cos 2y)} \;//$$

$$= \phi_1(y+x) \cdot x\phi_2(y+x) \cdot x^2\phi_3(y+x) \cdot$$

$$+ \frac{x e^x}{125\,25}\{2\sin 2y + \cos 2y\}.$$

Case. V:- particular Integral of Any function:-

when $f(x,y)$ is any function of x and y then the particular integral is

$$Z_p = P.I = \frac{1}{f(D,D')} f(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions

Considering $f(D,D')$ as a function of D alone and operate each partial fraction on $f(x,y)$ by using the formula.

$$\frac{1}{(D-mD')} f(x,y) = \int f(x, c-mx)\, dx$$

Here, we replace

$$y = c - mx \quad (\text{or}) \quad c = y + mx$$

problem:-1. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Sol^n:-

Given P.D.E is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

$\frac{\partial^2}{\partial x^2} = D^2$
$\frac{\partial^2}{\partial y^2} = D'^2$
$\frac{\partial^2}{\partial x \partial y} = DD'$

$\therefore$ the operator form is $f(D,D') = f(x,y)$

$$\Rightarrow (D^2 + DD' - 6D'^2)z = y\cos x$$

where $f(D,D') = D^2 + DD' - 6D'^2$, $f(x,y) = y\cos x$

The A.E is $f(m,1) = 0$

$$\Rightarrow m^2 + m - 6 = 0$$
$$m^2 + 3m - 2m - 6 = 0$$
$$m(m+3) - 2(m+3) = 0$$
$$(m+3)(m-2) = 0$$
$$m-2=0, m+3=0$$
$$m = 2, -3$$

The roots are real and different (distinct)

$$z_c = C.F = f_1(y+2x) + f_2(y-3x)$$
$$= \phi_1(y+2x) + \phi_2(y-3x)$$

Now, the particular integral is

$$z_p = P.I = \frac{1}{f(D,D')} f(x,y) \qquad f(D,D') = (D - mD')$$

$$= \frac{1}{D^2 + DD' - 6D'^2} y\cos x$$

$$z_p = \frac{1}{(D-2D')(D+3D')} y\cos x \qquad (D-(-3)D')$$

$$= \frac{1}{D-2D'}\left[\frac{1}{D+3D'} y\cos x\right] \qquad y = c - mx$$

$$= \frac{1}{D-2D'} \int (c+3x)\cos x\, dx \qquad y = c+3x$$
$$m = -3 \qquad c \to y-3x$$

$$= \frac{1}{D-2D'}\left[(c+3x)\int \cos x\, dx - \int\left(\frac{d}{dx}(c+3x)\int \cos x\, dx\right)dx\right]$$

$$z_p = \frac{1}{D-2D'}\left[(c+3x)\sin x - \int 3\sin x\, dx\right]_{c \to y-3x}$$

$$= \frac{1}{D-2D'}\left[c+3x \sin x + 3\cos x\right]_{c \to y-3x}$$

$$Z_p = \frac{1}{D-2D'}\left(y\sin x + 3\cos x\right)$$

$$= \int (c-2x)\sin x + 3\int (c-2x)\cos x$$

$$= \int \{(c-2x)\sin x + 3\cos x\}\, dx$$

$$= (c-2x)\int \sin x\, dx - \int\left\{\frac{d}{dx}(c-2x)\int \sin x\, dx\right\}dx$$

$$+ 3\int \cos x\, dx$$

$$Z_p = \left[(c-2x)(-\cos x)\right] - \int -2(-\cos x)\, dx$$

$$+ 3\sin x$$

$$= \left[(c-2x)(-\cos x) - 2\sin x\right]_{c \to y+2x}$$

$$Z_p = -y\cos x - 2\sin x$$

$\therefore$ The Complete Sol^n is

$$Z = Z_c + Z_p = f_2(y-3x) + f_1(y+2x)$$

$$- y\cos x - 2\sin x$$

Problems:-

(2). $(D^2 + DD' - 6D'^2)z = y\sin x$

(3). Solve $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

problem-3

Sol^n:

Given P.D.E is

$$4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y) \longrightarrow (1)$$

let us write the operator form of eq (1)

is

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

$$\Rightarrow f(D,D')z = F(x,y)$$

where $f(D,D') = 4D^2 - 4DD' + D'^2$

$$F(x,y) = 16\log(x+2y)$$

The Auxiliary equation is

$$f(m,1) = 0$$

$$\Rightarrow 4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$(2m-1)(2m-1) = 0$$

$\therefore$ $m = 1/2, 1/2$ are real and equal

$\therefore$ The C.F is

$$\boxed{Z_c = \phi_1(y + \tfrac{1}{2}x) + x\,\phi_2(y + \tfrac{1}{2}x)}$$

The P.I is

$$Z_p = \frac{1}{f(D,D')} \cdot F(x,y)$$

$$= \frac{1}{4D^2 - 4DD' + D'^2}\, 16\log(x+2y)$$

$$Z_p = \frac{1}{(D-\frac{1}{2}D')(D-\frac{1}{2}D')} \cdot 16\log(x+2y)$$

$$Z_p = \frac{1}{(D-\frac{1}{2}D')}\left[\frac{1}{(D-\frac{1}{2}D')} 16\log(x+2y)\right]$$

$$= \frac{1}{D-\frac{1}{2}D'} \int 16\log\left(x+2\left(c-\frac{1}{2}x\right)\right) dx$$

$$\frac{1}{D-\frac{1}{2}D'} 16\int \log\left(\cancel{x}+2c-\cancel{x}\right) dx$$

$$= \frac{16}{D-\frac{1}{2}D'} \int \log 2c \, dx$$

$$= \frac{16\left[\log 2c\right]_{c \to y+\frac{1}{2}x}^{x}}{D-\frac{1}{2}D'}$$

$$Z_p = \frac{16}{D-\frac{1}{2}D'} x\log(2y+x)$$

$$= 16\int x\log\left(2\left(c-\frac{1}{2}x\right)+x\right) dx$$

$$= 16\int x\log\left(2c-\cancel{x}+\cancel{x}\right) dx$$

$$= 16\log 2c \int x \, dx$$

$$= 16\left[\log 2c \; \frac{x^2}{2}\right]_{c \to y+\frac{1}{2}x}$$

$$= 16\log(2y+x) \cdot \frac{x^2}{2}$$

$$\boxed{Z_p = 8x^2\log(2y+x)}$$

$$\boxed{\begin{array}{l}\text{The G.S is} \\ Z = Z_p + Z_c\end{array}}$$

Non-Homogeneous Linear Equations :-

If in the equation $f(D,D')z = F(x,y)$ the polynomial expression $f(D,D')$ is not homogeneous, then it is called a non-homogeneous linear partial differential equation. As in the case of Homogeneous linear partial differential equations, its complete Solution = C.F + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations. But to find the C.F, we factorize $f(D,D')$ into factors of the form $D-mD'-c$. To find the Solution of $(D-mD'-c)z = 0$, we write it as $p-mq = cz$. $\longrightarrow (1)$

The Subsidiary Equation are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

$D = \frac{\partial z}{\partial x} = p$
$\frac{\partial z}{\partial y} = q$

* From first two fractions, we get $y+mx = a_1$ $\longrightarrow (2)$
* From first & last fractions, we get

$z = a_2 e^{cx}$ or $\frac{z}{e^{cx}} = a_2$ $\longrightarrow (3)$

$\therefore$ from eqs (2), (3),

we get $z = e^{cx}\phi(y+mx)$

$\phi(u,v) = 0$
$u = f(v)$ or $v = f(u)$

which is a Solution of eq (1)

* The Solution Corresponding to various factors added up, give the C.F of given P.D.E.

* Similarly the Solution of $(D - mD' - c)^2 = 0$ is

$$z = e^{cx} \phi_1(y+mx) + x e^{cx} \phi_2(y+mx)$$

problem.1 Solve $[(D+D'-2)(D+4D'-3)]z = 0$

Sol:- Given P.D.E is

$$[(D+D'-2)(D+4D'-3)]z = 0 \longrightarrow (1)$$

Consider $(D+D'-2)z=0$

$$\Rightarrow (D+(-1)D'-2)z = 0 \longrightarrow (2)$$

Comparing eq (2) into $(D-mD'-c)z=0$

we have $m=-1$, $c=2$

$$\therefore C.F_1 = e^{cx}\phi_1(y+mx)$$

$$C.F_1 = e^{2x}\phi_1(y-x) \longrightarrow (A)$$

and again Consider

$$(D+4D'-3)z = 0 \longrightarrow (3)$$

Comparing eq (3) into $(D-mD'-c)z=0$

we have $m=-4$, $c=3$

$$\therefore C.F_2 = e^{c_2 x}\phi_2(y+m_2 x)$$

$$= e^{3x}\phi_2(y-4x) \longrightarrow (B)$$

$\therefore$ The C.F from eq (A), eq (B) is

$$z = C.F = C.F_1 + C.F_2$$

$$z = z_c = e^{2x}\phi_1(y-x) + e^{3x}\phi_2(y-4x)$$

which is also a complete solution of equation (1).

problems :-

2). Solve $(D+3D'+4)^2 z = 0$.

3). Solve $(r+2s+t+2p+2q+z) = 0$

4). Solve $r-t+p-q = 0$

5). Solve $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$

6). Solve $(\cancel{DD}\, D-D'-1)(D-D'-2)z = e^{2x-y}$

7). Solve $(2DD'+D'^2-3D')z = 3\cos(3x-2y)$

8). Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = e^{-x}$

Unit-4

Vector Calculus

Divergence of a vector :-

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a vector point function f is denoted by $\text{div} f$ and is defined by the equation

$$\text{div}\,\bar{f} = \nabla\cdot\bar{f}$$

$$\text{div}\,\bar{f} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

(or)

$$\text{div}\,\bar{f} = \nabla\cdot\bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$$\boxed{\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Curl of vector point function :-

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ is a vector point function then the curl of vector point function $\bar{f}$ is defined as follows

$$\text{Curl}\,\bar{f} = \nabla\times\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times\left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$$\text{Curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\text{curl}\,\bar{f} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Solenoidal Vector :- A vector point function $\bar{f}$ is said to be solenoidal, if $\operatorname{div} \bar{f} = 0$.

Irrotational Vector :- A vector point function $\bar{f}$ is said to be irrotational, if $\operatorname{Curl} \bar{f} = 0$.

Scalar potential :- If $\bar{f}$ is irrotational then there exists ϕ such that $\bar{f} = \nabla\phi$, here ϕ is called scaller potential.

Q. If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$, find $\operatorname{div}\bar{f}$ at the point $(1,-1,1)$.

Sol :- Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$, $\longrightarrow (1)$

eq. (1) is Comparing to $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we get $f_1 = xy^2$, $f_2 = 2x^2y$, $f_3 = -3yz^2$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2y) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2 - 3y(2z)$$

$$\operatorname{div}\bar{f} = y^2 + 2x^2 - 6yz$$

$$\operatorname{div}\bar{f}_{(1,-1,1)} = (-1)^2 + 2(1)^2\cancel{(-1)} - 6(-1)(1)$$

$$= 1+2+6$$

$$\operatorname{div}\bar{f}_{(1,-1,1)} = 9$$

1). Evaluate div f at the point (1,2,3) given

(i). $\bar{f} = x^2y\,\bar{i} + xy^2z\,\bar{j} + xyz^2\,\bar{k}$

(ii). $\bar{f} = 3x^2\,\bar{i} + 5xy^2\,\bar{j} + 5xyz^3\,\bar{k}$

(iii). $\bar{f} = x^2y\,\bar{i} + y^3x\,\bar{j} + z^3x\,\bar{k}$

(iv). $\bar{f} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$

2). Find Curl $\bar{f}$ at the point (1,-1,1) if $\bar{f} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} + (-3yz^2)\,\bar{k}$

Sol^n:

Given $\bar{f} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} + (-3yz^2)\,\bar{k} \longrightarrow (1)$

eq. (1) is Comparing to $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = xy^2,\ f_2 = 2x^2yz,\ f_3 = -3yz^2$

$$\operatorname{Curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right)$$

$$+\bar{j}\left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right)$$

$$+\bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left[(-3z^2) - (2x^2y)\right] + \bar{j}\left[0-0\right] + \bar{k}\left[4xyz - 2xy\right]$$

$\text{Curl}\,\bar f = \bar i\,(-3x^2 - 2x^2y) + k\,(4xyz - 2xy)$

$\text{Curl}\,\bar f_{(1,-1,1)} = \bar i\,(-3+2) + k\,(-4+2)$

$\text{Curl}\,\bar f_{(1,-1,1)} = -\bar i - 2k = -\bar i + 0\bar j - 2k$

Q. Find curl at the point (1,2,3) given [illegible]

(i) [illegible]

(ii) [illegible]

(iii) [illegible]

(iv) [illegible]

Find div f when $\bar f = \text{grad}(x^3+y^3+z^3-3xyz)$

also find $\text{Curl}\,\bar f$

Sol^n :-

Given : $\bar f = \text{grad}\,(x^3+y^3+z^3-3xyz)$

let $\phi = x^3+y^3+z^3-3xyz$

$\frac{\partial\phi}{\partial x} = 3x^2-3yz$, $\frac{\partial\phi}{\partial y} = 3y^2-3xz$

$\frac{\partial\phi}{\partial z} = 3z^2-3xy$ $\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\phi$

$\bar f = \nabla\phi = \text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$= i\,(3x^2-3yz) + j\,(3y^2-3xz) + k\,(3z^2-3xy)$ ⟶ (1)

eq. (1) is comparing to

$\bar f = i f_1 + j f_2 + k f_3$

where $f_1 = 3x^2 - 3yz$, $f_2 = 3y^2 - 3xz$

$f_3 = 3z^2 - 3xy$

$\therefore \operatorname{div} \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$= 6x - 0 + 6y - 0 + 6z - 0$

$\operatorname{div} \bar{f} = 6x + 6y + 6z$

$\operatorname{curl} \bar{f} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$= \bar{i}(-3x + 3x) + \bar{j}(-3y + 3y) + \bar{k}(-3z + 3z)$

$\operatorname{curl} \bar{f} = 0$

6). Evaluate $\operatorname{div} \bar{f}$ and $\operatorname{curl} \bar{f}$ at the point $(1, 2, 3)$ given $\bar{f} = \operatorname{grad}(x^3y + y^3z + z^3x - x^2y^2z^2)$

7). Show that each of following vectors are Solenoidal ?

(i). $3y^4z^2\bar{i} + 4x^3z^2\bar{j} + 3x^2y^2\bar{k}$

(ii). $(x + 3y)\bar{i} + (y - 3z)\bar{j} + (x - 2z)\bar{k}$

(iii). $(-x^2 + yz)\bar{i} + (4y - z^2x)\bar{j} + (2xz - 4z)\bar{k}$

(i).

Sol^n:- Given

let $\bar{f} = 3y^4z^2\bar{i} + 4x^3z^2\bar{j} + 3x^2y^2\bar{k} \longrightarrow (1)$

eq. (1) is Comparing to $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

where $f_1 = 3y^4z^2$, $f_2 = 4x^3z^2$, $f_3 = 3x^2y^2$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^4z^2 & 4x^3z^2 & 3x^2y^2 \end{vmatrix}$$

$$= i(6x^2y - 8x^3z) + j(6xy^2 - 6y^4z)$$
$$+ k(12x^2z^2 - 12y^3z^2)$$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$\text{div } \bar{f} = 0 + 0 + 0$$

$$\text{div } \bar{f} = 0$$

$\therefore$ Hence the given vector is Solenoidal.

Hence proved.

(b). Find div f where $\bar{f} = r^n \bar{r}$, also find n if f is solenoidal.

Sol^n:

Given

$\bar{f} = r^n \bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}|$, $r^2 = x^2 + y^2 + z^2 \Rightarrow r = \sqrt{x^2+y^2+z^2}$

$\longrightarrow (1)$ $\quad r = \sqrt{f_1^2 + f_2^2 + f_3^2}$

P.D w.r.to x in eq (1)

we got

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$||^{ly}$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\bar{f} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$\bar{f} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k} \longrightarrow (2)$$

Comparing eq (2) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

where $f_1 = r^n x$, $f_2 = r^n y$, $f_3 = r^n z$

$$\therefore \operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \cancel{nr^n} + r^n + r^n = \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n)$$

~~div f = 3r^n~~

$$\operatorname{div} f = nr^{n-1}\frac{\partial r}{\partial x}\cdot x + r^n \cdot 1 + nr^{n-1}\frac{\partial r}{\partial y}\cdot y + r^n\cdot 1$$

$$+ nr^{n-1}\frac{\partial r}{\partial z}\cdot z + r^n$$

$$\operatorname{div}\bar{f} = nr^{n-1}\left[\frac{x}{r}\cdot x + \frac{y}{r}y + \frac{z}{r}\cdot z\right] + 3r^n$$

$$\operatorname{div}\bar{f} = \frac{nr^{n-1}}{r}\left[x^2+y^2+z^2\right] + 3r^n$$

$$= nr^{n-2}\left[x^2+y^2+z^2\right] + 3r^n$$

$$= nr^{n-2}\cdot r^2 + 3r^n$$

$$\operatorname{div}\bar{f} = nr^n + 3r^n$$

$$\operatorname{div}\bar{f} = (n+3)r^n \,//.$$

let $\bar{f} = r^n \bar{r}$ be Solenoidal

Then $\operatorname{div}\bar{f} = 0$

$$\Rightarrow (n+3)r^n = 0 \Rightarrow n+3 = 0.$$

$$\boxed{n = -3} \,//.$$

9). Show that $\frac{\bar{r}}{r^3}$ is solenoidal (or) Evaluate $\nabla\cdot\frac{\bar{r}}{r^3}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

Solⁿ:-

Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow r = |\bar{r}| = \sqrt{f_1^2+f_2^2+f_3^2}$

$$r^2 = x^2+y^2+z^2 \longrightarrow (1)$$

P. D. w. r. to x, y, z in eq (1)

we get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

^{ly} $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\frac{\bar{r}}{r^3} = \frac{x\bar{i}+y\bar{j}+z\bar{k}}{r^3} = \bar{f}(\text{say})$

$\bar{f} = xr^{-3}\bar{i} + yr^{-3}\bar{j} + zr^{-3}\bar{k}$ —— (2)

Comparing eq (2) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = xr^{-3}$, $f_2 = yr^{-3}$, $f_3 = zr^{-3}$

$\frac{\partial f_1}{\partial x} = -3r^{-3-1}\frac{\partial r}{\partial x}x + r^{-3}.1$, $\frac{\partial f_2}{\partial y} = y(-3)r^{-3-1}\frac{\partial r}{\partial y} + r^{-3}.1$

$\frac{\partial f_3}{\partial z} = -3r^{-3-1}\frac{\partial r}{\partial z}z + r^{-3}.1$

$\therefore \nabla\frac{\bar{r}}{r^3} = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)$ $\left(r^{-4}.r^{-1} = r^{-4-1} = r^{-5}\right)$

$\nabla\bar{f} = -3r^{-4}\frac{x}{r}.x + r^{-3} - 3r^{-4}\frac{y}{r}y + r^{-3}$
$-3r^{-4}\frac{z}{r}z + r^{-3}$

$= -3r^{-5}x^2 + r^{-3} - 3r^{-5}y^2 + r^{-3} - 3r^{-5}z^2 + r^{-3}$

$\nabla\frac{\bar{r}}{r^3} = -3r^{-5}(x^2+y^2+z^2) + 3r^{-3}$

$= -3r^{-5}r^2 + 3r^{-3}$

$= \cancel{-3r^{-3}} + \cancel{3r^{-3}}$

$\nabla\frac{\bar{r}}{r^3} = 0$

$\therefore$ Hence $\frac{\bar{r}}{r^3}$ is a Solenoidal.

Hence proved..

(10). prove that if r is the position vector of point in space, then $r^n\bar{r}$ is irrotational.

Soln:

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$

let $\bar{f} = r^n\bar{r}$ $\qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \bar{f} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$\bar{f} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k} \longrightarrow (1)$

Comparing eq.(1) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = r^n x$, $f_2 = r^n y$; $f_3 = r^n z$

$$\frac{\partial f_1}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x}\cdot x + r^n$$

$$\frac{\partial f_2}{\partial y} = nr^{n-1}\frac{\partial r}{\partial y}\cdot y + r^n$$

$$\frac{\partial f_3}{\partial z} = nr^{n-1}\frac{\partial r}{\partial z}\cdot z + r^n$$

$$\text{Curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(0-0)$$

$\text{curl}\,\bar{f} = 0$

$\therefore \text{Curl}(r^n\bar{r}) = 0$ Thus given $\bar{f}$ is irrotational

iii) Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol:-

Given let $\bar{f} = (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ $\longrightarrow (1)$

Comparing eq (1) into $\bar{f} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

we have $f_1 = x^2-yz$, $f_2 = y^2-zx$, $f_3 = z^2-xy$

Now, since
$$\text{Curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$\text{Curl}\,\bar{f} = \bar{i}(-x+x)-\bar{j}(-y+y)+\bar{k}(-z+z)$$

$$\text{Curl}\,\bar{f} = 0$$

$\therefore$ Given vector is irrotational.

Then, there exists ϕ such that

$$\bar{f} = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\phi$$

$$(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

Comparing on b.s,

we get
$$\frac{\partial\phi}{\partial x} = x^2-yz$$

Integrating on b.s

$$\phi = \frac{x^3}{3}-xyz+c_1 \longrightarrow (2)$$

$\frac{\partial\phi}{\partial y} = (y^2 - zx)$, $\frac{\partial\phi}{\partial z} = z^2 - xy$

Int. on b.s : Int. on b.s

$\phi = \frac{y^3}{3} - xyz + c_2$ $\longrightarrow (3)$, $\phi = z^3/3 - xyz + c_3$ $\longrightarrow (4)$

then from eq (2), (3) and eq (4)

we have $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - 3xyz +$ constant (K)

(17). S.T the vector field $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos(yz))\bar{j} + (2x^2yz + y\cos(yz))\bar{k}$ is irrotational and find scalar potential (or) potential function.

Soln:- Given

let $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos(yz))\bar{j} + (2x^2yz + y\cos(yz))\bar{k}$ $\longrightarrow (1)$

Comparing eq (1) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = 2xyz^2$, $f_2 = x^2z^2 + z\cos(yz)$

$f_3 = 2x^2yz + y\cos(yz)$

Now

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos(yz) & 2x^2yz + y\cos(yz) \end{vmatrix}$$

$$\text{Curl}\,\vec{f} = \hat{i}\left(\cancel{2x^2z} + y(-\sin(yz))z + \cancel{2xyz}\right)$$

$$-\hat{i}\left(-\cancel{2x^2z} + z\sin(yz)\,y - \cancel{\cos(yz)}\right)$$

$$-\hat{j}\left(\cancel{4xyz} + 0 - \cancel{4xyz}\right)$$

$$+\hat{k}\left(\cancel{2xz^2} + 0 - \cancel{2xz^2}\right)$$

$$\text{Curl}\,\vec{f} = 0.$$

Then, there exist ϕ such that

$$\vec{f} = \nabla\phi$$

$$(2xyz^2)\hat{i} + (x^2z^2 + z\cos(yz))\hat{j} + (2x^2yz + y\cos(yz))\hat{k}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing on b.s.

$\int \cos ax = \frac{\sin ax}{a}$

$\int \cos(zy) = \frac{\sin zy}{z}$

we get

$$\frac{\partial\phi}{\partial x} = 2xyz^2 \longrightarrow (2), \quad \frac{\partial\phi}{\partial y} = x^2z^2 + z\cos(yz) \longrightarrow (3)$$

$$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos(yz) \longrightarrow (4)$$

Integrating eq (2), (3) and (4) eq

we get

$$\phi = \cancel{2}yz^2\frac{x^2}{\cancel{2}} + c_1 \longrightarrow (5), \quad \phi = \frac{x^3z^2}{3} +$$

$$\phi = x^2z^2y + \frac{\cancel{z}\sin(yz)}{\cancel{z}} + c_2 \longrightarrow (6)$$

$$\phi = \cancel{2}x^2y\frac{z^2}{\cancel{2}} + \frac{y\sin yz}{y} \longrightarrow (7)$$

from eq (5), (6) and eq (7),

we have.

$$\phi = x^2yz^2 + x^2yz^2 + x^2yz^2 + \sin yz + \sin yz + c_1 + c_2 + c_3$$

$$\boxed{\phi = 3x^2yz^2 + 2\sin yz + K}$$

Q3. Find whether the function $\bar{f} = (x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k$ is irrotational and hence find potential function corresponding to it.

Sol:

Given

$$\bar{f} = (x^2 - y^3)\,\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k} \longrightarrow (1)$$

Comparing eq. (1) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = x^2 - y^3,\ f_2 = y^2 - zx,\ f_3 = z^2 - xy$

Now
$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^3 & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \bar{i}(-x + x) - \bar{j}(-y + y) + \bar{k}(-z + z)$$

$\text{curl}\,\bar{f} = 0$

$\therefore$ Given vector $\bar{f}$ is irrotational

then, there exist ϕ such that

$$\bar{f} = \nabla\phi$$

$$(x^2 - yz)\,\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Comparing on b.s

$$\frac{\partial\phi}{\partial x} = x^2 - yz \implies \int\frac{\partial\phi}{\partial x} = \int(x^2 - yz)\,dx$$

$$\phi = \frac{x^3}{3} - xyz \longrightarrow (2)$$

||ly

$$\phi = \frac{y^3}{3} - xyz \longrightarrow (3)$$

$$\phi = \frac{z^3}{3} - xyz \longrightarrow (4)$$

from eq (2), (3) and eq (4)

we get $\phi = x^3 + y^3 + z^3 - 3xyz$ //.

