Annamacharya Institute of Technology & Sciences: Tirupati

(Autonomous)

NUMERICAL METHODS

Subject Code: 20ABS9921

(AK20 Regulation)
(Common to AI&DS)

UNIT-1

10 Marks Questions

1. Find Relative maximum error in the function $u = \frac{5xy^2}{z^3}$ at $x = y = z = 1$ and $\Delta x = \Delta y = \Delta z = 0.001$
2. Find Relative maximum error in the function $u = xyz^2 + x^2y^2z$ at $x = 2, y = 4 \;\&\; z = 5$ and $\Delta x = \Delta y = \Delta z = 0.001$
3. Round off the numbers 865250 and 37.46235 to four significant figures and compute

E_A, E_R and E_P in each case (b) Round off the number X=2.2544 to 3 significant digits.

Compute the E_A,E_R and E_P in each case.

4. Find the sum of $\sqrt{6}+\sqrt{7}+\sqrt{8}$ correct to four significant digits. Find the Absolute error, Relative error and Percent Error.
5. Find the sum of (S) = $\sqrt{2}+\sqrt{5}+\sqrt{8}$ to 4 significant digits. Also find the Absolute error Relative error & Percent Error.
6. It was assumed that around 1,00,000 people would reach at a certain hill station in a month summer. But the exact number of people was 88,000. Calculate the Percentage error.

2 Marks

1. .Define Error and write types of Errors
2. Define types of Numerical Errors.
3. Define Absolute Error, Percent Error and Relative Error.
4. Find the Absolute error of the number 8.6 if both of digits are correct.

Unit-2

10 Marks Questions

1. Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in five stages.
2. Find out the square root of 25 given $x_0 = 2.0, x_1 = 7.0$ using bisection method.
3. Write the procedure for bisection method.
4. Write the procedure for newton-Raphson method.
5. Write the procedure for Regular False position method.
6. Find a root of the equation $x\log_{10}^{x} = 1.2$ using false position method.
7. Find a real root of the equation $e^x \sin x = 1$ using false position method.
8. Using Newton-Raphson method finds a positive root of $x^4 - x - 9 = 0$.
9. Using Newton-Raphson method finds a positive root of $xe^x - \cos x = 0$.
10. Solve the equations $2x + 3y + z = 9$, $x + 2y + 3z = 6$, $3x + y + 2z = 8$ factorization method.

11. Solve the following system of equations by triangularisation method $2x-3y+10z=3$, $-x+4y+2z=20$, $5x+2y+z=-12$.

12. Apply Gauss-Seidal iteration method to solve the equations $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$.

13. Solve the following system by Gauss-Seidal iteration method $10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=104$.

2 Marks Questions

14. Define Algebraic equation.
15. Define polynomial equation
16. Define Transcendental equation
17. Define root of an equation.
18. Write a formula of Newton-Raphson.
19. Write a formula of Bisection method.
20. Write the order of converges.
21. Write a formula of Regular False position.
22. Write the methods of iterative method based on number of guesses.
23. Write the merits of Newton Raphson method.
24. Write the De-merits of Newton Raphson method.

UNIT-3
(10 mark question)

1. State appropriate interpolation formula which is to be used to calculate the value of exp(1.75) from the following data and hence evaluate it from the given data

X	1.7	1.8	1.9	2.0
$Y=e^x$	5.474	6.050	6.686	7.389

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
F(x)	0.21	0.69	1.25	1.89	2.61

Obtain the value of f(x) when x=1.4.

3. For x=0,1,2,3,4; f(x)=1,14,15,5,6. Find f (3) using forward difference table.
4. The population of a town in the decimal census was given below. Estimate the population for the 1895 and 1925.

Year (x)	1891	1901	1911	1921	1931
Population y	46	66	81	93	101

5. Use Gauss's backward interpolation formula to find f (32) given that f(25)=0.2707, f(30)=0.3027, f(35)=0.3386, f(40)=0.3794.
6. Find y (25), given that $y_{20}=24$, $y_{24}=32$, $y_{28}=35$, $y_{32}=40$, using Gauss forward difference formula.
7. Using Gauss Backward difference formula, find y(8) from the following table.

X	0	5	10	15	20	25

y	7	11	14	18	24	32

8. Given f (2)=10, f(1)=8, f(0)=5, f(-1)=10 estimate f(1/2) by using Gauss's forward formula.
9. Find f(2.5) using the following table.

X	1	2	3	4
F(x)	1	8	27	64

10. Using Newton's interpolation formula, find the polynomial y=tan x satisfying the following data. Hence evaluate tan (0.12) and tan (0.28).

X	0.10	0.15	0.20	0.25	0.30
Y	0.1003	0.1511	0.2027	0.2533	0.3093

11. Use Stirling's formula to find y_9 given y_1=5225; y_6=4316; y_{11}=3256; y_{16}=1926; y_{21}=306.
12. Find the value of tan 16^0 from the table using Stirling's formula.

X	0	5	10	15	20	25
Tanx	0.00	0.0875	0.1763	0.2679	0.3640	0.4663

13. Use Stirling's formula to interpolate the value of y when x=13.8 from the following table:

X:	10	12	14	16	18
Y:	0.240	0.281	0.381	0.352	0.384

14. Use Bessel's formula to obtain y_{25} given y_{20}=24, y_{24}=32, y_{28}=35, y_{32}=40.
15. Using Stirling's formula find the polynomial of degree 4 which approximates the following data:

X	-2	-1	0	1	2
Y	19	3	1	1	15

16. Estimate f(1.2) using Bessel's formula, given

X	0.2	0.6	1.0	1.4	1.8
F(x)	0.39104	0.33322	0.24197	0.14973	0.07895

17. Apply Bessel's formula to find a polynomial that approximates the following data. And hence evaluate y(7).

X	4	6	8	10
Y	1	3	8	20

18. Evaluate f(10) given f(x)=168, 192, 336 at x=1,7,15 respectively. Use Lagrange interpolation.
    Using Lagrange's interpolation formula, find the value of y(10) from the following table:

X	5	6	9	11
Y	12	13	14	16

(or) Find y(10), Given that y(5)=12,y(6)=13, y(9)=14, y(11)=16 using Lagrange's formula.

19. Find the unique polynomial p(x) of degree 2 or less such that p(1)=1, p(3)=27, p(4)=64 using Lagrange interpolation formula.
20. Find the parabola passing through points (0,1), (1,3) and (3,55) using Lagrange's interpolation formula.
21. Given u_1=22, u_2=30, u_4=82,u_7=106, u_8=206, find u_6. Use Lagrange's interpolation formula.

Two marks questions

1. Write the Newton's forward and backward interpolation formulae.
2. Write the Gauss's forward and backward interpolation formulae.
3. Write the Stirling's and Bessel's formulae.
4. Write the Lagrange's interpolation formula.
5. Show that (a) $\Delta = E - 1$ (b) $\nabla = 1 - E^{-1}$ (c) $\Delta = \nabla E = E^{1/2}$ (d) $\mu^2 = 1 + \frac{1}{4}\delta^2$

6. Evaluate (a) $\Delta \cos x$ *(b)* $\Delta \tan^{-1} x$ *(c)* $\Delta^n e^{ax+b}$ (d) $\Delta \log f(x)$

7. If the interval of differencing is unity prove that $\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$

8. The following table gives set of values of x & y .then find $\Delta f(10)$.

x	10	15	20
f(x)	19.97	21.51	22.47

9.If $\mu_0 = 1, \mu_1 = 5, \mu_2 = 8, \mu_3 = 3$ *then*

10. Show that $\Delta f_i^2 = (f_i + f_{i+1})\Delta f_i$

UNIT-4

(10 mark questions)

1. Derive the normal equations to fit the parabola $y=a+bx+cx^2$.
2. Fit a second degree polynomial to the following data by the method of least squares:

X	10	12	15	23	20
Y	14	17	23	25	21

3. Fit a second degree polynomial to the following data by the method of least squares:

X	0	1	2	3	4
Y	1	1.8	1.3	2.5	6.3

4. Fit a polynomial of second degree to the data points given in the following table:

X	0	1.0	2.0
Y	1.0	6.0	17.0

5. Determine the constants a and b by the method of least squares such that $y=ae^{bx}$

X	2	4	6	8	10
Y	4.077	11.084	30.128	81.897	222.62

6. Find the curve of best fit of the type $y= ae^{bx}$ to the following data by the method of least squares.

X	1	5	7	9	12
Y	10	15	12	15	21

7. Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares.

X	2	3	4	5	6
Y	8.3	15.4	33.1	65.2	127.4

8. Fit the straight line to the following data

X:	0.0	0.2	0.4	0.6	0.8	1.3
Y:	-1.85	-1.20	-0.55	0.15	0.80	1.35

9. Fit a straight line to the following data.

X:	4	6	8	10	12
Y:	13.72	12.90	12.01	11.14	10.31

10. Fit a second degree parabola to the data

X:	0	1	2	3	4
Y:	1.0	1.8	1.3	2.5	6.3

11. For the below, find f'(1.76) and f'(1.72).

X:	1.72	1.73	1.74	1.75	1.76
F(x):	0.17907	0.17728	0.17552	0.17377	0.17204

12. Find the first and second derivatives of the function tabulated below at x=0.6.

X:	0.4	0.5	0.6	0.7	0.8
Y:	1.5836	1.7974	2.0442	2.3275	2.6511

13. Compute f'(1) using the data:

X	1.0	1.5	2.0	2.5	3.0
F(x)	27	106.75	324	783.75	1621

14. Using the table below, find f'(0) and $\int_0^9 f(x)dx$.

X:	0	2	3	4	7	9
F(x):	4	26	58	110	460	920

15. Given the following data, find f'(6).

X	0	2	3	4	7	9
Y	4	26	58	112	466	922

16. For the following data, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) x=1.1 (b) x=1.6.

X	1.0	1.1	1.2	1.3	1.4	1.5	1.6
Y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

17. Find the first and second derivatives of the function tabulated below, at the point
(a) X=1.2 (b) x=2.2 (c) x=1.6

X	1.0	1.2	1.4	1.6	1.8	2.0	2.2
F(x)	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0256

18. A slider, in a machine moves along a fixed straight rod. Its distance x cm .along the rod is given below for various values of the time t seconds. Find the velocity and acceleration of the slider when t=0.3 seconds.

t (sec)	0	0.1	0.2	0.3	0.4	0.5	0.6
x (cm)	30.13	31.62	32.87	33.64	33.95	33.81	33.24

19. Evaluate $\int_0^2 e^{-x^2}$ dx using Simpson's rule taking h=0.25.

20. Evaluate $\int_0^1 \frac{1}{1+x}dx$ By (i)Trapezoidal rule and Simpson's $\frac{1}{3}$ rule (ii) Using Simpson's $\frac{3}{8}$ rule.

21. Evaluate $\int_0^4 e^x$dx using Trapezoidal and Simpson's rule. Also compare your result with the exact value of the integral.

22. When a train is moving at 30 m/sec, steam is shut off and brakes are applied. The speed of the train per second after t seconds is given by

Time(t)	0	5	10	15	20	25	30	35	40
Speed(v)	30	24	19.5	16	13.6	11.7	10	8.5	7.0

Using Simpson's rule, determine the distance moved by the train in 40 seconds.

