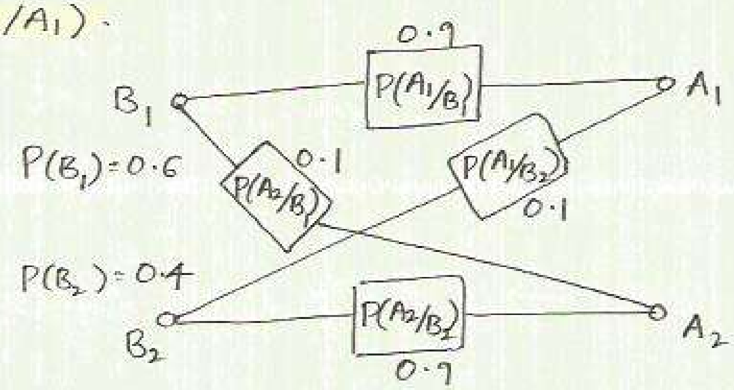
Unit – I

① a) A binary Symmetric Channel is as shown below. Find the probability of (i) $P(A_1)$ (ii) $P(A_2)$ (iii) $P(B_1/A_1)$ (iv) $P(B_2/A_2)$ (v) $P(B_1/A_2)$ (vi) $P(B_2/A_1)$.



Soln: A binary Symmetric Channel is as shown.

Given Data:

$P(B_1) = 0.6$

$P(B_2) = 0.4$.

$P(A_1/B_1) = 0.9$

$P(A_1/B_2) = 0.1$

$P(A_2/B_1) = 0.1$

$P(A_2/B_2) = 0.9$.

(i) We know that, From the total probability theorem

$$P(A_1) = P(A_1/B_1) \cdot P(B_1) + P(A_1/B_2) \cdot P(B_2)$$
$$= 0.9(0.6) + 0.1(0.4) = \underline{\underline{0.58}}$$

(ii)
$$P(A_2) = P(A_2/B_1) \cdot P(B_1) + P(A_2/B_2) \cdot P(B_2)$$
$$= 0.1(0.6) + 0.9(0.4) = \underline{\underline{0.42}}$$

(iii) From Baye's Theorem, we have,

$$P(B_1/A_1) = \frac{P(A_1/B_1) \cdot P(B_1)}{P(A_1)} = \frac{0.9(0.6)}{0.58} = \underline{\underline{0.931}}$$

(iv)
$$P(B_2/A_2) = \frac{P(A_2/B_2) \cdot P(B_2)}{P(A_2)} = \frac{0.9(0.4)}{0.42} = \underline{\underline{0.857}}$$

(v)
$$P(B_1/A_2) = \frac{P(A_2/B_1) \cdot P(B_1)}{P(A_2)} = \frac{0.1(0.6)}{0.42} = 0.143$$

(vi)
$$P(B_2/A_1) = \frac{P(A_1/B_2) \cdot P(B_2)}{P(A_1)} = \frac{0.1(0.4)}{0.58} = \underline{\underline{0.069}}$$

b) List the properties of conditional density function.

Ans: The conditional density function of a random variable 'X' is defined as the derivative of the conditional distribution function $F_X(x/B)$. It was denoted by $f_X(x/B)$ and is given by properties of conditional density function:

→ The conditional density function is always non negative i.e

$$f_X(x/B) \geq 0$$

→ The area under the conditional density is equal to 1 i.e

$$\int_{-\infty}^{\infty} f_X(x/B)\, dx = 1.$$

→ $$F_X(x/B) = \int_{-\infty}^{x} f_X(\lambda/B)\, d\lambda$$

→ $$P(x_1 < X \leq x_2/B) = \int_{x_1}^{x_2} f_X(x/B)\, dx.$$

② a) What is the Concept of Random Variable? Explain with a suitable example.

Ans: A Random Variable is defined as a real valued function of the elements of the Sample Space 'S'. Random Variables are usually represented by upper Case letters (X, Y, Z etc) and the values taken by the random variable are represented using lower Case letters (x, y, z, etc). Thus for a given experiment defined by a Sample Space 'S' with elements 's', we assign to every 's' a real number $X(s)$ according to some rule. A Random Variable 'X' can be considered to be a function that maps all elements of the Sample Space into a points on the real Line. Below example Shows the mapping of a random variable.

An experiment Consists of rolling a die and flipping a coin. The resulting sample Space is as shown. Here two random Variables are defined,

→ Coin Head and outcome corresponds to positive values of 'X' that are equal to the numbers that show upon the die.

→ Coin Tail and outcome Corresponds to negative values of 'X' that are equal in magnitude to the twice the number that show up on the die.

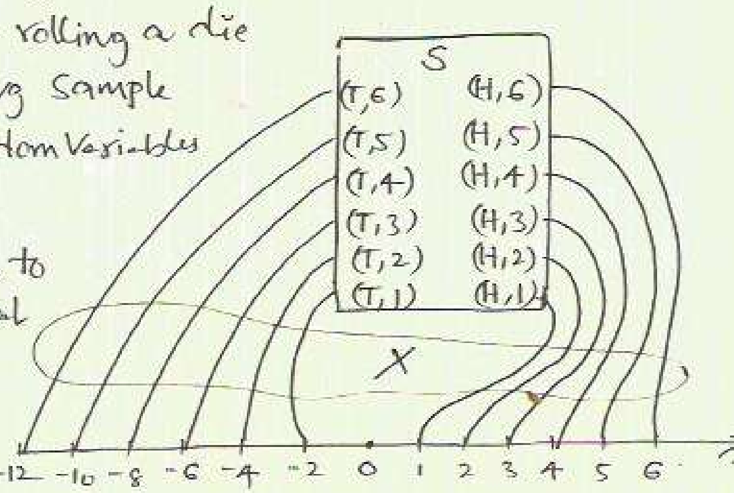


fig. A Random Variable Mapping Of a Sample Space.

b) A Random Variable 'X' has the distribution function

$$F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n).$$

Find the probabilities (i) $P(-\infty < X \leq 6.5)$ (ii) $P(X>4)$ (iii) $P(6<X\leq 9)$

Soln: The distribution function of a random variable 'X' is given by

$$F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n) \longrightarrow ①$$

(i) $P(-\infty < X \leq 6.5) = F_X(6.5)$

$$= \sum_{n=1}^{12} \frac{n^2}{650} u(6.5-n)$$

$u(x) = 1, x \geq 0$
0, elsewhere

$$= \sum_{n=1}^{6} \frac{n^2}{650} u(6.5-n)$$

$$= \frac{1^2}{650} + \frac{2^2}{650} + \frac{3^2}{650} + \frac{4^2}{650} + \frac{5^2}{650} + \frac{6^2}{650}$$

$$= \frac{91}{650} = 0.140.$$

(ii) $P(X>4) = 1 - P(X \leq 4)$

$$= 1 - F_X(4)$$

$$= 1 - \left(\sum_{n=1}^{4} \frac{n^2}{650} u(4-n) \right)$$

$$= 1 - \left(\frac{1^2}{650} + \frac{2^2}{650} + \frac{3^2}{650} + \frac{4^2}{650} \right)$$

$$= 1 - \frac{30}{650} = \frac{620}{650} = 0.9538$$

(iii) $P(6<X\leq 9) = F_X(9) - F_X(6)$

$$= \sum_{n=1}^{9} \frac{n^2}{650} - \sum_{n=1}^{6} \frac{n^2}{650}$$

$$= \frac{7^2}{650} + \frac{8^2}{650} + \frac{9^2}{650} = \frac{194}{650} = 0.2985$$

③ a) Give classical and Axiomatic definitions of probability.