(14). find div $\vec{f}$ for $\vec{f} = $ [illegible]

(15). [illegible] $\vec{f} = $ [illegible] $(x+y)\hat{k}$ then [illegible]

(16). P.T. curl$(\vec{r}) = \vec{0}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(17). P.T. $\vec{f} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ is irrotational

(18). prove that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$

Sol^n:

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow (1)$, $r^2 = x^2 + y^2 + z^2 \rightarrow (2)$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

P.D. w.r. to x, y and z in eq (2)

we have $2r\frac{\partial r}{\partial x} = 2x$, $2r\frac{\partial r}{\partial y} = 2y$, $2r\frac{\partial r}{\partial z} = 2z$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\operatorname{grad} r^n = \nabla r^n$

$$= \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n$$

$$= \hat{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= \hat{i}\, n r^{n-1}\frac{x}{r} + \hat{j}\, n r^{n-1}\frac{y}{r} + \hat{k}\, n r^{n-1}\frac{z}{r}$$

$$= \hat{i}\, n r^{n-2}x + \hat{j}\, n r^{n-2}y + \hat{k}\, n r^{n-2}z$$

$$= n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$\operatorname{grad} r^n = n r^{n-2}\vec{r}$

$\Rightarrow \operatorname{div}(\operatorname{grad} r^n) = \nabla\cdot(n r^{n-2}x\hat{i} + n r^{n-2}y\hat{j} + n r^{n-2}z\hat{k})$

$$= \frac{\partial}{\partial x}(x n r^{n-2}) + \frac{\partial}{\partial y}(y n r^{n-2})$$

$$+\frac{\partial}{\partial z}(z\, n\, r^{n-2})$$

$$\text{div}(\text{grad}\, r^n) = n(n-2)r^{n-3}\frac{\partial r}{\partial x}\cdot x + r^{n-2}\cdot n$$

$$+ n(n-2)r^{n-3}\frac{\partial r}{\partial y}\cdot y + r^{n-2}\cdot n$$

$$+ n(n-2)r^{n-3}\frac{\partial r}{\partial z}\cdot z + r^{n-2}\cdot n$$

$$= n(n-2)r^{n-3}\frac{x}{r}\cdot x + n(n-2)r^{n-3}\frac{y}{r}\cdot y$$

$$+ n(n-2)r^{n-3}\frac{z}{r}\cdot z + 3r^{n-2}n$$

$$= n(n-2)r^{n-4}(x^2+y^2+z^2) + 3r^{n-2}n$$

$$\text{div}(\text{grad}\, r^n) = n(n-2)r^{n-4}\, r^2 + 3r^{n-2}\cdot n$$

$$= n(n-2)r^{n-2} + 3r^{n-2}\cdot n$$

$$= n r^{n-2}(n - 2 + 3)$$

$$= n r^{n-2}(n+1)$$

$$\text{div}(\text{grad}\, r^n) = n(n+1)r^{n-2}$$ //

Hence proved.

Note:-

* $\nabla^2\phi = \nabla\cdot\nabla\phi = div(grad\phi)$

* $\nabla^2\phi = 0$ Then ϕ is Said to Satisfy Laplace eq

Also ϕ is called harmonic function.

* The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is Called Laplacian operator.

* $Curl(Curl\bar{f}) = \nabla\times(\nabla\times\bar{f}) = \nabla(\nabla\cdot\bar{f}) - \nabla^2\bar{f}$

$= grad\, div\bar{f} - \nabla^2\bar{f}$

operators :-

vector differential operator :-

The operator $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$\Rightarrow \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

If is a Scalar point function then

$\nabla\phi = grad\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Scalar point differential operator :

$\bar{a}\nabla = \bar{a}\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$

$= (\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial}{\partial z}$

$$(\bar{a}\cdot\nabla)\phi = (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\phi}{\partial z}$$

$||^{ly}$ $$(\bar{a}\cdot\nabla)\bar{f} = (\bar{a}\cdot\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\bar{f}}{\partial z}$$

Vector point differential operator:

$$\therefore\ \bar{a}\times\nabla = \bar{a}\times\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$= (\bar{a}\times\bar{i})\frac{\partial}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$$

$$\Rightarrow (\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$$

$||^{ly}$ $$(\bar{a}\times\nabla)\bar{f} = (\bar{a}\times\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\bar{f}}{\partial z}$$

$$\therefore\ \nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

(19). Prove that $\operatorname{div}(r^n R) = (n+3) r^n$. Hence show that R/r^3 is solenoidal. If $R = xi + yj + zK$ and ...

(20). If $u = x^2 + y^2 + z^2$ and $V = xi + yj + zK$, show that $\operatorname{div}(uV) = 5u$.

(21). If $R = xi + yj + zK$ and $r \neq 0$, show that

(i). $\operatorname{Curl}(r^n R) = 0$ (ii). $\operatorname{grad}\left(\operatorname{div} \frac{R}{r}\right) = \frac{-2R}{r^3}$

(iii). $\nabla \frac{1}{r^2} = -2R/r^4$ (iv). $\nabla \cdot (R/r^2) = 1/r^2$

(22). Calculate (i). $\operatorname{Curl}(\operatorname{grad} f)$, given $f(x, y, z) = x^2 y^2 ...$

(ii). $\operatorname{Curl}(\operatorname{Curl} A)$, given $A = x^2 y\, i + y^2 z\, j + ...$

(23). (a). If $f = (x^2 + y^2 + z^2)^{-n}$, find $\operatorname{div}(\operatorname{grad} f)$ and determine 'n' if $\operatorname{div}(\operatorname{grad} f) = 0$.

(b). Show that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1) r^{n-2}$

where $r^2 = x^2 + y^2 + z^2$

(24). If $u = x^2 y z$, $v = xy - 3z^2$ find (i). $\nabla(\nabla u \cdot \nabla v)$

(ii). $\nabla \cdot (\nabla u \times \nabla v)$.

(15). Given vector is

Sol^n:

$$\bar{f} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k} \longrightarrow (1)$$

Comparing eq (1) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $f_1 = x+y+1$, $f_2 = 1$, $f_3 = -x-y$

Now $\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$

$$= \bar{i}(-1-0) - \bar{j}(-1-0) + \bar{k}(0-1)$$

$$\text{Curl}\,\bar{f} = -\bar{i} + \bar{j} - \bar{k}$$

and also $\bar{f}.\text{Curl}\,\bar{f} = \{(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}\}.\{-\bar{i}+\bar{j}-\bar{k}\}$

$$= \cancel{-x} - \cancel{y} - \cancel{1} + \cancel{1} + \cancel{x} + \cancel{y}$$

$$\bar{f}.\text{Curl}\,\bar{f} = 0$$

Hence proved.

(17). let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow r^2 = x^2+y^2+z^2 \longrightarrow (1)$$

Since $R = x\bar{i} + y\bar{j} + z\bar{k}$

P.D. w.r.t. to x, y and z in eq (1)

we got $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $r^n R = r^n(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$$r^n R = r^n x\mathbf{i} + r^n y\mathbf{j} + r^n z\mathbf{k}$$

$$\Rightarrow \operatorname{div}(r^n R) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= nr^{n-1}\frac{\partial r}{\partial x} x + r^n \cdot 1 + nr^{n-1}\frac{\partial r}{\partial y} \cdot y + r^n$$
$$+ nr^{n-1}\frac{\partial r}{\partial z} z + r^n$$

$$= nr^{n-1}\frac{x}{r} \cdot x + nr^{n-1}\frac{y}{r} \cdot y + nr^{n-1}\frac{z}{r} \cdot z$$
$$+ 3r^n$$

$$= nr^{n-2}x^2 + nr^{n-2}y^2 + nr^{n-2}z^2$$
$$+ 3r^n$$

$$= nr^{n-2}(x^2+y^2+z^2) + 3r^n$$

$$= nr^{n-2}(r^2) + 3r^n$$

$$\operatorname{div}(r^n R) = nr^n + 3r^n$$

$$= r^n(n+3)$$

$$\boxed{\operatorname{div}(r^n R) = (n+3)r^n}$$

Now, we have to prove that

$\frac{R}{r^3}$ is a Solenoidal,

i.e $\operatorname{div}(R/r^3) = 0$.

$$\therefore \frac{R}{r^3} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3}$$

$$\frac{R}{r^3} = r^{-3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

$$\frac{R}{r^3} = r^{-3}x\,i + r^{-3}y\,j + r^{-3}z\,k$$

$$div\left(R/r^3\right) = \frac{\partial}{\partial x}\left(r^{-3}x\right) + \frac{\partial}{\partial y}\left(r^{-3}y\right) + \frac{\partial}{\partial z}\left(r^{-3}z\right)$$

$$= -3r^{-4}\cdot\frac{\partial r}{\partial x}\cdot x + r^{-3}\cdot 1 - 3r^{-4}\frac{\partial r}{\partial y}\cdot y + r^{-3}\cdot 1$$

$$- 3r^{-4}\frac{\partial r}{\partial z}\cdot z + r^{-3}\cdot 1$$

$$= -3r^{-4}\frac{x}{r}\cdot x + r^{-3} - 3r^{-4}\frac{y}{r}\cdot y + r^{-3}$$

$$- 3r^{-4}\frac{z}{r}\cdot z + r^{-3}$$

$$= -3r^{-5}x^2 - 3r^{-5}y^2 - 3r^{-5}z^2 + 3r^{-3}$$

$$= -3r^{-5}\left(x^2+y^2+z^2\right) + 3r^{-3}$$

$$div\left(R/r^3\right) = -3r^{-5}r^2 + 3r^{-3}$$

$$= \cancel{-3r^{-3}} \cancel{+3r^{-3}}$$

$$div\left(R/r^3\right) = 0$$

$\therefore$ R/r^3 is a Solenoidal

Hence proved.

(20)
39:

Given vectors is

$u = x^2+y^2+z^2$, $V = x\,i + y\,j + z\,k$

Now $uv = (x^2+y^2+z^2)(x\,i + y\,j + z\,k)$

$$= x^3 i + x^2 y\,j + x^2 z\,k + y^2 x\,i$$

$$+ y^3 j + y^2 z\,k + xz^2 i + yz^2 j$$

$$uV = (x^3 + xy^2 + xz^2)\,i + (y^3 + yx^2 + yz^2)\,j$$
$$+ (z^3 + zx^2 + zy^2)\,k$$

$$\operatorname{div}(uV) = \frac{\partial}{\partial x}(x^3 + xy^2 + xz^2)$$
$$+ \frac{\partial}{\partial y}(y^3 + yx^2 + yz^2)$$
$$+ \frac{\partial}{\partial z}(z^3 + zx^2 + zy^2)$$

$$= 3x^2 + y^2 + z^2 + 3y^2 + x^2 + z^2$$
$$+ 3z^2 + x^2 + y^2$$

$$= 3(x^2 + y^2 + z^2) + 2x^2 + 2y^2 + 2z^2$$

$$= 34 + 2(x^2 + y^2 + z^2)$$

$$= 34 + 24$$

$$\operatorname{div}(uV) = 54 \,//.$$

$\therefore$ Hence proved.

(21) (i). Given $R = xi + yj + zk \longrightarrow (1)$

let $\bar{r} = xi + yj + zk$

$r = |\bar{r}| = \sqrt{f_1^2 + f_2^2 + f_3^2} = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2 \longrightarrow (2)$

$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\text{Curl}(r^n R) = \text{Curl}\left(r^n(xi + yj + zk)\right)$

$\text{Curl}(r^n R) = \text{Curl}\left(xr^n i + yr^n j + zr^n k\right) \longrightarrow (3)$

Here $r^n R = xr^n i + yr^n j + zr^n k$

let $\bar{f} = r^n R = xr^n i + yr^n j + zr^n k$

$\text{Curl} \bar{f} = \text{Curl}(r^n R)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix} = i(0-0) - j(0-0) + k(0-0)$$

$\text{Curl} \bar{f} = 0.$

(ii).

Now $\nabla \frac{1}{r^2} = \nabla r^{-2}$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) r^{-2}$

$= i\frac{\partial}{\partial x} r^{-2} + j\frac{\partial}{\partial y} r^{-2} + k\frac{\partial}{\partial z} r^{-2}$

$$\nabla \frac{1}{r^2} = i(-2)r^{-2-1}\frac{\partial r}{\partial x} + j(-2)r^{-2-1}\frac{\partial r}{\partial y} + k(-2)r^{-2-1}\frac{\partial r}{\partial z}$$

$$= i(-2)r^{-3}\frac{x}{r} - 2jr^{-3}\frac{y}{r} - 2kr^{-3}\frac{z}{r}$$

$$= -2ir^{-4}x - 2jr^{-4}y - 2kr^{-4}z$$

$$\nabla \frac{1}{r^2} = -2r^{-4}(xi + yj + zk)$$

$$= -2r^{-4}R$$

$$\boxed{\nabla \frac{1}{r^2} = -2R/r^4}$$

(ii). Now $\frac{R}{r} = \frac{xi + yj + zk}{r}$

$$\frac{R}{r} = xr^{-1}i + yr^{-1}j + zr^{-1}k$$

$$\Rightarrow \mathrm{div}\left(\frac{R}{r}\right) = \mathrm{div}\left(xr^{-1}i + yr^{-1}j + zr^{-1}k\right)$$

$$= \frac{\partial}{\partial x}(r^{-1}x) + \frac{\partial}{\partial y}(r^{-1}y) + \frac{\partial}{\partial z}(r^{-1}z)$$

$$= x.(-1)r^{-1-1}\frac{\partial r}{\partial x} + r^{-1}.1$$

$$+ y(-1)r^{-1-1}\frac{\partial r}{\partial y} + r^{-1}.1$$

$$+ z(-1)r^{-1-1}\frac{\partial r}{\partial z} + r^{-1}.1$$

$$= -xr^{-2}\frac{x}{r} + 3r^{-1}$$

$$- yr^{-2}\frac{y}{r} - zr^{-2}\frac{z}{r}$$

$$\mathrm{div}(R/r) = -x^2r^{-3} - y^2r^{-3} - z^2r^{-3} + 3r^{-1}$$

$$= -r^{-3}(x^2+y^2+z^2) + 3r^{-1}$$

$$= -r^{-3}\, r^2 + 3r^{-1}$$

$$= -r^{-1} + 3r^{-1} = 2r^{-1}$$

$$\operatorname{div}\left(\frac{R}{r}\right) = \frac{2}{r}$$

$$\Rightarrow \operatorname{grad}\left(\operatorname{div}\left(\frac{R}{r}\right)\right) = \nabla\left(\operatorname{div}(R/r)\right)$$

$$= \nabla\left(\frac{2}{r}\right)$$

$$= I\frac{\partial}{\partial x}\frac{2}{r} + J\frac{\partial}{\partial y}\frac{2}{r} + K\frac{\partial}{\partial z}\frac{2}{r}$$

$$= 2I\frac{\partial}{\partial x}r^{-1} + 2J\frac{\partial}{\partial y}r^{-1} + 2K\frac{\partial}{\partial z}r^{-1}$$

$$= 2I(-1)r^{-2}\frac{\partial r}{\partial x} + 2J(-1)r^{-2}\frac{\partial r}{\partial y}$$

$$+ 2K(-1)r^{-2}\frac{\partial r}{\partial z}$$

$$= -2I\,r^{-2}\frac{x}{r} - 2J\,r^{-2}\frac{y}{r}$$

$$-2K\,r^{-2}\frac{z}{r}$$

$$= -2I\,r^{-3}x - 2J\,r^{-3}y - 2K\,r^{-3}z$$

$$= -2r^{-3}(xI + yJ + zK)$$

$$= -2r^{-3}R$$

$$\boxed{\operatorname{grad}\left(\operatorname{div}(R/r)\right) = -2R/r^3}$$

(20) (ii).

Given vector

$A = x^2y\,\bar{i} + y^2z\,\bar{j} + z^2y\,\bar{k} \longrightarrow (i)$

Comparing eq (i) into $A = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we get $f_1 = x^2y$, $f_2 = y^2z$, $f_3 = z^2y$.

Now $\text{Curl}(A) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2y \end{vmatrix}$

$= \bar{i}(z^2 - y^2) - \bar{j}(0-0) + \bar{k}(0 - x^2)$

$\text{Curl}(A) = \bar{i}(z^2 - y^2) \cancel{+} 0\bar{j} - x^2\bar{k}$

$\Rightarrow \text{Curl}(\text{curl}(A)) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-y^2 & 0 & -x^2 \end{vmatrix}$

$= \bar{i}(0-0) - \bar{j}(-2x^2 - 2z^2)$

$+ \bar{k}(0 - 2y)$

$\text{Curl}(\text{curl}(A)) = 0\bar{i} + 2(x^2+z^2)\bar{j} - 2y\bar{k}$

(02)(i). Given vector

$$\bar{f}(x,y,z) = x^2+y^2-z$$

Now $\text{grad} f = \nabla f = \nabla(x^2+y^2-z)$

$$= \bar{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \bar{k}\left(\frac{\partial}{\partial z}(x^2+y^2-z)\right)$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1)$$

$$\text{grad} \bar{f} = 2x\bar{i} + 2y\bar{j} - \bar{k}$$

$$\text{Curl}(\text{grad} \bar{f}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & -1 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(0-0)$$

$$\text{Curl}(\text{grad} \bar{f}) = 0$$

(03).(a). Given $f = (x^2+y^2+z^2)^{-n} \longrightarrow (1)$

we know that

$\bar{r} = xi+yj+zk \Rightarrow r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

from eq.(1) $f = (r^2)^{-n} = r^{-2n}$

Now $\text{div}(\text{grad} f) = \nabla\cdot(\nabla f)$

$$= \nabla\cdot\left(i\frac{\partial}{\partial x}r^{-2n} + j\frac{\partial}{\partial y}r^{-2n} + k\frac{\partial}{\partial z}r^{-2n}\right)$$

$$= \nabla\cdot\left(i(-2n)r^{-2n-1}\frac{\partial r}{\partial x}\right)$$

$$+ j\left(-2n r^{-2n-1}\frac{\partial r}{\partial y}\right)$$

$$+ k\left(-2n r^{-2n-1}\frac{\partial r}{\partial z}\right)$$

but we know that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \Big[i(-2n) r^{-2n-1}\frac{x}{r} + j(-2n) r^{-2n-1}\frac{y}{r} + k(-2n) r^{-2n-1}\frac{z}{r} \Big]$$

$$= \nabla \cdot \Big[-2n r^{-2n-2} x\, i - 2n r^{-2n-2} y\, j - 2n r^{-2n-2} z\, k \Big]$$

$$\operatorname{div}(\operatorname{grad} f) = \frac{\partial}{\partial x}\left(-2n r^{-2n-2} x\right) + \frac{\partial}{\partial y}\left(-2n r^{-2n-2} y\right) + \frac{\partial}{\partial z}\left(-2n r^{-2n-2} z\right)$$

$$= +2n(2n+2) r^{-2n-3}\frac{\partial r}{\partial x}\cdot x - 2n r^{-2n-2}\cdot 1$$

$$+ 2n(2n+2) r^{-2n-3}\frac{\partial r}{\partial y}\cdot y - 2n r^{-2n-2}\cdot 1$$

$$+ 2n(2n+2)\, r^{2n-3}\, \frac{\partial r}{\partial z} \cdot z - 2n\, r^{2n-2}$$

$$= 2n(2n+2)\, r^{2n-4}(x^2+y^2+z^2) - 6n\, r^{2n-2}$$

$$= 2n(2n+2)\, r^{2n-4}\, r^2 - 6n\, r^{2n-2}$$

$$\text{div}(\text{grad} f) = 2n(2n+2)\, r^{2n-2} - 6n\, r^{2n-2}$$

$$= r^{2n-2}(4n^2+4n-6n)$$

$$= r^{2n-2}(4n^2-2n)$$

$$\boxed{\text{div}(\text{grad} f) = 2n(2n-1)\, r^{2n-2}}$$

If $\text{div}(\text{grad} f) = 0$

$$\Rightarrow 2n(2n-1)\, r^{2n-2} = 0$$

$$2n(2n-1) = 0$$

$$2n = 0 \quad (\text{or}) \quad 2n-1 = 0$$

$$n = 0 \quad (\text{or}) \quad n = 1/2$$

$$\therefore \boxed{n = 1/2} \;//.$$

23) (b).

we have to prove that

$$\text{div}(\text{grad}\, r^n) = n(n+1)\, r^{n-2}$$

Now $\text{grad}\, r^n = \nabla r^n = i\frac{\partial}{\partial x} r^n + j\frac{\partial}{\partial y} r^n + k\frac{\partial}{\partial z} r^n$

$$= n(r^{n-1})\frac{\partial r}{\partial x}\, i + (n)(r^{n-1})\frac{\partial r}{\partial y}\, j$$

$$+ n(r^{n-1})\frac{\partial r}{\partial z}\, k$$

$$\text{grad } r^n = n r^{n-1}\frac{x}{r}\,\hat{i} + n r^{n-1}\frac{y}{r}\,\hat{j} + n r^{n-1}\frac{z}{r}\,\hat{k}$$

$$\text{grad } r^n = n r^{n-2} x\,\hat{i} + n r^{n-2} y\,\hat{j} + n r^{n-2} z\,\hat{k}$$

$$\text{div}(\text{grad } r^n) = \frac{\partial}{\partial x}\left(n r^{n-2}\right)x + \frac{\partial}{\partial y}\left(n r^{n-2}\right)y + \frac{\partial}{\partial z}\left(n r^{n-2}\right)z$$

$$= n(n-2) r^{n-3}\frac{\partial r}{\partial x}\cdot x + n r^{n-2}$$

$$+ n(n-2) r^{n-3}\frac{\partial r}{\partial y}\cdot y + n r^{n-2}$$

$$+ n(n-2) r^{n-3}\frac{\partial r}{\partial z}\cdot z + n r^{n-2}$$

$$= n(n-2) r^{n-4} x^2 + 3 r n^{n-2} +$$

$$n(n-2) r^{n-4} y^2 + n(n-2) r^{n-4} z^2$$

$$= n(n-2) r^{n-4}(x^2+y^2+z^2) + 3 r n^{n-2}$$

$$= n(n-2) r^{n-4} r^2 + 3 r^{n-2}$$

$$\text{div}(\text{grad } r^n) = n(n-2) r^{n-2} + 3 r n^{n-2}$$

$$= r^{n-2}(n^2 - 2n + 3n)$$

$$r^{n-2}(n^2 + n)$$

$$\text{div}(\text{grad } r^n) = n(n+1) r^{n-2}$$

//.

problem:-25 Show that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$ (or)

$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Sol:

we know that

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \longrightarrow (1)$, $\quad r = |\bar{r}|$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2 \longrightarrow (2)$

P. diff eq. (2) w.r.to x, y, z

we have $\quad 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\parallel^{ly}$ $\quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\nabla^2 f(r) = \nabla\cdot(\nabla f(r)) = \operatorname{div}(\operatorname{grad} f(r)) \longrightarrow (3)$

Consider $\operatorname{grad}(f(r)) = \nabla f(r)$

$= \bar{i}\frac{\partial}{\partial x} f(r) + \bar{j}\frac{\partial}{\partial y} f(r) + \bar{k}\frac{\partial}{\partial z} f(r)$

$= \sum^{3} \bar{i}\frac{\partial}{\partial x} f(r) = \sum \bar{i} f'(r)\frac{\partial r}{\partial x}$

$\operatorname{grad}(f(r)) = \sum \bar{i} f'(r)\frac{x}{r} = \bar{f}$ (say)

$\Rightarrow \bar{f} = \bar{i} f'(r)\frac{x}{r} + \bar{j} f'(r)\frac{y}{r} + \bar{k} f'(r)\frac{z}{r} \longrightarrow (4)$

Comparing eq (4) into $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

we have $\quad f_1 = f'(r)\frac{x}{r}, \quad f_2 = f'(r)\frac{y}{r}, \quad f_3 = f'(r)\frac{z}{r}$

$\therefore \operatorname{div}(\operatorname{grad} f(r)) = \operatorname{div}\bar{f}$

$= \frac{\partial}{\partial x} f'(r)\frac{x}{r} + \frac{\partial}{\partial y} f'(r)\frac{y}{r} + \frac{\partial}{\partial z} f'(r)\frac{z}{r}$

$$\operatorname{div}(\operatorname{grad} f(r)) = \frac{\partial}{\partial x}\left(\frac{f'(r)\cdot x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{f'(r)\cdot y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{f'(r)\cdot z}{r}\right) \quad \text{——} (5)$$

Consider

$$\frac{\partial}{\partial x}\left(\frac{f'(r)\cdot x}{r}\right) = \frac{r\frac{\partial}{\partial x}(f'(r)\cdot x) - f'(r)\cdot x\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r\left[f'(r)\frac{\partial x}{\partial x} + x\frac{\partial f'(r)}{\partial x}\right] - f'(r)\cdot x\frac{x}{r}}{r^2}$$

$$= \frac{r f'(r) + r x f''(r)\frac{\partial r}{\partial x} - f'(r)\frac{x^2}{r}}{r^2}$$

$$= \frac{1}{r^2}\left\{r f'(r) + r x f''(r)\frac{x}{\cancel{r}} - f'(r)\frac{x^2}{r}\right\}$$

$$= \frac{1}{r^2}\left\{r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right\}$$

$$\frac{\partial}{\partial x}\left(\frac{f'(r)\cdot x}{r}\right) = \frac{1}{r} f'(r) + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r)$$

$|||^{ly}$ $$\frac{\partial}{\partial y}\left(\frac{f'(r)\cdot y}{r}\right) = \frac{1}{r} f'(r) + \frac{y^2}{r^2} f''(r) - \frac{y^2}{r^3} f'(r)$$

$$\frac{\partial}{\partial z}\left(\frac{f'(r)\cdot z}{r}\right) = \frac{1}{r} f'(r) + \frac{z^2}{r^2} f''(r) - \frac{z^2}{r^3} f'(r)$$

from eq (5), we have.

$$\operatorname{div}(\operatorname{grad} f(r)) = \frac{3}{r} f'(r) + \frac{1}{r^2} f''(r)(x^2+y^2+z^2) - \frac{1}{r^3} f'(r)(x^2+y^2+z^2)$$

$$= \frac{3}{r} f'(r) + \frac{1}{r^2} f''(r)\, r^2 - \frac{1}{r^3} f'(r)\, r^2$$

$$= \frac{3}{r} f'(r) + f''(r) - \frac{1}{r} f'(r)$$

$$\operatorname{div}(\operatorname{grad} f(r)) = \frac{2}{r} f'(r) + f''(r)$$

(or)

$$\operatorname{div}(\operatorname{grad} f(r)) = \frac{2}{r}\frac{d f(r)}{dr} + \frac{d^2 f(r)}{dr^2}$$

$$= \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{d f(r)}{dr}$$

Hence proved.

Note:-

* $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$
* $\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$
* $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - \bar{a}(\bar{b}\cdot\bar{c})$
* $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

Problems:-

(26). prove that $(\bar{f}\times\nabla)\cdot\bar{r} = 0$

$\because (\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}\times(\bar{b}\cdot\bar{c})$

$\therefore \bar{a}\times(\bar{b}\bar{c}) = (\bar{a}\times\bar{b})\bar{c}$

Sol:- Now $(\bar{f}\times\nabla)\cdot\bar{r} = \bar{f}\times(\nabla\cdot\bar{r})$

$$= \bar{f}\times\left(i\frac{\partial}{\partial x}\bar{r} + j\frac{\partial}{\partial y}\bar{r} + k\frac{\partial}{\partial z}\bar{r}\right)$$

$$= (\bar{f}\times i)\frac{\partial\bar{r}}{\partial x} + (\bar{f}\times j)\frac{\partial\bar{r}}{\partial y} + (\bar{f}\times k)\frac{\partial\bar{r}}{\partial z} \longrightarrow (1)$$

Consider:

$$\bar{f}\times i = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 0 & 0 \end{vmatrix}$$

$= i(0-0) - j(0-0) + k(0-0) = \bar{0}$.