23. Evaluate $\int_0^{\pi/2} e^{\sin x}dx$ correct to four decimal places by Simpson's three-eighth rule.

24. Dividing the range into 10 equal parts, find the value of $\int_0^{\pi/2} \sin x \, dx$, using
(i) Trapezoidal rule (ii)Simpson's 1/3rd rule

25. Evaluate $\int_0^{\pi/2} \sin x \, dx$ by Simpon's $\frac{3}{8}$ rd rule and compare with exact value.

15. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule (iii) Simpson's 3/8 rule and compare the result in each case with its actual value.

26. Evaluate approximately, by Trapezoidal rule, $\int_0^1 (4x - 3x^2)dx$ by taking n=10. Compute the exact integral and find the absolute value.

ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES::TIRUPATI (AUTONOMOUS)

Year:I **Semester:II** **Branch of Study:AID**

COURSE CODE	COURSE TITLE	L	T	P	CREDITS
20ABS9921	Numerical Methods	3	0	0	3

Course Outcomes:

1) Analyze the concepts of Errors, Relative and Percentage Errors
2) Analyze the concepts of Algebraic & Transcendental Equations to solve different Engineering problems
3) Analyze Interpolation using the concepts of the Numerical Methods
4) Apply the concepts of Integration in Numerical Methods
5) Apply the concepts of O.D.E on Numerical Methods

Unit - I

Errors in Numerical computations: Errors and their Accuracy, Mathematical Preliminaries, Errors and their Analysis, Absolute, Relative and Percentage Errors, A general error formula, Error in a series approximation.

UNIT – II

Solution of Algebraic and Transcendental Equations: The Bisection Method – The Method of False Position– Newton-Raphson Method, Solution of linear simultaneous equation: Crout‘striangularisation method, Gauss - Seidal iteration method.

UNIT – III

Interpolation: Newton‘s forward and backward interpolation formulae – Lagrange‘s formulae. Gauss forward and backward formula, Stirling‘s formula, Bessel‘s formula.

UNIT – IV

Curve fitting: Fitting of a straight line – Second degree curve – Exponential curve-Power curve by method of least squares. Numerical Differentiation for Newton‘s interpolation formula. Numerical Integration: Trapezoidal rule – Simpson‘s 1/3 Rule – Simpson‘s 3/8 Rule.

UNIT –V

Numerical solution of Ordinary Differential equations: Solution by Taylor‘s series-Picard‘s Method of successive Approximations-Euler‘s Method- Runge - Kutta Methods. Numerical solutions of Laplace equation using finite difference approximation.

TEXT BOOKS:

1. Higher Engineering Mathematics, B.S.Grewal, Khanna publishers.
2. Introductory Methods of Numerical Analysis, S.S. Sastry, PHI publisher.

REFERENCES:

1. Engineering Mathematics, Volume - II, E. Rukmangadachari Pearson Publisher.
2. Mathematical Methods by T.K.V. Iyengar, B.Krishna Gandhi, S.Ranganatham and M.V.S.S.N.Prasad, S.Chand publication.
3. Higher Engineering Mathematics, by B.V.Ramana, McGraw Hill publishers.
4. Advanced Engineering Mathematics, by Erwin Kreyszig, Wiley India

Numerical Methods

Unit - 1

Error Analysis :-

There are different types of Errors to find an Error, they are

(i). Inherent Errors

(ii). Round-off Errors

(iii). Truncation Errors

(iv). Absolute Errors

(v). Relative Errors

(vi). percentage Errors and

(vii). General Formula for Error With Examples

* Accuracy of Numbers :-

we have two types

(a). Exact numbers :-

The numbers of the type $8, 1, 2, \frac{1}{2}, \frac{3}{4}, 9.75$ will be in Category of "Exact numbers".

(b). Approximate numbers:-

The numbers $\frac{1}{3} = 0.33333...$, $\pi = 3.141592...$ do not have finite decimal Expansion and hence they are approximated to some finite digits called as significant digits for the purpose of calculations.

Error in Numerical Computation:-

we have following types of Errors in numerical calculations:

(i). Inherent Errors :-

Errors which Exist in the problem either due to approximation given data, limitations of Computing aids, are called as inherent Errors. The inherent Error can be minimised by taking better data and using high precision computing aids.

(ii). Round-off Errors :-

These Errors are due to rounding off the numbers while process of Computations. The round off Errors Can be reduced by doing the Computations to more Significant digits than given in data at each stage of Computation.

Ex:- Some times we have numbers with a large number of digits Such as 7.5846712 which Can be round-off to a Significant figure ~~7.58~~ 7.585. This process is Called as rounding off the numbers.

$$\Rightarrow 7.5846712 - 7.585 = -0.0003288$$

(Round-off Error).

(iii). Truncation Errors :-

These Errors arise due to use of approximate formula in Computation or by truncating the infinite Series to Some approximate terms. The Study of this type of Error is is usually associated with the problem of Convergence of infinite Series.

Ex:- $y = y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) +$
$\frac{x^4}{4!} y^{iv}(0) + \dots + \frac{x^n}{n!} y^n(0) + \dots$

let $y = 1 + \frac{x}{2} + \frac{x^2}{2} + \frac{3x^2}{8} + \dots$

$\therefore y(0.1) = 1 + \frac{0.1}{2} + \frac{(0.1)^2}{2} + 3\frac{(0.1)^3}{8} + \dots$

$\therefore y(0.1) \approx 1.055375$
$y(0.1) \approx 1.0554$

$(0.1)^2 = 0.01$
$(0.1)^3 = 0.001$
$(0.1)^4 = 0.0001 \approx 0$
$(0.1)^5 = 0.00001 \approx 0$

(1b). **Absolute Error, Relative and percentage Errors:-**

If 'x' is true value of quantity and x' is its approximate value, than

(a). **Absolute Error:-**

It is denoted by 'E_A'.

$E_A = |x - x'|$

$= |7.5846712 - 7.585| = |-0.0003288|$

$E_A = 0.0003288$

(b). Relative Error :-

It is denoted by E_R.

$\Rightarrow E_R = \left|\frac{X - X'}{X}\right|$

$\Rightarrow E_R = \left|\frac{7.5846712 - 7.585}{7.5846712}\right|$

$= |-0.00004335059376|$

$E_R = 0.00004335059376$

(c). percentage Error :-

It is denoted by E_P.

$\Rightarrow E_P = \left|\frac{X - X'}{X}\right| \times 100$

$= |-0.00004335059376| \times 100$

$\therefore E_P = 0.004335059376$

$\rightarrow$ If ΔX be a number such that $|X - X'| \leq \Delta X$, then ΔX is an upper limit of the magnitude of Absolute Error and measures the absolute accuracy.

$\frac{\Delta X}{|X|} \approx \frac{\Delta X}{|X'|}$ measures the relative accuracy.

General Formula for Error :-

let $u = f(x, y, z)$ be the function of 3 vari[ables]

$$\therefore \Delta u = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

let $u = f(x_1, x_2, x_3 \ldots x_n)$ be the function of 'n' variables

$$\therefore \Delta u = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i$$

$$\Delta u = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

Then relative Error E_R in 'u' is given by

$$E_R = \frac{\Delta u}{u}$$

$\Delta X = X - X'$

Ex: (1). Find the Relative Error (maximum) in the function $\boxed{u = \frac{5xy^2}{z^3}}$ at $x = y = z = 1$ with $\Delta x = \Delta y = \Delta z = 0.001$.

Sol:- we have $u = \frac{5xy^2}{z^3}$ $R = \frac{\Delta X}{X}$

$$\therefore \frac{\partial u}{\partial x} = \frac{5y^2}{z^3}, \quad \frac{\partial u}{\partial y} = \frac{10xy}{z^3} \quad \therefore \frac{\partial u}{\partial y} = \frac{5x}{z^3}2y = \frac{10xy}{z^3}$$

$$\frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{5xy^2}{z^3}\right)$$

$$= \frac{5x}{z^3}\frac{\partial}{\partial y}(y^2)$$

$$\therefore \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z \longrightarrow (1)$$

$$(\Delta u)_{max} = \left|\frac{5y^2}{z^3}\Delta x\right| + \left|\frac{10xy}{z^3}\Delta y\right| + \left|\frac{15xy^2}{z^4}\Delta z\right|$$

$$(\Delta u)_{max} = 5(0.001) + 10(0.001) + 15(0.001)$$

$$= 0.03$$

$$(E_R)_{max} = \frac{(\Delta u)_{max}}{u} \qquad \therefore u = \frac{5(1)(1)^2}{(1)^3}$$

$$= \frac{0.03}{5} \qquad \boxed{u = 5.}$$

$$(E_R)_{max} = 0.006$$

(2). An approximate value of "π" is given by $x_1' = 22/7$ $= 3.1428571$ and its true value is $x = 3.1415926$. Find the absolute and relative Errors.

Sol:- Given approximate value of π is

$x_1' = 3.1428571$

true value of π is

$x = 3.1415926$

Now $E_A = |x - x_1'|$

$= |3.1428571 - 3.1415926|$

$= |-0.0012645| = 0.0012645$

and $E_R = \left|\frac{x - x'}{x}\right|$

$$= \left|\frac{0.0012645}{3.1415926}\right|$$

$$= |0.0004021|$$

$$E_R = 0.000402.$$

(3). Three approximate values of the number $1/3$ are given as 0.30, 0.33 and 0.34. which of these three values is the best approximation?

Sol^n:- we have $x = \frac{1}{3}$, $x_1^* = 0.30$, $x_2 = 0.33$, $x_3 = 0.34$

$$\left|\frac{1}{3} - 0.30\right| = \frac{1}{30}$$

$$\left|\frac{1}{3} - 0.33\right| = \frac{1}{300}$$

$$\left|\frac{1}{3} - 0.34\right| = \frac{1}{150}$$

$\therefore$ It follows that 0.33 is the best approximation for $\frac{1}{3}$.

(4). Find If Absolute Error 0.05. Find the E_R of the number 8.6

Sol:- we have $\varepsilon_A = 0.05 = |X - X'|$, $|X| = 8.6$

$$\varepsilon_R = \frac{0.05}{8.6} = 0.0058$$

(5). Evaluate the Sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to '4' Significant digits, and find its absolute and relative Errors.

Sol:- we have $\sqrt{3} = 1.732 = 1.732050808$, $\sqrt{5} = 2.236 = 2.236067977$

$\sqrt{7} = 2.646 = 2.645751311$

Hence $S = 6.6138$ then

$\varepsilon_A = 0.00005 + 0.00006 + 0.00024$
$|x - x'| + |x - x'| + |x - x'|$

$\varepsilon_A = +0.00014$

$$\varepsilon_R = \left| \frac{+0.00014}{6.6138} \right| = 0.00002$$

→ 4 S.D.

(1). 3.0045

0.0045

3 S.D.

5.33578

5.336

5.34.

9.23

5.336

5.3.

5.336

(4). Find the difference $\sqrt{6.37}-\sqrt{6.36}$ to three significant figures.

Sol^n:

we have

$\sqrt{6.37} = 2.523885893$

and $\sqrt{6.36} = 2.521904043$

$\therefore \sqrt{6.37}-\sqrt{6.36} = 0.001981850$

$= 0.00198$, Correct to three Significant figures.