Ans: Classical Definition of probability: The probability $P(A)$ of an event 'A' is defined as the ratio of the number of favourable outcomes of event 'A' to the total no. of possible outcomes, provided that all the outcomes are equally likely.

$$\therefore \quad P(A) = \frac{N_A}{N} = \frac{\text{No. of favourable outcomes}}{\text{Total no. of possible outcomes}}$$

Ex: When a fair coin is tossed, the possible outcomes are two. If event 'A' is " Show heads", then the favourable outcome is one.

$$\therefore \quad P(A) = \frac{1}{2}.$$

Axiomatic definition of probability: The probability P(A) of an event is a non negative real number which satisfies the following axioms.

Axiom 1: $P(A) \geq 0$ i.e the probability of an event 'A' is always non negative and a real number.

Axiom 2: $P(S) = 1$. Since the event 'S' is a universal set, and is sure event, it should have highest probability. $P(\phi) = 0$. Since the null is an impossible event it is having zero probability.

Axiom-3: let 'N' events A_n, $n = 1, 2, 3 \cdots N$, where 'N' may be possibly infinite, defined on a sample space 'S' and satisfies the property, $A_m \cap A_n = \phi$, $\forall\, m \neq n$ i.e all events are mutually exclusive then axiom-3 states that,

$$P\left(\bigcup_{i=1}^{N} A_n\right) = \sum_{i=1}^{N} P(A_n), \quad \text{if } A_m \cap A_n = \phi. \text{ i.e}$$

The probability of union of any no. of mutually exclusive events is equal to the sum of the individual event probabilities.

(b) In a single through of two dice, what is the probability of obtaining a sum atleast 10.

Ans: In an experiment of throwing two dice, the sample space 'S' contains 36, outcomes as shown.

$$\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

let the event 'A' be getting sum atleast '10'

The outcomes favourable to event 'A' are,

(5,5), (6,4), (4,6), (6,5), (5,6), (6,6)

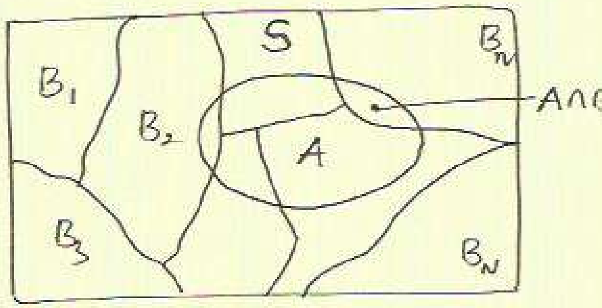
$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}.$

④ a) State and prove Bayes Theorem.

<u>Ans:</u> Bayes theorem states that if the sample space 'S' has 'N' mutually exclusive events B_n, $n = 1,2,3 \ldots N$, such that $B_m \cap B_n = \phi$, for $m \neq n$ as shown in fig. and let 'A' be any event defined on this sample space then the conditional probability of B_n given 'A' can be written as,

$$P(B_n/A) = \frac{P(A/B_n) \cdot P(B_n)}{P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + \cdots + P(A/B_n) \cdot P(B_n)}.$$

<u>proof:</u> The proof can be derived from the total probability theorem and the definition of conditional probabilities.



we know that the conditional probability is given as

$$P(B_n/A) = \frac{P(B_n \cap A)}{P(A)}, \quad P(A) \neq 0. \longrightarrow ①$$

also $P(B_n \cap A) = P(A/B_n) \cdot P(B_n)$ and $\longrightarrow ②$

from the total probability theorem, we have,

$$P(A) = \sum_{n=1}^{N} P(B_n \cap A)$$

$$P(A) = \sum_{n=1}^{N} P(A/B_n) \cdot P(B_n) \longrightarrow ③$$

Substitute Equations ② & ③ in Equ ① we get one form of Baye's Theorem as,

$$P(B_n/A) = \frac{P(A/B_n) \cdot P(B_n)}{\sum_{n=1}^{N} P(A/B_n) \cdot P(B_n)} \quad \text{or}$$

$$P(B_n/A) = \frac{P(A/B_n) \cdot P(B_n)}{P(A/B_1) \cdot P(B_1) + P(A/B_2) P(B_2) + \cdots\cdots + P(A/B_N) P(B_N)}$$

In Bayes theorem, the probabilities $P(B_n)$ are usually referred to as 'a priori probabilities' since events B_n have probabilities $P(B_n)$ before the performance of the experiment. Similarly the probabilities $P(B_n/A)$ are known as 'posteriori probabilities since they apply after the performance of the experiment.

b) A Random Variable 'X' has the density function $f_X(x) = \frac{1}{2} u(x) \exp(\frac{-x}{2})$. Evaluate the probabilities of the events. $A = \{1 < X \leq 3\}$, $B = \{X \leq 2.5\}$. $C = A \cap B$.

Soln:

given the density function of the R.V. 'X' as

$$f_X(x) = \tfrac{1}{2} u(x) e^{-x/2}$$

$\therefore$ The distribution function of the random variable is given as

$$F_X(x) = \int_{-\infty}^{x} f_X(\lambda) \cdot d\lambda$$

$$= \int_{-\infty}^{x} \tfrac{1}{2} u(\lambda) e^{-\lambda/2} d\lambda$$

$$= \tfrac{1}{2} \int_{0}^{x} e^{-\lambda/2} d\lambda$$

$$= \tfrac{1}{2} \cdot \frac{e^{-\lambda/2}}{(-\frac{1}{2})} \Big|_0^x$$

$$= -(e^{-x/2} - 1), \quad x \geq 0$$

$$\therefore \quad F_X(x) = (1 - e^{-x/2}) \cdot u(x)$$

(i) $A = \{1 < X \leq 3\}$

$$\therefore \quad P(A) = P(1 \leq X \leq 3) = F_X(3) - F_X(1)$$

$$= \cancel{1} - e^{-3/2} - (\cancel{1} - e^{-1/2})$$

(ii) $B = \{X \leq 2.5\}$

$\therefore \quad P(B) = P\{X \leq 2.5\}$

$= F_X(2.5)$

$= (1 - e^{-2.5/2}) = 0.7135$

(iii) $A \cap B = C$

$\therefore \quad C = \{1 < X \leq 2.5\}$

$= F_X(2.5) - F_X(1)$

$= (\cancel{1} - e^{-2.5/2}) - (\cancel{1} - e^{-1/2})$

$= e^{-1/2} - e^{-2.5/2} = 0.3200$

5 a) Compute the joint and conditional probabilities based on the given data. In a box there are 100 resistors having resistances and tolerance as shown in below table. Define three events as "A: Draw a 47Ω resistor", "B: Draw a 5% tolerance resistor", "C: Draw a 100Ω resistor.

Resistance (Ω)	Tolerance 5%	Tolerance 10%	Total
22	10	14	24
47	28	16	44
100	24	08	32
Total	62	38	100

Soln: In a box there are 100 resistors having resistances and tolerances as shown. Define three events as

A: Draw a 47Ω resistor

B: Draw a 5% tolerance resistor

C: Draw a 100Ω resistor.

→ Event probabilities: $P(A) = \frac{44}{100}$

$P(B) = \frac{62}{100}$

$P(C) = \frac{32}{100}$.

→ Joint probabilities:

$P(A \cap B) = \frac{28}{100}$

$P(B \cap C) = \frac{24}{100}$

$P(C \cap A) = \frac{0}{100} = 0$

→ The conditional probabilities are given by

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{28/106}{62/106} = \frac{28}{62}$$

$$P(B/C) = \frac{P(B\cap C)}{P(C)} = \frac{24/106}{32/106} = \frac{24}{32}$$

$$P(C/A) = \frac{P(C\cap A)}{P(A)} = \frac{0}{44/100} = 0$$

$$P(B/A) = \frac{P(B\cap A)}{P(A)} = \frac{28/106}{44/106} = \frac{28}{44}$$

$$P(C/B) = \frac{P(C\cap B)}{P(B)} = \frac{24/106}{62/106} = \frac{24}{62}$$

$$P(A/C) = \frac{P(A\cap C)}{P(C)} = \frac{0}{32/100} = 0$$

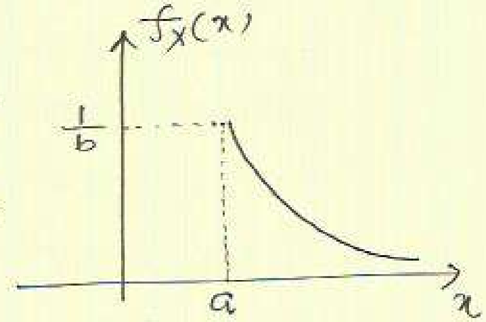
(b) Define and explain the following distribution and densities with an application.