$\bar{f}\times i = (f_1 i + f_2 j + f_3 k)\times i$

$= f_1(i\times i) + f_2(j\times i) + f_3(k\times i)$

$(\bar{f}\times\nabla)\cdot\bar{r} = \bar{0}+\bar{0}+\bar{0}$

$= \bar{0}$

$$(\bar{f}\times\nabla)\cdot\bar{r} = (\bar{f}\times\bar{i})\frac{\partial\bar{r}}{\partial x} + (\bar{f}\times\bar{j})\frac{\partial\bar{r}}{\partial y} + (\bar{f}\times\bar{k})\frac{\partial\bar{r}}{\partial z} \longrightarrow (1)$$

Consider

$$\frac{\partial\bar{r}}{\partial x} = \frac{\partial}{\partial x}(x\bar{i}+y\bar{j}+z\bar{k})$$

$$= \bar{i}$$

$\text{||}^{ly}\ \frac{\partial\bar{r}}{\partial y} = \bar{j}\ ,\ \frac{\partial\bar{r}}{\partial z} = \bar{k}$

$$(\bar{f}\times\bar{i}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ f_1 & f_2 & f_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$\bar{f} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$\bar{i} = 1\bar{i}+0\bar{j}+0\bar{k}$

$$= \bar{i}(0-0) - \bar{j}(0-f_3) + \bar{k}(0-f_2)$$

$$(\bar{f}\times\bar{i}) = f_3\bar{j} - f_2\bar{k}$$

$$\text{||}^{ly}\ (\bar{f}\times\bar{j}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ f_1 & f_2 & f_3 \\ 0 & 1 & 0 \end{vmatrix} = \bar{i}(-f_3) - \bar{j}(0-0) + \bar{k}(f_1)$$

$$= -f_3\bar{i} + f_1\bar{k}$$

$$\therefore (\bar{f}\times\bar{k}) = f_2\bar{i} - f_1\bar{j}$$

from eq (1)

$$(\bar{f}\times\nabla)\cdot\bar{r} = (f_3\bar{j} - f_2\bar{k})\cdot\bar{i} + (-f_3\bar{i}+f_1\bar{k})\cdot\bar{j} + (f_2\bar{i}-f_1\bar{j})\cdot\bar{k}$$

$$= \cancel{f_3\bar{j}\cdot\bar{i}} - \cancel{f_2\bar{k}\cdot\bar{i}} - \cancel{f_3\bar{i}\cdot\bar{j}} + \cancel{f_1\bar{k}\cdot\bar{j}} + \cancel{f_2\bar{i}\cdot\bar{k}} - \cancel{f_1\bar{j}\cdot\bar{k}}$$

$$(\bar{f}\times\nabla)\cdot\bar{r} = 0$$

//

(iv).(i). Given

$u = x^2yz$, $V = xy - 3z^2$

Now $\nabla u = i\frac{\partial}{\partial x} x^2yz + j\frac{\partial}{\partial y} x^2yz + k\frac{\partial}{\partial z} x^2yz$

$= i\, 2xyz + j\, x^2z + k\, x^2y$

and

$\nabla V = i\frac{\partial}{\partial x}(xy - 3z^2) + j\left(\frac{\partial}{\partial y}(xy - 3z^2)\right.$

$+ k\frac{\partial}{\partial z}(xy - 3z^2)$

$= i(y) + j(x) + k(-6z)$

$\nabla V = y\bar{i} + x\bar{j} - 6z\bar{k}$

$\therefore \nabla u \cdot \nabla V = \left[(2xyz)\bar{i} + (x^2z)\bar{j} + (x^2y)\bar{k}\right] \cdot$

$\left[y\bar{i} + x\bar{j} - 6z\bar{k}\right]$

$\nabla u \cdot \nabla V = 2xy^2z + x^3z - 6x^2yz$

$\Rightarrow \nabla(\nabla u \cdot \nabla V) = i\frac{\partial}{\partial x}(2xy^2z + x^3z - 6x^2yz)$

$+ j\frac{\partial}{\partial y}(2xy^2z + x^3z - 6x^2yz)$

$+ k\frac{\partial}{\partial z}(2xy^2z + x^3z - 6x^2yz)$

$\nabla(\nabla u \cdot \nabla V) = \bar{i}(2y^2z + 3x^2z - 12xyz)$

$+ j(4xyz + 0 - 6x^2z)$

$+ k(2xy^2 + x^3 - 6x^2y)$

(24)(ii). Given $u = x^2yz$, $v = xy - 3z^2$

Now, $\nabla u = i\frac{\partial}{\partial x} x^2yz + j\frac{\partial}{\partial y} x^2yz + k\frac{\partial}{\partial z} x^2yz$

$= i(2xyz) + j(x^2z) + k(x^2y)$

and $\nabla v = i\frac{\partial}{\partial x}(xy - 3z^2) + j\frac{\partial}{\partial y}(xy - 3z^2)$

$+ k\frac{\partial}{\partial z}(xy - 3z^2)$

$\nabla v = i(y) + j(x) + k(-6z)$

$\therefore \nabla u \times \nabla v = [i(2xyz) + j(x^2z) + k(x^2y)] \times$

$[yi + xj - 6zk]$

$$= \begin{vmatrix} i & j & k \\ xy-3z^2 & xy-3z^2 & xy-3z^2 \\ y & x & -6z \end{vmatrix}$$

$\nabla u \times \nabla v = i(-6xyz + 18z^3 - x^2y + 3xz^2)$

$- j(-6xyz + 18z^3 - xy^2 + 3yz^2)$

$+ k(x^2y - 3xz^2 - xy^2 + 3yz^2)$

$\nabla \cdot (\nabla u \times \nabla v) = i\frac{\partial}{\partial x}(-6xyz + 18x^3 - x^2y + 3xz^2)$

$+ j\frac{\partial}{\partial y}(-6xyz + 18z^3 - xy^2 + 3yz^2)$

$+ k\frac{\partial}{\partial z}(~~-6xyz + 18z^3 - x^2y + 3xz^2~~$

$x^2y - 3xz^2 - xy^2 + 3yz^2$

$$= -6yz + 54x^2 - 27xy + 3z^2$$

$$+ 6yz + 54x - 27xy \mp 3z^2$$

$$+ 0 - 6xz - 0 + 6yz$$

$$\nabla \cdot (\nabla \phi \times \nabla \psi) = 54x^2, \; i(-6yz + 54x^2 - 27xy + 3z^2) + j(6yz + 27xy - 3z^2) + k(-6xz + 6yz)$$

Vector Identities :-

1. prove that for any vector functions f and g

$$\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$$

(2). prove that $(\bar{f} \times \nabla) \times \bar{r} = -2\bar{f}$

(3). If $\bar{a}$ is a vector function and u is a scalar function then $\text{div}(u\bar{a}) = u \,\text{div}\,\bar{a} + (\text{grad}\, u) \cdot \bar{a}$

(4). prove that $\text{curl}(\text{grad}\,\phi) = \nabla \times \nabla \phi = 0$

(5). P.T $\text{grad}(\text{div}\,\bar{A}) = \text{curl}(\text{curl}\,\bar{A}) + \nabla^2 \bar{A}$

(or)

(6). $\text{curl}(\text{curl}\,\bar{A}) = \text{grad}(\text{div}\,\bar{A}) - \nabla^2 \bar{A}$

(or).

$$\nabla \times (\nabla \times \bar{A}) = \text{grad}(\text{div}\,\bar{A}) - \nabla^2 \bar{A}$$

(6). P.T $\text{div}(\text{curl}\,\bar{A}) = 0$

(7). P.T $\text{curl}(\bar{a} \times \bar{b}) = (b \cdot \nabla)\bar{a} - (\bar{a} \cdot \nabla)\bar{b} + \bar{a}\,\text{div}\,b - b\,\text{div}$

(8). For a solenoidal vector $\bar{f}$, P.T $\nabla \times (\nabla \times (\nabla \times (\nabla \times \bar{f}))) = \nabla^4 \bar{f}$

(or)

$$\text{curl}(\text{curl}(\text{curl}(\text{curl}\,\bar{f}))) = \nabla^4 \bar{f}$$

(9). P.T $\nabla\left(\nabla\cdot\frac{\bar r}{r}\right)=-\frac{2}{r^3}\cdot\bar r$ (or) grad $\left(\text{div}\,\frac{\bar r}{r}\right)=-\frac{2}{r^3}$

(1)
Ans: we have to prove that

$$\nabla\cdot(\bar f\times\bar g)=\bar g\cdot(\nabla\times\bar f)-\bar f\cdot(\nabla\times\bar g)$$

Now $\nabla\cdot(\bar f\times\bar g)=\bar i\frac{\partial}{\partial x}(\bar f\times\bar g)+\bar j\frac{\partial}{\partial y}(\bar f\times\bar g)+\bar k\frac{\partial}{\partial z}(\bar f\times\bar g)$

$$=\sum\bar i\,\frac{\partial}{\partial x}(\bar f\times\bar g)$$

$$=\sum\bar i\left(\frac{\partial\bar f}{\partial x}\times\bar g\right)+\sum\left(\bar i\left(\bar f\times\frac{\partial\bar g}{\partial x}\right)\right)$$

$(\bar a\times\bar b=-\bar b\times\bar a)$

$$=\sum\bar i\left(\frac{\partial\bar f}{\partial x}\times\bar g\right)-\sum\bar i\left(\frac{\partial\bar g}{\partial x}\times\bar f\right)$$

$(\bar a(\bar b\times\bar c)=(\bar a\times\bar b)\cdot\bar c)$

$$=\sum\left(\bar i\times\frac{\partial\bar f}{\partial x}\right)\cdot\bar g-\sum\left(\bar i\times\frac{\partial\bar g}{\partial x}\right)\cdot\bar f$$

$$=\left(\bar i\times\frac{\partial\bar f}{\partial x}+\bar j\times\frac{\partial\bar f}{\partial y}+\bar k\times\frac{\partial\bar f}{\partial z}\right)\cdot\bar g$$

$$-\left(\bar i\times\frac{\partial\bar g}{\partial x}+\bar j\times\frac{\partial\bar g}{\partial y}+\bar k\times\frac{\partial\bar g}{\partial y}\right)\cdot\bar f$$

$$\nabla\cdot(\bar f\times\bar g)=(\nabla\times\bar f)\cdot\bar g-(\nabla\times\bar g)\cdot\bar f$$

$\therefore(\bar a\times\bar b)\cdot\bar c=\bar a\cdot($ $\boxed{\bar a\cdot\bar b=\bar b\cdot\bar a}$

$$\boxed{\nabla\cdot(\bar f\times\bar g)=\bar g\cdot(\nabla\times\bar f)-\bar f\cdot(\nabla\times\bar g)}$$

(c) we have to prove that

$$(\vec{f}\times\nabla)\times\vec{r} = -2\vec{f}$$

Now $\vec{f}\times\nabla = \vec{f}\times\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)$

$$=(\vec{f}\times i)\frac{\partial}{\partial x}+(\vec{f}\times j)\frac{\partial}{\partial y}+(\vec{f}\times k)\frac{\partial}{\partial z}$$

and

$$\Rightarrow (\vec{f}\times\nabla)\times\vec{r} = (\vec{f}\times i)\times\frac{\partial\vec{r}}{\partial x}+(\vec{f}\times j)\times\frac{\partial\vec{r}}{\partial y}+(\vec{f}\times k)\times\frac{\partial\vec{r}}{\partial z} \longrightarrow (1)$$

we know that

$$\vec{r} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$\frac{\partial\vec{r}}{\partial x} = \bar{i},\quad \frac{\partial\vec{r}}{\partial y} = \bar{j},\quad \frac{\partial\vec{r}}{\partial z} = \bar{k}$$

from eq (1)

$$(\vec{f}\times\nabla)\times\vec{r} = (\vec{f}\times\bar{i})\times\bar{i}+(\vec{f}\times\bar{j})\times\bar{j}+(\vec{f}\times\bar{k})\times\bar{k}$$

$$=(\vec{f}\cdot\bar{i})\bar{i}-\vec{f}(\bar{i}\cdot\bar{i})+(\vec{f}\cdot\bar{j})\bar{j}-\vec{f}(\bar{j}\cdot\bar{j})$$

$$+(\vec{f}\cdot\bar{k})\bar{k}-\vec{f}(\bar{k}\cdot\bar{k})$$

$$\left(\vec{f}\cdot\bar{i} = (f_1\bar{i}+f_2\bar{j}+f_3\bar{k})\cdot\bar{i} = f_1+0+0\right)$$

$$\Rightarrow (\vec{f}\times\nabla)\times\vec{r} = f_1\bar{i}-\vec{f}+f_2\bar{j}-\vec{f}+f_3\bar{k}-\vec{f}$$

$$=\underbrace{f_1\bar{i}+f_2\bar{j}+f_3\bar{k}}-3\vec{f}$$

$$=\vec{f}-3\vec{f}$$

$$\boxed{(\vec{f}\times\nabla)\times\vec{r} = -2\vec{f}}$$

(3).
Soln:

Given

a is a vector function

i.e $\bar{a} = a_1 i + a_2 j + a_3 k$

and u is a Scalar function.

$u\bar{a} = u(a_1 \bar{i} + a_2 \bar{j} + a_3 k)$

$u\bar{a} = ua_1 \bar{i} + ua_2 \bar{j} + ua_3 k$

$\Rightarrow div(u\bar{a}) = \frac{\partial}{\partial x}(ua_1) + \frac{\partial}{\partial y} ua_2 + \frac{\partial}{\partial z} ua_3$

$~~= u\frac{\partial a_1}{\partial x} + u\frac{\partial a_2}{\partial y} + u\frac{\partial a_3}{\partial z}~~$

$div(u\bar{a}) = u\frac{\partial a_1}{\partial x} + a_1\frac{\partial u}{\partial x} + u\frac{\partial a_2}{\partial y} + a_2\frac{\partial u}{\partial y}$

$+ u\frac{\partial a_3}{\partial z} + a_3\frac{\partial u}{\partial z}$

$div(u\bar{a}) = \left(u\frac{\partial a_1}{\partial x} + u\frac{\partial a_2}{\partial y} + u\frac{\partial a_3}{\partial z}\right)$

$+ \left(a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + a_3\frac{\partial u}{\partial z}\right)$

$= \left(u i\frac{\partial}{\partial x} + u j\frac{\partial}{\partial y} + u k\frac{\partial}{\partial z}\right) \cdot (a_1 i + a_2 \bar{j} + a_3 \bar{k})$

$+ ~~a_1 i + a~~ \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right) \cdot (a_1 i + a_2 \bar{j} + a_3 k)$

$= u\nabla \cdot \bar{a} + ~~div~~ u \cdot \bar{a}$

$= u\nabla \cdot \bar{a} + \nabla u \cdot \bar{a}$

$\boxed{div(u\bar{a}) = u\,div\,\bar{a} + grad\,u \cdot \bar{a}}$

(4).

Sol: Given

we have to prove that

$$\text{Curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi = 0$$

Now $\text{grad}\,\phi = \nabla\phi$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\times\nabla\phi = \text{Curl}(\text{grad}\,\phi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - j\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)$$

$$+ k\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$\text{Curl grad}\,\phi = \nabla\times\nabla\phi = 0$$

(5).

we have to prove that

$$\text{grad div}\,\bar{A} = \text{Curl Curl}\,\bar{A} + \nabla^2\bar{A}$$

(or)

$$\text{Curl curl}\,\bar{A} = \text{grad div}\,\bar{A} - \nabla^2\bar{A}$$

(or)

$$\nabla\times(\nabla\times\bar{A}) = \text{grad div}\,\bar{A} - \nabla^2\bar{A}$$

let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

then $\text{curl}\,\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$\text{curl}\,\vec{A} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \sum \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)$$

$\therefore \text{curl}\,\text{curl}\,\vec{A} = \nabla\times(\nabla\times\vec{A})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right]$$

$$= \sum \hat{i}\left[\frac{\partial^2 A_2}{\partial y\,\partial x} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right]$$

$$+ \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial x^2}$$

$$\text{curl curl}\bar{A} = \sum \bar{i}\left[\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right]$$

$$= \sum \bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A_1\right]$$

$$= \sum \bar{i}\left[\frac{\partial}{\partial x}(\text{div}\bar{A}) - \nabla^2 A_1\right]$$

$$= \bar{i}\left[\frac{\partial}{\partial x}(\text{div}\bar{A}) - \nabla^2 A_1\right] + \bar{j}\left[\frac{\partial}{\partial y}(\text{div}\bar{A}) - \nabla^2 A_2\right] + \bar{k}\left(\frac{\partial}{\partial z}(\text{div}\bar{A}) - \nabla^2 A_3\right)$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\text{div}\bar{A} - \nabla^2(A_1\bar{i} + A_2\bar{j} + A_3\bar{k})$$

$$= \nabla\, \text{div}\bar{A} - \nabla^2\bar{A}$$

$$\text{Curl curl}\bar{A} = \text{grad div}\bar{A} - \nabla^2\bar{A} \quad //.$$

(6). we have to prove that

$\text{div}(\text{curl}\bar{A}) = 0$ we know that $\bar{A} = \bar{i}A_1 + \bar{j}A_2 + \bar{k}A_3$

Now

$$\text{curl}\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\text{curl}\,\bar{A} = i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\text{div}(\text{curl}\,\bar{A}) = \cancel{\frac{\partial^2 A_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 A_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 A_1}{\partial z \partial y}}$$

$$\text{div}(\text{curl}\,\bar{A}) = 0$$

(7). we have to prove that

$$\text{Curl}(\bar{a}\times\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b} + \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a}$$

Now $\text{Curl}(\bar{a}\times\bar{b}) = \nabla\times(\bar{a}\times\bar{b})$

$$= \bar{i}\times\frac{\partial}{\partial x}\times(\bar{a}\times\bar{b}) + \bar{j}\times\frac{\partial}{\partial y}\times(\bar{a}\times\bar{b}) + \bar{k}\times\frac{\partial}{\partial z}\times(\bar{a}\times\bar{b})$$

$$= \sum i\times\left(\frac{\partial}{\partial x}\times(\bar{a}\times\bar{b})\right)$$

$$= \sum i\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left[i\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + i\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)\right]$$

$$\operatorname{Curl}(\bar a\times\bar b) = \sum \bar i\times\left(\frac{\partial \bar a}{\partial x}\times\bar b\right) + \sum \bar i\times\left(\bar a\times\frac{\partial \bar b}{\partial y}\right)$$

$$\left(\because\ \bar a\times(\bar b\times\bar c) = (\bar a\cdot\bar c)\bar b - \bar a(\bar b\cdot\bar c)\right)$$

$$\operatorname{Curl}(\bar a\times\bar b) = \sum\left[(\bar i\cdot\bar b)\frac{\partial \bar a}{\partial x} - \bar i\left(\frac{\partial \bar a}{\partial x}\cdot\bar b\right)\right]$$
$$+ \sum\left[\left(\bar i\cdot\frac{\partial \bar b}{\partial x}\right)\bar a - \bar i\left(\bar a\cdot\frac{\partial \bar b}{\partial x}\right)\right]$$

$$\operatorname{Curl}(\bar a\times\bar b) = \sum(\bar i\cdot\bar b)\frac{\partial \bar a}{\partial x} - \sum \bar i\left(\frac{\partial \bar a}{\partial x}\cdot\bar b\right)$$
$$+ \sum\left(\bar i\cdot\frac{\partial \bar b}{\partial x}\right)\bar a - \sum \bar i\left(\bar a\cdot\frac{\partial \bar b}{\partial x}\right)$$

$$\cancel{= \bar b\cdot\sum \bar i\frac{\partial \bar a}{\partial x} - \bar b\sum \bar i\frac{\partial \bar a}{\partial x}}$$
$$\cancel{+ \bar a\sum \bar i\cdot\frac{\partial \bar b}{\partial x} - \bar a\sum \bar i\frac{\partial \bar b}{\partial x}}$$

$$\cancel{= \bar b\operatorname{div}\bar a - \bar b}$$

$$= \sum\left(\bar b\cdot\bar i\frac{\partial}{\partial x}\right)\bar a - \bar b\sum \bar i\cdot\frac{\partial \bar a}{\partial x}$$
$$+ \bar a\sum \bar i\cdot\frac{\partial \bar b}{\partial x} - \sum \bar a\cdot\left(\bar i\cdot\frac{\partial}{\partial x}\right)\bar b$$

$$= (\bar b\cdot\nabla)\bar a - \bar b\operatorname{div}\bar a + \bar a\operatorname{div}\bar b - (\bar a\cdot\nabla)\bar b$$

$$\operatorname{Curl}(\bar a\times\bar b) = (\bar b\cdot\nabla)\bar a - (\bar a\cdot\nabla)\bar b + \bar a\operatorname{div}\bar b - \bar b\operatorname{div}\bar a$$

(8). we have to prove that

$$\nabla \times (\nabla \times (\nabla \times (\nabla \times \bar{f}))) = \nabla^4 \bar{f}$$

(a)

$$\text{Curl Curl Curl Curl } \bar{f} = \nabla^4 \bar{f}$$

we know that

$$\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$$

Since $\bar{f}$ is a solenoidal then

$$\text{div} \bar{f} = 0 \longrightarrow (1)$$

also we know that

$$\text{Curl Curl } \bar{f} = \text{grad div} \bar{f} - \nabla^2 \bar{f}$$

$$\text{Curl Curl } \bar{f} = \text{grad}(0) - \nabla^2 \bar{f} \quad (\text{from } (1))$$

$$\text{Curl Curl } \bar{f} = -\nabla^2 \bar{f} = -\bar{a} \text{ (say)}$$

$\longrightarrow (2)$

Now

$$\text{Curl Curl Curl Curl } \bar{f} = \text{Curl Curl } (-\bar{a})$$

$$= -\text{Curl Curl } \bar{a}$$

$$= -\text{grad div } \bar{a} + \nabla^2 \bar{a}$$

$$\text{Curl Curl Curl Curl } \bar{f} = -\text{grad div } \bar{a} + \nabla^2 (\nabla^2 \bar{f})$$

$\longrightarrow (3)$

$= -\text{grad div}(\nabla^2 \bar{f}) + \nabla^2(\nabla^2 \bar{f})$ $\longrightarrow$ (3)

Now Consider

$$\nabla^2 \bar{f} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k})$$

$$= \sum \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right)\bar{i}$$

$$= \sum \nabla^2 f_1 \bar{i}$$

$$\nabla^2 \bar{f} = \nabla^2 f_1 \bar{i} + \nabla^2 f_2 \bar{j} + \nabla^2 f_3 \bar{k}$$

$$\Rightarrow \text{div}(\nabla^2 \bar{f}) = \frac{\partial}{\partial x}\nabla^2 f_1 + \frac{\partial}{\partial y}\nabla^2 f_2 + \frac{\partial}{\partial z}\nabla^2 f_3$$

$$= \nabla^2 \frac{\partial f_1}{\partial x} + \nabla^2 \frac{\partial}{\partial y} f_2 + \nabla^2 \frac{\partial}{\partial z} f_3$$

$$= \nabla^2 \text{div} \bar{f}$$

$$\text{div}(\nabla^2 \bar{f}) = 0 \longrightarrow (4)$$

from eq (4), we get

eq (3)

$$\text{Curl Curl Curl Curl } \bar{f} = -\text{grad}(0) + \nabla^4 \bar{f}$$

$$\text{Curl curl curl curl } \bar{f} = \nabla^4 \bar{f} \quad //.$$

(or)

$$\nabla \times (\nabla \times (\nabla \times (\nabla \times \bar{f}))) = \nabla^4 \bar{f} \quad //.$$

(9).

Sol^n: we have to prove that

$$\nabla\left[\nabla\cdot\frac{\bar r}{r}\right] = \frac{-2}{r^3}\cdot\bar r$$

(or)

$$\text{grad div}\ \frac{\bar r}{r} = \frac{-2}{r^3}\bar r$$

Now we know that

$$\bar r = x\bar i + y\bar j + z\bar k$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\frac{\bar r}{r} = \left[r^{-1}\bar r\right)$

$$= r^{-1}x\bar i + r^{-1}y\bar j + r^{-1}z\bar k$$

$$\text{div}\left(\frac{\bar r}{\partial r}\right) = \frac{\partial}{\partial x}\left(r^{-1}x\right) + \frac{\partial}{\partial y}\left(r^{-1}y\right) + \frac{\partial}{\partial z}\left(r^{-1}z\right)$$

$$\text{div}\left(\frac{\bar r}{r}\right) = -1r^{-2}\frac{\partial r}{\partial x}\cdot x + r^{-1} - 1r^{-2}\frac{\partial r}{\partial y}\cdot y + r^{-1} - 1r^{-2}\frac{\partial r}{\partial z}\cdot z + r^{-1}$$

$$= -r^{-3}\left(x^2+y^2+z^2\right) + 3r^{-1}$$

$$= -r^{-3}r^2 + 3r^{-1}$$

$$= -r^{-1} + 3r^{-1}$$

$$\text{div}\left(\frac{\bar r}{r}\right) = 2r^{-1}$$

$$\Rightarrow \nabla\left(\operatorname{div}\frac{\vec{r}}{r}\right) = 2\left\{\hat{i}\frac{\partial}{\partial x}r^{-1} + \hat{j}\frac{\partial}{\partial y}r^{-1} + \hat{k}\frac{\partial}{\partial z}r^{-1}\right\}$$

$$= 2\left\{\hat{i}(-1)r^{-2}\frac{\partial r}{\partial x} + \hat{j}(-1)r^{-2}\frac{\partial r}{\partial y} + \hat{k}(-1)r^{-2}\frac{\partial r}{\partial z}\right\}$$

$$= 2\left\{-\hat{i}r^{-2}\frac{x}{r} - \hat{j}r^{-2}\frac{y}{r} - \hat{k}r^{-2}\frac{z}{r}\right\}$$

$$= -2\left\{r^{-3}(x\hat{i} + y\hat{j} + z\hat{k})\right\}$$

$$= -2r^{-3}\cdot\vec{r}$$

$$\boxed{\nabla\left(\nabla\cdot\frac{\vec{r}}{r}\right) = \frac{-2}{r^3}\vec{r}}$$

Hence proved ~~that~~.

(10). Show That $\nabla\left(\frac{1}{r^3}\right) = \frac{-3\vec{r}}{r^5}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol^n:-

Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = \sqrt{x^2+y^2+z^2}$$

Now, we have to prove that

$$\nabla\left(\frac{1}{r^3}\right) = \frac{-3\vec{r}}{r^5}$$

Now, $\nabla\left(\frac{1}{r^3}\right) = \nabla\left(r^{-3}\right)$

$$= \hat{i}\frac{\partial}{\partial x}r^{-3} + \hat{j}\frac{\partial}{\partial y}r^{-3} + \hat{k}\frac{\partial}{\partial z}r^{-3}$$

$$\nabla\left(\frac{1}{r^3}\right) = \bar{i}\,(-3)\,r^{-4}\frac{\partial r}{\partial x} + \bar{j}\,(-3)\,r^{-4}\frac{\partial r}{\partial y} + \bar{k}\,(-3)\,r^{-4}\frac{\partial r}{\partial z}$$

$$= -3\bar{i}\,r^{-4}\frac{x}{r} - 3\bar{j}\,r^{-4}\frac{y}{r} - 3\bar{k}\,r^{-4}\frac{z}{r}$$

$$= -3r^{-5}\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^5}\,\bar{r}$$ //.