Alternatively, we have

$$\sqrt{6.37}-\sqrt{6.36} = \frac{6.37-6.36}{\sqrt{6.37}+\sqrt{6.36}}$$

$$= \frac{0.01}{2.524+2.522}$$

$$= 0.198 \times 10^{-2},$$

which is the Same result as obtained before.

(5). Find the value of $s = \frac{a^2\sqrt{b}}{c^3}$ where $a = 6.54 \pm 0.01$, $b = 48.64 \pm 0.02$ and $c = 13.5 \pm 0.03$ also find the relative Error in the result

Sol: we have

$a = 6.54 \pm 0.01$, $b = 48.64 \pm 0.02$

$c = 13.5 \pm 0.03$

$\Rightarrow a^2 = (6.54)^2 = 42.7716.$

$\sqrt{b} = \sqrt{48.64} = 6.9742$

$c^3 = (13.5)^3 = 2460.375$

$\therefore \quad s = \frac{42.7716 \times 6.9742}{2460.375}$

$\log a^n = n \log a$

$= 0.12124... = 0.121$

$s = 0.121$

$\log \frac{a}{b} = \log a - \log b$

$\log ab = \log a + \log b.$

$\log s = \log\left(\frac{a^2\sqrt{b}}{c^3}\right)$

$\Rightarrow \log s = \log(a^2\sqrt{b}) - \log c^3 \Rightarrow \log a^2 + \log b^{1/2} - \log c^3$

$\log s = \log a^2 + \log\sqrt{b} - \log c^3$

$\log s = 2\log a + \frac{1}{2}\log b - 3\log c$

Now

$\Rightarrow \left|\frac{\Delta s}{s}\right| \leq 2\left|\frac{\Delta a}{a}\right| + \frac{1}{2}\left|\frac{\Delta b}{b}\right| - 3\left|\frac{\Delta c}{c}\right|$

$\leq 2\left(\frac{0.01}{6.54}\right) + \frac{1}{2}\left(\frac{0.02}{48.64}\right) + 3\left(\frac{0.03}{13.5}\right)$

$\left|\frac{\Delta s}{s}\right| = 0.009931 = \varepsilon_R$

$$\Delta S = \frac{\partial S}{\partial a}\Delta a + \frac{\partial S}{\partial b}\Delta b + \frac{\partial S}{\partial c}\Delta c$$

$$= \frac{2a\sqrt{b}}{c^3}\Delta a + \frac{a^2}{2\sqrt{b}c^3}\Delta b - \frac{3a^2\sqrt{b}}{c^4}\Delta c$$

$$= \frac{2 \times 6.54 \times 6.9742}{2460.375} \times 0.01$$

$$+ \frac{42.7716}{2(6.9742)(2460.375)} \times 0.02$$

$$- \frac{3 \times 42.7716 \times 6.9742}{38215.0625} \times 0.03$$

$$= 0.0004 + 0 - 0.0008$$

$$\Delta S = -0.0004$$

$$\therefore E_R = \frac{\Delta S}{S} = \frac{-0.0004}{0.121} = 0.00033$$

$$E_R = 0.00033$$

Error in a Series Approximation :-

The truncated error committed in a Series approximation can be evaluated by using Taylor's Series. If x_i and x_{i+1} are two successive values of 'x', than we have.

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots + \frac{h^n}{n!} f^n(x_i) + O(h^{n+1}) \longrightarrow (1)$$

where $O(h^{n+1})$ means that the truncation Error is of the order of h^{n+1}

let the Series be truncated after the first term. This gives the Zero-order approximation

$$f(x_{i+1}) = f(x_i) + O(h^1) \longrightarrow (2)$$

which means that halving the interval length 'h' will also halve the Error in the approximate Solution.

Similarly, the First-order Taylor-Series approximation is given by

$$f(x_{i+1}) = f(x_i) + h f'(x_i) + O(h^2) \quad \rightarrow (3)$$

which means that halving the interval length, 'h' will quarter the Error in the approximation. (Second)

In such a case we say that approximation has a Second order of Convergence.

Example:- 1 Evaluate $f(1)$ using Taylor's Series for $f(x)$, where $f(x) = x^3 - 3x^2 + 5x - 10$

Sol^n:- In general, it is easily evaluate $f(1)$

$\Rightarrow$ i.e $f(1) = -7$.

but it will be instructive to see how the Taylor Series approximations of orders '0' to '3' improve the accuracy of "$f(1)$" gradually.

let $h = 1$, $x_i = 0$ and $x_{i+1} = 1$.

~~we~~ ~~then~~ the derivatives of $f(x)$ are given by

$f'(x) = 3x^2 - 6x + 5$; $f'' = 6x - 6$, $f''' = 6$

$f^{iv}(x)$ and higher derivatives being all zero.

Hence $f'(x_i) = f'(0) = 5$, $f''(x_i) = f''(0) = -6$

$$f'''(x_i) = f'''(0) = 6$$

also $f(x_i) = f(0) = -10$

Hence, Taylor's Series gives

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{6} f'''(x_i) \longrightarrow (1)$$

from eq (1), The zero-order approximation is given by

$$f(x_{i+1}) = f(x_i) + O(h)$$

$$= -10 + O(h) \approx -10$$

and $\Rightarrow f(1) = f(0) + O(h) \approx -10$

there Error in which is $-7 + 10$ i.e 3 units

For the first approximation;-

we have

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + O(h^2)$$

$$\Rightarrow f(1) = f(0) + hf'(0) + O(h^2)$$

$$f(1) = -10 + 5 + O(h^2) \approx -5$$

$$f(1) \approx -5$$

then error in which is $-7+5$ i.e -2 units

Again, the second order Taylor approximation is given by

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + O(h^3)$

$f(1) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + O(h^3)$

$f(1) = -10+5 + \frac{1}{2}(-6) + O(h^3) \approx -8$

$f(1) \approx -8$

in which the Error is $-7+8$ i.e $+1$ unit.

Finally, the third order Taylor approximation is given by

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(x_i) + \cancel{O(h^3)} \frac{h^3}{6} f'''(x_i)$

$f(1) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \cancel{O(h^3)}$ $h^3/6\, f'''f(x_i)$

$= -10+5+\frac{1}{2}(-6)+\frac{1}{6}(6)$

$f(1) = -7$

$\therefore$ which is the Exact value of $f(1)$.

Unit - I

Solution of Algebraic and Transcendental Equations:

Determination of roots of an equation of the form $f(x)=0$ has great importance in the fields of Science and Engineering.

(1). polynomial functions-

A function $f(x)$ is said to be a polynomial function, if $f(x)$ is a polynomial in 'x'.

i.e $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n$, where $a_0\neq 0$ the co-efficients $a_0, a_1, \dots a_n$ are real constants and 'n' is a non-negative integer.

(2). Algebraic function:-

A function which is a sum (or) difference or product of two polynomials is called an "algebraic function"; otherwise, the function is called a "transcendental" or "non-algebraic function".

→ If $f(x)$ is an algebraic function, then the equation $f(x)=0$ is called an algebraic equation.

→ If $f(x)$ is a transcendental function, then the function $f(x)=0$ is called a "transcendental equation".

Ex:- $f(x) = C_1e^{x} + C_2e^{-x} = 0$;

$f(x) = 2\log x - \frac{\pi}{4} = 0$;

$f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$.

$f(x) = xe^{x} - 2 = 0$

(3). Root of an Equation:-

The roots of the equation $f(x)=0$ can be obtained by the following two methods.

(i). Iterative Methods

(ii). Direct Methods.

(i). Iterative Methods:-

Suppose, we have to find a root "α" of the equation $f(x)=0$.

let 'x_0' be an approximation to 'α'. using x_0, we generate a sequence of numbers $x_1, x_2, \ldots$ Under Certain Conditions this sequence Converges to the root 'α'.

The method of generating better and better approximation from an initial guess is called an "Iteration method".

(ii). Direct Method:-

we are familiar with the solution of the polynomial equations such as linear equation, $ax+b=0$, and quadratic equation $ax^2+bx+c=0$, using direct methods (or) analytical methods.

(4). Bisection Method (or) Bolzano Method :-

* Suppose we know an equation of the form $f(x)=0$ has exactly one real root b/w two real numbers x_0, x_1.
* The number is choosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign.
* let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0+x_1}{2}$.
* if $f(x_2)=0$ then x_2 is a root. If $f(x_1)$ and $f(x_2)$ have same sign then the root lies b/w x_0 and x_2.

* The interval is taken as $[x_0, x_2]$. otherwise the root lies in the interval $[x_2, x_1]$

(1). Find a positive root of $x^3 - x - 1 = 0$ Correct to two decimal places by bisection method.

Soln:

let $f(x) = x^3 - x - 1 = 0$

Consider $x_0 = 1$ and $x_1 = 2$

$\Rightarrow f(1) = -1 < 0$, $f(2) = 5 > 0$.

By bisection method,

the next approximation x_2 is,

$$\therefore x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = \frac{3}{2}$$

$$\boxed{x_2 = 1.5}$$

$\Rightarrow f(1.5) = (1.5)^3 - (1.5) - 1 = f(x_2)$

$f(1.5) = 0.875 > 0$

$\therefore$ The root lies b/w $\overset{x_0}{1}$ and $\overset{x_2}{1.5}$

$$x_3 = \frac{x_0 + x_2}{2} = \frac{1+1.5}{2} = 1.25.$$

$\Rightarrow f(1.25) = (1.25)^3 - (1.25) - 1 = f(x_3)$

$f(1.25) = -0.2968 < 0$

Thus the root lies b/w $\overset{x_2}{1.25}$ and $\overset{x_3}{\cancel{1.375}\ 1.5}$

Now $x_4 = \frac{x_2 + x_3}{2} = \frac{1.25 + 1.5}{2}$

$x_4 = 1.375$ and $f(1.375) = 0.224 > 0$
$f(A) > 0$.

Thus, the root lies b/w $\underset{x_3}{1.25}$ and $\underset{x_4}{1.375}$.

Now $x_5 = \frac{x_3 + x_4}{2} = \frac{1.25 + 1.375}{2}$

$x_5 = 1.3125 \Rightarrow f(x_5) < 0$

Thus, the root lies b/w $\underset{x_4}{1.375}$ and $\underset{x_5}{1.3125}$

Now $x_6 = \frac{x_4 + x_5}{2} = \frac{1.375 + 1.3125}{2}$

$x_6 = 1.34375$

$\therefore f(1.34375) = 0.0826 > 0$.

Now the root lies b/w $\underset{x_5}{1.3125}$ and $\underset{x_6}{1.34375}$.

So $x_7 = \frac{x_5 + x_6}{2} = \frac{1.3125 + 1.34375}{2}$

$x_7 = 1.3281$

$\therefore f(1.3281) = 0.01457 > 0$.

Now, the root lies b/w $\underset{x_5}{\overset{1.3125}{\cancel{1.34375}}}$ and $\underset{x_7}{1.3281}$.

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{\overset{1.3125}{\cancel{1.34375}} + 1.3281}{2}$

$x_8 = 1.3203$

$\therefore f(1.32) = .0.0187$

Hence, the root is 1.32 //

(2). Find a root of the equation $x^3-5x+1=0$ using the Bisection method in '5' stages.