(i) Exponential

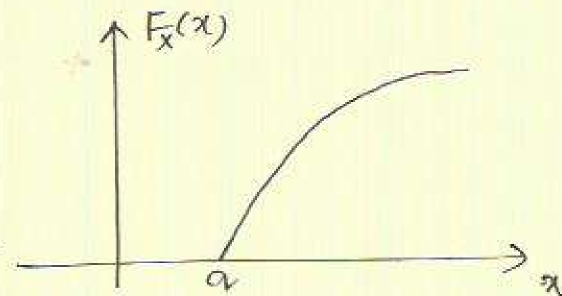
(ii) Uniform.

Ans: Exponential distribution: The exponential density and distribution functions of a random Variable 'X' are defined as,

$$f_X(x) = \begin{cases} \frac{1}{b}\left(e^{-(x-a)/b}\right), & x > a \\ 0, & x < a \end{cases}$$



$$F_X(x) = \begin{cases} 1 - e^{-(x-a)/b}, & x > a \\ 0, & x < a \end{cases}$$



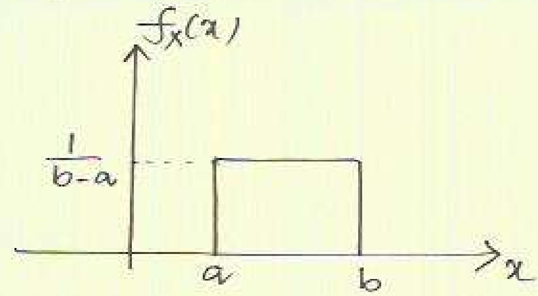
The plots are as shown in fig.

Applications: → It is useful in describing raindrop sizes when a large no. of rainstroam measurements are made

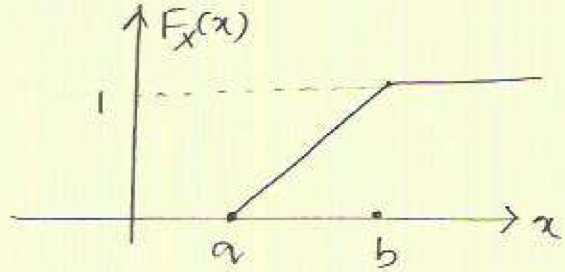
→ This distribution is used to describe the fluctuations in Signal Strength received by radar from Certain types

Uniform distribution: The uniform probability density and distribution functions are defined as

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$



$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b \end{cases}$$



The plots of uniform distribution and density are as shown.

Applications: Quantization of signal samples in digital comm. systems. The errors introduced in the rounding off process are uniformly distributed.

⑥ a) Define conditional distribution function and list its properties.

Ans: Conditional distribution function: let A be defined as the event $\{X \leq x\}$ for the random variable 'X'. The probability $P\{X \leq x/B\}$ is defined as the conditional distribution function of 'X' denoted by $F_X(x/B)$.

Thus,

$$F_X(x/B) = P(X \leq x/B) = \frac{P(X \leq x \cap B)}{P(B)}$$

The joint event $\{X \leq x\} \cap B$ consists of all outcomes of 'S' such that

$$X(S) \leq x \text{ and } S \in B.$$

properties:

→ $F_X(-\infty/B) = 0$

→ $F_X(\infty/B) = 1$

→ $0 \leq F_X(x/B) \leq 1$

→ $F_X(x_1 < X \leq x_2/B) = F_X(x_2/B) - F_X(x_1/B)$

=

(b) A continuous random variable 'X' has a PDF $f_X(x) = 3x^2, 0 \leq x \leq 1$.
Find the constants a & b such that

(i) $P\{X \leq a\} = P(X > a)$

(ii) $P\{X > b\} = 0.05$.

Soln: Given the density function of a R.V. 'X' is

$$f_X(x) = 3x^2, \ 0 \leq x \leq 1.$$

∴ The distribution function is thus given by

$$F_X(x) = \int_{-\infty}^{x} f_X(\lambda)\, d\lambda$$

$$= \int_{0}^{x} 3\lambda^2\, d\lambda = \lambda^3 \Big|_0^x = x^3, \quad 0 \leq x \leq 1.$$

(i)
$$P(X \leq a) = P(X > a)$$
$$= 1 - P(X \leq a)$$

$$\therefore 2.P(X \leq a) = 1$$
$$2.F_X(a) = 1$$
$$2.a^3 = 1 \Rightarrow a^3 = \tfrac{1}{2} \Rightarrow a = \sqrt[3]{1/2}$$

(ii)
$$P(X > b) = 0.05$$
$$1 - P(X \leq b) = 0.05$$
$$1 - F_X(b) = 0.05$$
$$1 - b^3 = 0.05$$
$$b^3 = 1 - 0.05 = 0.95$$
$$b = \sqrt[3]{0.95}$$

⑦ a) Define probability density function. List its properties.

Ans: The probability density function of a random variable 'X' is denoted by $f_X(x)$ and is defined as the derivative of the distribution function $F_X(x)$. i.e

$$f_X(x) = \frac{d}{dx} F_X(x). \longrightarrow ①$$

Equ ① is often called as the density function of the random variable 'X' and exists only if the derivative of $F_X(x)$ exists.

properties of density function:

→ The density function is a non negative function of 'x' i.e

$$f_X(x) \geq 0 \quad \forall x.$$

→ Area under the probability density function is always unity. i.e

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

→ The distribution function is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi$$

→ $$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x).dx.$$

(b) Consider the experiment of tossing four fair coins. The Random Variable 'X' is associated with the no. of tails showing. Compute and Sketch the CDF of 'X'.

Soln: In an experiment of tossing four coins, The Sample Space 'S' consists of '16' outcomes, as

$$S = \{ HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, THHT, HTHT, THTH, TTTH, TTHT, THTT, HTTT, TTTT \}.$$

R.V 'X' is associated with the no. of tails showing.

∴ 'X' takes values. 0, 1, 2, 3, 4.

$P(X=0) = \frac{1}{16}$ {HHHH}

$P(X=1) = \frac{4}{16}$, {HHHT, HHTH, HTHH, THHH}

$P(X=2) = \frac{6}{16}$, {HHTT, TTHH, THTH, HTHT, THHT, HTTH}

$P(X=3) = \frac{4}{16}$, {TTTH, TTHT, THTT, HTTT}

$P(X=4) = \frac{1}{16}$ {TTTT}

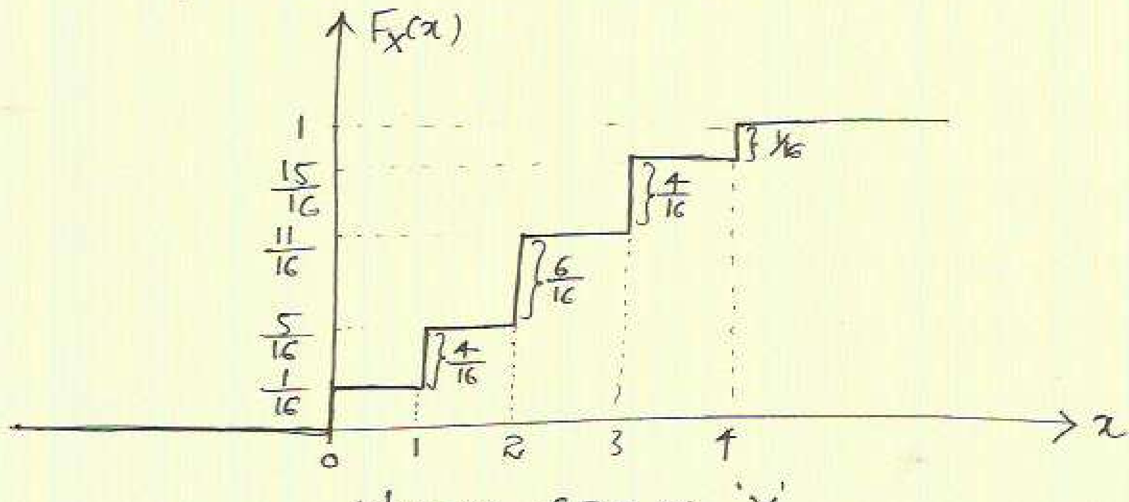
$\therefore$ The CDF of the Random Variable 'X' is given by,