(11). Prove that $\nabla r^n = n r^{n-2} R$, where $R = x\bar{i} + y\bar{j} + z\bar{k}$

(12). If $\nabla u = 2r^4 R$, find u.

properties of partial Derivatives :-

(1). $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$

(2). If λ is a Constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$

(3). If $\bar{c}$ is a Constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$

(4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

(5). $\frac{\partial}{\partial t}(\bar{a} \cdot \bar{b}) = \frac{\partial \bar{a}}{\partial t} \cdot \bar{b} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial t}$

(6). $\frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$

(7). let $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$, where f_1, f_2, f_3 are differential Scalar functions of more than one variable. Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions).

Scalar point function :-

Consider a region in three dime-nsional Space. To each point $P(x,y,z)$, Suppose we associate a unique real number (called 'Scalar') say ϕ. This $\phi(x,y,z)$ is called a "Scallar point function" defined on the region. Similarly, if to each point $P(x,y,z)$ we associate a unique vector $\vec{f}(x,y,z)$, f is called a "vector point function."

Vector Differential operator :-

The vector differential operator ∇ (read as del) is defined as $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities know as "gradient", "divergence" and "Curl" involving this operator ∇.

~~we must~~ .

Gradient of a Scalar point function :-

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ (or) $\nabla\phi$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Properties :-

(1). If f and g are two Scalar functions then

$$\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$$

(2). The necessary and Sufficient Condition for a Scalar point function to be Constant is that

$$\nabla f = \bar{0}$$

(3). $\text{grad}(fg) = f(\text{grad} g) + g(\text{grad} f)$

(4). If c is a Constant, $\text{grad}(cf) = c(\text{grad} f)$

(5). $\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g}, \ (g \neq 0)$

(6). let $\bar{r} = xi + yj + zk$. then $d\bar{r} = dx\,i + dy\,j + dz\,k$

If ϕ is any Scalar point function, then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right) \cdot (i\,dx + j\,dy + k\,dz)$$

$$= \nabla\phi \cdot d\bar{r}$$

Directional Derivative :-

let $\phi(x, y, z)$ be a scalar function defined throughout some region of space. let this function have a value ϕ at a point 'p' whose position referred to the origin 'O' is $\overline{OP} = \bar{r}$

* The directional derivative of a scalar point function ϕ at a point $p(x, y, z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\phi = \bar{e} \cdot \nabla\phi$

* The Greatest value of directional derivative of ϕ at a point 'p' $= |\text{grad}\phi|$ at that point 'p'.

Gradient of a function of a function :-

let $v = f(u)$ where $u = u(x, y, z)$. Then

$$\nabla v = \nabla(f(u)) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(u)$$

$$= \bar{i}\frac{\partial}{\partial x} f(u) + \bar{j}\frac{\partial}{\partial y} f(u) + \bar{k}\frac{\partial}{\partial z} f(u)$$

$$\nabla(f(u)) = f'(u)\left\{\bar{i}\frac{\partial u}{\partial x} + \bar{j}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z}\right\}$$

problems:- (1). If $a = x+y+z$, $b = x^2+y^2+z^2$,
$c = xy+yz+zx$, prove that [grad a, grad b, grad c] = 0

Sol:- Given $a = x+y+z$, $b = x^2+y^2+z^2$

$c = xy+yz+zx$

$\therefore \frac{\partial a}{\partial x} = 1, \frac{\partial a}{\partial y} = 1, \frac{\partial a}{\partial z} = 1$

$\text{grad}\, a = \nabla a = \bar{i}\frac{\partial a}{\partial x} + \bar{j}\frac{\partial a}{\partial y} + \bar{k}\frac{\partial a}{\partial z}$

$\text{grad}\, a = \nabla a = \bar{i}+\bar{j}+\bar{k}$

and $\frac{\partial b}{\partial x} = 2x, \frac{\partial b}{\partial y} = 2y, \frac{\partial b}{\partial z} = 2z$

$\Rightarrow \text{grad}\, b = \bar{i}\frac{\partial b}{\partial x} + \bar{j}\frac{\partial b}{\partial y} + \bar{k}\frac{\partial c}{\partial z}$

$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

also $\frac{\partial c}{\partial x} = (y+z), \frac{\partial c}{\partial y} = (x+z), \frac{\partial c}{\partial z} = y+x$

$\Rightarrow \text{grad}\, c = \bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \bar{k}\frac{\partial c}{\partial z}$

$= (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$

Now $[\text{grad}\, a, \text{grad}\, b, \text{grad}\, c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$

$= 2y^2 + 2xy - 2xz - 2z^2$
$- 2xy - 2x^2 + 2x^2 + 2xz$
$- 2y^2 - 2yz + 2yz + 2z^2$

$= 0$

(2). Find the direction derivative of $f = xy + yz + zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1, 2, 0)$

Sol:- Given $f = xy + yz + zx$.

$$\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{f}}{|\bar{f}|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{9}}$$

$$\bar{e} = \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

$\therefore$ Directional derivative of f along the given direction $= \bar{e} \cdot \nabla f$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k}) \cdot (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$$

$$= \left(\frac{y+z}{3} + \frac{2x+2z}{3} + \frac{2y+2x}{3}\right)_{(1,2,0)}$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{6}{3}$$

$$= \frac{10}{3} \text{ //.}$$

Q) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a directional of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

Given

~~let $f(x,y,z)=xyz^2+xz$~~

let $f(x,y,z)=3xy^2+y-z=0$

let us find the unit normal $\bar{e}$ to the surface at $(0,1,1)$ then

$$\frac{\partial f}{\partial x}=3y^2,\ \frac{\partial f}{\partial y}=6xy+1,\ \frac{\partial f}{\partial z}=-1$$

$$\therefore \text{grad} f=\nabla f=\bar{i}(3y^2)+\bar{j}(6xy+1)+\bar{k}(-1)$$

$$=3y^2\,\bar{i}+(6xy+1)\bar{j}-\bar{k}$$

$$(\nabla f)_{(0,1,1)}=3\bar{i}+\bar{j}-\bar{k}=\bar{n}$$

$$\therefore \bar{e}=\frac{\bar{n}}{|\bar{n}|}=\frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{9+1+1}}=\frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}$$

let $g(x,y,z)=xyz^2+xz$ then

$$\frac{\partial g}{\partial x}=yz^2+z,\ \frac{\partial g}{\partial y}=xz^2,\ \frac{\partial g}{\partial z}=2xy+x$$

$$\nabla g=(yz^2+z)\bar{i}+(xz^2)\bar{j}+(2xy+x)\bar{k}$$

$$(\nabla g)_{(1,1,1)}=2\bar{i}+\bar{j}+3\bar{k}$$

Directional derivative of the given function in the direction of $\bar{a}$ at $(1,1,1) = \hat{e} \cdot \nabla g$

$$= \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}} \cdot 2\bar{i}+\bar{j}+3\bar{k}$$

$$= \frac{6}{\sqrt{11}} + \frac{1}{\sqrt{11}} - \frac{3}{\sqrt{11}}$$

$$= \frac{4}{\sqrt{11}} \text{ //.}$$

(4). Find the directional derivative of $\phi = xyz$ along the direction of the normal to the surface. $x^2z+y^2x+yz^2=3$ at the point $(1,1,1)$.

(5). Find the directional derivative of $2xy+z^2$ at $(1,-1,3)$ in the direction of $\bar{i}+2\bar{j}+3\bar{k}$.

(6). Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$

(7). Find the directional derivative of $f = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$.

Q). Find the direction derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P = (1,2,3)$ in the direction of the PQ where $Q = (5,0,4)$.

→ The position vectors of 'P' and 'Q' with respect to the origin are $\overline{OP} = i + 2j + 3k$ and $\overline{OQ} = 5i + 4k$

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = 4i - 2j + k$

let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$ then $\bar{e} = \dfrac{4i - 2j + k}{\sqrt{21}}$

$\text{grad} f = 2xi - 2yj + 4zk$

$\therefore$ The directional derivative of f at $P(1,2,3)$ in the direction of $\overline{PQ} = \bar{e} \cdot \nabla f$

$$= \left\{ \frac{1}{\sqrt{21}} (8x + 4y + 4z) \right\}_{(1,2,3)}$$

$$= \frac{1}{\sqrt{21}} (8 + 8 + 12)$$

$$= \frac{28}{\sqrt{21}} \; //$$

(9). Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at the point $(1,1,1)$.

Sol^n: Given let $f = xy^2+yz^2+zx^2$

$$\text{grad} f = y^2\bar{i} + z^2\bar{j} + x^2\bar{k}$$

$$\text{grad} f = \bar{i}(y^2+2xz) + \bar{j}(2xy+z^2) + \bar{k}(2yz+x^2)$$

at $(1,1,1)$

$$(\nabla f)_{(1,1,1)} = 3\bar{i} + 3\bar{j} + 3\bar{k}$$

let $\bar{r}$ be the position vector of any point on the curve $x=t$, $y=t^2$, $z=t^3$, then

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$\frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} \quad \text{at } (1,1,1)$$

$$= \bar{i} + 2\bar{j} + 3\bar{k}$$

we know that $\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve.

unit vector along the tangent $= \hat{t} = \dfrac{i+2j+3k}{\sqrt{1+4+9}}$

$$\hat{t} = \frac{i+2j+3k}{\sqrt{14}}$$

$\therefore$ Directional Derivative along the tangent

$$= \hat{t} \cdot \nabla f$$

$$= \frac{1}{\sqrt{14}}(i+2j+3k) \cdot 3i+3j+3k$$

$$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}} \;//$$

ii). Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at $(2,1,-1)$

iii). In what direction from the point $(-1,1,2)$ is the directional derivative of $\phi = xy^2z^3$ a maximum. What is the magnitude of this maximum?

iv). In what direction from $(3,1,-2)$ is the directional derivative of $f(x,y,z) = x^2y^2z^4$ maximum and what is its magnitude?

(13). Find a unit normal vector to the given Surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

(14). Find a unit normal vector to the Surface $x^2+y^2+2z^2=$ at the point $(2,2,3)$

(15). Find the values of 'a' and 'b' So that the Surfaces $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally at the point $(1,-1,2)$.

Soln. let the given Surface be

$$f(x,y,z)=ax^2-byz-(a+2)x=0 \longrightarrow (1)$$

and

$$g(x,y,z)=4x^2y+z^3-4=0 \longrightarrow (2)$$

given the two Surfaces meet at the point $(1,-1,2)$

Substituting the point in eq (1),

we get $(a-2b)-(a+2)=0$

$$-2b-2=0$$

$$b=-2/2=-1$$

Now $\frac{\partial f}{\partial x}=2ax-a-2=2ax-(a+2)$

$\frac{\partial f}{\partial y}=-bz$, and $\frac{\partial f}{\partial z}=-by$

$\nabla f = \text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}(2ax - a - 2) + \hat{j}(-bz) + \hat{k}(by)$

$(\nabla f)_{(1,-1,2)} = (2a - a - 2)\hat{i} - 2b\hat{j} + b\hat{k}$

$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k} = \bar{n}_1,$

normal vector to Surface 1.

also $\frac{\partial g}{\partial x} = 8xy, \frac{\partial g}{\partial y} = 4x^2, \frac{\partial g}{\partial z} = 3z^2$

$\therefore \nabla g = \text{grad} g = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$(\nabla g)_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k} = \bar{n}_2,$

normal vector to Surface 2.

Given the Surfaces f, g are orthogonal at the point $(1,-1,2)$.

$\therefore [\nabla f]\cdot[\nabla g] = 0$

$\Rightarrow ((a-2)\hat{i} - 2\hat{j} + \hat{k})\cdot(-8\hat{i} + 4\hat{j} + 12\hat{k}) = 0$

$\Rightarrow -8a + 16 - 8 + 12 = 0$

$\Rightarrow a = 5/2$

Hence $a = 5/2, b = 1$.

(16). Evaluate the angle b/w the normals to the Surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

(17). Find the angle b/w the normals to the Surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.

(18). Find the angle of intersection of the Spheres $x^2+y^2+z^2 = 29$ and $x^2+y^2+z^2+4x-6y-8z-47 = 0$ at the point $(4,-3,2)$.

(19). Find the angle b/w the Surfaces $x^2+y^2+z^2 = 9$ and x^2+y^2-3 at the point $(2,-1,2)$.

(20). If $\bar{a}$ is Constant vector then prove that

$$\text{grad}\,(\bar{a}\cdot\bar{r}) = \bar{a}$$

(21). If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ

Unit-[illegible]

Vector Integral Calculus

Introduction :-

Vector integral calculus extends the concepts of integral calculus to vector functions. It has applications in fluid flows, design of underwater transmission cables, heat flow in stars, study of satellites.

* Line integrals are useful in the calculation of work done by variable force along path in space and the rates at which fluids flow along curves (Circulation) and across boundaries (flux).

Line Integral :-

If a vector function is defined at every point on the curve 'C' from the point 'A' to point 'B' then the evaluation of integral of a vector function $\bar{f}$ along paths in $\bar{f}$, curve (or) path 'C' is called the line integral of a vector function '$\bar{f}$' and it is denoted by $\int_C \bar{f} \cdot d\bar{r}$

Here $\bar{f} = f_1 i + f_2 j + f_3 k$, $\bar{r} = xi + yj + zk$

$d\bar{r} = dx\,i + dy\,j + dz\,k$

$$\therefore \int_C \bar{f} \cdot d\bar{r} = \int_C (f_1 i + f_2 j + f_3 k) \cdot (dx\,i + dy\,j + dz\,k)$$

$$= \int_C f_1\,dx + f_2\,dy + f_3\,dz$$

A natural application of the line integral is to define the work done by a force $\bar{F}$ in moving a particle along a curve 'C' from point 'A' to point 'B' as

$$\text{Work done} = \int_A^B \bar{F} \cdot d\bar{r}$$

If $\bar{v}$ represents the velocity of a fluid particle and 'c' is a closed curve then the integral $\oint_c \bar{v} \cdot d\bar{r}$ is called circulation of a vector function $\bar{v}$.

* If $\oint_c \bar{v} \cdot d\bar{r} = 0$ then field $\bar{v}$ is called conservative.
* If $\bar{v}$ is a conservative field then the line integral from 'A' to 'B' is independent of path and it depends on the end points 'A' and 'B' only.
* If $\bar{v}$ is a conservative then there exist ϕ such that $\bar{v} = \nabla\phi$, ϕ is called scalar potential.

problems:

01. If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ Evaluate $\int \bar{F}\cdot d\bar{r}$, where [illegible] in the xy-plane $y = 2x^2$ [illegible]

Sol^n:

Given $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$

and the curve 'C' is $y = 2x^2$ in the xy-plane $\left(\text{i.e } z = 0 \Rightarrow dz = 0\right)$

we know that

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

Now $\bar{F}\cdot d\bar{r} = (3xy\,\bar{i} - y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2\,dy \qquad \left(\because \bar{i}\cdot\bar{i} = 1 = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k}\right)$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2\,dy \longrightarrow (1)$$

Given curve is $y = 2x^2 \Rightarrow dy = 4x\,dx$

and from the points $(0,0)$ & $(1,2)$

The limits 'x' varies from $x = 0$ to $x = 1$

Substitute above values in eq (1),

we have

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 3x(2x^2)\,dx - (2x^2)^2(4x)\,dx$$

$$= \int_0^1 6x^3\,dx - 4x^4\,4x\,dx$$

$$= \left(6\frac{x^4}{4} - \frac{16x^6}{6}\right)_0^1$$

$$\int \bar{F}\cdot d\bar{r} = \frac{6}{4} - \frac{16}{6} = \frac{36-64}{24} = \frac{-28}{24} = -7/6$$

(2)
Ans:-

Given force field

$\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ and

the points joining the straight line curve

$A(0,0,0)$ and $B(2,1,3)$

we know that

The work done by force $= \int_A^B \bar{F} \cdot d\bar{r} \longrightarrow (1)$

Here $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \bar{F} \cdot d\bar{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$

Now, the equation of line joining the points $A(0,0,0)$ and $B(2,1,3)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t \text{ (say)}$$

$x = 2t, \quad y = t, \quad z = 3t$

$dx = 2dt, \quad dy = dt, \quad dz = 3dt$

put $x=0 \Rightarrow t=0$ and $x=2 \Rightarrow t=1$

't' varies from '0' to 1

from eq (ii), we have

$$\int_A^B \bar{F}\cdot d\bar{r} = \int_A^B 3x^2 dx + (2xz-y)dy + z\,dz$$

$$= \int_0^1 3x^2 dx + (2xz-y)dy + z\,dz$$

$$= \int_0^1 6t^2\, 2dt + (2(2t)(3t)-t)dt + 3t(3dt)$$

$$= \int_0^1 (\cancel{24}t^2 + 12t^2 + 8t)\,dt = \int_0^1 (\underset{36}{\cancel{24}}t^2 + \cancel{8}t)\,dt$$

$$= \left(\frac{36t^3}{3} + \cancel{8}\frac{t^2}{2}\right)_0^1 = \cancel{\frac{12}{3} + \frac{20}{2} = \frac{24+60}{6}} = \cancel{\frac{84}{6}} = \frac{42}{3}$$

$$\int_A^B \bar{F}\cdot d\bar{r} = \cancel{\frac{42}{3}}\ 12+4 = 16$$

(3)
Ans –

Given force field $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

and the points are $A(0,0,0)$ and $B(1,1,1)$

we know that

The work done by force $= \int_A^B \bar{F}\cdot d\bar{r} \longrightarrow (1)$

Here $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\therefore \bar{F}\cdot d\bar{r} = (3x^2 - 6yz)dx + (2y+3xz)dy + (1-4xyz^2)dz$

Now, the give curve 'C' is $x=t,\ y=t^2,\ z=t^3$

$dx = dt,\quad dy = 2t\,dt,\quad dz = 3t^2 dt$

$\Rightarrow \bar F\cdot d\bar r = \left(3(t^2)-6(t^2)(t^3)\right)dt + \left(2t^2+3t(t^3)\right)2t\,dt$
$+ \left(1-4(t)(t^2)(t^6)\right)3t^2\,dt$

$= \left(3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{11}\right)dt$

$\bar F\cdot d\bar r = \left(6t^2 - 12t^{11} - 6t^5 + 6t^5 + 4t^3\right)dt$

t varies from 0 to 1

from eq (1) $\int_A^B \bar F\cdot d\bar r = \int_0^1 \left(6t^2 - 12t^{11} - 6t^5 + 6t^5 + 4t^3\right)dt$

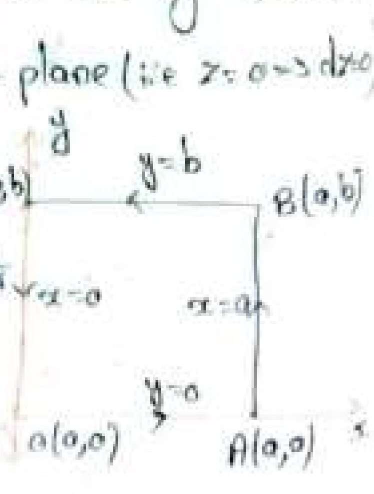
$= \left(\frac{6t^3}{3} - \frac{12t^{12}}{12} - \frac{6t^6}{6} + \frac{6t^6}{6} + \frac{4t^4}{4}\right)_0^1$

$= 2 - 1 - 1 + 1 + 1 = 2$

* If $\bar F = (x^2+y^2)\bar i - 2xy\bar j$, evaluate $\oint_C \bar F\cdot d\bar r$ where curve C is the rectangle in xy-plane bounded by $y=0$, $y=b$, $x=0$, $x=a$.

Sol^n:- Given $\bar F = (x^2+y^2)\bar i - 2xy\bar j$ and the points of the rectangular is $O(0,0)$, $A(a,0)$, $B(a,b)$, $C(0,b)$

here the curve 'C' is a rectangular having four (4) lines OA, AB, BC, CO in the xy-plane (i.e $z=0 \Rightarrow dz=0$)



Now $\oint_C \bar F\cdot d\bar r = \int_{OA}\bar F\cdot d\bar r + \int_{AB}\bar F\cdot d\bar r + \int_{BC}\bar F\cdot d\bar r + \int_{CO}\bar F\cdot d\bar r \longrightarrow (1)$

we know that

$\bar F\cdot d\bar r = f_1\,dx + f_2\,dy + f_3\,dz$

$\bar F\cdot d\bar r = (x^2+y^2)\,dx + (-2xy)\,dy$

Along OA: The points of the line joining are $O=(0,0)$ and $A=(a,0)$ from the points x varies from $x=0$ to $x=a$ and $y=0 \Rightarrow dy=0$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \frac{a^3}{3} \longrightarrow (2)$$

Along AB: The points of the line joining are $A=(a,0)$ and $B=(a,b)$ from the points y varies from $y=0$ to $y=b$ and $x=a \Rightarrow dx=0$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_{y=0}^{b} (a^2+y^2)0 + 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b = \cancel{2}a\,\frac{b^2}{\cancel{2}} = ab^2$$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = ab^2 \longrightarrow (3)$$

Along BC: The points of the line joining are $B=(a,b)$ and $C=(0,b)$ from the points x varies from $x=a$ to $x=0$ and $y=b \Rightarrow dy=0$

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_a^0 (x^2+b^2)\,dx + 0 = \left(\frac{x^3}{3}+b^2x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = -\frac{a^3}{3} - ab^2 \longrightarrow (4)$$

The points of the line joining are $C=(0,b)$ and $O=(0,0)$ from the points y varies from $y=b$ to $y=0$ and $x=0 \Rightarrow dx=0$

$$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_{CO} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_b^0 0 + 2 \cdot 0 \cdot y = \int_b^0 0 + 0$$

$$\int_{CO} \bar{F}\cdot d\bar{r} = 0 \longrightarrow (5)$$

From equation (2), eq (1), eq (3), eq (4) and eq (5)

we get

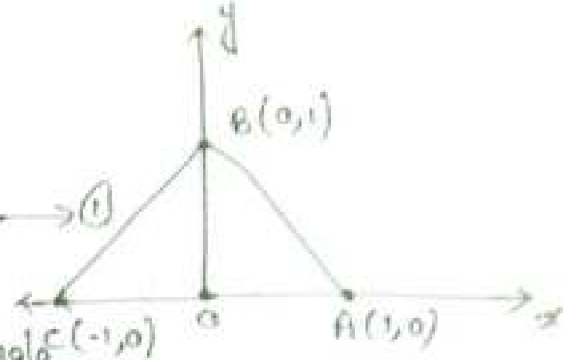
$$\oint_C \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + \cancel{ab^2} - \cancel{\frac{a^3}{3}} - \cancel{ab^2}$$

$$\oint_C \bar{F}\cdot d\bar{r} = 0$$

Hence $\bar{F}$ is Conservative field.

Evaluate the line integral $\int (y^2 dx - x^2 dy)$ around the triangle whose vertices are [illegible]

Sol^n:- Given



let $\int_C \bar{F}\cdot d\bar{r} = \int_C (y^2 dx - x^2 dy) \longrightarrow (1)$

and the points of the triangle

A(1,0), B(0,1) and C(-1,0)

Here the curve 'C' is a triangle having the lines AB, BC and CA.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CA} \bar{F}\cdot d\bar{r} \longrightarrow (2)$$

Line Integral along AB :-

The points are $A = \underset{x_1,\,y_1}{(1,0)}$, $B = \underset{x_2,\,y_2}{(0,1)}$

The line equation along AB is

$$y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$$

$$y - 0 = \frac{x - 1}{0 - 1}(1 - 0) \Rightarrow y = 1 - x$$

$$\Rightarrow dy = -dx.$$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (y^2 dx - x^2(-dx))$$

$$= \int_1^0 ((1-x)^2 + x^2)\,dx = \int_1^0 (1 + x^2 - 2x + x^2)\,dx$$

$$= \int_1^0 (1 + 2x^2 - 2x)\,dx = \left(x + \frac{2x^3}{3} - \frac{2x^2}{2}\right)_1^0$$

$$= -\cancel{1} - \frac{2}{3} + \cancel{1} = \cancel{-2} - \frac{2}{3} = -\frac{2}{3}$$

Sol:

Given that.

$\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} - (3x - 2y - 5z)\bar{k}$

and the path of integration is a circle in xy-plane i.e $x^2 + y^2 = 4$ and $z = 0 \Rightarrow dz = 0$

we know that

The circulation of vector function

i.e $\oint_C \bar{F} \cdot d\bar{r} = \oint_C F_1\,dx + F_2\,dy + F_3\,dz$

$= \oint_C (2x - y + 2z)dx + (x + y - z)dy$

$- (3x - 2y - 5z)dz \longrightarrow (1)$

given circle is

$x^2 + y^2 = 4$, Here $r = 2$

put $x = 2\cos\theta$, $y = 2\sin\theta$

then $dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$

The limits of integration θ varies from 0 to 2π (because of circle $360°$)

From eq (1)

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_0^{2\pi} \Big(2(2\cos\theta) - 2\sin\theta + 0\Big)(-2\sin\theta)\,d\theta$$

$$+ \Big(2\cos\theta + 2\sin\theta - 0\Big)(2\cos\theta)\,d\theta$$

$$- \Big(3(2\cos\theta) - 2(2\sin\theta) - 0\Big)0$$

$$= \oint_0^{2\pi} \Big\{ (4\cos\theta - 2\sin\theta)(-2\sin\theta) + 4\cos^2\theta + 4\sin\theta\cos\theta \Big\}\,d\theta$$

$$= \oint_0^{2\pi} \Big\{ -8\cos\theta\sin\theta + 4\sin^2\theta + 4(\cos^2\theta + \sin^2\theta) + 4\cos^2\theta + 4\sin\theta\cos\theta \Big\}\,d\theta$$

$$= \oint_0^{2\pi} \left(-8\cos\theta\sin\theta + 4\sin\theta\cos\theta\right)d\theta + 4\oint_0^{2\pi} d\theta$$

$$= \oint_0^{2\pi} -4\sin\theta\cos\theta\,d\theta + 4\oint_0^{2\pi} d\theta$$

$$= \int_0^{2\pi} -2\sin 2\theta\,d\theta + 4(\theta)_0^{2\pi}$$

$$= +2\left(\frac{\cos 2\theta}{2}\right)_0^{2\pi} + 4(2\pi)$$

$$= \cos 4\pi - \cos 0 + 8\pi$$

$$= 1 - 1 + 8\pi = 8\pi.$$

$$\oint_C \vec{F}\cdot d\vec{r} = 8\pi$$ //.

(7). Find the total workdone by the force $\vec{F} = 3xy\,\hat{i} - y\,\hat{j} + 2zx\,\hat{k}$ in moving a particle around the circle $x^2+y^2=4$

Ans: 0

(8). Find the workdone in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the curve defined by $x^2=4y$, $3x^3=8z$ from $x=0$ to $x=2$.