Sol^n:-

let $f(x) = x^3 - 5x + 1$.

Consider $x_0 = 0$, $x_1 = 1$

we note that, $f(0) > 0$ and $f(1) < 0$

$\therefore$ one root lies b/w '0' and '1'

By Bisection method,

the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0+1}{2} = 0.5$$

$\Rightarrow f(1.5) = -1.375 < 0$

$\therefore$ one root lies b/w 0 and 0.5

the next approximation is

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0.5+0}{2} = 0.25$$

$\Rightarrow f(x_3) = f(0.25) = -0.234375 < 0$

$\therefore$ one root lies b/w 0 and 0.25

the next approximation is,

$$x_4 = \frac{x_0 + x_3}{2} = \frac{0+0.25}{2} = 0.125$$

$\Rightarrow f(x_4) = f(0.125) = 0.37495 > 0$

$\therefore$ one root lies b/w 0.25 and 0.125

$\therefore f(x_4) = f(0.125) = 0.37495 > 0$

$\Rightarrow x_5 = \dfrac{0.25 + 0.125}{2} = 0.1875$

$f(x_5) = 0.06910 > 0$

$\therefore$ one root lies b/w 0.1875 and 0.25

$\therefore$ one root lies b/w " "

$x_6 = \dfrac{0.25 + 0.1875}{2} = 0.21875$

$\because$ we are asked to do upto 5 stages

$\therefore$ Hence we stop here. 0.21875 is taken as an approximate value of the root and it lies b/w '0' and '1'.

(3). Find out Square root of 25 given $x_0 = 2.0$, $x_1 = 7.0$ using Bisection method.

Soln:- H.W.

(4). Find a positive root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages.

(5). Find a real root of the equation $x^3 - 6x - 4 = 0$ by bisection method.

False position Method (Regula Falsi Method) :-

→ * In the false position method we will find the root of the equation $f(x)=0$. Consider two initial approximation values $\overset{a}{x_0}$ and $\overset{b}{x_1}$ near the required root so that $f(\underset{a}{x_0})$ and $f(\underset{b}{x_1})$ have different signs. This implies that a root lies b/w '$\underset{a}{x_0}$' and '$\underset{b}{x_1}$'.

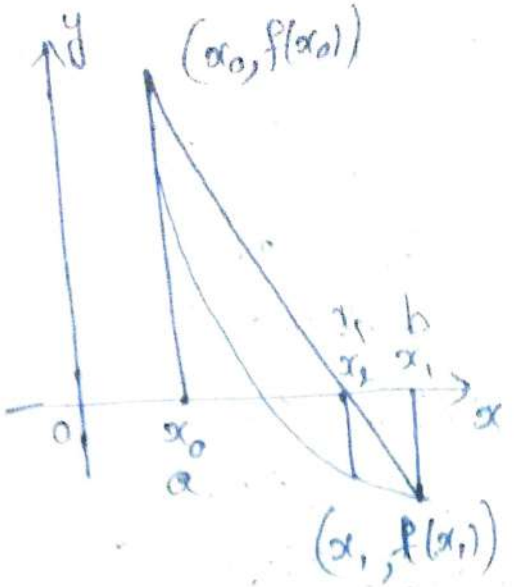
→ * Consider the point $A = (\overset{a}{x_0}, \overset{f(a)}{f(x_0)})$ and $B = (\overset{b}{x_1}, \overset{f(b)}{f(x_1)})$ on the graph and Suppose they are Connected by a straight line.

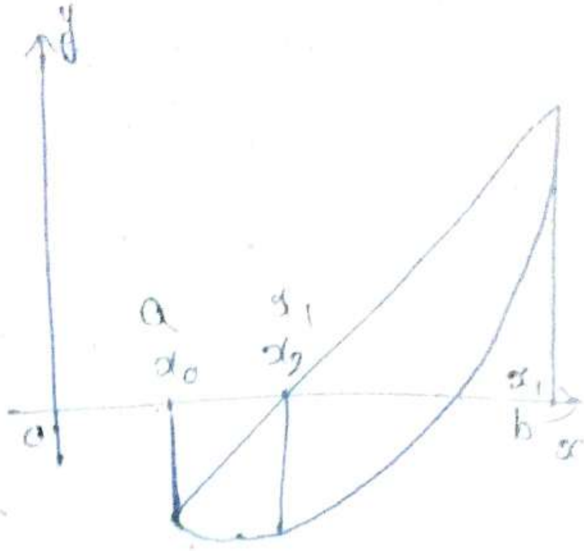
→ * Suppose this line Cuts x-axis at x_2.

→ We Calculate the value of $f(x_2)$ at the point.

→ If $f(x_0)$ and $f(x_2)$ are of opposite Signs, then the root lies b/w $\overset{a}{x_0}$ and $\overset{b}{x_2}$ and value x_1 is replaced by '$\overset{a}{x_2}$' (See Fig. (1)).

→ * otherwise the root lies b/w $\overset{x_1}{x_2}$ and $\overset{a}{x_0}$ and the value of $\underset{a}{x_0}$ is replaced by $\underset{a_1}{x_2}$ (See Fig. (2)).





To obtain the equation to find the next approximation to the root.

let $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ be the points on the curve $y = f(x)$. then the equation to the chord AB is

$$y - y_1 = m(x - x_1) \qquad \therefore m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - f(x_0) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_0)$$

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \longrightarrow (1)$$

we have $f(x) = 0$ i.e $y = 0$

$$\therefore \frac{-f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$-\frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0) = x - x_0$$

$$x = x_0 - \frac{x_1 f(x_0) - f(x_0) x_0}{f(x_1) - f(x_0)}$$

$$= \frac{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_0) + x_0 f(x_0)}{f(x_1) - f(x_0)}$$

$$x' = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \longrightarrow (2)$$

If the new value of 'x'' is taken as x_2 then eq (2) becomes

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \longrightarrow (3)$$

Now we decide whether the root lies b/w x_0 and x_2 (or) x_2 and x_1.

we name that interval as (x_1, x_2).

- The line joining (x_1, y_1), (x_2, y_2) meets x-axis at x_3 is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

- This will in general, be nearer to the exact root we continue this procedure till the root is found to the desired accuracy.

- The iteration process based on eq (3) is known as the "Method of False position".

- The Successive intervals where the root lies, in the above procedure are named as (x_0, x_1), (x_1, x_2), (x_2, x_3) --- where $x_i < x_{i+1}$ and $f(x_i)$, $f(x_{i+1})$ are of opposite signs.

Also
$$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

(or).

$$x_{i+1} = \frac{b f(a) - a f(b)}{f}$$

$$\boxed{x_{i+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

(1). Find the root of the equation $x \log_{10}(x) = 1.2$ 0... False position method.

Soln.-

let $f(x) = x \log_{10} x - 1.2$

Then Consider $x_0 = 2$ and $x_1 = 3$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794 < 0$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 > 0$

$\therefore$ f(2) and f(3) have opposite signs, the root lies b/w '2' and '3'.

By False position method.

the next approximation is

$\cancel{x_2 = \frac{x_0 + x_1}{2} = \frac{2+3}{2} = \frac{5}{2} =}$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{2(0.23136) - 3(-0.59794)}{0.23136 + 0.59794}$$

$x_2 = 2.7210$

$\Rightarrow f(x_2) = f(2.7210) = 2.7210 \log_{10} 2.7210 - 1.2$

$= -0.0171 < 0$

Now the root lies b/w 2.721 and 3

$\therefore x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$= \dfrac{3(-0.0171) - 2.7210(0.23136)}{-0.0171 - 0.23136}$

$x_3 = 2.740$

$\Rightarrow f(x_3) = f(2.740) = 2.740 \times \log_{10}(2.740) - 1.2$

$f(x_3) = -0.00056 < 0$

Now, the root lies b/w 2.740 and 3

$\therefore x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$= \dfrac{2.740(-0.00056) - 2.740(-0.0171)}{-0.00056 + 0.0171}$

$x_4 = 2.7406$

$\therefore$ Hence the root is $x = 2.74$ //

(2). Find out the roots of the equation $x^3 - x - 4 = 0$ using false position.

Sol^n: Given let $f(x) = x^3 - x - 4$

Consider $a = 1$ and $b = 2$

$f(a) = f(1) = 1 - 1 - 4 = -4 < 0$

$f(b) = f(2) = 2^3 - 2 - 4 = 8 - 2 - 4 = 2 > 0$

$\therefore f(a), f(b)$ have opposite signs

The root lies b/w 1 and 2

By false position method

The next approximation is

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{1(2) - 2(-4)}{2+4} = \frac{2+8}{6} = \frac{10}{6} = 1.6667$$

$x_2 = 1.6667$

$f(x_2) = (1.6667)^3 - (1.6667) - 4$

$= -1.0368 < 0$

$\therefore$ The root lies b/w 1.6667 and 2

The next approximation is

$$x_3 = \frac{b f(x_2) - x_2 f(b)}{f(b) - f(x_2)}$$

$$= \frac{2(-1.0368) - (1.667)(2)}{\cancel{2+}\,1.0368 - 2}$$

$x_3 = 0.5863$

$f(x_3) = (0.5863)^3 - (0.5863) - 4 = -4.3848 < 0$

The root lies b/w x_3 and b.

$$\therefore x_4 = \frac{(0.5863)2 - 2(-4.3848)}{2+4.3848}$$

$= 1.5572$

$f(x_4) = -1.7812 < 0$

* Find the root of $x^3 - x - 2$ using By Bisection, Regular Falsi method, Newton-Raphson method :-

Sol:- let $f(x) = x^3 - x - 2$

Consider $x_0 = 1$ and $x_1 = 2$

$f(x_0) \cdot f(1) = -2 < 0$, $f(x_1) = f(2) = 8 - 2 - 2 = 4 > 0$

$f(x_0) < 0$ & $f(x_1) > 0$

(1). Bisection Method :-

- The root lies b/w '1' and '2'

First Approximation :-

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = 1.5$$

$x_2 = 1.5$

$\Rightarrow f(x_2) = 1.5^3 - 1.5 - 2 = -0.125 < 0$

Second Approximation :-

- The root lies b/w '1.5' and '2'

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1.5+2}{2} = 1.25$$

$f(x_3) = -1.2969 < 0$

Third Approximation :-

- The root lies b/w '1.25' and '2'

$$x_4 = \frac{x_1 + x_3}{2} = \frac{3.25}{2} = 1.625$$

$f(x_3) = 0.6660 > 0$

Fourth Approximation :-

The root lies b/w 1.625 and 1.25

$$x_4 = \frac{1.625 + 1.25}{2} = 1.4375$$

$f(x_4) = -0.4670 < 0$

Sixth Approximation :-

The root lies b/w 1.4375 and 1.625

$$x_5 = \frac{1.4375 + 1.625}{2} = 1.5313$$

$f(x_5) = 0.0594 > 0$

Seventh Approximation :-

The root lies b/w 1.5313 and 1.4375

$x_6 = 1.4844$

$f(x_6) = -0.2136 < 0$

eigth Approximation :-

The root lies b/w 1.4844 and 1.5313

$x_7 = 1.5079$

$f(x_7) = -0.0793 < 0$

9^{th} Approximation :-

The root lies b/w 1.5079 and 1.5313

$x_8 = 1.5196$

$f(x_8) = -0.0106 < 0$

10^{th} Approximation :-

The root lies b/w 1.5196 and 1.5313

$x_9 = 1.5255$

$f(x_9) = 0.0246 > 0$

11^{th} Approximation :-

The root lies b/w 1.5255 and 1.5196.