$$F_X(x) = \sum_{i=0}^{4} P(X=x_i) \cdot u(x-x_i) \qquad x_i = 0,1,2,3,4$$

$$= P(X=0)\,u(x) + P(X=1)\,u(x-1) + P(X=2)\,u(x-2) + P(X=3)\,u(x-3) + P(X=4)\,u(x-4)$$

$$F_X(x) = \frac{1}{16}u(x) + \frac{4}{16}u(x-1) + \frac{6}{16}u(x-2) + \frac{4}{6}u(x-3) + \frac{1}{16}u(x-4)$$

The plot of CDF is as shown.



plot of CDF of 'X'.

(8) a) Write and plot, probability density function and probability distribution function for the following random Variables.

(i) Uniform Random Variable

(ii) Exponential Random Variable

(iii) Laplace Random Variable

(iv) Rayleigh Random Variable.

Ans: for (i) and (ii)

(iii) Laplace Random Variable: A Random Variable 'X' is laplacian if its density and distribution functions are of the form,

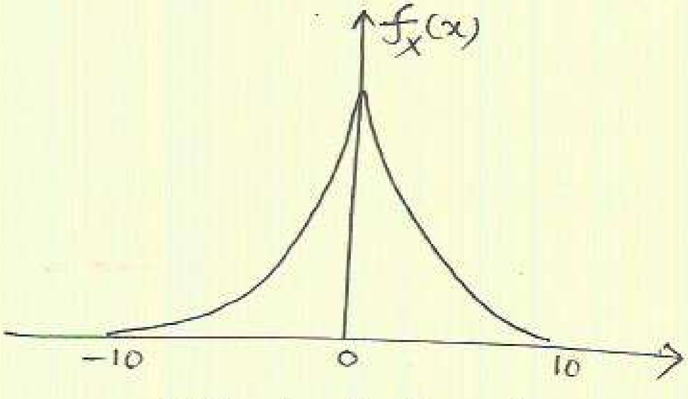
$$f_X(x) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right) \text{ and}$$

$$F_X(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & \text{if } x<\mu \\ 1-\frac{1}{2}\exp\left(-\frac{(x-\mu)}{b}\right) & \text{if } x \geq \mu \end{cases}$$

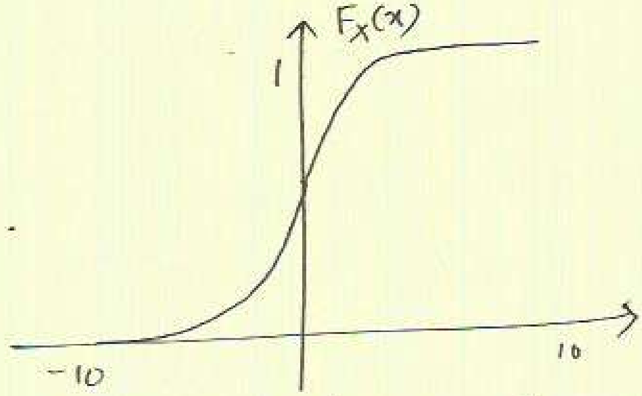
Where $\mu \rightarrow$ location (real)

$b>0$ is a Const. (real).

The plots of distribution and density functions of Laplace random variables are as shown. for $\mu=0$ and $b=1$



prob. density function



Cumulative distribution function

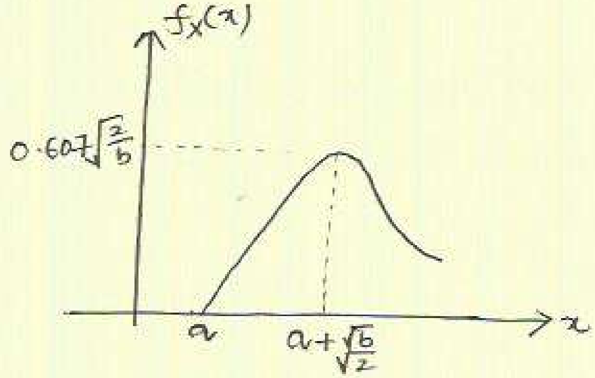
(IV) Rayleigh Random Variable: The Rayleigh density and distribution functions are defined by

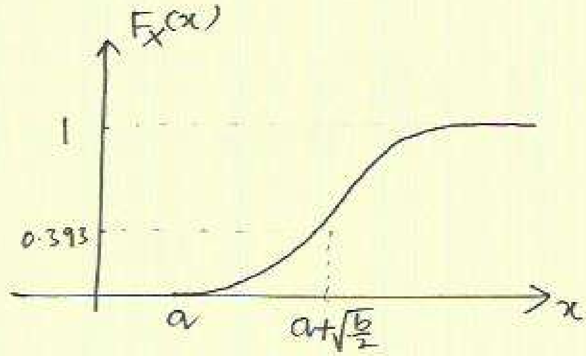
$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}, & x \geq a \\ 0, & x<a \end{cases}$$

$$F_X(x) = \begin{cases} 1-e^{-(x-a)^2/b}, & x \geq a \\ 0, & x<a. \end{cases}$$

where, $-\infty<a<\infty$ and $b>0$ are real Constants.

The plots of rayleigh density and distribution functions





(b) A Random Variable 'X' is defined as below over the interval (0,1). Find its conditional CDF of X given that $X < \frac{1}{2}$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x > 1 \end{cases}$$

Soln: The cumulative distribution function of a R.V. 'X' is given by,

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x > 1. \end{cases}$$

The conditional distribution function of 'X' given $\{X \leq \frac{1}{2}\}$ is given by,

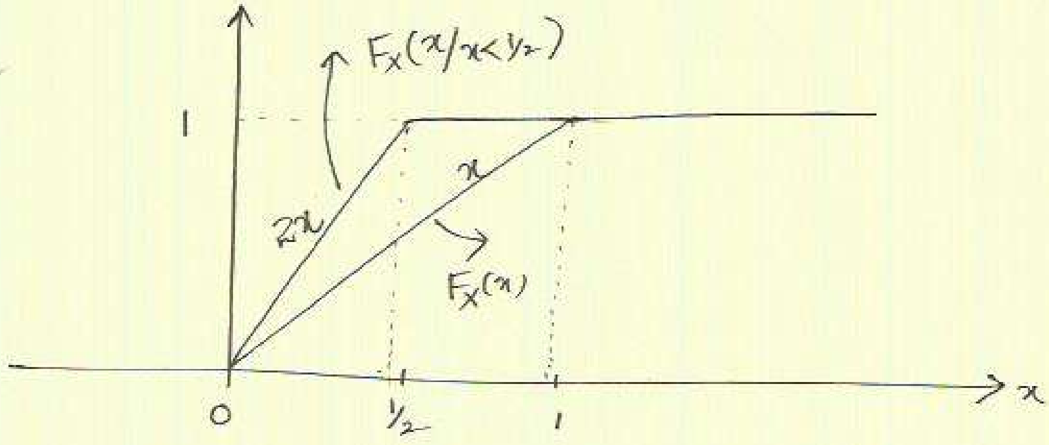
$$F_X(x/X < \tfrac{1}{2}) = \begin{cases} \dfrac{F_X(x)}{F_X(b)}, & x < \frac{1}{2} \\ 0, & x \geq \frac{1}{2} \end{cases} \qquad \because b = \tfrac{1}{2}$$

$$F_X(\tfrac{1}{2}) = \int_{-\infty}^{1/2} f_X(\xi)\,d\xi = \tfrac{1}{2}$$

$$\therefore \quad F_X(x/X < \tfrac{1}{2}) = \begin{cases} \dfrac{F_X(x)}{1/2}, & x < \frac{1}{2} \\ 0, & x \geq \frac{1}{2} \end{cases}$$

$$\therefore \quad F_X(x/X < \tfrac{1}{2}) = \begin{cases} 2 \cdot F_X(x), & x < \frac{1}{2} \\ 0, & x \geq \frac{1}{2} \end{cases}$$