Ans: 16

(9). Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where 'c' is the square formed by the lines $y=\pm 1$ and $x=\pm 1$

An: 0

(10). If $\vec{F} = (5xy-6x^2)\hat{i} + (2y-4x)\hat{j}$, Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve 'C' in the xy-plane, $y=x^3$ from the point $(1,1)$ to $(2,8)$

Ans: 35

(11). Using line integral, Compute work done by the force $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$

Ans:- 303

(12). Using the line integral, find the work done by the force $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x=2t^2$, $y=t$, $z=t^3$

Ans:- $8\frac{8}{35}$

(13). Evaluate
helix x
(1,0,0)

(14). Evaluate
curve

Surface Integr

(13).
Soln:- Giv

and
$x = \cos$
$dx = -$
put x
put x
then, along
$\int_C \vec{F}\cdot d\vec{r}$

(13). Evaluate $\int_C (xy+z^2)\, d\bar{r}$ where C is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ which joins the points $(1,0,0)$ and $(-1,0,\pi)$.

(14). Evaluate $\int_C \bar{f} \cdot d\bar{r}$ where $\bar{f} = 2xy^2z + x^2y$ and 'c' is the curve $x = t$, $y = t^2$, $z = t^3$ from $t = 0$ to 1.

Ans: $\frac{19}{45}\bar{i} + \frac{11}{15}\bar{j} + \frac{75}{77}\bar{k}$

<u>Surface Integrals</u> :-

(13).

Sol^n:- Given that

$$\bar{f} \cdot d\bar{r} = (xy+z^2) \cdot d\bar{r}$$

$$= (xy+z^2)(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$\bar{f} \cdot d\bar{r} = (xy+z^2)\,dx\,\bar{i} + (xy+z^2)\,dy\,\bar{j} + (xy+z^2)\,dz\,\bar{k}$$

and given helix,

$x = \cos t$, $y = \sin t$, $z = t$

$dx = -\sin t\, dt$, $dy = \cos t\, dt$, $dz = dt$

put $x = 1 \Rightarrow 1 = \cos t$

$\cos 0 = \cos t \Rightarrow \boxed{t = 0}$

put $x = -1 \Rightarrow -1 = \cos t$

$\cos \pi = \cos t \Rightarrow \boxed{t = \pi}$

then, along the curve 'c', is

$$\int_C \bar{f} \cdot d\bar{r} = \int_{t=0}^{\pi} (\cos t \sin t + t^2)(-\sin t)\,dt\,\bar{i}$$

$$+ (\cos t \sin t + t^2)(\cos t\,dt)\,\bar{j}$$

$+ (\cos t \sin t + t^2)\, dt\, \bar{k}$

$$\int_C \bar{f}\cdot d\bar{r} = \int_0^{\pi} \{ \cos t \sin^2 t + (t^2 \sin t) \}\, dt\, \bar{i}$$

$$+ \int_0^{\pi} \{ \cos^2 t \sin t + t^2 \cos t \}\, dt\, \bar{j}$$

$$+ \int_0^{\pi} (\cos t \sin t + t^2)\, dt\, \bar{k}$$

$$\int_C \bar{f}\cdot d\bar{r} = \underbrace{\int_0^{\pi} -\cos t \sin^2 t\, dt\, \bar{i}}_{I_1} - \underbrace{\int_0^{\pi} t^2 \sin t\, dt\, \bar{i}}_{I_2}$$

$$+ \underbrace{\int_0^{\pi} \cos^2 t \sin t\, dt\, \bar{j}}_{I_3} + \underbrace{\int_0^{\pi} t^2 \cos t\, dt\, \bar{j}}_{I_4}$$

$$+ \underbrace{\int_0^{\pi} \cos t \sin t\, dt\, \bar{k}}_{I_5} + \underbrace{\int_0^{\pi} t^2\, dt\, \bar{k}}_{I_6} \longrightarrow (1)$$

Consider $I_1 = \int_0^{\pi} -\cos t \sin^2 t\, dt\, \bar{i} = \int_0 f(x) f^n(x)\, dx = \frac{f^{n+1}(x)}{n+1}$

$$I_1 = -\bar{i}\left(\frac{\sin^3 t}{3}\right)_0^{\pi} = 0$$

and $I_2 = +\int_0^{\pi} t^2 \sin t\, dt\, \bar{i}$

$$= -\left\{ \left(\sin t \frac{t^3}{3}\right)_0^{\pi} + \int_0^{\pi} 2t \cos t\, dt \right\} \bar{i}$$

$$-+ \left\{ (-t^2 \cos t)_0^{\pi} - \int_0^{\pi} 2t \right.$$

$$I_2 = \int_0^{\pi} t^2 \sin t \, dt \, \hat{i}$$

$$= \left\{ t^2 \int_0^{\pi} \sin t \, dt - \int_0^{\pi} \left(\frac{d}{dt} t^2 \int \sin t \, dt \right) dt \right\} \hat{i}$$

$$= \left\{ \left(t^2 (-\cos t) \right)_0^{\pi} + \int_0^{\pi} 2t \cos t \, dt \right\} \hat{i}$$

$$= \left(-\pi^2 \cos \pi - 0 \right) + 2 \left\{ t \int_0^{\pi} \cos t \, dt - \int_0^{\pi} \left(\frac{d}{dt} t \int \cos t \, dt \right) dt \right\} \hat{i}$$

$$= \left\{ \pi^2 + 2 \left\{ \left(t \sin t \right)_0^{\pi} - \left(-\cos t \right)_0^{\pi} \right\} \hat{i}$$

$$= \pi^2 + 2(0) + 2(\cos \pi - \cos 0) \, \hat{i}$$

$$I_2 = \left(\pi^2 + 0 - 2 - 2 \right) \hat{i} = (\pi^2 - 4) \hat{i}$$

$$I_3 = \int_0^{\pi} \cos^2 t \sin t \, dt \, \hat{j} = \left(\frac{\cos^3 t}{3} \right)_0^{\pi} = -\frac{1}{3} - \frac{1}{3} = \frac{-2}{3}$$

$$I_4 = \int_0^{\pi} t^2 \cos t \, dt \, \hat{j} = $$

$$= \left(t^2 \int_0^{\pi} \cos t \, dt - \int_0^{\pi} 2t \left(\sin t \right) dt \right\} \hat{j}$$

$$= \left(t^2 \sin t \right)_0^{\pi} - 2 \left\{ t \int_0^{\pi} \sin t \, dt - \int_0^{\pi} -\cos t \, dt \right\}$$

$$= (0 - 0) - 2 \left\{ \left(-t \cos t \right)_0^{\pi} + \left(\sin t \right)_0^{\pi} \right\} \hat{j}$$

$$= -2(\pi - 0) + (0 - 0)$$

$$= -2\pi$$

$$I_5 = \int_0^{\pi} \cos t \sin t \, dt \, \bar{k} = \left(\frac{\sin^2 t}{2}\right)_0^{\pi} = 0$$

$$I_6 = \int_0^{\pi} t^2 \, dt \, \bar{k} = \left(\frac{t^3}{3}\right)_0^{\pi} = \left(\frac{\pi^3}{3} - 0\right)\bar{k}$$

$$= \frac{\pi^3}{3}$$

Substituting $I_1, \ldots, I_6$ in eq ①

$$\int_C \bar{F} \cdot d\bar{r} = 0 - (\pi^2 - 4)\bar{i} + \left(-\frac{2}{3}\right)\bar{j} + (-2\pi)\bar{j} + 0 + \frac{\pi^3}{3}\bar{k}$$

$$\int_C \bar{F} \cdot d\bar{r} = -(\pi^2 - 4)\bar{i} - \left(\frac{2}{3} + 2\pi\right)\bar{j} + \frac{\pi^3}{3}\bar{k}$$

properties of partial Derivatives :-

(1). $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi\frac{\partial \bar{a}}{\partial t}$

(2). If λ is a Constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda\frac{\partial \bar{a}}{\partial t}$

(3). If $\bar{c}$ is a Constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$

(4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

(5). $\frac{\partial}{\partial t}(\bar{a}\cdot\bar{b}) = \frac{\partial \bar{a}}{\partial t}\cdot\bar{b} + \bar{a}\cdot\frac{\partial \bar{b}}{\partial t}$

(6). $\frac{\partial}{\partial t}(\bar{a}\times\bar{b}) = \frac{\partial \bar{a}}{\partial t}\times\bar{b} + \bar{a}\times\frac{\partial \bar{b}}{\partial t}$

(7). let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, where f_1, f_2, f_3 are differential Scalar functions of more than one variable. Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions).

Scalar point function :-

Consider a region in three dime-nsional Space. To each point $P(x,y,z)$, Suppose we associate a unique real number (called 'Scalar') say ϕ. This $\phi(x,y,z)$ is called a "Scalar point function" defined on the region. Similarly, if to each point $P(x,y,z)$ we associate a unique vector $\vec{f}(x,y,z)$, f is called a "vector point function."

Vector Differential operator :-

The vector differential operator ∇ (read as del) is defined as $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

This operator possesses properties analogous to those of ordinary vectors as well as different-iation operator. we will define now some quantities know as "gradient", "divergence", and "Curl" involving this operator ∇.

~~we meet~~ .

Gradient of a Scalar point function :-

let $\phi(x,y,z)$ be a Scalar point function of position defined in some region of Space. Then the vector function $i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ (or) $\nabla\phi$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Properties :-

(1). If f and g are two Scalar functions then

$\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$

(2). The necessary and Sufficient Condition for a Scalar point function to be Constant is that

$\nabla f = \bar{0}$

(3). $\text{grad}(fg) = f(\text{grad} g) + g(\text{grad} f)$

(4). If c is a Constant, $\text{grad}(cf) = c(\text{grad} f)$

(5). $\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g}$, $(g \neq 0)$

(6). let $\bar{r} = xi + yj + zk$. then $d\bar{r} = dx\,i + dy\,j + dz\,k$

If ϕ is any Scalar point function, then

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)$

$\cdot\,(i\,dx + j\,dy + k\,dz)$

$= \nabla\phi \cdot d\bar{r}$

Directional Derivative :-

let $\phi(x, y, z)$ be a Scalar function defined throughout Some region of Space. let this function have a value ϕ at a point 'p' whose position referred to the origin 'O' is $\overline{OP} = \overline{r}$

* The directional derivative of a Scalar point function ϕ at a point $p(x, y, z)$ in the direction of a unit vector $\overline{e}$ is equal to $\overline{e} \cdot \text{grad}\phi = \overline{e} \cdot \nabla\phi$

* The Greatest value of directional derivative of ϕ at a point 'p' $= |\text{grad}\phi|$ at that point 'p'.

Gradient of a function of a function :-

let $V = f(u)$ where $u = u(x, y, z)$. Then

$$\nabla V = \nabla(f(u)) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(u)$$

$$= \bar{i}\frac{\partial}{\partial x} f(u) + \bar{j}\frac{\partial}{\partial y} f(u) + \bar{k}\frac{\partial}{\partial z} f(u)$$

$$\nabla(f(u)) = f'(u)\left\{\bar{i}\frac{\partial u}{\partial x} + \bar{j}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z}\right\}$$

problems :- (1). If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$, prove that $[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = 0$

Sol :-

Given $a = x+y+z$, $b = x^2+y^2+z^2$

$c = xy+yz+zx$

$\therefore \frac{\partial a}{\partial x} = 1, \frac{\partial a}{\partial y} = 1, \frac{\partial a}{\partial z} = 1$

$\text{grad}\,a = \nabla a = \bar{i}\frac{\partial a}{\partial x} + \bar{j}\frac{\partial a}{\partial y} + \bar{k}\frac{\partial a}{\partial z}$

$\text{grad}\,a = \nabla a = \bar{i} + \bar{j} + \bar{k}$

and $\frac{\partial b}{\partial x} = 2x, \frac{\partial b}{\partial y} = 2y, \frac{\partial b}{\partial z} = 2z$

$\Rightarrow \text{grad}\,b = \bar{i}\frac{\partial b}{\partial x} + \bar{j}\frac{\partial b}{\partial y} + \bar{k}\frac{\partial c}{\partial z}$

$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

also $\frac{\partial c}{\partial x} = (y+z), \frac{\partial c}{\partial y} = (x+z), \frac{\partial c}{\partial z} = y+x$

$\Rightarrow \text{grad}\,c = \bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \bar{k}\frac{\partial c}{\partial z}$

$= (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$

Now $[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$

$= \cancel{2y^2} + \cancel{2xy} - \cancel{2xz} - \cancel{2z^2}$

$- \cancel{2xy} - \cancel{2x^2} + \cancel{2x^2} + \cancel{2xz}$

$- \cancel{2y^2} - \cancel{2yz} + \cancel{2yz} + \cancel{2z^2}$

$= 0$

(2). Find the direction derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point $(1,2,0)$

Sol:- Given $f = xy+yz+zx$.

$$\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$, then

$$\bar{e} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{f}}{|\bar{f}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{9}}$$

$$\bar{e} = \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

$\therefore$ Directional derivative of f along the given direction $= \bar{e}\cdot\nabla f$

$$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})\cdot(y+z)\bar{i}+(x+z)\bar{j} + (y+x)\bar{k}$$

$$= \left(\frac{y+z}{3} + \frac{2x+2z}{3} + \frac{2y+2x}{3}\right)_{(1,2,0)}$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{6}{3}$$

$$= \frac{10}{3} \text{ //.}$$

(2) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a directional of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

Given

~~let $f(x,y,z)=xyz^2+xz$~~

let $f(x,y,z)=3xy^2+y-z=0$

let us find the unit normal $\bar{e}$ to the surface at $(0,1,1)$ then

$$\frac{\partial f}{\partial x}=3y^2,\ \frac{\partial f}{\partial y}=6xy+1,\ \frac{\partial f}{\partial z}=-1$$

$$\therefore \text{grad} f=\nabla f=\bar{i}(3y^2)+\bar{j}(6xy+1)+\bar{k}(-1)$$

$$=3y^2\,\bar{i}+(6xy+1)\bar{j}-\bar{k}$$

$$(\nabla f)_{(0,1,1)}=3\bar{i}+\bar{j}-\bar{k}=\bar{n}$$

$$\therefore \bar{e}=\frac{\bar{n}}{|\bar{n}|}=\frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{9+1+1}}=\frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}$$

let $g(x,y,z)=xyz^2+xz$ then

$$\frac{\partial g}{\partial x}=yz^2+z,\ \frac{\partial g}{\partial y}=xz^2,\ \frac{\partial g}{\partial z}=2xyz+x$$

$$\nabla g=(yz^2+z)\bar{i}+(xz^2)\bar{j}+(2xyz+x)\bar{k}$$

$$(\nabla g)_{(1,1,1)}=2\bar{i}+\bar{j}+3\bar{k}$$

Directional derivative of the given function in the direction of $\bar{a}$ at $(1,1,1) = \hat{e} \cdot \nabla g$

$$= \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}} \cdot 2\bar{i}+\bar{j}+3\bar{k}$$

$$= \frac{6}{\sqrt{11}} + \frac{1}{\sqrt{11}} - \frac{3}{\sqrt{11}}$$

$$= \frac{4}{\sqrt{11}} \text{ //.}$$

(4). Find the directional derivative of $\phi = xyz$ along the direction of the normal to the surface. $x^2z + y^2x + yz^2 = 3$ at the point $(1,1,1)$.

(5). Find the directional derivative of $2xy + z^2$ at $(1,-1,3)$ in the direction of $\bar{i}+2\bar{j}+3\bar{k}$.

(6). Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$

(7). Find the directional derivative of $f = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$.

Q). Find the direction derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P = (1,2,3)$ in the direction of the PQ where $Q = (5,0,4)$.

A: The position vectors of 'P' and Q with respect to the origin are $\overline{OP} = i + 2j + 3k$ and

$\overline{OQ} = 5i + 4k$

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = 4i - 2j + k$

let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$ then $\bar{e} = \dfrac{4i - 2j + k}{\sqrt{21}}$

$\text{grad} f = 2xi - 2yj + 4zk$

$\therefore$ The directional derivative of f at $P(1,2,3)$ in the direction of $\overline{PQ} = \bar{e} \cdot \nabla f$

$$= \left\{ \frac{1}{\sqrt{21}} \left(8x + 4y + 4z \right) \right\}_{(1,2,3)}$$

$$= \frac{1}{\sqrt{21}} \left(8 + 8 + 12 \right)$$

$$= \frac{28}{\sqrt{21}}$$ //

(9). Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at the point $(1,1,1)$.

Sol^n: Given let $f=xy^2+yz^2+zx^2$

$$\text{grad} f = y^2\bar{i}+z^2\bar{j}+x^2\bar{k}$$

$$\text{grad} f = \bar{i}(y^2+2xz)+\bar{j}(2xy+z^2)+\bar{k}(2yz+x^2)$$

at $(1,1,1)$

$$(\nabla f)_{(1,1,1)} = 3\bar{i}+3\bar{j}+3\bar{k}$$

let $\bar{r}$ be the position vector of any point on the curve $x=t$, $y=t^2$, $z=t^3$, then

$$\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$$

$$\bar{r}=t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\frac{d\bar{r}}{dt}=\bar{i}+2t\bar{j}+3t^2\bar{k} \quad \text{at } (1,1,1)$$

$$=\bar{i}+2\bar{j}+3\bar{k}$$

we know that $\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve.

unit vector along the tangent $= \hat{t} = \dfrac{i+2j+3k}{\sqrt{1+4+9}}$

$\hat{t} = \dfrac{i+2j+3k}{\sqrt{14}}$

$\therefore$ Directional Derivative along the tangent

$= \hat{t} \cdot \nabla f$

$= \dfrac{1}{\sqrt{14}}\left(i+2j+3k\right) \cdot 3i+3j+3k$

$= \dfrac{3+6+9}{\sqrt{14}} = \dfrac{18}{\sqrt{14}}$ //

(i). Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at $(2,1,-1)$

(ii). In what direction from the point $(-1,1,2)$ is the directional derivative of $\phi = xy^2z^3$ a maximum. what is the magnitude of this maximum?

(iii). In what direction from $(3,1,-2)$ is the directional derivative of $f(x,y,z) = x^2y^2z^4$ maximum and what is its magnitude?

(13). Find a unit normal vector to the given surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

(14). Find a unit normal vector to the surface $x^2+y^2+2z^2=$ at the point $(2,2,3)$

(15). Find the values of 'a' and 'b' so that the surfaces $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally at the point $(1,-1,2)$.

Solⁿ. let the given surface be

$$f(x,y,z)=ax^2-byz-(a+2)x=0 \longrightarrow (1)$$

and

$$g(x,y,z)=4x^2y+z^3-4=0 \longrightarrow (2)$$

given the two surfaces meet at the point $(1,-1,2)$

Substituting the point in eq (1),

we get $(a-\cancel{2b})-(a+2)=0$

$$-2b-2=0$$

$$b=\frac{2}{2}=1$$

Now $\frac{\partial f}{\partial x}=2\cancel{a}x\ a-2=2ax-(a+2)$

$\frac{\partial f}{\partial y}=-bz$, and $\frac{\partial f}{\partial z}=-by$

$\nabla f = \text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}(2ax - a - 2) + \hat{j}(-bz) + \hat{k}(by)$

$(\nabla f)_{(1,-1,2)} = (2a - a - 2)\hat{i} - 2b\hat{j} + b\hat{k}$

$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k} = \bar{n}_1$,

normal vector to Surface 1.

also $\frac{\partial g}{\partial x} = 8xy$, $\frac{\partial g}{\partial y} = 4x^2$, $\frac{\partial g}{\partial z} = 3z^2$

$\therefore \nabla g = \text{grad} g = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$(\nabla g)_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k} = \bar{n}_2$,

normal vector to Surface 2.

Given the Surfaces f, g are orthogonal at the point $(1,-1,2)$.

$\therefore [\nabla f] \cdot [\nabla g] = 0$

$\Rightarrow ((a-2)\hat{i} - 2\hat{j} + \hat{k}) \cdot (-8\hat{i} + 4\hat{j} + 12\hat{k}) = 0$

$\Rightarrow -8a + 16 - 8 + 12 = 0$

$\Rightarrow a = 5/2$

Hence $a = 5/2$, $b = 1$.

(16). Evaluate the angle b/w the normals to the Surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

(17). Find the angle b/w the normals to the Surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.

(18). Find the angle of intersection of the Spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$.

(19). Find the angle b/w the Surfaces $x^2+y^2+z^2=9$ and x^2+y^2-3 at the point $(2,-1,2)$.

(20). If $\bar{a}$ is Constant vector then prove that

$$\text{grad}\,(\bar{a}\cdot\bar{r}) = \bar{a}$$

(21). If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ

Unit - V

Vector Integral Calculus

Introduction :-

Vector integral calculus extends the concepts of integral calculus to vector functions. It has application in fluid flows, design of underwater transmission cables, heat flow in stars, study of satellites.

* Line integrals are useful in the calculation of work done by variable force along path in space and the rates at which fluids flow along curves (Circulation) and across boundaries (flux).

Line Integral :-

If a vector function is defined at every point on the curve 'C' from the point 'A' to point 'B' then the evaluation of integral of a vector function $\bar{f}$ along paths in $\bar{f}$, curve (or) path 'C' is called the line integral of a vector function '$\bar{f}$' and it is denoted by $\int_C \bar{f} \cdot d\bar{r}$

Here $\bar{f} = f_1 i + f_2 j + f_3 k$, $\bar{r} = xi + yj + zk$

$d\bar{r} = dx\, i + dy\, j + dz\, k$

$$\int_C \bar{f} \cdot d\bar{r} = \int_C (f_1 i + f_2 j + f_3 k) \cdot (dx\, i + dy\, j + dz\, k)$$

$$= \int_C f_1\, dx + f_2\, dy + f_3\, dz$$

A natural application of the line integral is to define the work done by a force $\bar{F}$ in moving a particle along a curve 'C' from point 'A' to point 'B' as

$$\text{Work done} = \int_A^B \bar{F} \cdot d\bar{r}$$

If $\bar{v}$ represents the velocity of a fluid particle and 'c' is a closed curve then the integral $\oint_C \bar{v} \cdot d\bar{r}$ is called circulation of a vector function $\bar{v}$.

* If $\oint_C \bar{v} \cdot d\bar{r} = 0$ then field $\bar{v}$ is called conservative.
* If $\bar{v}$ is a conservative field then the line integral from 'A' to 'B' is independent of path and it depends on the end points 'A' and 'B' only.
* If $\bar{v}$ is a conservative then there exist ϕ such that $\bar{v} = \nabla\phi$, ϕ is called scalar potential.

problems:

If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ Evaluate $\int \bar{F}\cdot d\bar{r}$, where ... in the xy-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$

Solⁿ:

Given $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$

and the curve 'C' is $y = 2x^2$ in the xy-plane (i.e $z = 0$ $\Rightarrow dz = 0$)

we know that

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

Now $\bar{F}\cdot d\bar{r} = (3xy\,\bar{i} - y^2\,\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2\,dy \qquad (\because \bar{i}\cdot\bar{i} = 1 = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k})$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2\,dy \longrightarrow (1)$$

Given curve is $y = 2x^2 \Rightarrow dy = 4x\,dx$

and from the points $(0,0)$ & $(1,2)$

The limits x varies from $x = 0$ to $x = 1$

Substitute above values in eq (1),

we have

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 3x(2x^2)\,dx - (2x^2)^2(4x)\,dx$$

$$= \int_0^1 6x^3\,dx - 4x^4\,4x\,dx$$

$$= \left(6\frac{x^4}{4} - \frac{16x^6}{6}\right)_0^1$$

$$\int \bar{F}\cdot d\bar{r} = \frac{6}{4} - \frac{16}{6} = \frac{36-64}{24} = \frac{-28}{24} = -7/6$$

(2)

Ans:- Given force field

$\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ and

the points joining the straight line curve

$A(0,0,0)$ and $B(2,1,3)$

we know that

The work done by force $= \int_A^B \bar{F} \cdot d\bar{r} \longrightarrow (1)$

Here $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \bar{F} \cdot d\bar{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$

Now, the equation of line joining the points $A(0,0,0)$ and $B(2,1,3)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t \text{ (say)}$$

$x = 2t, \quad y = t, \quad z = 3t$

$dx = 2dt, \quad dy = dt, \quad dz = 3dt$

put $x=0 \Rightarrow t=0$ and $x=2 \Rightarrow t=1$

't' varies from 0 to 1

from eq (1), we have

$$\int_A^B \bar{F}\cdot d\bar{r} = \int_A^B 3x^2 dx + (2xz-y)\,dy + z\,dz$$

$$= \int_0^1 3x^2 dx + (2xz-y)\,dy + z\,dz$$

$$= \int_0^1 6t^2\, 2\,dt + (2(2t)(3t)-t)\,dt + 3t(3\,dt)$$

$$= \int_0^1 (24t^2 + 12t^2 + 8t)\,dt = \int_0^1 (\underset{36}{\cancel{24}}t^2 + \cancel{8}t)\,dt$$

$$= \left(\frac{36t^3}{3} + 8\frac{t^2}{2}\right)_0^1 = \cancel{\frac{12}{3} + \frac{20}{2} = \frac{24+60}{6}}$$

$$\cancel{= \frac{84}{6}/_3 = \frac{42}{3}}$$

$$\int_A^B \bar{F}\cdot d\bar{r} = \cancel{\frac{42}{3}}\ 12+4 = 16$$

(3)

Ans - Given force field $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

and the points are $A(0,0,0)$ and $B(1,1,1)$

we know that

The work done by force $= \int_A^B \bar{F}\cdot d\bar{r} \longrightarrow (1)$

Here $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

and $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \bar{F}\cdot d\bar{r} = (3x^2 - 6yz)\,dx + (2y+3xz)\,dy + (1-4xyz^2)\,dz$

Now, the give curve C: is $x=t,\ y=t^2,\ z=t^3$

$dx = dt,\quad dy = 2t\,dt,\quad dz = 3t^2\,dt$

$\Rightarrow \bar{F}\cdot d\bar{r} = \left(3(t^2) - 6(t^2)(t^3)\right)dt + \left(2t^2 + 3t(t^3)\right)2t\,dt$

$\quad + \left(1 - 4(t)(t^2)(t^6)\right)3t^2\,dt$

$= \left(3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{11}\right)dt$

$\bar{F}\cdot d\bar{r} = \left(6t^2 - 12t^{11} - 6t^5 + 6t^5 + 4t^3\right)dt$

t varies from 0 to 1

from eq (1) $\displaystyle\int_A^B \bar{F}\cdot d\bar{r} = \int_0^1 \left(6t^2 - 12t^{11} - 6t^5 + 6t^5 + 4t^3\right)dt$

$= \left(\frac{6t^3}{3} - \frac{12t^{12}}{12} - \frac{6t^4}{4} + 6\frac{t^6}{6} + 4\frac{t^4}{4}\right)_0^1$

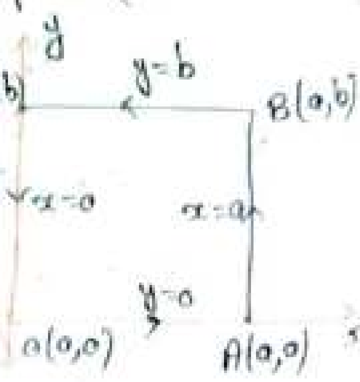
$= 2 - 1 - \frac{6}{4} + 1 + 1 = 2$

If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$, evaluate $\oint_C \bar{F}\cdot d\bar{r}$ where curve C is the rectangle in xy-plane bounded by $y=0$, $y=b$, $x=0$, $x=a$.