$x_{10} = 1.5226.$

$f(x_{10}) = 0.0073 > 0$

12^{th} Approximation :-

The root lies b/w 1.5226 and 1.5196

$x_{11} = 1.5211$

$f(x_{11}) = -0.0017 < 0$

13^{th} Approximation :-

The root lies b/w 1.5211 and 1.5226

$x_{12} = 1.5219 \approx 1.522$

$f(x_{12}) = 0.0031 > 0$

14^{th} Approximation:-

The root lies b/w 1.5219 and 1.5211

$x_{13} = 1.5215 \approx 1.522$

$f(x_{13}) = 0.0007 > 0$

Since $x_{12} = x_{13} \approx 1.522$

Hence

The real positive root is 1.522

(2). Regular Falsi possition:-

The root lies b/w 1 and 2

First Approximation:-

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$i = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$f(1) \quad x_2 = \frac{1(4) - 2(2)}{2 - 1} = 8$$

$f(x_2) = 502 > 0$

2nd Approximation :-

$i = 2$ $\quad x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

The root lies b/w $x_1 = 8$ and $x_2 = 1$

$(x_3) = \dfrac{8(-2) - 1(502)}{502+2} = -1.0278$

$x_3 = -1.0278$

$f(x_3) = -2.0579 < 0$

3rd Approximation :-

The root lies b/w -1.0278 and 8

8-4-5
27-6-5
27-11-
-15

---

* $f(x) = x^3 - 2x - 5 = 0$

$x_0 = 2 \Rightarrow f(2) = -1 < 0$

$x_1 = 3 \Rightarrow f(3) = 16 > 0$

The root lies b/w 2 and 3

$\therefore x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

first Approximation :-

$i = 1$, $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$= \dfrac{2(16) - 3(-1)}{16+1} = 2.0588$

$f(x_2) = -0.3911 < 0$

(11). Regular Falsi Method :-

* Find a real root of $xe^x = 3$ using R.F.M

Sol^n:- Given

let $f(x) = xe^x - 3$

Consider $x_0 = 1$ & $x_1 = 2$

Now $f(x_0) = f(1) = 1 \times e^1 - 3 = -0.2817 < 0$

$f(x_1) = f(2) = 2e^2 - 3 = 11.7781 > 0$

The root lies b/w '1' and '2'

1st Approximation :-

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{\cancel{1(10.2817) - 2(1\cdot}}{}$$

$$x_2 = \frac{1(11.7781) - 2(-0.2817)}{11.7781 + 0.2817}$$

$$= \frac{12.3415}{12.0598}$$

$x_2 = \cancel{1.3295}\ 1.0234$

$f(x_2) = \cancel{2.0244}\ -0.1522 < 0$

The root lies b/w ~~1.020~~ x_2 and x_1

2nd Approximation :-

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2(-0.1522) - (1.0234)(11.7781)}{-0.1522 - 11.7781}$$

$$x_3 = \frac{-12.3581}{-11.9303} = 1.03585$$

$$f(x_3) = -0.0812 - 0.0815$$

$$f(x_3) = -0.0815 < 0$$

The root lies b/w $\underset{x_2}{\overset{\shortparallel}{x_3}}$ and $\underset{x_3}{\overset{\shortparallel}{x_1}}$

3rd Approximation :-

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} \quad \left(\frac{x_2 f(x_3) -}{f(x_3) - f} \right)$$

$$= \frac{(1.03585)(11.7781) - 2(-0.0815)}{11.7781 + 0.0815}$$

$$x_4 = 1.0425 = \frac{12.3633}{11.8593} \approx 1.043$$

$$f(x_4) = 1 - \cancel{0.04} (1.0425) e^{1.0425} - 3$$

$$f(x_4) = -0.0432 < 0$$

The root lies b/w $\underset{x_3}{\overset{\shortparallel}{x_4}}$ and $\underset{x_4}{\overset{\shortparallel}{x_1}}$

4th Approximation :-

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = \frac{(1.0425)(11.7781) - 2(-0.0432)}{11.7781 + 0.0815}$$

$$= \frac{12.3651}{11.8596} = 1.0426 \approx 1.043$$

Hence $x_4 = x_5 = 1.043$

* Find out the root of the equation $x^3 - x - 4 = 0$ using false position method

Sol^n:

$$f(x) = x^3 - x - 4$$

let $x_0 = 1 = a$, $x_1 = 2 = b$ (say)

$f(x_0) = f(1) = -4 < 0$, $f(x_1) = f(2) = 2 > 0$

The root lies b/w 1 and 2

we know that
$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

1st Approximation :-

$i = 1$, $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{1(2) - 2(-4)}{2+4}$$

$$x_2 = \frac{10}{6} = 1.6667 = x_1$$

$f(x_2) = -1.0368 < 0 = f(x_1)$

The root lies b/w x_2 and x_1

x_2 and x_1
" x_1 " x_2

2nd Approximation :—

$i=2, \quad x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \dfrac{x_1 f(b) \ldots}{f(b) \ldots}$

$= \dfrac{\cancel{2(-1.0368) - (1.6667)\ldots}}{\cancel{-1.0368 - 1.6667}}$

$= \dfrac{(1.6667)\,2 - 2(-0.0368)}{2 + 0.0368}$

$x_3 = \dfrac{3.4070}{2.0368} = 1.7403 = x_2$

$f(x_3) = -0.4696 = x_2 < 0$

The root lies b/w 1.7403 and 2
$\qquad\qquad\qquad\qquad\qquad\quad \shortparallel \qquad\quad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\quad x_2 \qquad\quad x_3$

3rd Approximation :—

$i=3, \quad x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$= \dfrac{(1.7403)(2) - 2(-0.4696)}{2 + 0.4696}$

$x_4 = 1.7897 = x_3 \text{ (say)}$

$f(x_4) = (1.7897)^3 - 1.7897 - 4$

$= -0.0572 < 0 = f(x_3)$

∴ The root lies b/w 1.7897 and 2

4th Approximation :-

$$i=4, \quad x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = \frac{(1.7897)(2) - 2(-0.0572)}{2+0.0572}$$

$$x_5 = 1.7955 = x_4 \text{ (Say)}$$
$$\simeq 1.796$$

$$f(x_5) = -0.0071 < 0 = f(x_4) \text{ (Say)}$$

∴ The root lies b/w $\underset{x_4}{\overset{\shortparallel}{1.7955}}$ and $\underset{x_5}{\overset{\shortparallel}{2}}$

5th Approximation :-

$$i=5, \quad x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$$

$$= \frac{(1.7955)(2) - 2(-0.0071)}{2+0.0071}$$

$$x_6 = 1.7962.$$

∴ Since $x_5 = x_6 \simeq 1.7962$

Hence The real positive root is 1.7962.

* Find the root of the equation $x\sin x + \cos x = 0$ using Newton - Raphson method

Sol^n:-

Let given

$$f(x) = x\sin x + \cos x$$

$$\Rightarrow f'(x) = (x\cos x + \sin x) - \sin x$$

$$f'(x) = x\cos x$$

let $a = 2.7$, $b = 2.8$

$\Rightarrow f(a) = f(2.7) = 0.2499 > 0$

$\Rightarrow f(b) = f(2.8) = -0.0043 < 0$

choose $x_0 = \frac{a+b}{2} = \frac{2.7+2.8}{2} = 2.75$

$$\boxed{x_0 = 2.75}$$

$\therefore x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

First Approximation:-

$f(x_0) = 0.1258$, $f'(x_0) = -2.5418$

$i = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.75 - \frac{0.1253}{-2.5418}$$

$$x_1 = 2.7993$$

2^{nd} Approximation :-

$i=1, \quad x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x_1) = -0.0024, \quad f'(x_1) = -2.6369$

$x_2 = 2.7993 - \dfrac{0.0024}{2.6369}$

$x_2 = 2.7984$

3^{rd} Approximation :-

$i=2, \quad x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

$= 2.7984 - \dfrac{0}{-2.6352}$

$x_3 = 2.7984$

Since $\boxed{x_2 = x_3 \approx 2.7984}$

$\therefore$ Hence ~~the~~ Then real possitive root is

2.7984

* Solve the following system by the method of factorisation $x+3y+8z=4$, $x+4y+3z=-2$, $x+3y+4z=1$.

Sol^n: Given that

$$\left.\begin{array}{l} x+3y+8z=4 \\ x+4y+3z=-2 \\ x+3y+4z=1 \end{array}\right\} \longrightarrow (1)$$

The above systems is of the form $AX=B$

where

$$A=\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B=\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

let $A=LU \longrightarrow (2)$

where

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{12} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

from eq (2)

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{12} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}=\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ l_{12}a_{11} & l_{12}a_{12}+a_{22} & l_{12}a_{13}+a_{23} \\ l_{31}a_{11} & l_{31}a_{12}+l_{32}a_{22} & l_{31}a_{13}+l_{32}a_{23}+a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Comparing squar matrix on b.s

we have

$\boxed{a_{11} = 1}$, $\boxed{a_{12} = 3}$, $\boxed{a_{13} = 8}$:

$l_{12}a_{11} = 1$, $l_{12}a_{12} + a_{22} = 4$, $l_{12}a_{13} + a_{23} = 3$

$\boxed{l_{12} = 1}$ $\quad 3 + a_{22} = 4$, $\quad 8 + a_{23} = 3$

$\boxed{a_{22} = 1}$ $\quad \boxed{a_{23} = 5}$

$l_{31}a_{11} = 1$, $l_{31}a_{12} + l_{32}a_{22} = 3$;

$\boxed{l_{31} = 1}$ $\quad 1(3) + l_{32}(5) = 3$.

$5l_{32} = 0$

$\boxed{l_{32} = 0}$

$l_{31}a_{13} + l_{32}a_{23} + a_{33} = 4$

$1(8) + 0 + a_{33} = 4$

$\boxed{a_{33} = -4}$

Then

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{bmatrix}$$

Consider $LY = B$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore y_1 + 3y_2 + 8y_3 = 4 \longrightarrow (3)$$

$$y_2 + 5y_3 = -2 \longrightarrow (4)$$

$$-4y_3 = 1$$

$$\boxed{y_3 = -1/4}$$

from eq (4), we get

$$y_2 + (-5/4) = -2$$

$$y_2 = -2 + \frac{5}{4} = \frac{-8+5}{4}$$

$$\boxed{y_2 = -3/4}$$

from eq (3), we have

$$y_1 - \frac{9}{4} - \frac{8}{4} = 4 \Rightarrow y_1 - \frac{9+8}{4} = 4$$

$$\Rightarrow 4y_1 = 4 + 17 \Rightarrow \boxed{y_1 = 21/4}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{y_1 = 4} \qquad y_1 + y_2 = -2 \Rightarrow y_2 = -2-4$$

$$y_1 + y_3 = 1 \Rightarrow y_3 = 0$$

$$y_1 + y_2 = -2$$

$$4 + y_2 = -2 \Rightarrow y_2 = -6$$

$$y_1 + y_3 = 1$$

$$\boxed{y_3 = -3}$$

and let $Ux = y$

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -3 \end{bmatrix}$$

$$x + 3y + 8z = 4,$$

$$y + z = -6 \Rightarrow y = -6 - \frac{3}{4}$$

$$-4z = -3 \qquad y = -\frac{27}{4}$$

$$\boxed{z = 3/4}$$

$$x = 4 - \frac{81}{4} + \frac{24}{4}$$

$$= \frac{16 - 81 + 24}{4} = \frac{40 - 81}{4}$$

$$\boxed{x = -39/4}$$

Newton's Forward Method (or) Formulae :-

Let the function $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots$ Corresponding, to the values $x_0, x_1, x_2 \ldots$ of 'x'. Suppose it is required to evaluate $f(x)$ = for $x = x_0 + ph$ where 'p' is any real number. and $h = x_n - x_{n-1}$; $p = \frac{x - x_0}{h}$

For any real number 'p', Such that

$$\boxed{y_p = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots}$$

It is called N. F. I. F., h = step interval.