The plots of $F_X(x)$ and $F_X(x/X<\frac{1}{2})$ are as follows.



$$\therefore \quad F_X(x/X<\tfrac{1}{2}) = \begin{cases} 0, & x<0 \\ 2x, & 0<X\leq \frac{1}{2} \\ 1, & x\geq \frac{1}{2} \end{cases}$$

(9) a) For the random variable 'X' whose density is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a\leq x\leq b \\ 0, & \text{otherwise.} \end{cases}$$

Determine the mean and variance.

Soln: given the pdf of a R.V. 'X' as

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a\leq x\leq b \\ 0, & \text{otherwise.} \end{cases}$$

(i) $$m_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b$$

$$\text{mean} = \overline{X} = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}\,2} = \frac{a+b}{2}.$$

(ii) The variance of 'x' is given by

$$\sigma^2 = E(x^2) - \overline{X}^2$$

The mean square value is given by

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{3(b-a)} \; x^3 \Big|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}} = \frac{a^2+b^2+ab}{3}$$

$$\therefore \quad \sigma_X^2 = E(X^2) - \overline{X}^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$\sigma_X^2 = \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

(b) A Random variable 'X' is known to have a distribution function $F_X(x) = u(x)\left[1 - e^{-x^2/b}\right]$, $b>0$ is a constant. Find its density function.

Ans: The distribution function of a R.V. 'X' is given by

$$F_X(x) = u(x)\left[1 - e^{-x^2/b}\right], \quad b>0 \text{ is a const.}$$

The density function is thus given by,

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

$$= \frac{d}{dx}\left(u(x)\left[1 - e^{-x^2/b}\right]\right)$$

$$= u(x) \cdot \frac{d}{dx}\left(1 - e^{-x^2/b}\right) + \left(1 - e^{-x^2/b}\right)\frac{d}{dx}u(x).$$

$$f_X(x) = \frac{2x}{b} e^{-x^2/b} u(x) + (\cancel{1} - \cancel{e^{0}}) \qquad f(x)\cdot\delta(x) = f(0)$$

$$\therefore \; f_X(x) = \frac{2x}{b} e^{-x^2/b} \cdot u(x)$$

⑩ a) What is a poisson random Variable. Sketch the distribution and density function of poisson random Variable with $b=4$.

Ans: The density and distribution function of a poisson random Variable 'X' are defined as,

$$f_X(x) = e^{-b} \sum_{K=0}^{\infty} \frac{b^K}{K!} \delta(x-k) \longrightarrow ①$$

$$F_X(x) = e^{-b} \sum_{K=0}^{\infty} \frac{b^K}{K!} u(x-k) \longrightarrow ②$$

Where $b>0$ is a constant.

The poisson random Variable applies to a wide variety of counting type applications. It describes the no. of defective units in a sample taken from a production line, the no. of telephone calls made during a period of time. etc.

If the time interval of duration is 'T' and the events being counted are known to occur at an average rate 'λ' and have a poisson distribution, then 'b' in Eqn ① is given as

"$b = \lambda T$"

plots of density and distribution functions of poisson R.V. with $b=4$ are obtained as follows

$$\rightarrow \quad f_X(x) = e^{-4} \sum_{K=0}^{\infty} \frac{4^K}{K!} \delta(x-k)$$

$$f_X(x) = e^{-4}\delta(x) + \frac{4e^{-4}}{1!}\delta(x-1) + \frac{4^2 e^{-4}}{2!}\delta(x-2) + \frac{4^3 e^{-4}}{3!}\delta(x-3) + \frac{4^4 e^{-4}}{4!}\delta(x-4) + \cdots$$

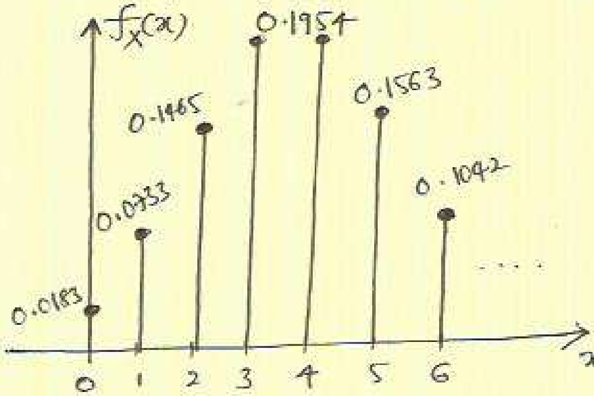
$$f_X(x) = 0.0183\,\delta(x) + 0.0733\,\delta(x-1) + 0.1465\,\delta(x-2) + 0.1954\,\delta(x-3) + 0.1954\,\delta(x-4) + 0.1563\,\delta(x-5) + \cdots \longrightarrow ③$$

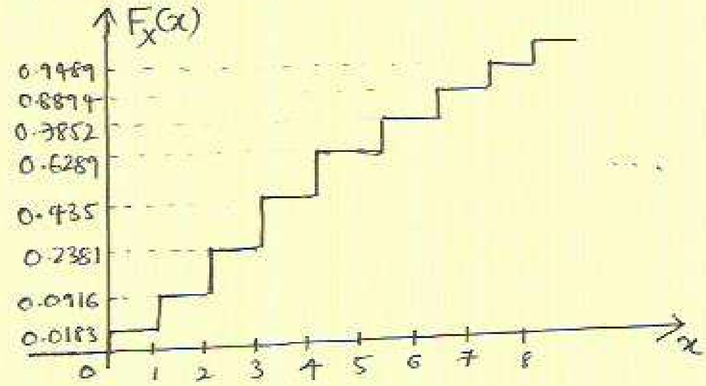
$$\therefore \; F_X(x) = e^{-4} \sum_{K=0}^{\infty} \frac{4^K}{K!} u(x-k)$$

$$= 0.0183\,u(x) + 0.0733\,u(x-1) + 0.1465\,u(x-2) + 0.1954\,u(x-3) \longrightarrow ④$$

$$+ \cdots 0.1563\,u(x-5) + \cdots$$

Equations ③ and ④ are plotted as shown.





(b) Assume an automobile arrives at a gasoline Station are poisson and occur at an average rate of 50 per hour. The Station has only one gas pump. If all cars are assumed to require 1 min. to obtain the fuel. What is the probability that a waiting Line will occur at the pump.

Soln: An automobile arrives at a gasoline Station are poisson and occur at an average rate of 50 per hour i.e

$$\lambda = 50/\text{hour} = \frac{50}{60} \text{ per minute} = \frac{5}{6} \text{ cars/minute}.$$

All cars are assumed to require one minute to obtain fuel i.e

$$T = 1 \text{ min}.$$

$$\therefore \quad b = \lambda \cdot T = \frac{5}{6}.$$

A waiting Line will occur if two or more cars arrive in any one minute interval. The probability of this event is one minus the probability that either none or one car arrives.

$$\therefore \text{ probability of waiting Line} = 1 - F_X(1) - F_X(0)$$

$$F_X(0) = e^{-5/6} \sum_{K=0}^{\infty} \frac{(5/6)^K}{K!} u(-K) = 0 \qquad \therefore u(-K) = 0$$

$$F_X(1) = e^{-5/6} \sum_{K=0}^{\infty} \frac{(5/6)^K}{K!} u(1-K)$$

$$= e^{-5/6} \left(1 + 5/6\right)$$

$$\therefore \text{ probability of waiting Line} = 1 - e^{-5/6}\left(\frac{11}{6}\right)$$

$$= 0.2032$$

$\therefore$ We expect a waiting Line at the pump about 20.32 % of time

Short Answer Questions

① What are the conditions for a function to be a random Variable.