Solⁿ :- Given $\bar{F} = (x^2+y^2)\bar{i} + 2xy\bar{j}$ and the points of the rectangular is $O(0,0)$, $A(a,0)$, $B(a,b)$, $C(0,b)$

here the curve 'C' is a rectangular having four (4) lines OA, AB, BC, CO in the xy-plane (i.e $z=0 \Rightarrow dz=0$)

Now $\displaystyle\oint_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CO}\bar{F}\cdot d\bar{r}$ $\longrightarrow$ (1)



we know that

$\bar{F}\cdot d\bar{r} = f_1\,dx + f_2\,dy + f_3\,dz$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)\,dx + (2xy)\,dy$

Along OA: The points of the line joining are $O = (0,0)$ and $A = (a,0)$ from the points x varies from $x = 0$ to $x = a$ and $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \frac{a^3}{3} \longrightarrow (2)$$

Along AB: The points of the line joining are $A = (a,0)$ and $B = (a,b)$ from the points y varies from $y = 0$ to $y = b$ and $x = a \Rightarrow dx = 0$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2+y^2)\,dx + 2xy\,dy$$

$$= \int_{y=0}^{b} (a^2+y^2)0 + 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b = \cancel{2}a\frac{b^2}{\cancel{2}} = ab^2$$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = ab^2 \longrightarrow (3)$$

Along BC: The points of the line joining are $B = (a,b)$ and $C = (a,b)$ from the points x varies from $x = a$ to $x = 0$ and $y = b \Rightarrow dy = 0$

$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+y^2)\,dx + 2xy\,dy$

$= \int_a^0 (x^2+b^2)\,dx + 0 = \left(\frac{x^3}{3} + b^2x\right)_a^0$

$= -\frac{a^3}{3} - ab^2$

$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = -\frac{a^3}{3} - ab^2 \longrightarrow (4)$

The points of the line joining are $C=(0,b)$ and $O=(0,0)$ from the points y varies from $y=b$ to $y=0$ and $x=0 \Rightarrow dx=0$

$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_{CO} (x^2+y^2)\,dx + 2xy\,dy$

$= \int_b^0 0 + 2(0)y = \int_b^0 0+0$

$\int_{CO} \bar{F}\cdot d\bar{r} = 0 \longrightarrow (5)$

From equation (2), eq (1), eq (3), eq (4) and eq (5)

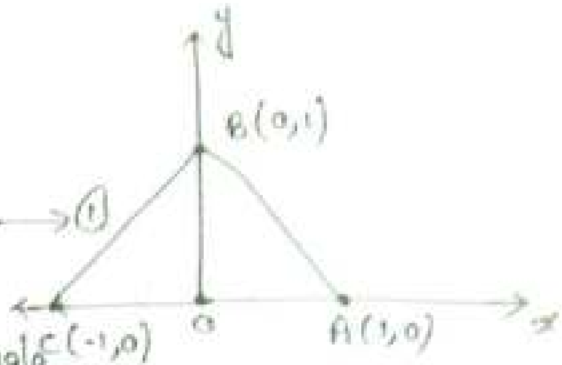
we get $\oint_C \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + \cancel{ab^2} - \cancel{\frac{a^3}{3}} - \cancel{ab^2}$

$\oint_C \bar{F}\cdot d\bar{r} = 0$

Hence $\bar{F}$ is Conservative field.

Sol:-

Given



let $\int_C \bar{F}\cdot d\bar{r} = \int_C (y^2 dx - x^2 dy) \longrightarrow (1)$

and the points of the triangle

$A(1,0)$, $B(0,1)$ and $C(-1,0)$

Here the curve 'C' is a triangle having the lines AB, BC and CA.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CA} \bar{F}\cdot d\bar{r} \longrightarrow (2)$$

Line Integral along AB :-

The points are $A = \underset{x_1, y_1}{(1,0)}$, $B = \underset{x_2, y_2}{(0,1)}$

The line equation along AB is

$$y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$$

$$y - 0 = \frac{x - 1}{0 - 1}(1 - 0) \Rightarrow y = 1 - x$$

$$\Rightarrow dy = -dx.$$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (y^2 dx - x^2(-dx))$$

$$= \int_1^0 ((1-x)^2 + x^2)\, dx = \int_1^0 (1 + x^2 - 2x + x^2)\, dx$$

$$= \int_1^0 (1 + 2x^2 - 2x)\, dx = \left(x + \frac{2x^3}{3} - \frac{2x^2}{2}\right)_1^0$$

Sol:

• Given that.

$\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} - (3x - 2y - 5z)\bar{k}$

and the path of integration is a Circle in xy - plane i.e $x^2 + y^2 = 4$ and $z = 0 \Rightarrow dz = 0$

we Know that

The Circulation of vector function

i.e $\oint_C \bar{F} \cdot d\bar{r} = \oint_C F_1\, dx + F_2\, dy + F_3\, dz$

$= \oint_C (2x - y + 2z)dx + (x + y - z)\, dy - (3x - 2y - 5z)\, dz$ $\longrightarrow (1)$

given Circle is

$x^2 + y^2 = 4$, Here $r = 2$

put $x = 2\cos\theta$, $y = 2\sin\theta$

then $dx = -2\sin\theta\, d\theta$, $dy = 2\cos\theta\, d\theta$

∴ the limits of integration θ varies from 0 to 2π (because of Circle $360°$)

From eq (1)

$$\oint_C F \cdot dr = \oint_0^{2\pi} \big(2(2\cos\theta) - 2\sin\theta + 0\big)(-2\sin\theta)\,d\theta$$

$$+ (2\cos\theta + 2\sin\theta - 0)(2\cos\theta)\,d\theta$$

$$- \big(3(2\cos\theta) - 2(2\sin\theta) - 0\big)\,0$$

$$= \oint_0^{2\pi} \Big\{ (4\cos\theta - 2\sin\theta)(-2\sin\theta) + 4\cos^2\theta + 4\sin\theta\cos\theta \Big\}\,d\theta$$

$$= \oint_0^{2\pi} \Big\{ -8\cos\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta \Big\}\,d\theta$$

$$= \oint_0^{2\pi} (-8\cos\theta\sin\theta + 4\sin\theta\cos\theta)\,d\theta + 4\oint_0^{2\pi} d\theta$$

$$= \oint_0^{2\pi} -4\sin\theta\cos\theta\,d\theta + 4\oint_0^{2\pi} d\theta$$

$$= \int_0^{2\pi} -2\sin 2\theta\,d\theta + 4(\theta)_0^{2\pi}$$

$$= +2\left(\frac{\cos 2\theta}{2}\right)_0^{2\pi} + 4(2\pi)$$

$$= \cos 4\pi - \cos 0 + 8\pi$$

$$= 1 - 1 + 8\pi = 8\pi.$$

$$\oint_C F \cdot dr = 8\pi \quad //$$

UNIT-5
Vector Integral Calculus

Introduction:- Vector integral calculus extends the concepts of integral calculus to vector functions. It has applications in fluid flow, design of underwater transmission cables, heat flow in stars, study of satellites. Line integrals are useful in the calculation of work done by variable forces along paths in space and the rates at which fluids flow along curves (circulation) and across boundaries (flux).

Line integral:-

If a vector function is defined at every point on the curve 'c' from the point A to point B then the evaluation of integral of a vector function "$\bar{F}$" along the curve (or) path 'c' is called the line integral of a vector function "$\bar{F}$" and it is denoted as

$$\int_C \bar{F}\cdot d\bar{r}$$

Here $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C F_1dx + F_2dy + F_3dz$$

Thus in a line integral, the integrand $\bar{F}$ is evaluated along a curve (or) line. The curve 'c' is known as path of integration.

Work done by a force:- A natural application of the line integral is to define the work done by a force $\bar{F}$ in moving a particle along a curve 'c' from point A to point B as

$$\text{work done} = \int_A^B \bar{F}\cdot d\bar{r}$$

Circulation:-

If $\bar{v}$ represents the velocity of a ~~(fluid)~~ fluid particle and 'c' is a closed curve then the integral $\oint_c \bar{v}\cdot d\bar{r}$ is called circulation of a vector function $\bar{v}$.

→ If $\oint_c \bar{v}\cdot d\bar{r} = 0$ then field $\bar{v}$ is called conservative

→ If $\bar{v}$ is a conservative field then the line integral from A to B is independent of path and it depends on the end points A and B only.

→ If $\bar{v}$ is a conservative then there exist ϕ such that $\bar{v} = \nabla\phi$. ϕ is called scalar potential.

Problems:

1) If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$. evaluate $\int_C \bar{F}\cdot d\bar{r}$, where 'c' is the curve in the xy-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$

Sol:- Given $F = 3xy\bar{i} - y^2\bar{j}$ and the curve 'c' is $y = 2x^2$ in the xy-plane $(z=0 \Rightarrow dz=0)$

w.k.T $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

Now $\bar{F}\cdot d\bar{r} = (3xy\,\bar{i} - y^2\bar{j} + 0\bar{k})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2 dy \qquad (\because \bar{i}\cdot\bar{i} = 1 = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k})$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2 dy \longrightarrow (1)$$

The given curve is $y = 2x^2 \Rightarrow dy = 4x\,dx$

and from the points $(0,0)$ & $(1,2)$

x varies $x=0$ to $x=1$

substitute these values in Eq^n (1), we get

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 3x(2x^2)dx - (2x^2)^2 4x\,dx$$

$$= \int_{x=0}^{1} (6x^3 - 16x^5)dx = \left[\cancel{6}^{3}\frac{x^4}{\cancel{4}_2} - \cancel{16}^{8}\frac{x^6}{\cancel{6}_3}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = -7/6}$$

2) Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol:- Given force $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ and

the points joining the straight line are $A(0,0,0)$ and $B(2,1,3)$

W.K.T the work done by force $= \int_A^B \bar{F}\cdot d\bar{r} \longrightarrow ①$

Here $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ and $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$

$\bar{F}\cdot d\bar{r} = (3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k})\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$

$\bar{F}\cdot d\bar{r} = 3x^2dx + (2xz-y)dy + zdz$

Now the Eq^n of line joining the points $A\underset{x_1\,y_1\,z_1}{(0,0,0)}$ and $B\underset{x_2\,y_2\,z_2}{(2,1,3)}$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = (t \text{ Say})$$

$\Rightarrow \frac{x}{2}=t \;;\; \frac{y}{1}=t \;;\; \frac{z}{3}=t$

$x=2t \;;\; y=t \;;\; z=3t$

$dx=2dt \;;\; dy=dt \;;\; dz=3dt$

put $x=0$ in $t=\frac{x}{2}$

$\boxed{t=0}$

put $x=2$ in $t = x/2 = 2/2 = 1$

$\boxed{t=1}$

Subs these values in Eq^n ①, we get

$$\int_A^B \bar{F}\cdot d\bar{r} = \int_A^B 3x^2dx + (2xz-y)dy + zdz$$

$$= \int_{t=0}^1 3(2t)^2 2dt + (2\times 2t\times 3t - t)dt + 3t\times 3dt$$

$$= \int_{t=0}^1 (24t^2+12t^2-t+9t)dt = \int_{t=0}^1 (36t^2+8t)dt$$

$$\text{work done} = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = 12+4 = \underline{\underline{16}}$$

3) Find the work done by the force $\bar{F} = (3x^2-6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$ in moving a particle from the point $(0,0,0)$ to $(1,1,1)$ along the curve C; $x=t$, $y=t^2$, $z=t^3$

Sol: W.K.T the work done by force $= \int_c \bar{F}\cdot d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz \longrightarrow (1)$

Here $\bar{F} = (3x^2-6yz)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = f_1 dx + f_2 dy + f_3 dz$

$$= (3x^2-6yz)dx + (2y+3xz)dy + (1-4xyz^2)dz$$

The given points are $(0,0,0)$ & $(1,1,1)$ and the

the curve C: $x=t$; $y=t^2$; $z=t^3$

$dx = dt$; $dy = 2t\,dt$; $dz = 3t^2 dt$

put $x=0$ in $t=x$

$\boxed{t=0}$

put $x=1$ in $t=x$

$\boxed{t=1}$

Subs these values in Eqn (1), we get

$$\text{workdone} = \int_c \bar{F}\cdot d\bar{r} = \int_c (3x^2-6yz)dx + (2y+3xz)dy + (1-4xyz^2)dz$$

$$= \int_{t=0}^{1} (3t^2 - 6\times t^2\times t^3)dt + (2t^2+3\times t\times t^3)2t\,dt + (1-4t\times t^2\times t^3)3t\,dt$$

$$= \int_{t=0}^{1} (3t^2 - \cancel{6t^5} + 4t^3 + \cancel{6t^5} + 3t - 12t^7)dt$$

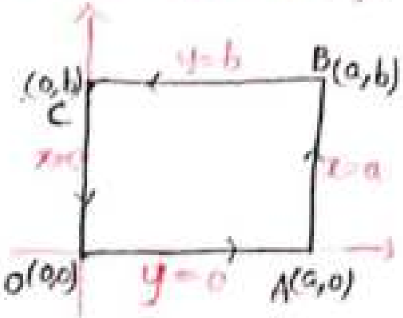
$$= \int_{t=0}^{1} (3t^2+4t^3+3t-12t^7)dt$$

$$= \left[\cancel{3}\frac{t^3}{\cancel{3}} + \cancel{4}\frac{t^4}{\cancel{4}} + 3\frac{t^2}{2} - \cancel{12}^{3}\frac{t^8}{\cancel{8}_{2}}\right]_0^1$$

$$= \left[1 + 1 + \frac{3}{\cancel{2}} - \frac{3}{\cancel{2}}\right] = 2$$

$$\boxed{\therefore \text{work done} = \int_c \bar{F}\cdot d\bar{r} = 2.}$$

4) If $\bar{F} = (x^2+y^2)\bar{i} + 2xy\bar{j}$. evaluate $\oint_c \bar{F}\cdot d\bar{r}$ where curve c is the rectangle in xy-plane bounded by $y=0, y=b, x=0, x=a$



Sol:- Given $\bar{F} = (x^2+y^2)\bar{i} + 2xy\bar{j}$ and the points of the rectangle is $O(0,0), A(a,0), B(a,b), C(0,b)$
($f_1 = x^2+y^2$, $f_2 = 2xy$)

Here the curve 'c' is a rectangle having '4' lines OA, AB, BC & CO in the xy-plane, i.e $z=0 \Rightarrow dz=0$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r} \longrightarrow (1)$$

Now $\bar{F}\cdot d\bar{r} = f_1\,dx + f_2\,dy + f_3\,dz$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx + 2xy\,dy$

Along OA:- The points of the line joining are $O=(0,0)$ and $A=(a,0)$ from the points x varies from $x=0$ to $x=a$ and $y=0 \Rightarrow dy=0$

$$\therefore \int_{OA} \bar{F}\,d\bar{r} = \int_{OA} (x^2+y^2)dx + 2xy\,dy = \int_{x=0}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \longrightarrow (2)$$

Along AB:- Along AB, $x=a \Rightarrow dx=0$ and the points $A=(a,0)$ & $B=(a,b)$ from the points y varies $y=0$ to $y=b$.

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2+y^2)dx + 2xy\,dy = \int_{y=0}^{b} (a^2+y^2)0 + 2ay\,dy = \int_{y=0}^{b} 2ay\,dy = \left[\cancel{2}a\,\frac{y^2}{\cancel{2}}\right]_0^b$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = ab^2 \longrightarrow (3)$$

Along BC:- Along BC, $y=b \Rightarrow dy=0$ and the point $B:(a,b)$ & $C=(0,b)$ from the points x varies $x=a$ to $x=0$

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+y^2)dx + 2xy\,dy = \int_{x=a}^{0} (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_a^0 = -\left[\frac{a^3}{3} + b^2a\right] \longrightarrow (4)$$

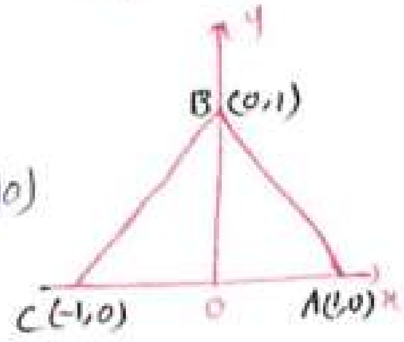
Along CO:- Along CO, $x=0 \Rightarrow dx=0$ and $C=(0,b)$, $O=(0,0)$, Here $y=b$ to $y=0$

$$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_{CO} (x^2+y^2)dx + 2xy\,dy = \int_{CO} (0+y^2)0 + 2\cdot 0\cdot y\,dy = 0 \longrightarrow (5)$$

from Eqn's (1), (2), (3), (4) & (5)

$$\boxed{\oint_c \bar{F}\,d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} - ab^2 = 0 \quad \text{Hence } \bar{F} \text{ is conservative field.}}$$

5) Evaluate the line integral $\int (y^2 dx - x^2 dy)$ about the triangle whose vertices are $(1,0)$, $(0,1)$ and $(-1,0)$



Sol:- Let $\int_c \bar{F}\cdot d\bar{r} = \int_c y^2 dx - x^2 dy \longrightarrow (1)$

and the points of the triangle $A(1,0)$, $B(0,1)$ & $C(-1,0)$

Here the curve 'c' is a triangle having the lines AB, BC, CA.

$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CA} \bar{F}\cdot d\bar{r} \longrightarrow (2)$

Line integral along AB:- The points are $A = \underset{x_1\,y_1}{(1,0)}$ & $B = \underset{x_2\,y_2}{(0,1)}$

The line Eq^n along AB is $y - y_1 = \dfrac{x - x_1}{x_2 - x_1}(y_2 - y_1)$

$y - 0 = \dfrac{x-1}{0-1}(1-0) \Rightarrow y = \dfrac{x-1}{-1} = 1 - x$

$dy = -dx$

$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} y^2 dx - x^2 dy$

$= \int_{x=1}^{0} (1-x)^2 dx - x^2(-dx) = \int_{x=1}^{0} (1 + x^2 - 2x + x^2) dx$

$= \int_{x=1}^{0} (1 + 2x^2 - 2x) dx = \left[x + 2\dfrac{x^3}{3} - 2\dfrac{x^2}{2}\right]_1^0 = \dfrac{-2}{3} \longrightarrow (3)$

Line integral along BC:-

The points are $B = \underset{x_1\,y_1}{(0,1)}$ & $C = \underset{x_2\,y_2}{(-1,0)}$

the line Eq^n with points BC is $y - y_1 = \dfrac{x - x_1}{x_2 - x_1}(y_2 - y_1)$

$y - 1 = \dfrac{x - 0}{-1 - 0}(0 - 1) \Rightarrow y - 1 = x$

$y = x + 1 \Rightarrow dy = dx.$

$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} y^2 dx - x^2 dy$

$= \int_{x=0}^{-1} (x+1)^2 dx - x^2 dx = \int_{x=0}^{-1} (\cancel{x^2} + 1 + 2x - \cancel{x^2}) dx = \left[x + 2\dfrac{x^2}{2}\right]_0^{-1} = -1 + (-1)^2 = 0 \longrightarrow (4)$

Line integral along CA: The points are $C = (-1,0)$ and $A = (1,0)$

from the points x varies from $x = -1$ to $x = 1$ and $y = 0 \Rightarrow dy = 0$

$\therefore \int_{CA} \bar{F}\cdot d\bar{r} = \int_{CA} y^2 dx - x^2 dy = \int_{x=-1}^{1} 0\, dx - x^2 \cdot 0 = 0 \longrightarrow (5)$

From Eq^n (1), (2), (3), (4) & (5) $\Rightarrow \int_c \bar{F}\cdot d\bar{r} = \int_c y^2 dx - x^2 dy$

$$\boxed{\int_c \bar{F}\cdot d\bar{r} = \dfrac{-2}{3} + 0 + 0 = \dfrac{-2}{3}}$$

6) Find the circulation of $\bar{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} - (3x-2y-5z)\bar{k}$ along the circle $x^2+y^2=4$ in the xy plane.

Sol:- Given that $\bar{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} - (3x-2y-5z)\bar{k}$

and the path of integration is a circle in xy- plane

i.e $x^2+y^2=4$ and $z=0 \Rightarrow dz=0$

W.K.T The circulation of a vector function.

$$\text{i.e} \quad \oint_C \bar{F}\cdot d\bar{r} = \oint_C F_1dx + F_2dy + F_3dz$$

$$= \oint_C (2x-y+2(0))dx + (x+y-0)dy + (-3x+2y+5z)0$$

$$\text{circulation} = \oint_C (2x-y)dx + (x+y)dy \longrightarrow 1$$

given circle is $x^2+y^2=4$, here $r=2$

put $x = 2\cos\theta$ and $y = 2\sin\theta$

then $dx = -2\sin\theta d\theta$; $dy = 2\cos\theta d\theta$ and the limits of integration $\theta=0$ to $\theta=2\pi$

Sub these values in Eq^n (1)

$$\text{circulation} = \int_{\theta=0}^{2\pi} (4\cos\theta - 2\sin\theta)(-2\sin\theta d\theta) + (2\cos\theta + 2\sin\theta)2\cos\theta d\theta$$

$$= \int_{\theta=0}^{2\pi} (-8\sin\theta\cdot\cos\theta + 4\sin^2\theta)d\theta + (4\cos^2\theta + 4\sin\theta\cos\theta)d\theta$$

$$= \int_0^{2\pi} (-4\sin 2\theta + 4\sin^2\theta + 4\cos^2\theta + 2\sin 2\theta)d\theta$$

$$= \int_0^{2\pi} (4 - 2\sin 2\theta)d\theta$$

$$= \left[4\theta - \cancel{2}\frac{-\cos 2\theta}{\cancel{2}}\right]_0^{2\pi} = (4\times 2\pi + \cos 4\pi) - (0 + \cos 0)$$

$$= 8\pi + \cancel{1} - \cancel{1} = 8\pi$$

$$\boxed{\therefore \text{ circulation} = 8\pi}$$

7) Find the total workdone by the force $\bar{F} = 3xy\bar{i} - y\bar{j} + 2zx\bar{k}$ in moving a particle around the circle $x^2+y^2=4$

Sol:- Given that $\bar{F} = 3xy\bar{i} - y\bar{j} + 2zx\bar{k}$ and the curve c is a circle $x^2+y^2=4$

W.K.T workdone by force $= \int_c \bar{F}.d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz \longrightarrow (1)$

and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \bar{F}.d\bar{r} = F_1 dx + F_2 dy + F_3 dz = 3xy\,dx + (-y)dy + 2zx\,dz$

The given circle with radius $r=2$ is $x^2+y^2=4$ it is in xy-plane

put $x = 2\cos\theta$ and $y = 2\sin\theta$ $(z=0, dz=0)$

$dx = -2\sin\theta d\theta$; $dy = 2\cos\theta\, d\theta$.

The limits of integration are $\theta=0$ to $\theta=2\pi$ (comple circle)

Subs these values in Eq^n (1), we get

$$\text{workdone} = \int_c \bar{F}.d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz$$

$$= \int_c 3xy\,dx + (-y)dy + 2zx\,dz$$

$$= \int_{\theta=0}^{2\pi} 3\times2\cos\theta\times2\sin\theta.(-2\sin\theta d\theta) - 2\sin\theta\times2\cos\theta d\theta$$

$$= \int_{\theta=0}^{2\pi} (-24\cos\theta.\sin^2\theta - 4\sin\theta.\cos\theta)d\theta$$

$$= -24\int_{\theta=0}^{2\pi} \underset{f(x)}{\cos\theta}.\underset{f(x)^n}{(\sin\theta)^2}d\theta - 2\int_{\theta=0}^{2\pi} \sin2\theta\, d\theta$$

$$= -24\left[\frac{(\sin\theta)^3}{3}\right]_0^{2\pi} - \cancel{2}\left[\frac{-\cos2\theta}{\cancel{2}}\right]_0^{2\pi}$$

$$\text{work done} = 0 + (\cos4\pi - \cos0) = 1-1 = 0$$

Problems :-

1) Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$. $\rightarrow$ Ans: 16

2) Evaluate the line integral $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where 'c' is the square formed by the lines $y = \pm 1$ and $x = \pm 1$ $\rightarrow$ Ans: '0'

3) If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the curve 'c' in the xy-plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$. $\longrightarrow$ 35

4) Evaluate $\int_C (xy + z^2) d\bar{r}$ where 'c' is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ which joins the points $(1,0,0)$ and $(-1,0,\pi)$.