(1). Find the polynomial for the following data by N. F. I. F.

(a).

x	0	1	2	3	4
y	1	5	11	19	29

Ans : 2.75

(b).

x	0	1	2	3	4
y	3.	6	11	18	27

Ans : 4.25 and hence

find the value of y at x = 0.5.

(2). The population of a town in the decimal Census was given below. Estimate the population for the year 1895.

x (year)	1891	1901	1911	1921	1931
Population y (Thousand)	46	66	81	93	101

$h = 10$.

Ans 54.45 $\approx$ 55 Thou

(3). For $x = 0, 1, 2, 3, 4$; $f(x) = 1, 14, 15, 5, 6$ Find $f(3)$ using Forward difference table.

Ans: 5.

Newton's Backward Formulae :-

Let the formula function $y = f(x)$ take the values $y_0, y_1, y_2 \ldots$ Corresponding to the values $x_0, x_1, x_2 \ldots$ of 'x'. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$ where 'p' is any real number. Then we have

$$y_p = f(x_n + ph) = y_n + p \Delta y_n + \frac{p(p+1)}{2!} \Delta^2 y_n + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_n + \cdots$$

$\textcircled{1}$ is Called Newton's Backward Interpolation formula.

problem :- 1 :- Find the cubic polynomial which takes the following values:

x	0	1	2	3
$f(x)$	1	2	1	10

and hence find the f(4)

Sol^n:-

The difference table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0 x_0	1 y_0			
		1 Δy_0		
1 x_1	2 y_1		-2 $\Delta^2 y_0$	
		-1 Δy_1		12 $\Delta^3 y_0$ $\Delta^3 y_n$
2 x_2	1 y_2		10 $\Delta^2 y_n$	
		$+9$ Δy_n		
3 (x_n)	10 y_n			

Step interval $h=1$, $\boxed{x=4, \; x_n = 3}$

$x = x_n + ph \Rightarrow 4 = 3 + p(1)$

$\boxed{p = 1}$

Now, the Newton's Backward Interpolation formula

$$y_p = f(x_n + ph) = y_n + p\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_n + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_n$$

$$f(x) = 10 + x(9) + \frac{x(x+1)}{2!}10 + \frac{x(x+1)(x+2)}{3!}12 + \frac{x(x-1)(x-2)(x-3)}{24}$$

$= 10 + 9x + \frac{x^2+x}{2}(10) + \frac{(x^2+x)(x+2)}{6}(12)$

$= 10 + 9x + 5x^2 + 5x + (x^3 + 2x^2 + x^2 + 2x)2$

$= 10 + 9x + 5x^2 + 5x + 2x^3 + 6x^2 + 4x$

$\boxed{f(x) = 2x^3 + 11x^2 + 18x + 10}$

Since $x = 4$

$f(4) = 2(64) - 16 + 32 + 10$

$= 128 + 42 - 16$

$= 112 + 42$

$\boxed{f(4) = 154} = y$

19
29
41
82

2). In the table below, the values of 'y' are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series:

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Central difference interpolation :-

i) x takes the values $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, x_4 \ldots$ and the corresponding values of $y = f(x)$ are $y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, y_4 \ldots$ then we can write the difference table in the two notations as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
---	---					
x_{-4}	y_{-4}					
		$\Delta y_{-4} = y_{-3} - y_{-4}$				
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$			
		Δy_{-3}		$\Delta^3 y_{-4}$		
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$	
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^4 y_{-4}$
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$	
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^4 y_{-3}$
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^4 y_{2}$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$	
		Δy_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^2 y_1$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					
---	---					

Gauss's Forward Interpolation Formula :-

The Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\ldots$$

we have

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$$\Delta^2 y_0 = \Delta^3 y_{-1} + \Delta^2 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \text{ etc.}$$

also

$$\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$$

$$\Delta^3 y_{-1} = \Delta^4 y_{-2} + \Delta^3 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.}$$

^{lly} Substituting for $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0 \ldots$ ~~from~~ in (1), we get, $x = x_0 + ph$

$$f(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called Gauss's forward Interpolation formulae.

Example 1: Find $f(22)$ from the Gauss forward formula

x :	20	25	30	35	40	45
$f(x)$: y	354	332	291	260	231	204

Sol: The difference table is as follows:

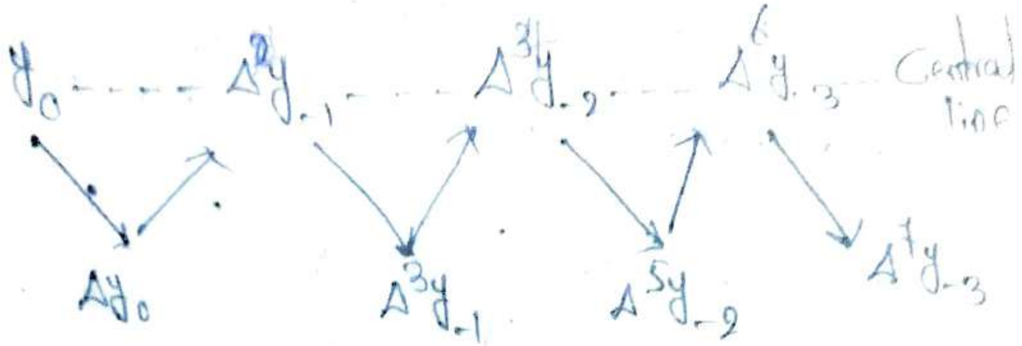
x	$f(x)=y_p$	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
20 (x_0)	354 y_{-1} (y_0)				
		$\Delta y_{-1}=-22$			
25	332 y_0		$\Delta^2 y_{-1}=-19$		
		$\Delta y_0=-41$		$\Delta^3 y_{-1}=29$	
30	291 y_1		$\Delta^2 y_0=10$		$\Delta^4 y_{-1}=-37$
		$\Delta y_1=-31$		$\Delta^3 y_0=-8$	
35	260 y_2		$\Delta^2 y_1=2$		$\Delta^4 y_0=8$
		$\Delta y_2=-29$		$\Delta^3 y_1=0$	
40	231 y_3		$\Delta^2 y_2=2$		
		$\Delta y_3=-27$			
45	204 y_4				

let $x_0=20$, $x=22$, $h=5$, $y_0=354$

$x=x_0+ph \Rightarrow 22-20=p(5)$

$2=\cancel{0.6p}$ $p=0.5$

Note:- (1). It employs odd differences just below the Central line and even difference on the central line as shown below:



(2). This formula is used to interpolate the values of 'y' for $(0<p<1)$ measured forwardly from the origin.

Now, Gauss Forward Interpolation formula is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+1)(p+2)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$= 332 + (-0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2}(-19)$$
$$+ \frac{(-0.6)(-0.6+1)(-0.6-1)(-0.6-2)}{24}(0) + 0.$$

$f(22) = y_p = 332 + 24.6 - 9.12 + 0 + 0$

$$\boxed{f(22) = 347.48.}$$

(2). from the following table values of x and y interpolate values of y when $x = 1.91$.

x	1.7	1.8	1.9	2.0	2.1	2.2
$e^x = y$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Sol: The Central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.7	5.4739					
		0.5757				
1.8	6.0496		0.0606			
		0.6363		0.0063		
1.9 x_0	6.6859 y_0		0.0669		0.0007	0.000
		0.7032		0.0070		
2	7.3891		0.0739		0.0008	
		0.7771		0.0078		
2.1	8.1662		0.0817			
		0.8588				
2.2	9.0250					

Since $h = 0.1$, $x = 1.91$, $y_0 = 6.6859$, $x_0 = 1.9$

$x = x_0 + ph$

$1.91 = 1.9 + p(0.1)$

$\boxed{p = 0.1}$

now, G.F.D.F is given by

$$y_p = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$\frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+1)(p+2)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$= 6.6859 + (0.1)(0.7032) + \frac{(0.1)(0.1-1)}{2}(0.0669)$$

$$+ \frac{(0.1+1)(0.1)(0.1-1)}{6}(0.0070) + \frac{(0.1+1)(0.1)(0.1-1)(0.1-2)}{24}0.0007$$

$$+ \frac{(0.1+1)(0.1+2)(0.1)(0.1-1)(0.1-2)}{120} 0$$

$= 6.6859 + 0.07032 - 0.00301 - 0.000115$

$+ 0.00000$

$\boxed{f(1.91) = 6.7531}$

Gauss

(3). using G.F.I.F to find $f(30)$ given that $f(21)=18.4708$, $f(25)=17.8144$, $f(29)=17.10$ $f(33)=16.3432$, $f(37)=15.5154$.

H.W

Ans:- 16.921

Gauss Backward Interpolation formula :-

The Newton's forward interpolation for is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \cdots \rightarrow$$

we have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$

Similarly

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \text{ etc.}$$

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

^{ll}y

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.}$$

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ in (i) in (1)

we get.

$$y_p = y_0 + p\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{p(p-1)}{1.2}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right)$$

$$+ \frac{p(p-1)(p-2)}{1.2.3}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{1.2}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{1.2.3}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1} + \ldots$$

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$\cancel{+ \frac{(p+1)(p(p-1))(p-2)}{}}\,\Delta^4 y_{-2} + \frac{(p+1)(p+2)p(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$+ \frac{(p+1)(p+2)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called Gauss's Backward Interpolation formula.

Note: (1). This formula contains odd difference above the central line and even difference on the central line as shown below:

(8). It is used to interpolate the value of y for a negative value of p lying b/w -1 and 0 (i.e $-1 < p < 0$)

Example: 1

Use Gauss's Backward Interpolation Formula find $f(32)$ given that $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35) = 0.3386$, $f(40) = 0.3794$.