Ans: A Random variable may be almost any function that should satisfy the following conditions.

→ The function should not be multivalued

→ The set $\{X \leq x\}$ must be an event for any real number

→ The probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$ is zero

$$P(X = \infty) = 0 \text{ and } P(X = -\infty) = 0.$$

② Define Gaussian random Variable.

Ans: A Random Variable 'X' is said to be gaussian if its density function has the form,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(x-a_X)^2}{2\sigma_X^2}}$$

where $\sigma_X > 0$ and $-\infty < a_X < \infty$ are real constants.

③ Define probability of the event with an example.

Ans: "Refer 3(a) Answer The definition of classical probability" (page no. 3).

④ Determine the value of 'k' such that the given density is valid.

$$f_X(x) = \begin{cases} K, & a < x < b \\ 0, & \text{elsewhere.} \end{cases}$$

Ans: we know that the area under probability density function is unity. i.e

$$\int_{-\infty}^{\infty} f_X(x).dx = 1 \Rightarrow \int_a^b K \cdot dx = 1$$

$$K \cdot x \Big|_a^b = 1$$

$$K(b-a) = 1$$

$$\boxed{K = \frac{1}{b-a}}$$

⑤ State the properties of conditional density function.

⑥ What is the importance of Rayleigh distribution function.

Ans: The Rayleigh density describes the envelope of one type of noise when passed through a band pass filter.

The Rayleigh distribution plays a vital role in the analysis of errors in various measurement systems.

⑦ What are the conditions to be satisfied for the statistical independance of three events A, B and C.

Ans: Three events A, B and C are statistically independent if they satisfies the following conditions.

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(B \cap C) = P(B) \cdot P(C)$$

$$P(C \cap A) = P(C) \cdot P(A) \text{ and}$$

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

⑧ Show that $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$.

Ans: It is know that the events $\{X \leq x_1\}$ and $\{x_1 < X \leq x_2\}$ are mutually exclusive, so, the probability of the event $\{X \leq x_2\}$ can be written as,

$$P(X \leq x_2) = P[\{X \leq x_1\} \cup \{x_1 < X \leq x_2\}]$$

$$= P(X \leq x_1) + P(x_1 < X \leq x_2)$$

$$\therefore P(x_1 < X \leq x_2) = P(X \leq x_2) - P(X \leq x_1)$$

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

if $x_2 = \infty$ & $x_1 = x$ then

$$P(x < X \leq \infty) = F_X(\infty) - F_X(x)$$

$$P(X > x) = 1 - F_X(x) = 1 - P(X \leq x)$$

⑨ Three coins are tossed in succession. Find out the probabilities of occurence of two consecutive Heads.

Ans: When three coins are tossed the sample space

The outcomes where Two Consecutive heads will appear are

$\{HHT, THH, HHH\}$ i.e 3

$\therefore \quad P(A) = \frac{3}{8}$

(10) State Bayes Theorem.

Ans: Bayes Theorem States that if a Sample Space 'S' has 'N' mutually exclusive events B_n, $n = 1, 2 \ldots N$. Such that $B_m \cap B_n = \phi$ for $m \neq n = 1, 2, 3 \ldots N$ and let A be any event defined on this Sample space, then the Conditional probability Of B_n given A Can be written as

$$P(B_n/A) = \frac{P(A/B_n) \cdot P(B_n)}{\sum_{i=1}^{N} P(A/B_i) \cdot P(B_i)}$$

(11) Write the Axioms of probability.

Ans: 3(a) Answer. Axiomatic Definition page. no. 2

Unit - II

① a) State and explain Central Limit theorem.

Ans: Central Limit theorem broadly says that, the distribution function of sum of Large no. of random Variables approaches a gaussian distribution. This theorem is particularly Suitable for Statistically independent random Variables.

Central Limit theorem: Let $\bar{X}_i$ and $\sigma_{X_i}^2$ be the means and Variances of 'N' random variables X_i, $i = 1,2,3 \cdots N$, which may have arbitrary probability density functions. The Central Limit theorem says that the Sum $Y_N = X_1 + X_2 + \cdots + X_N$, which has mean $\bar{Y}_N = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_N$ and variance $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$ has a probability distribution that asymptotically approaches gaussian as $N \to \infty$ i.e

$$f_{Y_N}(y) \cong \mathcal{N}_{N\to\infty}(\bar{Y}_N, \sigma_{Y_N}^2) \longrightarrow ①$$

The necessary conditions for the theorem Validity are difficult to State, but sufficient conditions are known to be,

$$\text{Variance}: \quad 0 < B_1 < \sigma_{X_i}^2, \quad i = 1,2,3 \cdots N$$

$$\text{Skew}: \quad E\left(|X_i - \bar{X}_i|^3\right) < B_2, \quad i = 1,2,3 \cdots N$$

where B_1 & B_2 are positive numbers.

The central Limit theorem, guarantees only that, the distribution of sum of random Variables becomes gaussian. It does not follow that the probability density is always gaussian.

For discrete Random Variables X_i, Y_N is also discrete, and hence the density is not gaussian even though the distribution approaches gaussian.

The Approximation will be quite accurate even for relatively small Values of 'N' in the Central region of gaussian Curve near mean. However approximation can be very inaccurate in the tail regions away from mean even for large Values of 'N'

(b) given the function

$$f_{XY}(x,y) = \begin{cases} b(x+y)^2, & -2<x<2, -3<y<3 \\ 0, & \text{elsewhere.} \end{cases}$$

(i) Find a constant 'b' such that this is a valid density function.

(ii) Determine the marginal density functions $f_X(x), f_Y(y)$.

Soln: Given data:

The function $f_{XY}(x,y)$ is given as

$$f_{XY}(x,y) = \begin{cases} b(x+y)^2, & -2<x<2, -3<y<3 \\ 0, & \text{elsewhere.} \end{cases}$$

(i). we know that from the properties of joint density,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$$

$$\int_{-2}^{2}\int_{-3}^{3} b(x+y)^2\,dx\,dy = 1$$

integrate w.r.t. 'x'

$$\int_{-3}^{3} b.\frac{(x+y)^3}{3}\Big|_{-2}^{2} dy = 1.$$

$$\frac{b}{3}\int_{-3}^{3}\left((y+2)^3 - (y-2)^3\right)dy = 1.$$

$$\frac{b}{3}\left(\frac{(y+2)^4}{4} - \frac{(y-2)^4}{4}\right)\Big|_{-3}^{3} = 1$$

$$\frac{b}{3}\left(\left(\frac{5^4}{4} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{5^4}{4}\right)\right) = 1$$

$$\frac{b}{3}\left(\frac{5^4}{4} - \frac{1}{4} - \frac{1}{4} + \frac{5^4}{4}\right) = 1$$

$$\frac{b}{3}\left(\frac{5^4}{2} - \frac{1}{2}\right) = 1$$

$$\frac{b}{3}\left(\frac{625-1}{2}\right) = 1$$

$$\boxed{b = \frac{6}{624} = \frac{1}{104}}$$

(ii) The marginal density function $f_X(x)$ is given by

$$\therefore f_X(x) = \int_{-3}^{3} b.(x+y)^2 dy$$

$$= \frac{1}{104} \left. \frac{(x+y)^3}{3} \right|_{-3}^{3}$$

$$= \frac{1}{312}\left((x+3)^3 - (x-3)^3\right)$$

$$\therefore f_X(x) = \frac{(x+3)^3 - (x-3)^3}{312}, \quad -2 < x < 2.$$

Similarly the marginal density function $f_Y(y)$ is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_{-2}^{2} b.(x+y)^2 dx$$

$$= \frac{1}{104} \left. \frac{(x+y)^3}{3} \right|_{-2}^{2}$$

$$= \frac{1}{312}\left((y+2)^3 - (y-2)^3\right)$$

$$\therefore f_Y(y) = \frac{(y+2)^3 - (y-2)^3}{312}, \quad -3 < y < 3.$$

(2) a) Briefly explain about jointly gaussian random variables. What are the properties of jointly gaussian random variables.