Sol:- Given that $f \cdot d\bar{r} = (xy + z^2)d\bar{r}$

$$= (xy + z^2)(dx\bar{i} + dy\bar{j} + dz\bar{k})$$

and $x = \cos t$, $y = \sin t$ and $z = t$

$dx = -\sin t\, dt$; $dy = \cos t\, dt$; $dz = dt$

put $x = 1$ in $x = \cos t$: $1 = \cos t \Rightarrow \cos 0 = \cos t$ $\boxed{t = 0}$

put $x = -1$ in $-1 = \cos t$: $\cos \pi = \cos t \Rightarrow \boxed{t = \pi}$

then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k} = -\sin t\, dt\, \bar{i} + \cos t\, dt\, \bar{j} + dt\, \bar{k}$

$$d\bar{r} = (-\sin t\, \bar{i} + \cos t\, \bar{j} + \bar{k})dt$$

Now along the curve 'c', $f = (\sin t \cdot \cos t + t^2)$.

$$\therefore \int_C f \cdot d\bar{r} = \int_{t=0}^{\pi} (\sin t \cdot \cos t + t^2) \cdot (-\sin t\, \bar{i} + \cos t\, \bar{j} + \bar{k})dt$$

$$= \int_{t=0}^{\pi} \left[-\sin^2 t \cdot \cos t\, \bar{i}\, dt - t^2 \sin t\, \bar{i}\, dt + \sin t \cdot \cos^2 t\, \bar{j}\, dt + t^2 \cos t\, \bar{j}\, dt + \sin t \cdot \cos t\, \bar{k}\, dt + t^2 \bar{k}\, dt\right]$$

$$= \bar{i}\underbrace{\int_{t=0}^{\pi} (-\cos t)(\sin t)^2 dt}_{I_1} - \bar{i}\underbrace{\int_{t=0}^{\pi} t^2 \sin t\, dt}_{I_2} + \bar{j}\underbrace{\int_{t=0}^{\pi} \sin t (\cos t)^2 dt}_{I_3} + \bar{j}\underbrace{\int_0^{\pi} t^2 \cos t\, dt}_{I_4}$$

$$+ \bar{k}\underbrace{\int_0^{\pi} \frac{1}{2}\sin 2t\, dt}_{I_5} + \bar{k}\underbrace{\int_0^{\pi} t^2 dt}_{I_6}$$

(I_1: $f(x)$, $f(x)^n$; I_2: u, v; I_3: $f(x)$, $f(x)^n$; I_4: u, v)

Consider

$$I_1 = -\int_{t=0}^{\pi} (+\cos t)(\sin t)^2 dt$$

$f(x)$ $f(x)^n$

$$I_1 = -\left[\frac{(\sin t)^3}{3}\right]_0^{\pi} = 0$$

$$I_2 = \int_0^{\pi} t^2 \sin t\, dt = \left[t^2(\cos t) - 2t\,(-\sin t) + 2\cos t\right]_0^{\pi}$$

$u\ v_1\ -u'\ v_2\ +u''\ v_3$

$$= \left[(-\pi^2 \cos\pi + 0) + (2\pi \sin\pi - 0) + (2\cos\pi - 2\cos 0)\right]$$

$$= -\pi^2(-1) + 0 + 2(-1) - 2 \quad = \pi^2 - 4$$

$$I_3 = -\int_0^{\pi} (-\sin t)(\cos t)^2 dt = \left[\frac{(\cos t)^3}{3}\right]_0^{\pi} = \frac{1}{3}\left[(\cos\pi)^3 - (\cos 0)^3\right] = \frac{1}{3}(-1-1) = \frac{-2}{3}$$

$f'(x)\ \ f(x)^n$

$$I_4 = \int_0^{\pi} t^2 \cos t\, dt = \left[t^2 \sin t - 2t(-\cos t) + 2(-\sin t)\right]_0^{\pi}$$

$$= (\pi^2 \sin\pi - 0) + (2\pi\cos\pi - 0) - 2(\sin\pi - \sin 0)$$

$$= -2\pi$$

$$I_5 = \int_0^{\pi} \frac{1}{2}\sin 2t\, dt = \frac{1}{2}\left(-\frac{\cos 2t}{2}\right)_0^{\pi} = -\frac{1}{4}(\cos 2\pi - \cos 0) = \frac{-1}{4}(1-1) = 0$$

$$I_6 = \int_0^{\pi} t^6 dt = \left[\frac{t^7}{7}\right]_0^{\pi} = \frac{\pi^7}{7}$$

$$\therefore \int f\cdot d\bar{r} = \int (xy + z^2)\, d\bar{r} = [0 + (\pi^2 - 4)]\,\bar{i} + \left[-\frac{2}{3} - 2\pi\right]\bar{j} + \left[0 + \frac{\pi^7}{7}\right]\bar{k}$$

$$\boxed{\int (xy + z^2)\, d\bar{r} = (4 - \pi^2)\,\bar{i} + \left(-\frac{2}{3} - 2\pi\right)\bar{j} + \frac{\pi^7}{7}\bar{k}}$$

5) Evaluate $\int_C f\cdot d\bar{r}$ where $f = 2xy^2z + x^2y$ and 'c' is the curve $x = t,\ y = t^2,\ z = t^3$ from $t = 0$ to 1 $\rightarrow \int_c f\cdot d\bar{r} = \frac{19}{45}\bar{i} + \frac{11}{15}\bar{j} + \frac{75}{77}\bar{k}$

6) using line integral, compute workdone by the force $\bar{F} = 3xy\,\bar{i} - 5z\,\bar{j} + 10x\,\bar{k}$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$ $\rightarrow 303$

7) using the line integral, Find the work done by the force $\bar{F} = (2y+3)\,\bar{i} + xz\,\bar{j} + (yz - x)\,\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2,\ y = t,\ z = t^3$ $\rightarrow 8\frac{8}{35}$

Surface Integrals :-

If a vector function "$\bar{F}$" is define at every point on the surface 'S' then evaluation of integral of '$\bar{F}$' over the surface 'S' is called surface integral and it is denoted as

$$\boxed{\iint_S (\bar{F}\cdot\bar{n})\,ds} \quad (or) \quad \boxed{\int_S (\bar{F}\cdot\bar{n})\,ds},$$ where $\bar{n}$ is outward drawn unit normal vector to the surface

Evaluation of Surface integral :-

A surface integral is evaluated by reducing it to a double integral by projecting the given surface 'S' onto on of the co-ordinate planes.

★ Let R_1 be the projection of 'S' onto the xy-plane then

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\cdot\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} \quad (\text{first octant})$$

★ Let R_2 be the projection of 'S' onto the yz-plane then

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \bar{F}\cdot\bar{n}\cdot\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

★ Let R_3 be the projection of 'S' onto the zx-plane then

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \bar{F}\cdot\bar{n}\cdot\frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$$

where '$\bar{n}$' is ~~unit~~ outward drawn unit normal vector to the surface.

Surface Area of a curved Surface :- Let 'S' be a surface represented by the Equation $F(x,y,z)=0$, Then the unit normal to the surface 'S' is given by $\bar{n} = \frac{\nabla F}{|\nabla F|} = \frac{F_x\bar{i} + F_y\bar{j} + F_z\bar{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$, where F_x, F_y, F_z are the partial derivatives of 'F' w.r.to x, y, z.

Let R be the projection of S onto the xy plane then

Surface area of 's' $= \iint_S ds = \iint_R \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$= \iint_R \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}\cdot dx\,dy$$

Since $\bar{n}\cdot\bar{k} = \dfrac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}$

Flux :-

The normal component $\bar{F}\cdot\bar{n}$ is a scalar. Let 'ρ' be the density, $\bar{V}$ be the velocity of a fluid and $\bar{F} = \rho\bar{V}$. Then flux of $\bar{F}$ represents the total quantity of fluid flowing in unit time across the surface 's' in the positive direction.

The flux of $\bar{F}$ across 's' is given by the flux integral.

$$\boxed{\text{Flux of } \bar{F} \text{ across 's'} = \iint_S \bar{F}\cdot\bar{n}\,ds}$$

Problems:

1) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$ and 's' is the portion of the plane $2x+3y+6z=12$ in the first octant.

Sol:- Given that $\bar{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$

and the plane $\phi = 2x+3y+6z-12$ lies in the first octant

W.K.T the unit normal vector to the surface 's' is $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

i.e $\nabla\phi = \bar{i}\dfrac{\partial\phi}{\partial x} + \bar{j}\dfrac{\partial\phi}{\partial y} + \bar{k}\dfrac{\partial\phi}{\partial z}$

$$= \bar{i}\frac{\partial}{\partial x}(2x+3y+6z-12) + \bar{j}\frac{\partial}{\partial y}(2x+3y+6z-12) + \bar{k}\frac{\partial}{\partial z}(2x+3y+6z-12)$$

$\nabla\phi = \bar{i}(2) + \bar{j}(3) + \bar{k}(6)$ and $|\nabla\phi| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$

$$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k}) = \frac{2}{7}\bar{i} + \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}$$

Now $\bar{F}\cdot\bar{n} = (6z\bar{i} - 4\bar{j} + y\bar{k})\left(\frac{2}{7}\bar{i} + \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}\right) = \frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y$

$$\bar{F}\cdot\bar{n} = \frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y$$

∴ By Taking projection on xy-plane (first octant)

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{6/7} \longrightarrow (1)$$

where R is the region of projection of 'S' on xy-plane. R is bounded by x-axis, y axis and the lines $2x+3y=12$; $z=0$. In order to evaluate double integral y varies from 0 to 4 (on the y-axis $x=0 \Rightarrow 2x+3y=12$, $3y=12$, $y=4$)

and x varies from 0 to $x=\frac{12-3y}{2}$ (from $2x+3y=12$)

Subs these values in eqn (1)

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \left(\frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y\right)\frac{dx\,dy}{6/7}$$

$$= \iint_R \frac{\cancel{6}}{\cancel{7}}(2z-2+y)\times\frac{\cancel{7}}{\cancel{6}}\,dx\,dy$$

$$= \int_{y=0}^{4}\int_{x=0}^{\frac{12-3y}{2}} \left[2\left(2-\frac{x}{3}-\frac{y}{2}\right)-2+y\right]dx\,dy$$

$$= \int_{y=0}^{4}\int_{x=0}^{\frac{12-3y}{2}} \left[4-\frac{2}{3}x-\cancel{y}-2+\cancel{y}\right]dx\,dy$$

$$= \int_{y=0}^{4}\left[\int_{x=0}^{\frac{12-3y}{2}}\left(2-\frac{2}{3}x\right)dx\right]dy$$

$$= \int_{y=0}^{4}\left[2(x)-\frac{\cancel{2}}{3}\left(\frac{x^2}{\cancel{2}}\right)\right]_0^{\frac{12-3y}{2}}dy$$

$$= \int_{y=0}^{4}\left[\cancel{2}\left(\frac{12-3y}{\cancel{2}}\right)-\frac{1}{3}\left(\frac{12-3y}{2}\right)^2\right]dy$$

$$= \int_{y=0}^{4}\left[12-3y-\frac{1}{12}\left(144+9y^2-72y\right)\right]dy$$

$$= \int_{y=0}^{4}\left[12-3y-\frac{\cancel{9}^{3}}{\cancel{12}_{4}}\left(16+y^2-8y\right)\right]dy$$

$$= \left[12y-3\frac{y^2}{2}-\frac{3}{4}\left(16y+\frac{y^3}{3}-\cancel{8}^{4}\frac{y^2}{\cancel{2}}\right)\right]_0^4$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \left[12(4)-\frac{3}{2}(4)^2-\frac{3}{4}\left(16\times4+\frac{4^3}{3}-4^3\right)\right]$$

$$= 48-24-3\left(\cancel{16}+\frac{16}{3}-\cancel{16}\right)$$

$$= 48-24-16 = 8$$

$$\boxed{\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = 8}$$

2) Find $\iint_S \phi.\bar{n}\,ds$ where $\phi = \frac{3}{8}xzy$ and 'S' is the surface of cylinder $x^2+y^2=16$ included in the first octant between $z=0$ to $z=5$.

Sol:- Given that $\phi = \frac{3}{8}xzy$ and $x^2+y^2=16$

let $f = x^2+y^2-16$

W.K.T the unit normal vector to the surface 'S' is $\bar{n} = \frac{\nabla f}{|\nabla f|}$

i.e $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \bar{i}\frac{\partial(x^2+y^2-16)}{\partial x} + \bar{j}\frac{\partial(x^2+y^2-16)}{\partial y} + \bar{k}\frac{\partial(x^2+y^2-16)}{\partial z}$

$\nabla f = \bar{i}(2x)+\bar{j}(2y) \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2}$

$$\bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{i}+2y\bar{j}}{2\sqrt{x^2+y^2}} = \frac{2x\bar{i}+2y\bar{j}}{2\sqrt{16}} = \frac{x}{4}\bar{i}+\frac{y}{4}\bar{j}$$

Now $\phi.\bar{n} = \frac{3}{8}xzy\left(\frac{x}{4}\bar{i}+\frac{y}{4}\bar{j}\right) = \frac{3}{32}\left(x^2yz\,\bar{i}+xy^2z\,\bar{j}\right)$

$\therefore$ By taking projection on xz-plane,

$$\iint_S \phi\bar{n}\,ds = \iint_R \phi\bar{n}\,\frac{dx\,dz}{|\bar{n}.\bar{j}|} = \iint_R \phi\bar{n}\,\frac{dx\,dz}{(y/4)}$$

$\left[\bar{n}.\bar{j} = \left(\frac{x}{4}\bar{i}+\frac{y}{4}\bar{j}\right).\bar{j} = y/4\right]$

where R is the region of projection of 'S' on on xz-plane. R is bounded by x axis from $x=0$ to 4 and z-axis from $z=0$ to 5

Hence $\iint_S \phi\bar{n}\,ds = \iint_R \frac{3}{32}(x^2yz\,\bar{i}+xy^2z\,\bar{j}).\frac{dx\,dz}{y/4}$

$$= \iint_R \frac{3}{\cancel{32}_8}(x^2yz\,\bar{i}+xy^2z\,\bar{j})\frac{\cancel{4}}{y}\,dx\,dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4}\left(x^2z\,\bar{i}+xz\sqrt{16-x^2}\,\bar{j}\right)dx\,dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left[\int_{x=0}^{4}\left(x^2z\,\bar{i}-\frac{z}{2}(-2x)\sqrt{16-x^2}\,\bar{j}\right)dx\right]dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left[\frac{x^3}{3}.z\,\bar{i}-\frac{z}{\cancel{2}}\frac{(16-x^2)^{3/2}}{3/\cancel{2}}\,\bar{j}\right]_0^4 dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left[\frac{4^3}{3}.z\,\bar{i}+z\frac{(16)^{3/2}}{3}z\,\bar{j}\right]dz$$

$$\iint_S \phi\,\bar{n}\,ds = \frac{3}{8}\int_{z=0}^{5}\left[\frac{64}{3}\cdot z\,\bar{i} + \frac{64}{3}z\,\bar{j}\right]dz$$

$$= \frac{\cancel{3}}{8}\times\frac{\overset{8}{\cancel{64}}}{\cancel{3}}\left[\frac{z^2}{2}\bar{i} + \frac{z^2}{2}\bar{j}\right]_0^5$$

$$= 8\left[\frac{5^2}{2}\bar{i} + \frac{5^2}{2}\bar{j}\right]$$

$$= 4\left[25\bar{i} + 25\bar{j}\right]$$

$$\therefore \iint_S \phi\,\bar{n}\,ds = 100\bar{i} + 100\bar{j}$$

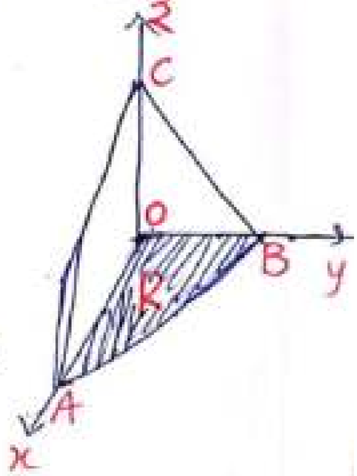
3) Find the flux of the vector field $\bar{A} = (x-2z)\bar{i} + (x+3y+z)\bar{j} + (5x+y)\bar{k}$ through the upper side of the triangle ABC with vertices at the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$

Sol:- Given that $\bar{A} = (x-2z)\bar{i} + (x+3y+z)\bar{j} + (5x+y)\bar{k}$ and the points are $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$

W.K.T the Eq^n of the plane passing through the three points.

i.e $x+y+z=1 \Rightarrow$ Let $\phi = x+y+z-1$

Now the unit normal vector $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|} = \dfrac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$



then $\bar{A}\cdot\bar{n} = \left[(x-2z)\bar{i} + (x+3y+z)\bar{j} + (5x+y)\bar{k}\right]\cdot\left(\dfrac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}\left[(x-2z) + (x+3y+z) + (5x+y)\right]$$

$$\bar{A}\cdot\bar{n} = \frac{1}{\sqrt{3}}(7x+4y-z)$$

Let AOB be the projection of ABC onto the xy-plane

then $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \sqrt{3}\,dx\,dy$.

$\therefore$ Flux across the triangle ABC $= \iint_S \bar{A}\cdot\bar{n}\,ds$

$$\Rightarrow \iint_S \bar{A}\cdot\bar{n}\,ds = \iint_{AOB} \bar{A}\cdot\bar{n}\,ds$$

$$= \iint_{AOB} \frac{7x+4y-3}{\sqrt{3}}\cdot\sqrt{3}\,dx\,dy$$

$$= \iint_{AOB} (7x+4y-3)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} [7x+4y-(1-x-y)]\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x}(8x+5y-1)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[8xy+5\frac{y^2}{2}-y\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left[8x(1-x)+\frac{5}{2}(1-x)^2-(1-x)\right] dx$$

$$= \int_{x=0}^{1}\left[8x-8x^2+\frac{5}{2}+\frac{5}{2}x^2 \cancel{-5x-1}+x\right] dx$$

$$= \int_{x=0}^{1}\left[\cancel{9x}\; 4x-\frac{\cancel{11}}{2}x^2+\frac{3}{2}\right] dx$$

$$= \left[\cancel{9}\,4\frac{x^2}{2}-\frac{\cancel{11}}{2}\frac{x^3}{3}+\frac{3}{2}x\right]_0^1 = \cancel{\frac{9}{2}}-\cancel{\frac{7}{2}}+\cancel{\frac{3}{2}} =$$

$$\iint_S \bar{A}\cdot\bar{n}\,ds = \left[2-\frac{11}{2}\times\frac{1}{3}+\frac{3}{2}\right] = \frac{12-11+9}{6} = \frac{21-11}{6} = \frac{10}{6} = \frac{5}{3}$$

$$\boxed{\therefore \iint_S \bar{A}\cdot\bar{n}\,ds = \frac{5}{3}}$$

Prob:- Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i}+x\bar{j}+3y^2z\bar{k}$ and 'S' is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Volume integrals:-

If `V` is the volume of the region bounded by closed surface `S` and $\phi(x,y,z)$, $\bar{F}(x,y,z)$ are defined at every point on the surface `S` then the volume integrals of $\phi(x,y,z)$ and $\bar{F}(x,y,z)$ are given by

$$\iiint_V \bar{F}\,dV \quad (\text{or}) \quad \iiint_V \phi\,dV$$

where $dV = dx\,dy\,dz$.

Problems:

1) If $\phi = xyz$, evaluate $\iiint_V \phi \cdot dv$ and `V` is volume of the region bounded by $x=0$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Sol:- Given that $\phi = xyz$ and $x=0, y=0, y=6, z=x^2, z=4$.

from $z=x^2 \Rightarrow x=\sqrt{z} \Rightarrow x=\sqrt{4} \Rightarrow \boxed{x=2}$

$$\therefore \iiint_V \phi \cdot dv = \iiint_V xyz\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=4}^{x^2} xyz\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[xy\int_{z=4}^{x^2} z\,dz\right]dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} xy\left(\frac{z^2}{2}\right)_4^{x^2} dx\,dy$$

$$= \int_0^2\int_0^6 \frac{xy}{2}\left[(x^2)^2 - 4^2\right]dx\,dy$$

$$= \int_0^2\int_0^6 \frac{xy}{2}\left[x^4 - 16\right]dx\,dy$$

$$= \int_{y=0}^{6}\left[\int_{x=0}^{2}\frac{y}{2}\left[x^5 - 16x\right]dx\right]dy$$

$$= \int_{y=0}^{6}\frac{y}{2}\left[\frac{x^6}{6} - \cancel{16}^{8}\frac{x^2}{\cancel{2}}\right]_0^2 dy = \int_{y=0}^{6}\frac{y}{2}\left[\frac{2^6}{6} - 8\times 2^2\right]dy$$

$$\iiint_V \phi\, dv = \int_{y=0}^{6} \frac{y}{2}\left[\frac{\cancel{64}^{32}}{\cancel{6}_3} - 32\right] dy$$

$$= \int_{y=0}^{6} \frac{y}{2}\left[\frac{32-96}{3}\right] dy$$

$$= \int_{y=0}^{6} \frac{y}{\cancel{2}}\left(-\frac{\cancel{64}^{32}}{3}\right) dy$$

$$= \frac{-32}{3}\left[\frac{y^2}{2}\right]_0^6 = \frac{-32}{3}\left[\frac{6^2}{2}\right] = \frac{-32}{\cancel{3}}(\cancel{18}^{6}) = -192$$

$$\boxed{\therefore \iiint_V \phi\, dv = 192}$$

2) If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$, evaluate $\iiint_V \text{div}\,\bar{F}\, dv$ where 'V' is the region bounded by the planes $x=0, y=0, x=2, y=6, z=x^2, z=4$.

Sol:- Given that $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ and the region bounded by

$x=0, x=2;\ y=0, y=6;\ z=x^2, z=4$

$$\therefore \iiint_V \bar{F}\, dv = \iiint_V (2xz\bar{i} - x\bar{j} + y^2\bar{k})\, dx\, dy\, dz$$

$$= \bar{i}\underbrace{\iiint_V 2xz\, dx\,dy\,dz}_{I_1} - \bar{j}\underbrace{\iiint_V x\, dx\,dy\,dz}_{I_2} + \bar{k}\underbrace{\iiint_V y^2\, dx\,dy\,dz}_{I_3}.$$

Let $I_1 = \iiint_V 2xz\, dx\,dy\,dz$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4} 2x \cdot z\, dz\right] dx\, dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} 2x\left[\frac{z^2}{\cancel{2}}\right]_{x^2}^{4} dx\, dy$$

$$= \int_0^2\int_0^6 x\,(4^2 - x^4)\, dx\, dy$$

$$I_1 = \int_0^2\int_0^6 (16x - x^5)\, dx\, dy$$

$$= \int_0^6 \left[\cancel{16}^{8}\frac{x^2}{\cancel{2}} - \frac{x^6}{6}\right]_0^2 dy$$

$$= \int_0^6 \left[8 \times 2^2 - \frac{2^6}{6}\right] dy$$

$$I_1 = \frac{64}{3}\left[y\right]_0^6 = \frac{64}{\cancel{3}} \times \cancel{6}^{2} = 64 \times 2 = 128$$

$$I_2 = \iiint_V x\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4} x\cdot 1\,dz\right]dx\,dy$$

$$= \int_{x=0}^{2}\int_{0}^{6} x\,[z]_{x^2}^{4}\,dx\,dy$$

$$= \int_0^2\int_0^6 (4x - x^3)\,dx\,dy$$

$$= \int_0^6 \left[4\frac{x^2}{2} - \frac{x^4}{4}\right]_0^2 dy$$

$$I_2 = \int_0^6 \left[2\times 2^2 - \frac{2^4}{4}\right]dy = 4\,[y]_0^6 = 4\times 6 = 24$$

$$I_3 = \iiint_V y^2\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[\int_{z=x^2}^{4} y^2\cdot 1\,dz\right]dx\,dy$$

$$= \int_0^2\int_0^6 y^2[z]_{x^2}^{4}\,dx\,dy$$

$$= \int_0^2\int_0^6 y^2(4-x^2)\,dx\,dy$$

$$= \int_0^6 y^2\left(4x - \frac{x^3}{3}\right)_0^2 dy$$

$$= \int_0^6 y^2\left[4\times 2 - \frac{2^3}{3}\right]dy$$

$$= \left(8 - \frac{8}{3}\right)\int_0^6 y^2\,dy$$

$$I_3 = \frac{24-8}{3}\left[\frac{y^3}{3}\right]_0^6 = \frac{16}{3}\left(\frac{6^3}{3}\right) = \frac{16}{3}\times\frac{216}{3} = 384$$

$$\therefore \iiint_V \bar{F}\,dv = I_1\bar{i} - I_2\bar{j} + I_3\bar{k}$$

$$\boxed{\iiint_V \bar{F}\,dv = 128\,\bar{i} - 24\,\bar{j} + 384\,\bar{k}}$$

★ 3) Evaluate $\iiint_V \phi\,dv$, where $\phi = 45x^2y$ and V is the region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Vector Integral Theorems

Green's Theorem:- Transformation between line integral and double integral (surface integral):-

Statement:- If R is a closed region in the xy-plane bounde by a simple closed curve 'c' and if $M(x,y)$ and $N(x,y)$ are continuous functions of 'x' and 'y' having continuous derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where 'c' is traversed in the positive direction and it is called. cartesian co-ordinates Green's Theorem.

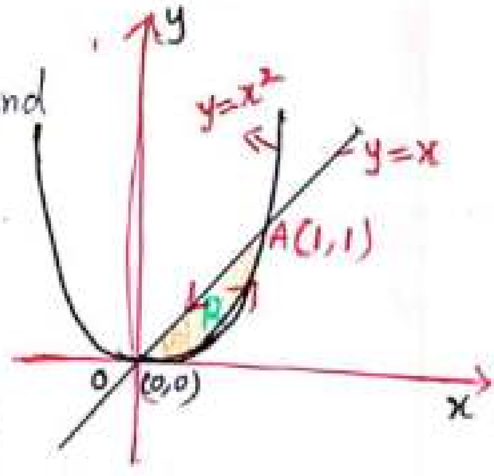
Note:- 1. Vector notation of Green's theorem is

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_R (\nabla\times\bar{A})\cdot\bar{K}\,d\bar{r}$$

2. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then by Green's theorem $\oint_C (M\,dx + N\,dy) = 0$

Problems:-

1) Verify Green's theorem in the plane for $\int_C (xy+y^2)dx + x^2 dy$ where 'c' is the closed curve of the region bounded by $y=x$ and $y=x^2$.



Sol:- Given integral is $\int_C (xy+y^2)dx + x^2 dy$ and

The curves $y = x$ — ①, $y = x^2$ — ②

W.K.T Green's theorem states that

Line integral = double integral

i.e $\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$

The line integral:-

a) The L.H.S of the Green's theorem result is the line integral.

i.e $\int_C (xy+y^2)dx + x^2 dy$, Here 'c' consists of the curves (or) lines OA and AO, So

$$\int_C (xy+y^2)dx + x^2dy = \int_{OA} (xy+y^2)dx + x^2dy + \int_{AO} (xy+y^2)dx + x^2dy \longrightarrow ③$$

The given curves are $y=x$ and $y=x^2$

Subn $y=x$ in $y=x^2$

$$x=x^2$$

$$x^2-x=0 \Rightarrow x(x-1)=0$$

$$x=0 \ \& \ x=1$$

put $x=0 \Rightarrow y=x$, $y=0$ | put $x=1$ in $y=x$, $y=1$

$\therefore$ The intersection points of the curves are $(0,0)$ and $(1,1)$

Along OA: Along OA the curve is $y=x^2$

$$dy=2x\,dx$$

and x varies $x=0$ to $x=1$

$$\therefore \int_{OA}(xy+y^2)dx+x^2dy = \int_{OA}[x.x^2+(x^2)^2]dx + x^2.2x\,dx$$

$$=\int_0^1 (x^3+x^4+2x^3)dx$$

$$\int_{OA}(xy+y^2)dx+x^2dy = \int_0^1(3x^3+x^4)dx = \left[\frac{3}{4}x^4+\frac{x^5}{5}\right]_0^1 = \frac{3}{4}+\frac{1}{5} = \frac{19}{20} \longrightarrow (4)$$

Along AO: Along AO the curve is $y=x \Rightarrow dy=dx$

and the limits of 'x' are $x=1$ and $x=0$

$$\therefore \int_{AO}(xy+y^2)dx+x^2dy = \int_{x=1}^{0}(x.x+x^2)dx+x^2dx$$

$$\int_{AO}(xy+y^2)dx+x^2dy = \int_{x=1}^{0}(2x^2+x^2)dx = \int_1^0 3x^2dx = \left[\cancel{3}\frac{x^3}{\cancel{3}}\right]_1^0 = -1 \longrightarrow (5)$$

$\therefore$ From Eq^n (3), (4) & (5), The line integral along 'c' is

$$\int_C (xy+y^2)dx+x^2dy = \frac{19}{20}-1 = -\frac{1}{20}$$

$$\boxed{\therefore \text{ The L.H.S } \int_C Mdx+Ndy = \frac{-1}{20}}$$

The double integral

b) The R.H.S of the Green's theorem is double integral

i.e $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

The given integral is $\int_C (xy + y^2)dx + x^2 dy$

it is of the form $\int_C M\, dx + N\, dy$

where $M = xy + y^2$; $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$; $\frac{\partial N}{\partial x} = 2x$

from Eqn ① $y = x$ and from Eqn ② $y = x^2$

and from the points $x=0$ and $x=1$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{x} [2x - (x+2y)]\, dx\, dy$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x - 2y)\, dy\right] dx$$

$$= \int_0^1 \left[xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_{x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\, dx$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 (x^4 - x^3)\, dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

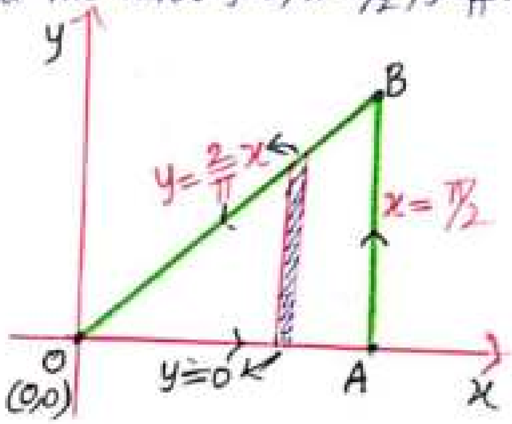
$\therefore$ L.H.S = R.H.S

i.e $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Hence Green's theorem is verified.