Sol:-

The following difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
25	0.2707			
		0.032		
30	0.3027		0.0039	
		0.0359		0.0010
35 (x_0)	0.3386 (y_0)		0.0049	
		0.0408		
40	0.3794			

C.L

let us take $x_0 = 35$, $y_0 = 0.3386$, $h = 5$

let $x_p = 32 = x$, then $x_p = x_0 + ph$

$32 = 35 + p(5)$

$p = \frac{32 - 35}{5} = -0.6$

By Gauss Backward Interpolation formula.

$$f(x) = y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)(p)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$f(32) = y_p = 0.3386 + (-0.6)(0.0354) + \frac{(-0.6+1)(-0.6)}{2}(0.0040)$$
$$+ \frac{(-0.6+1)(0.6)(-0.6-1)}{6}(-0.0010) + 0\ (0.0010)$$

$$= 0.3386 - 0.0215 - 0.0005 - 0.00006$$

$$\boxed{f(32) = 0.3165}$$

(2). Using Gauss Backward difference formula, find $f(8)$ (or) $y(8)$ from the following table

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Ans: 13.9056

(3). Find by G.B.D.F, the values of y at $x = \cancel{1936}$ 2000 using the following table.

x	1990	1993	1996	1999	2002
y	110	90	101	98	89

Stirling Formula:-

- The Stirling formula is given by

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)$$

$$+ \frac{p^2(p^2-1)(p^2-2^2)}{6!}\Delta^6 y_{-3} + \cdots$$

Note:- (1). This formula involves Means of the odd differences just above and below the central line and even differences on the central line as shown below.

$$\text{---} y_0 \text{---} \left\{\begin{matrix}\Delta y_{-1} \\ \Delta y_0\end{matrix}\right\} \text{---} \Delta^2 y_{-1} \text{---} \left\{\begin{matrix}\Delta^3 y_{-2} \\ \Delta^3 y_{-1}\end{matrix}\right\} \text{---} \Delta^4 y_{-2}$$

$$\text{---} \left\{\begin{matrix}\Delta^5 y_{-3} \\ \Delta^5 y_{-2}\end{matrix}\right\} \text{---} \Delta^6 y_{-3} \text{---} \text{ Central Line}$$

(2). It is used to interpolate the values of 'y' for

$-0.5 \le p \le 0.5$

(1). From the following table find y when $x = 38$

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

(2). Given

θ	0	5	10	15	20	25	30
$\tan\theta$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

using Stirling formula, estimate the value of $\tan 16^\circ$

Ans: 0.28676

(3). Employ Stirling formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$)

x	10	11	12	13	14
$10^5 y_x$	23967	28060	31788	35209	38368

Ans:

Bessel's Formula :-

The Bessel's formula is given by

$$f(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{(p-\frac{1}{2})(p)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)(p)(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$+ \frac{(p+1)(p-\frac{1}{2})p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(p+1)(p+2)p(p-1)(p-2)}{6!}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right)$$

$$+ \cdots$$

Note :-

* This is a very useful formula for practical purposes. It involves odd differences below the Central line and means of even differences of ~~above~~ and below Central line as shown below:

$$y_0 \cdots \Delta y_0 \begin{Bmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{Bmatrix} \cdots \Delta^3 y_0 \begin{Bmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{Bmatrix} \cdots \Delta^5 y_{-2} \begin{Bmatrix} \Delta^6 y_{-3} \\ \Delta^6 y_{-2} \end{Bmatrix}$$

$\cdots$ Central line.

* This formula is used to interpolate the values of y for p $\left(\frac{1}{4} < p < \frac{3}{4}\right)$ measured forwardly from the origin.

problem 1:-

Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.

Sol^n:-

x	y_x	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$
20	2854 y_{-1}			
		308 Δy_{-1}		
(x_0) 24	3162 (y_0)		74 $\Delta^2 y_{-1}$	
		382 Δy_0		-8
28	3544 y_1		66 $\Delta^2 y_0$	
		448 Δy_1		
32	3992 y_2			

at $x = 25$, $x_0 = 24$, $y_0 = 3162$, $h = 4$

$x = x_0 + ph$

$25 = 24 + p(4)$

$\frac{1}{4} = p$

$0.25 = p$ i.e. $0.25 \leq p < 0.75$

Now Bessel's formula is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \frac{(p-\frac{1}{2})p}{3!}\left(\Delta^3 y_{-1}\right)$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} - \Delta^4 y_{-1}}{2}\right)$$

$$+ \frac{(p-\frac{1}{2})(p+1)(p)(p-1)(p-2)}{5!}\Delta^5 y$$

$$+ \cdots$$

$$y_{0.25} = 3162 + (0.25)(382) + \frac{(0.25)(0.25-1)}{2!}\left(\frac{74+66}{2}\right) + \frac{(0.25-\frac{1}{2})(0.25)(0.25-1)}{6}(-8)$$

$$= 3162 + 95.5 - 6.5625 - 0.0625$$

$$- 6.5625$$

$$y_{0.25} = 3250.8750$$

2). apply Bessel's formula to find the value of $f(27.5)$ from the table:

x:	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

Ans: 3.585

3). Apply B.D.F to obtain y_{25} given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$

~~4). Apply B.D.F to find the value of $f(2$~~

* Lagrange's Interpolation Formula :

let $x_0, x_1, x_2 \ldots x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. let $y_0, y_1, y_2 \ldots y_n$ be the corresponding values of $y = f(x)$. let the polynomial of degree for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0))\ (x_1, f(x_1)) \ldots (x_n, f(x_n))$ be in the following form.

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} f(x_2)$$

$$+ \ldots$$

$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

(1). A Curve passes through the point (0, 18), (1, 10), (3, -18) and (6, 90). Find the slope of the curve at $x = 2$

Ans :- $(2x^3 - 10x^2 + 18)$, -16

(2). Using Lagrange's formula, Express the function $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions.

$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$

(3). Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively. Use L.I.F.

(4). Using L.I.F. find the value of $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

Unit - III

Numerical Solution of Ordinary Differential Equations :-

Taylor Series Method :-

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \longrightarrow (1)$$

$y' = f(x, y)$

given the initial condition $y(x_0) = y_0$

$y(x)$ can be expanded about the point x_0 in a Taylor's Series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots$$

$$+ \frac{(x - x_0)^n}{n!} y^n(x_0) + \cdots \longrightarrow (2)$$

where $y^i(x_0)$ is the i^{th} derivative of $y(x)$ at x_0

If we let $x - x_0 = h$ (i.e. $x = x_1 = x_0 + h$),

we can write the Taylor's series as

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots \longrightarrow (3)$$

i.e. $$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

one finding the value y_1 for $x = x_1$ using eq (2) (or) eq (3), y', y'', y''' etc. can be found at $x = x_1$ by means of eq (1), then y can be expanded about $x = x_1$

Thus $$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

III^{ly} expanding $y(x)$ in a Taylor's series about the point x_1, we will get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

III^{ly} $$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \cdots$$

where $y_n^r = \left(\frac{d^r y}{dx^r}\right)_{(x_n, y_n)}$.

Q: Using Taylor's Series method, Solve the equation $\frac{dy}{dx} = x^2+y^2$ for $x = 0.4$, given that $y_0 = 0$ when $x_0 = 0$

Sol: Given equation

$$\frac{dy}{dx} = x^2+y^2 \qquad \text{where } f(x,y) = x^2+y^2$$

$$\Rightarrow y' = x^2+y^2 = f(x,y)$$

Differentiating repeatedly w.r.t. to x

We get

$2y \cdot \frac{dy}{dx}$ $\quad$ $\frac{d}{dx} y'^2 =$

$2y \cdot y'$ $\quad$ $2y' \frac{dy'}{dx}$

$2y' \cdot y''$

$$y'(x) = \frac{dy}{dx} = x^2+y^2$$

$$\Rightarrow y'' = 2x + 2y \cdot y', \quad y''' = 2 + 2y(y'') + 2y' y' = 2 + 2yy'' + 2y'^2$$

$$y^{iv} = \frac{d^4y}{dx^4} = 2y'y'' + 2yy''' + 4y'y''$$

at $x = 0$, $y = 0$ So, we have

$y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 2$, $y^{iv}(0) = 0$

The Taylor Series for $y(x)$ near $x = 0$

is given by

$$y(x) = y(0) + \frac{x}{1!} y'(0) + y''(0) \cdot \frac{x^2}{2!} + \frac{x^3}{3!} y'''(0) + \ldots$$

$$= \frac{x^3}{3!} \cdot 2 + (\text{neglected})$$

$3! = 3 \times 2 \times 1$
$= 6.$

$$= \frac{x^3}{\cancel{6}\,3} \times \cancel{2}$$

$$y(x) = \frac{x^3}{3}$$

Hence at $x = 0.4$

$$y(0.4) = \left(\frac{(0.4)^3}{3}\right) = \frac{0.064}{3}$$

$$\boxed{y(0.4) = 0.02133}$$

(2). Using Taylor's Series method, find an approximation value of y at $x = 0.2$ for the diff eq $y' - 2y = 3e^{x}$, $y(0) = 0$. Compare the numerical Soln obtained with Exact Soln.

Soln:

(3). Solve $y' = x^2 - y$, $y(0) = 1$ using Taylor's Series method and Compute $y(0.1)$, $y(0.2)$, $y(0.3)$ and $y(0.4)$ (correct to 4 decimal places).

(4). Tabulated $y(0.1)$, $y(0.2)$, and $y(0.3)$ using Taylor's Series method given that $y' = y^2 + x$ and $y(0) = 1$.

(5). Use Taylor's Series method to find the approximate value of y when $x = 0.1$ given $y(0) = 1$ and $y' = 3x + y^2$.

Picard's method of Successive Approximations :-

Consider the differential equation

$\frac{dy}{dx} = f(x,y) \longrightarrow (1)$ given that $y = y_0$ for $x = x_0$ $\longrightarrow (2)$

It is required to obtain the Solution of eq (1) Subject to the Condition (2)

The equation is $dy = f(x,y)\,dx$ $\longrightarrow (3)$

Integrating eq (3), in the interval (x_0, x) we get

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$(y)_{x_0}^{x_0} = \int_{x_0}^{x} f(x,y)\,dx$$

i.e $y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$

$$\boxed{y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx}$$

$$\Rightarrow y(0) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \longrightarrow (4)$$

we find that the R.H.S of eq (A) Contains the unknown y under the ingral Sign. An eq of this Kind is Called an integral eq. and it Can be Solved by a process of Successive approximations.

Picard's method gives a Sequence of functions $y^{(1)}(x)$, $y^{(2)}(x)$, $y^{(3)}(x)$ ....

To get the first approximation $y^{(1)}(x)$ put $y = y_0$ in the integrated eq (A), we get

$$y^{(1)}(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$|||^{ly}$ $$y^{(2)}(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

and $$y^{(3)}(x) = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Proceeding in this way, we get n^{th} appr $y^{(n)}$ for y as

$$y^{n} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\, dx$$

(or)

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx, \quad \text{for } n = 1,2,3\ldots$$

Ex: Find an approximate value of 'y' for $x = 0.1$, $x = 0.2$ if $\frac{dy}{dx} = x + y$ and $y = 1$ at $x = 0$ using picard's method.

(2). Find the value of y for $x = 0.4$ by picard's method, given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$

(3). Solve $y' = y - x^2$, $y(0) = 1$ by picard's method upto the fourth approximation. Hence, find the value of $y(0.1)$, $y(0.2)$.

(4). Given that $\frac{dy}{dx} = 1 + xy$ and $y(0) = 1$, Compute $y(0.1)$ and $y(0.2)$ using picard's method.

Euler's Method :-

Suppose we wish to solve the equation $\frac{dy}{dx} = f(x,y)$ subject to the condition that $y(x_0) = y_0$.