Ans: Two random variables 'X' and 'Y' are said to be jointly gaussian if their joint density function is of the form,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-e^2}} \exp\left[\frac{-1}{2(1-e^2)}\left(\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2e(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right)\right] \rightarrow (1)$$

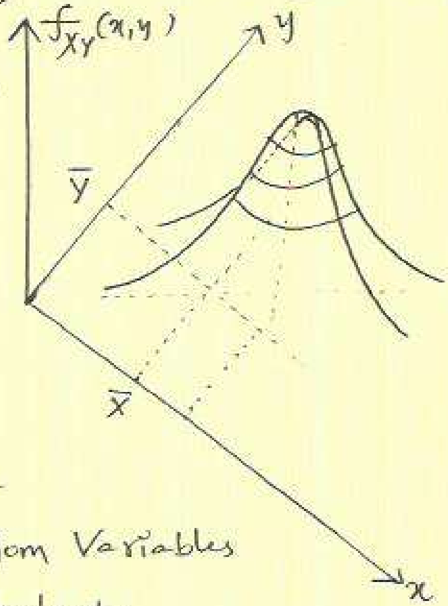
Equ (1) is some times called as bivariate gaussian density function. Here, $\bar{X}$ & $\bar{Y}$ are means of X & Y and σ_X^2 & σ_Y^2 are variances of X & Y respectively.

$e = \frac{C_{XY}}{\sigma_X\sigma_Y}$ is called Correlation Coefficient.

The maximum value of $f_{XY}(x,y)$ is located at the point $(\bar{X}, \bar{Y})$ and is obtained as,

$$f_{XY}(x,y) \leq f_{XY}(\bar{X},\bar{Y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-e^2}}$$

The appearance of joint gaussian density is as shown.



→ if $\rho = 0$ i.e for uncorrelated random variables X, & Y, we have $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ → ②

where $f_X(x)$ & $f_Y(y)$ are marginal gaussian density functions.

→ from Eqn ② it is clear that uncorrelated gaussian random variables are also statistically independent.

Properties of jointly gaussian random variables:

The properties exhibited by 'N' jointly gaussian random variables $X_1, X_2 \ldots X_N$ are given by,

→ gaussian random variables are completely defined through only their first and second order moments i.e means, variences, etc.

→ If the random variables are uncorrelated, they are also statistically independent.

→ Random variables produced by a Linear transformation of X_1, X_2 $X_3 \ldots X_N$ will also be gaussian.

→ The K-dimensional (K-variate) marginal density function obtined from, N-dimensional density function by integrating out N-k random variables will also be gaussian.

→ The conditional density $f_{X_1 X_2 \ldots X_K}(x_1, x_2 \ldots x_K / X_{K+1} = x_{K+1} \ldots X_N = x_N)$ is also gaussian.

(b) A random variable 'x' has $\bar{X} = -3$, $\overline{X^2} = 11$ and $\sigma_X^2 = 2$. For a new random variable $Y = 2X - 3$, Find (i) $\bar{Y}$ (ii) $\overline{Y^2}$ (iii) σ_Y^2.

Soln:

A Random 'X' has

$$\bar{X} = -3 = E(X)$$

$$\overline{X^2} = 11 = E(X^2)$$

$$\sigma_X^2 = 2 = E(X^2) - [E(X)]^2$$

A new random variable Y is defined as

(i) $\bar{Y} = E(Y) = E(2X-3)$

$= 2 \cdot E(X) - E(3)$

$= 2(-3) - 3 = -6-3 = -9$

$\therefore \boxed{\bar{Y} = -9}$

(ii) $\overline{Y^2} = E(Y^2) = E((2X-3)^2)$

$= E(4X^2 - 12X + 9)$

$= 4E(X^2) - 12E(X) + E(9)$

$= 4(11) - 12(-3) + 9$

$= 44 + 36 + 9 = 89$

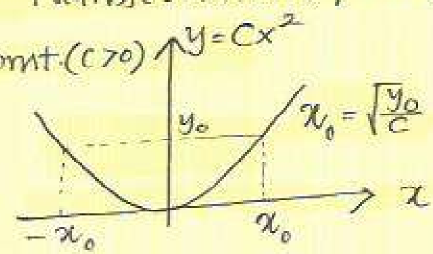
$\therefore \boxed{\overline{Y^2} = 89}$

(iii) $\sigma_Y^2 = E(Y^2) - \bar{Y}^2$

$= 89 - (-9)^2 = 89 - 81 = 8$

$\therefore \boxed{\sigma_Y^2 = 8}$

③ a) Find $f_Y(y)$ for the square law Transformation $Y = T(X) = CX^2$ shown below. where 'c' is a real Const. ($c > 0$)



Solu: Given a random variable 'x' with probability density function $f_X(x)$, The square Law transformation is as shown and is given by $Y = T(X) = CX^2$, $C > 0$ is a Const.

$\therefore \quad y = cx^2$

or $x = \pm\sqrt{\frac{y}{c}}$

at $y_0 \Rightarrow x_0 = \pm\sqrt{\frac{y_0}{c}}$ or we have two roots $+\sqrt{\frac{y_0}{c}}$ and $-\sqrt{\frac{y_0}{c}}$.

$\therefore$ The range of the event $\{Y \leq y\}$ transform into the range $\left\{-\sqrt{\frac{y_0}{c}} \leq x \leq \sqrt{\frac{y_0}{c}}\right\}$. Therefore the probability density function of 'y' is given by,

$f_Y(y) = \frac{d}{dy}\left(\int_{\{X/y \leq y\}} f_X(x) \cdot dx\right), \quad y \geq 0$

$$\therefore f_Y(y) = \frac{d}{dy}\int_{-\sqrt{\frac{y}{c}}}^{\sqrt{\frac{y}{c}}} f_X(x)\,dx$$

By using Leibniz's rule, we have,

$$f_Y(y) = f_X\left(\sqrt{\tfrac{y}{c}}\right)\frac{d}{dy}\sqrt{\tfrac{y}{c}} - f_X\left(-\sqrt{\tfrac{y}{c}}\right)\cdot\frac{d}{dy}\left(-\sqrt{\tfrac{y}{c}}\right)$$

$$f_Y(y) = \frac{f_X\sqrt{\frac{y}{c}} + f_X\left(-\sqrt{\frac{y}{c}}\right)}{2\sqrt{cy}}, \quad y \geq 0$$

(or)

Soln: we know that

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x=x_n}}$$

we have $X = \pm\sqrt{\frac{Y}{C}}$, $y \geq 0$ so $x_1 = -\sqrt{y/c}$, $x_2 = \sqrt{y/c}$

and $\frac{d}{dx}T(x) = \frac{d}{dx}(CX^2) = 2CX$

$$\therefore \left|\frac{dT(x)}{dx}\right|_{x=x_1} = 2Cx_1 = 2C\left(-\sqrt{y/c}\right) = -2\sqrt{cy}$$

$$\left|\frac{dT(x)}{dx}\right|_{x=x_2} = 2Cx_2 = 2C\cdot\sqrt{\frac{y}{c}} = 2\sqrt{cy}$$

$$\therefore f_Y(y) = \frac{f_X(x_1) + f_X(x_2)}{2\sqrt{cy} \qquad 2\sqrt{cy}}$$

$$f_Y(y) = \frac{1}{2\sqrt{cy}}\left(f_X\left(-\sqrt{\tfrac{y}{c}}\right) + f_X\left(\sqrt{\tfrac{y}{c}}\right)\right)$$

(b) Find whether the two random variables 'X', and 'Y' are statistically independent or not if the joint pdf is given by $f_{XY}(x,y) = \frac{1}{12}u(x).u(y)\,e^{-(x/4)-(y/3)}$.