2. Apply Green's theorem to evaluate $\int_C (y-\sin x)dx + \cos x\,dy$, where 'C' is the triangle formed by $y=0$, $x=\pi/2$, $y=\frac{2}{\pi}x$.

Sol:- Given that $\int_C (y-\sin x)dx + \cos x\,dy$, and the lines $y=0$, $x=\pi/2$, $y=\frac{2}{\pi}x$



By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = y-\sin x$; $N = \cos x$

$$\frac{\partial M}{\partial y} = 1 \quad , \quad \frac{\partial N}{\partial x} = -\sin x$$

The curve 'c' composed of the line segments $y=0$, $x=\pi/2$ and $y=\frac{2}{\pi}x$ and the region R is the triangle bounded by these lines

from the region the limits of 'x' and 'y' are

x is from '0' to $\pi/2$

& y is from 0 to $\frac{2}{\pi}x$

$$\therefore \int_C (y-\sin x)dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2}{\pi}x} (-\sin x - 1)\,dx\,dy$$

$$= -\int_0^{\pi/2} (1+\sin x)\left[y\right]_0^{\frac{2}{\pi}x} dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} (x + x\sin x)\,dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - 1(-\sin x)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{1}{2}\frac{\pi^2}{4} - \frac{\pi}{2}\cos\pi/2 + \sin\pi/2\right]$$

$$= -\frac{2}{\pi}\left(\frac{1}{2}\frac{\pi^2}{4} + 1\right) = -\left(\pi/4 + \frac{2}{\pi}\right)$$

$$\boxed{\therefore \int_C (y-\sin x)dx + \cos x\,dy = -\left(\pi/4 + 2/\pi\right)}$$

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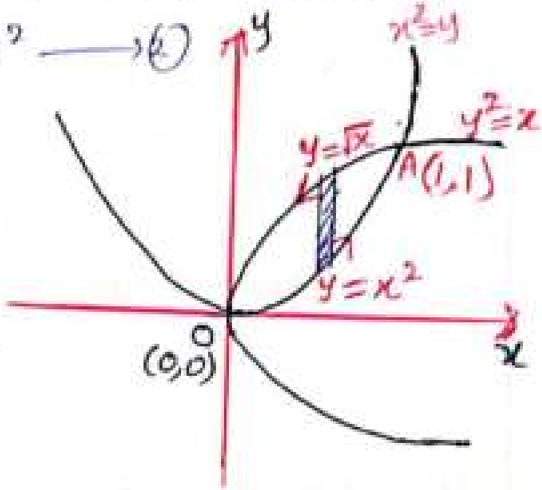
3) Verify Green's theorem in the plane $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$ where 'c' is the boundary of the region defined by $y=\sqrt{x}$, $y=x^2$

Sol:- Given that $\oint_C (3x^2-8y^2)\,dx+(4y-6xy)dy$. and the curves are parabolas $y=\sqrt{x}$ ⟶ ① and $y=x^2$ ⟶ ②
$y^2=x$



put Eqⁿ ① in Eqⁿ ②, we get
$$\sqrt{x}=x^2$$
$$x=x^4$$
$$x^4-x=0$$
$$x(x^3-1)=0$$
$$x=0,\ x=1$$

put $x=0$ in ① $y=0$
put $x=1$ in ① $y=1$ | ∴ The intersection points of ① & ② are (0,0) & (1,1)

By Green's theorem, we have
$$\int_C M\,dx+N\,dy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx.dy$$

a) L.H.S (line integral)

The L.H.S of the Green's theorem is the line integral

i.e $\int_C M\,dx+N\,dy=\int_C(3x^2-8y^2)dx+(4y-6xy)dy$

Here 'C' consists of the curves OA and AO

Along OA:- Along OA the curve is $y=x^2\Rightarrow dy=2x\,dx$ and the limits of x are $x=0$ to $x=1$

$$\therefore \int_{OA}(3x^2-8y^2)dx+(4y-6xy)dy=\int_{x=0}^{1}(3x^2-8x^4)dx+(4x^2-6x^3)\,2x\,dx$$
$$=\int_0^1(3x^2+8x^3-20x^4)dx$$
$$\int_{OA}M\,dx+N\,dy=\left[\cancel{3}\frac{x^3}{\cancel{3}}+\cancel{8}^{2}\frac{x^4}{\cancel{4}}-\cancel{20}^{4}\frac{x^5}{\cancel{5}}\right]_0^1=1+2-4=-1$$

Along AO:- Along AO the curve is $x=y^2 \Rightarrow dx = 2y\,dy$

and the limits of y are from '1' to '0'.

$$\therefore \int_{AO} M\,dx+N\,dy = \int_{AO} (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{y=1}^{0} (3y^4-8y^2)\,2y\,dy + (4y-6y^2 \cdot y)\,dy$$

$$= \int_{1}^{0} \left(6y^5-22y^3+4y\right) dy$$

$$= \left(6\frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right)_1^0$$

$$\int_{AO} M\,dx+N\,dy = -\left(1-\frac{11}{2}+2\right) = -\left(\frac{2-11+4}{2}\right) = \frac{5}{2}$$

$$\therefore \int_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AO} M\,dx+N\,dy$$

$$\boxed{\int_C M\,dx+N\,dy = -1+\frac{5}{2} = \frac{3}{2}}$$

b) The double integral:-

The R.H.S of the Green's theorem is double integral

i.e $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

The given integral is $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

Here $M = 3x^2-8y^2$; $N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y$; $\frac{\partial N}{\partial x} = -6y$

and the limits of integration are from Eqn ② $y=x^2$ and from ① $y=\sqrt{x}$

x limits are $x=0$ to $x=1$

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\, dx\,dy$

$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y \, dx\,dy$

$= \int_0^1 \left[\cancel{10}^{5} \frac{y^2}{\cancel{2}} \right]_{x^2}^{\sqrt{x}} dx$

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 5 \int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$

$\therefore$ L.H.S = R.H.S

i.e $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Hence Green's theorem is verified.

Prob 4) Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where 'c' is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$ $\rightarrow \frac{4}{3}a^3$

5) using Greens theorem, evaluate $\int_C (x^2 y\,dx + x^2 dy)$ where 'c' is the boundary described counter colock-wise of the triangle with vertices (0,0), (1,0) and (1,1). $\longrightarrow \frac{5}{12}$

6) Evaluate $\int_C (\cos x \sin y - xy)\,dx + (\sin x \cdot \cos y)\,dy$ by Green's theorem where 'c' is the circle $x^2 + y^2 = 1$

Gauss Divergence Theorem:- (Relation between Volume and Surface integral)

Statement: Suppose V is the volume bounded by a closed piecewise smooth surface 'S'. Suppose $\bar{F}(x,y,z)$ is a vector function which is continuous and has continuous first order partial derivatives in V.

Then $$\iiint_V \nabla\cdot\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds \quad (\text{or}) \quad \iiint_V \operatorname{div}\bar{F}\cdot dv = \iint_S \bar{F}\cdot\bar{n}\,ds$$

$$\Rightarrow \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint \bar{F}\cdot\bar{n}\,ds.$$

where $\bar{n}$ is the outward drawn unit normal to the surface 's'

Problems:-

1) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 4xy\bar{i} + yz\bar{j} - 3x\bar{k}$ and S is the surface of the cube bounded by the planes $x=0, x=2$, $y=0, y=2, z=0, z=2$.

Sol:- Given $\bar{F} = 4xy\bar{i} + yz\bar{j} - 3x\bar{k}$ and the planes

$x=0, x=2$; $y=0, y=2$; $z=0, z=2$

By Gauss Divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV = \iiint_V \left(\frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(-3x)\right) dV$$

$$= \iiint_V (4y + z - x)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} (4y+z-x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} \left[4yz + \frac{z^2}{2} - xz\right]_0^2 dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} \left[8y + 2 - 2x\right] dy\,dx = \int_{x=0}^{2} \left[\frac{8y^2}{2} + 2y - 2xy\right]_0^2 dx$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2} (20 - 4x)\,dx = \left[20x - 4\frac{x^2}{2}\right]_0^2 = (40-8) = 32$$

2) Verify Gauss divergence theorem for the surface $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ taken over the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

Sol:- Given that $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ and the planes

$$x=0, x=1, y=0, y=1, z=0, z=1$$

By Gauss divergence theorem, we have

$$\iiint_V \nabla\cdot\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds$$

L.H.S:

W.K.T $\nabla\cdot\bar{F} = \operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y$$

$$\nabla\cdot\bar{F} = 4z - y$$

and the limits of integration are $x=0, x=1$

$y=0, y=1$

& $z=0, z=1$

$$\therefore \iiint_V \nabla\cdot\bar{F}\,dv = \iiint_V (4z-y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dx\,dy\,dz$$

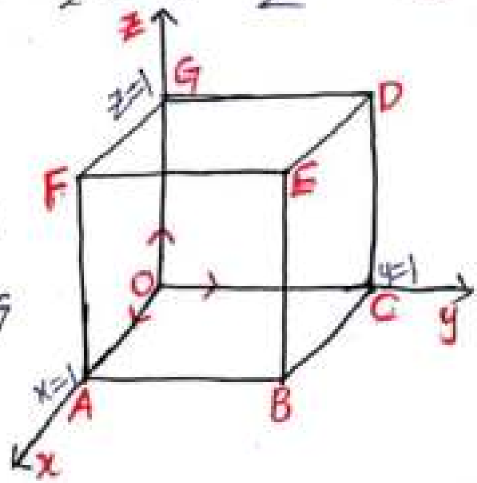
$$= \int_{x=0}^{1}\int_{y=0}^{1} \left(\cancel{4}^{2}\frac{z^2}{\cancel{2}} - yz\right)_0^1 dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1} (2-y)\,dy\right]dx$$

$$\iiint_V \nabla\cdot\bar{F}\,dv = \int_{x=0}^{1}\left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_{x=0}^{1}\left(2-\frac{1}{2}\right)dx = \frac{3}{2}[x]_0^1 = \frac{3}{2} \longrightarrow (1)$$

R.H.S:- Here we have to find $\iint_S \bar{F}\cdot\bar{n}\,ds$

Here 'S' is the surface of the cube bounded by the six plane surfaces OABC, BCDE, DEFG, AOGF, OCDG and ABEF



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i.e $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{OABC} \bar{F}\cdot\bar{n}\,ds + \iint_{BCDE} \bar{F}\cdot\bar{n}\,ds + \iint_{DEFG} \bar{F}\cdot\bar{n}\,ds + \iint_{AOGF} \bar{F}\cdot\bar{n}\,ds + \iint_{OCDG} \bar{F}\cdot\bar{n}\,ds + \iint_{ABEF} \bar{F}\cdot\bar{n}\,ds$

Along OABC :- over the face $z=0 \Rightarrow dz=0$, $x=0$ to $x=1$ & $y=0$ to $y=1$

$\bar{n}=-\bar{k}$ and $ds=dx\,dy$

$$\therefore \iint_{OABC} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{k})\,dx\,dy$$

$$\iint_{OABC} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} -yz\,dx\,dy = 0$$

over the face BCDE :- Here $y=1 \Rightarrow dy=0$, $\bar{n}=\bar{j}$ and $ds=dx\,dz$

The limits of integration is $x=0$ & $x=1$
$z=0$ & $z=1$

$$\therefore \iint_{BCDE} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1} (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{j}\,dx\,dz$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} -y^2\,dx\,dz$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} -1\,dx.dz$$

$$\iint_{BCDE} \bar{F}\cdot\bar{n}\,ds = -\int_{x=0}^{1} [z]_0^1\,dx = -\int_{x=0}^{1} 1.dx = -[x]_0^1 = -1$$

over the face DEFG :- Here $z=1 \Rightarrow dz=0$, $\bar{n}=\bar{k}$, $ds=dx\,dy$

The limits of integration $x=0$ to $x=1$
& $y=0$ to $y=1$

$$\therefore \iint_{DEFG} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{k}\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} yz\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{1} y\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{y^2}{2}\right]_0^1 dx$$

$$\iint_{DEFG} \bar{F}\cdot\bar{n}\,ds = \frac{1}{2}\int_0^1 1.dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

Over the face AOGF: Here $y=0 \Rightarrow dy=0$, $\bar{n}=-\bar{j}$, $ds=dx\,dz$

The limits of integration is $x=0$ to $x=1$

& $z=0$ to $z=1$

$$\therefore \iint_{AOGF} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1}(4xz\bar{i}-y^2\bar{j}+yz\bar{k})\cdot -\bar{j}\,dx\,dz$$

$$= \int_{x=0}^{1} -y^2\,dx\,dz = 0$$

Over the face OCDG: Here $x=0 \Rightarrow dx=0$, $\bar{n}=-\bar{i}$, $ds=dy\,dz$

The limits of integration is $y=0$ to $y=1$

& $z=0$ to $z=1$

$$\therefore \iint_{OCDG} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1}(4xz\bar{i}-y^2\bar{j}+yz\bar{k})\cdot -\bar{i}\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} -4xz\,dy\,dz = 0$$

Over the face ABEF:- Here $x=1 \Rightarrow dx=0$, $n=\bar{i}$, $ds=dy\,dz$

The limits of integration is $y=0$ to 1

$z=0$ to 1

$$\therefore \iint_{ABEF} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1}(4xz\bar{i}-y^2\bar{j}+yz\bar{k})\cdot\bar{i}\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} +4xz\,dy\,dz$$

$$\iint_{ABEF} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{1}\left[\int_{z=0}^{1} 4z\,dz\right]dy = \int_{y=0}^{1}\left[\frac{4z^2}{2}\right]_0^1 dy = \int_{y=0}^{1} 2\,dy = 2[y]_0^1 = 2$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{OABC} + \iint_{BDE} + \iint_{DEFG} + \iint_{AOGF} + \iint_{OCDG} + \iint_{ABEF}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 0-1+\frac{1}{2}+0+0+2 = \frac{3}{2} \longrightarrow (2)$$

$$\boxed{\therefore \iiint_V \nabla\cdot\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds = \frac{3}{2}}$$

Hence Gauss divergence theorem verified.

3) Use Gauss divergence theorem to evaluate the surface integral $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ where S is the portion of the plane $x+2y+3z=6$ which lies in the first octant

Sol:- Given that $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$

Here $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and $\bar{n} = dy\,dz\,\bar{i} + dz\,dx\,\bar{j} + dx\,dy\,\bar{k}$

In the first octant the values of $x, y \& z$ are 0

i.e $x=0, y=0, z=0$

& the given plane is $x+2y+3z=6$ —— ①

on the x-axis $y=0$ & $z=0 \Rightarrow$ ①

$x=6$

→ in the xy plane $z=0 \Rightarrow$ ① $x+2y=6$

$2y = 6-x$

$y = \frac{6-x}{2}$

→ from Eqⁿ ① $3z = 6-x-2y \Rightarrow z = \frac{6-x-2y}{3}$

∴ By Gauss divergence theorem $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)dv$$

$$= \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z}\right]dx\,dy\,dz$$

$$= \int_{x=0}^{6} \int_{y=0}^{\frac{6-x}{2}} \int_{z=0}^{\frac{6-x-2y}{3}} 3\,dx\,dy\,dz$$

$$= 3\int_{x=0}^{6} \int_{y=0}^{\frac{6-x}{2}} [z]_0^{\frac{6-x-2y}{3}}\,dx\,dy$$

$$= 3\int_{x=0}^{6} \int_{y=0}^{\frac{6-x}{2}} \frac{6-x-2y}{3}\,dy.dx$$

$$= \int_{x=0}^{6} \left[\int_{y=0}^{\frac{6-x}{2}} (6-x-2y)\,dy\right]dx$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{6}\left[(6-x)y - x\frac{y^2}{2}\right]_0^{\frac{6-x}{2}} dx$$

$$= \int_{x=0}^{6}\left[(6-x)\frac{(6-x)}{2} - \frac{(6-x)^2}{2^2}\right]dx$$

$$= \int_{x=0}^{6}\frac{(6-x)^2}{4}\,dx$$

$$= \frac{1}{4}\int_{x=0}^{6}(6-x)^2\,dx$$

$$= \frac{1}{4}\left[\frac{(6-x)^3}{-3}\right]_0^6$$

$$= \frac{-1}{12}\left[(6-6)^3 - (6-0)^3\right]$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \frac{-1}{12}\left[-216\right] = 18$$

$$\boxed{\therefore \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = 18}$$

4) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ with the help of Gauss theorem for $\bar{F} = 6z\bar{i} + (2x+y)\bar{j} - x\bar{k}$ ~~then~~ taken over the region 'S' bounded by the surface of the cylinder $x^2+z^2=9$ included between $x=0, y=0, z=0$ and $y=8$. $\longrightarrow 18\pi$

5) Verify divergence theorem for $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by the cylinder $x^2+y^2=4$, $z=0, z=3$ $\longrightarrow 84\pi$

Stoke's Theorem :-

Statement: If $\bar{F}$ is any continuously differentiable vector point function and 'S' is a surface bounded by a curve 'c', then

$$\int_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

(or)

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

where $\bar{n}$ is unit normal vector at any point of the surface 'S'

Prob: 1) Verify stoke's theorem for the function $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by $x=\pm a, y=0, y=b$

Sol: Given $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ and $x=\pm a, y=0, y=b$

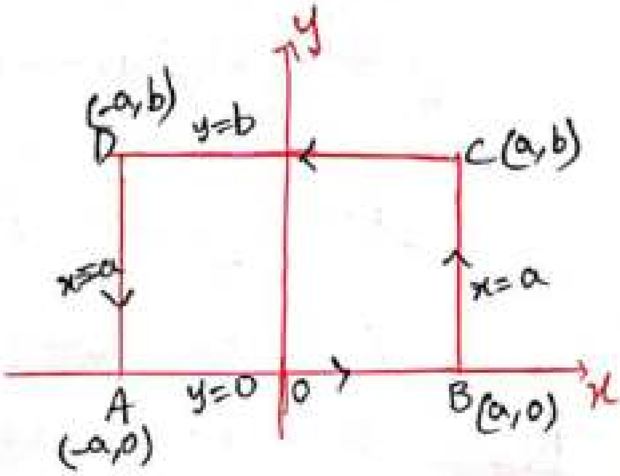
By stoke's theorem, Here we have to verify

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

W.K.T $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$

$$\bar{F}\cdot d\bar{r} = \left[(x^2+y^2)\bar{i} - 2xy\bar{j}\right]\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$$

$$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$$



L.H.S: In the L.H.S of stoke's theorem we have line integral

i.e $\int_C \bar{F}\cdot d\bar{r} = \int_C (x^2+y^2)dx - 2xy\,dy$

Here the region bounded by four lines i.e AB, BC, CD & DA

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CD}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r}$$

Along AB: Along AB the points are $A(-a,0)$ and $B(a,0)$

From the points $y=0 \Rightarrow dy=0$ and the x varies from $-a$ to a.

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{x=-a}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

Along BC: Here $B(a,0)$, $C(a,b)$

from the points $x=a$ and y varies 0 to b
$dx=0$

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{y=0}^{b} (a^2+y^2)0 - 2ay\,dy = \int_0^b -2ay\,dy = -\cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_0^b = -ab^2$$

Along CD: Here $C(a,b)$, $D(-a,b)$

from the points $y=b \Rightarrow dy=0$ and x varies from a to $-a$

$$\therefore \int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{x=a}^{-a} (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = \frac{-a^3}{3} - b^2a - \frac{a^3}{3} - b^2a = -2\left(\frac{a^3}{3}+b^2a\right)$$

Along DA: Here $D(-a,b)$, $A(-a,0)$

from the points $x=-a \Rightarrow dx=0$ and y varies from b to 0.

$$\therefore \int_{DA} \bar{F}\cdot d\bar{r} = \int_{DA} (x^2+y^2)dx - 2xy\,dy = \int_{y=b}^{0} -2(-a)y\,dy = \left[\cancel{2}a\frac{y^2}{\cancel{2}}\right]_b^0 = -ab^2$$

Hence $\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2b^2a - ab^2 = -4ab^2 \longrightarrow (1)$$

R.H.S: In the R.H.S of stoke's theorem, we have double integral

i.e $\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$, Here $\bar{F} = (x^2+y^2)\bar{i} - 2xy\,\bar{j}$

$$\text{Now curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = 0\bar{i} + 0\bar{j} + (-2y-2y)\bar{k} = -4y\bar{k}$$

The region bounded in xy-plane, so $\bar{n} = \bar{k}$, $ds = dx\,dy$ and the limits of integration are $x = -a$ to a

$y = 0$ to b

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \iint_S -4y\bar{k} \cdot \bar{k}\, dx\,dy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{b} -4y\, dy\, dx$$

$$= -\overset{2}{\cancel{4}} \int_{x=-a}^{a} \left[\frac{y^2}{\cancel{2}} \right]_0^b dx = -2b^2 \int_{-a}^{a} 1.dx = -2b^2 (x)_{-a}^{a}$$

$$= -4ab^2$$

$$\iint_S (\bar{\nabla} \times \bar{F}) \cdot \bar{n}\, ds = -4ab^2 \longrightarrow (2)$$

$\therefore$ from Eq^n (1) & (2) L.H.S = R.H.S

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = -4ab^2}$$

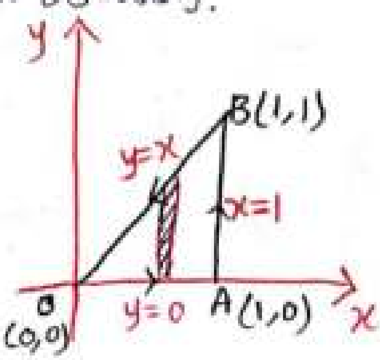
Hence stoke's theorem verified

2) Evaluate $\oint \bar{F} \cdot d\bar{r}$ by stoke's theorem, where $\bar{F} = y^2\bar{i} + x^2\bar{j} - (x+z)\bar{k}$ and 'c' is the boundary of the triangle with vertices at (0,0,0) (1,0,0) and (1,1,0).

Sol: Given $\bar{F} = y^2\bar{i} + x^2\bar{j} - (x+z)\bar{k}$ and (0,0,0), (1,0,0) and (1,1,0)

Since z-coordinates of each point of the triangle is zero, therefore the triangle lies in the xy-plane, so $\bar{n} = \bar{k}$ and $ds = dx\,dy$.

By the stoke's theorem we have



$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

The limits of integration is $x = 0$ to $x = 1$

and $y = 0$ to $y = x$

Now $\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$

$$= 0 + \bar{j}(1) + \bar{k}(2x-2y) = \bar{j} + 2(x-y)\bar{k}$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$$

$$= \iint_S \left(\bar{j} + 2(x-y)\bar{k}\right)\cdot\bar{k}\, dx dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\, dx\, dy$$

$$= 2\int_0^1 \left(xy - \frac{y^2}{2}\right)_0^x dx$$

$$\int_C \bar{F}\cdot d\bar{r} = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right)dx = \cancel{2}\int_0^1 \frac{x^2}{\cancel{2}}dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \frac{1}{3}}$$

3) Verify stoke's theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ where 's' is the upper half surface of the sphere $x^2+y^2+z^2=1$ and 'c' is its boundary.

Sol:- Given $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ and $x^2+y^2+z^2=1$

Since the boundary 'c' of 's' is a circle in the xy-plane of radius '1' and centre at the origin.

By stoke's theorem, we have

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$$

L.H.S: In the L.H.S we have a line integral

i.e $\int_C \bar{F}\cdot d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz$

Here the curve C is the circle $x^2+y^2=1$ and $z=0 \Rightarrow dz=0$

put $x=\cos\theta$, $y=\sin\theta$ and $\theta=0$ to 2π

$dx=-\sin\theta d\theta, dy=\cos\theta d\theta$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)dx - yz^2 dy - y^2 z dz$$

$$= \int_C (2x-y)dx$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta) - \sin\theta d\theta$$

$$= \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)d\theta$$

$$= \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2} - \cancel{2}\sin 2\theta\right)d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2} - \frac{-\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{2}}\cancel{2}\pi - \frac{1}{4}\sin 4\pi + \frac{1}{2}(\cos 4\pi - \cos 0)$$

$$\int_C \bar{F}\cdot d\bar{r} = \pi + \frac{1}{2}(1-1) = \pi \longrightarrow (1)$$

R.H.S: In the R.H.S of stoke's theorem, we have double integral

i.e $\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$, Here $\bar{F}=(2x-y)\bar{i} - yz^2\bar{j} - y^2 z\bar{k}$,

$$\text{Now } \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = \bar{i}(-2yz+2yz)+\bar{j}(0)+\bar{k}(1)$$

$$= \bar{k}$$

The surface of the sphere is upper half only, so $x^2+y^2=1, \bar{n}=\bar{k}$

and $ds=dxdy$, put $y=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

from $x^2+y^2=1 \Rightarrow y = \pm\sqrt{1-x^2} \Rightarrow y=-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$$\therefore \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_S \bar{k}\cdot\bar{k}\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\cdot dx\,dy$$

$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} 1\cdot dx\,dy$$

$$= 4\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= 4\left[\frac{1}{2}\sqrt{1-1}+\frac{1}{2}\sin^{-1}(1)-\frac{0}{2}\sqrt{1}+\frac{1}{2}\sin^{-1}(0)\right]$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = 4\left[\frac{1}{2}\,\frac{\pi}{2}\right] = \pi$$

$$\boxed{\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times F)\cdot\bar{n}\,ds = \pi}$$

Hence stokes theorum verified.

4) Verify stoke's theorum for the function $\bar{F} = x^2\bar{i} + xy\bar{j}$ integrated round the square whose sides are $x=0, y=0, x=a, y=a$ in the plane $z=0$

$\longrightarrow \frac{a^3}{2}$

5) Apply stoke's theorem evaluate $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$ where 'c' is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ $\longrightarrow 21$