In general, we obtain a recursive relation as

$h + x_0 = x_1$
$h = x_1 - x_0$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}, \quad n = 0, 1, 2, 3 \ldots$$

Ex: use Euler method to find $y(0.1)$, $y(0.2)$ given $y' = (x^3 + xy^2)e^{-x}$, $y(0) = 1$.

Soln: here $h = 0.1$, $f(x,y) = (x^3 + xy^2)e^{-x}$

$x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$

By E. algorithm,

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h(x_0^3 + x_0 y_0^2)e^{-x_0}$$
$$= 1 + (0.1)(0+0)e^{-0} = 1.$$

$$y_2 = 1.0091.$$

(2). Using Euler's method, Solve for y at $x = 2$ from $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, Taking step Size

(i). $h = 0.5$ (ii). $h = 0.25$

(3). Given $y' = x^2 - y$, $y(0) = 1$, find Correct to four decimal places the value of $y(0.1)$, by using E.M.

(4). Solve numerically using Eulers method $y' = y^2 + x$, $y(0) = 1$. find $y(0.1)$ and $y(0.2)$.

(5). Compute y at $x = 0.25$ by Euler's method given $y' = 2xy$, $y(0) = 1$.

Runge - Kutta Method :-

(1). first order R.K.M :

we know that, by E.M

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= y_0 + h y_0' \qquad \because \left(y' = f(x,y) \right)$$

Expanding L.H.S by Taylor's

$$y_1 = y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

(2). Second order R-K-M :-

The modified E-M gives

$$y_1^{(1)} = y_0 + h\,f(x_0, y_0)$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left\{ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right\}$$

$$= y_0 + \frac{h}{2}\left\{ f_0 + f(x_0+h, y_0+hf_0) \right\} \longrightarrow (1)$$

where $f_0 = f(x_0, y_0)$

we set

$$K_1 = hf_0 \,, \quad K_2 = h\left(f(x_0+h, y_0+K_1)\right)$$

from eq (1)

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

where $k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + h, y_0 + k_1)$

(3). Third order R-K Method:-

it is defined by the equation

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = h f(x_0, y_0)$

$$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$$

(4). Fourth order R-K-M:-

It is defined by

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and $k_4 = h f(x_0 + h, y_0 + k_3)$. Then

$$\boxed{y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

(1). Solve the following using R-K fourth method $y' = y - x$, $y(0) = 2$, $h = 0.2$. Find $y(0.2)$

(2). Solve $\frac{dy}{dx} = xy$ using R-K method for $x = 0.2$ given $y(0) = 1$, taking $h = 0.2$.

(3). Apply the fourth order R-K method to find $y(0.1)$ and $y(0.2)$, given $y' = xy + y^2$, $y(0) = 1$.

(4). Solve $y' = x - y$ given that $y(1) = 0.4$. Find $y(1.2)$ using R-K 4th order.

(2). Find $y(0.1)$ and $y(0.2)$ using Euler's modified given that $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$.

Sol:-

Given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$

$\Rightarrow y' = x^2 - y$, $x_0 = 0$, $y_0 = 1$

Here $f(x, y) = x^2 - y$, $h = 0.1$

By modified Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)(x_0^2 - y_0)$$

$$y_1^{(0)} = 1 + (0.1)(0^2 - 1)$$

$$= 1 - 0.1$$

$$= 0.9$$

, $f(x_0, y_0) = 0^2 - 1$
$= -1$

To find y_1 i.e $y(0.1)$:

Now $x = x_1 = x_0 + h$
$= 0 + 0.1$
$x = x_1 = 0.1$

$$y_1^{(1)} = y(0.1) = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(0)})\}$$

$$= 1 + \frac{0.1}{2}\{-1 + (0.1)^2 - 0.9\}$$

$$y(0.1) = 0.9055$$

$$y_1^{(2)} = y(0.1) = y_0 + \frac{h}{2}\left\{ f(x_0, y_0) + f(x_1, y_1') \right\}$$

$$= 1 + \frac{0.1}{2}\left\{ -1 + (0.1)^2 + 0.9955 \right\}$$

$$= \cancel{1.0003}\ 0.9052$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left\{ f(x_0, y_0) + f(x_1, y_1') \right\}$$

$$= 1 + \frac{0.1}{2}\left\{ -1 + (0.1)^2 + \cancel{1.0003}\ 0.9052 \right\}$$

$$y_1^{(3)} = 0.9052$$

$$\therefore\ y_1^{(3)} = y(0.1) = 0.9052$$

Since the value of $y_1^{(2)}$ and $y_1^{(3)}$ are equal, we take

$$y_1 = y(0.1) = \cancel{0.9468}\ 0.9052$$

$$\therefore\ \boxed{y(0.1) = 0.9052 = y_1} \quad \text{approximate}$$

To find y_2 i.e $y(0.2)$:

Now $x = x_2 = x_1 + h$

$= 0.1 + 0.1$

$x = x_2 = 0.2$, $y_1 = 0.9052$

$$y_1^{(1)} = y(0.2) = y_0 + \frac{h}{2}\left\{ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right\}$$

$$= 1 + \frac{0.1}{2}\left\{ -1 + (0.1)^2 - 0.9052 \right\}$$

$$f(x_1, y_1) = f(0.1, 0.9052)$$

$$= (0.1)^2 - 0.9052$$

$$= \cancel{-0.8952} \; -0.8952$$

By Euler's formula

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$= 0.9052 + (0.1)(x_1^2 - y_1)$$

$$= 0.9052 + (0.1)\left((0.1)^2 - 0.9052\right)$$

$$y_2^{(0)} = 0.8157 = y(0.2)$$

$$y_2^{(1)} = y(0.2) = y_1 + \frac{h}{2}\left\{ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right\}$$

$$= 0.9052 + \left(\frac{0.1}{2}\right)\left\{ -0.8952 + (0.2)^2 - 0.8157 \right\}$$

$$= 0.8217$$

$$y_2^{(2)} = y(0.2) = y_1 + \frac{h}{2}\left\{ f(x_1, y_1) + f(x_2, y_2^{(1)}) \right\}$$

$$= 0.9052 + \left(\frac{0.1}{2}\right)\left\{ -0.8952 + (0.2)^2 - 0.8217 \right\}$$

$$= \cancel{0.8214}$$

$$= 0.8214$$

$$y_2^{(3)} = 0.9052 + \left(\frac{0.1}{2}\right)\left\{-0.8952 + (0.2)^2 - 0.8214\right\}$$

$$= 0.8214 = y(0.2)$$

Since the value of $y_2^{(2)}$ and $y_2^{(3)}$ are equal, we take $\boxed{y_2 = y(0.2) = 0.8214}$ approximatly

Hence we Conclude that the value of 'y' when $x = 0.1$ is 0.9052 and the value of 'y' when $x = 0.2$ is 0.8214.

(1). Solve the following using R.K. fourth order method

$y' = y - x$, $y(0) = 2$, $h = 0.2$. Find $y(0.2)$

Sol:

Given $y' = y - x$, $y(0) = 2$

Here $f(x,y) = y - x$, $x_0 = 0$, $y_0 = 2$

$h = 0.2$

By 4^{th} order R-K-method

$k_1 = h f(x_0, y_0)$

$= (0.2) f(0,2)$

$= (0.2)(2) = 0.4$

$k_1 = 0.4$

$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$= (0.2) f\left(\frac{0.2}{2}, 2 + \frac{0.4}{2}\right)$

$= (0.2) f(0.1, 2.2)$

$= (0.2)(2.2 - 0.1)$

$k_2 = 0.42$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.2) f\left(\frac{0.2}{2}, 2 + \frac{0.42}{2}\right)$

$= 0.422$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= (0.2) f(0.2, 2 + 0.422)$

$k_4 = 0.444$

$$y_1 = y(x_0+h) = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y(0.2) = 2 + \frac{1}{6}\{0.4 + 2(0.42) + 2(0.422 + 0.444\}$$

$$\boxed{y(0.2) = 2.4214}$$ //

(2). Apply the fourth order R-K-method to find $y(0.$ and $y(0.2)$, given $y' = xy + y^2$, $y(0) = 1$.

Solⁿ:-

Given $y' = xy + y^2$, $y(0) = 1$

Here $f(x,y) = xy + y^2$, $x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $x_2 = 0.2$, $h = \cancel{0}x_2 - x_1$

$h = 0.1$

By 4^{th} order R-K-method.

$$K_1 = hf(x_0, y_0)$$

$$= (0.1) f(0,1) = (0.1)(0+1)$$

$$K_1 = (0.1)$$

$$K_2 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}K_1)$$

$$= (0.1) f\left(\frac{0.1}{2},\ 1 + \frac{0.1}{2}\right)$$

$$K_2 = 0.1155$$

$K_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)$

$= (0.1)\, f\left(\frac{0.1}{2},\ 1 + \frac{0.1155}{2}\right)$

$k_3 = 0.1171$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

$= (0.1)\, f(0.1,\ 1 + 0.1171)$

$k_4 = 0.136$

$$y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\left(0.1 + 2(0.1155) + 2(0.1171) + 0.136\right)$$

$y_1 = 1.1169$ at $x_1 = 0.1$

To find y_2 i.e $y_2(0.2)$:

$k_1 = hf(x_1, y_1)$

$= (0.1)\, f(0.1,\ 1.1169)$

$k_1 = \cancel{1.3592}\ 0.1359$

$k_2 = (0.1)\, f\left(0.1 + \frac{0.1}{2},\ 1.1169 + \frac{\overset{0.1359}{\cancel{1.3592}}}{2}\right)$

$= (0.1)\, f(0.15,\ \overset{}{\underset{1.1849}{\cancel{1.1865}}})$

$= 0.3497$

$k_2 = 0.1582$

$$K_3 = (0.1) f\left(0.1 + \frac{0.1}{2},\ 1.1169 + \frac{0.1582}{2}\right)$$

$$= (0.1) f(0.15,\ 1.1960)$$

$$K_3 = 0.1610$$

and

$$K_4 = (0.1) f(0.1 + 0.1,\ 1.1169 + 0.1610)$$

$$= (0.1) f(0.2,\ 1.2779)$$

$$K_4 = 0.1889$$

$$y_2 = y(0.2) = y(x_1 + h) = y_1 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$= 1.1169 + \frac{1}{6}\{0.1359 + 2(0.1582) + 2(0.1610) + 0.1889\}$$

$$y_2 = y(0.2) \approx 1.2775$$

$$y_2 = 1.2775 \text{ at } x_2 = 0.2$$

(2). Find $y(0.1)$ and $y(0.2)$ using R-K-4th order formula given that $y' = x^2 - y$ and $y(0) = 1$.

(3). Solve the following using R-K 4th method $y' = y - x$, $y(0) = 2$, $h = 0.2$ find $y(0.2)$.

(4). Apply the fourth order R-K method to find $y(0.1)$ and $y(0.2)$, given $y' = xy + y^2$, $y(0) = 1$.

(5). Solve $\frac{dy}{dx} = xy$ using R-K ~~meth~~ 4^{th} formula for $x = 0.2$ given $y(0) = 1$ taking $h = 0.2$.

(6). Solve $y' = x - y$ given that $y(1) = 0.4$. Find $y(1.2)$ using R-K 4^{th} order formula.