Soln: given the joint density function

$$f_{XY}(x,y) = \frac{1}{12}u(x)\,u(y).\,e^{-(x/4)-(y/3)} \longrightarrow (1)$$

Two random variables 'X' and 'Y' are said to be statistically independent if and only if

∴ The marginal density functions are given by,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y).\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{12} u(x).\,u(y).\,e^{-x/4 - y/3}\,dy$$

$$= \frac{1}{12} u(x).\,e^{-x/4}.\int_0^{\infty} e^{-y/3}\,dy$$

$$= \frac{1}{12} u(x).\,e^{-x/4}.\left.\frac{e^{-y/3}}{(-1/3)}\right|_0^{\infty}$$

$$= \frac{-\cancel{3}}{\cancel{12}_4} u(x).\,e^{-x/4}(0-1)$$

$$f_X(x) = \frac{1}{4} u(x).\,e^{-x/4} \longrightarrow (3)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y).\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{12} u(x)\,u(y).\,e^{-x/4 - y/3}\,dx$$

$$= \frac{1}{12} u(y)\,e^{-y/3}\int_0^{\infty} e^{-x/4}\,dx$$

$$= \frac{1}{12} u(y)\,e^{-y/3}.\left.\frac{e^{-x/4}}{-1/4}\right|_0^{\infty}$$

$$= \frac{-\cancel{4}}{\cancel{12}_3} u(y).\,e^{-y/3}(0-1)$$

$$f_Y(y) = \frac{1}{3} u(y).\,e^{-y/3} \longrightarrow (4)$$

$$\therefore\ f_X(x).\,f_Y(y) = \frac{1}{4}u(x).\,e^{-x/4}.\,\frac{1}{3}u(y)\,e^{-y/3} = \frac{1}{12}u(x)u(y)e^{-x/4-y/3}$$

$$\therefore\ f_{XY}(x,y) = f_X(x).\,f_Y(y) \Rightarrow X \,\&\, Y \text{ are independent.}$$

(4) a) Find the pdf. of a random variable 'W' defined as Sum of X, Y with densities Shown below.

$$f_X(x) = \frac{1}{a}\left(u(x) - u(x-a)\right)$$

$$f_Y(y) = \frac{1}{b}\left(u(y) - u(y-b)\right) \quad \text{with } a<b.$$

Solw: The probability density functions of two random variables 'X' and 'Y' are given by a new random variable 'w' is defined as

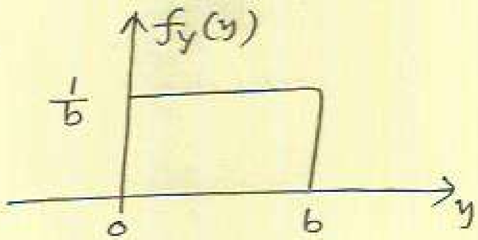
$$W = X + Y.$$

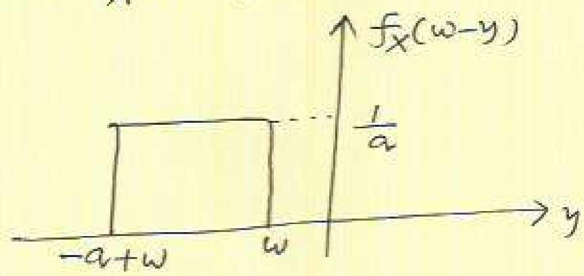
The density of sum of two random variables is given by the convolution of the densities of individual random variables.

$$\therefore f_W(w) = f_X(x) * f_Y(y)$$

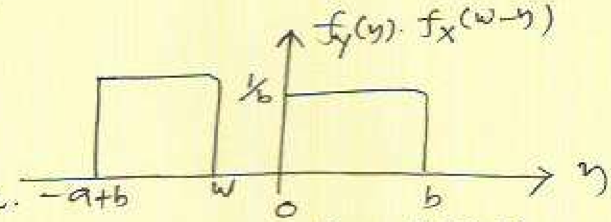
$$\therefore f_W(w) = \int_{-\infty}^{\infty} f_Y(y) . f_X(w-y)\, dy.$$

The functions $f_Y(y)$ and $f_X(w-y)$ are as follows.

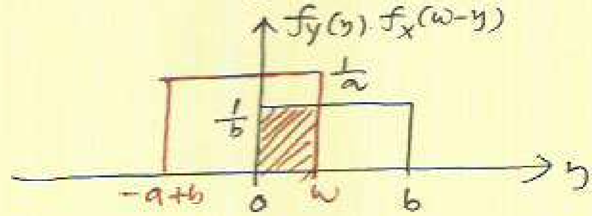




Now consider the product for different intervals of 'w'.

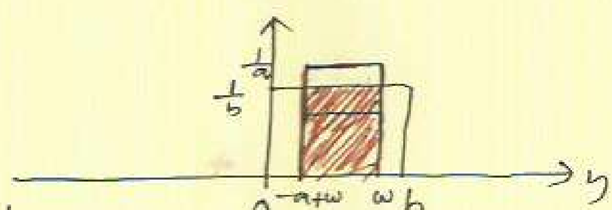


→ $w < 0$, $f_W(w) = 0$, since no common area.



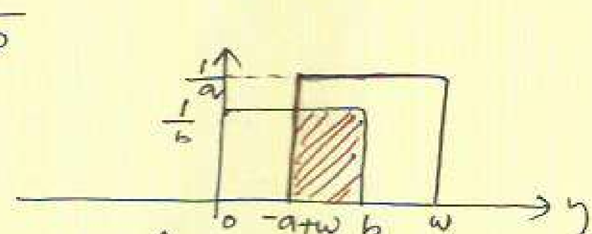
→ $0 \leq w < a$: $f_W(w) = \int_0^w \frac{1}{a} \cdot \frac{1}{b} \cdot dw$

$$= \frac{1}{ab}(w-0) = \frac{w}{ab}$$



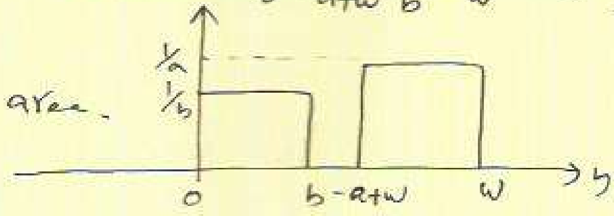
→ $a \leq w < b$, $f_W(w) = \int_{-a+w}^{w} \frac{1}{a} \cdot \frac{1}{b} \cdot dw$

$$= \frac{1}{ab}(\cancel{w} + a - \cancel{w}) = \frac{1}{b}$$



→ $b \leq w < a+b$, $f_W(w) = \int_{-a+w}^{b} \frac{1}{a} \cdot \frac{1}{b} \cdot dw$

$$= \frac{1}{ab}(a+b-w).$$



→ $w > a+b$, $f_W(w) = 0$, no common area.

(b) An exponential random variable has a pdf as shown below. $f_X(x) = b e^{-bx} u(x)$ with mean value $\frac{1}{b}$. Find its coefficient of skewness and kurtosis.

Soln: The pdf of an exponential R.V is given by

$$f_X(x) = b e^{-bx} u(x), \quad E(X) = \bar{X} = \frac{1}{b} = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx \rightarrow (1)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) \cdot dx = \int_0^{\infty} x^2 b e^{-bx}\, dx$$

$$\therefore E(X^2) = b \cdot \left[x^2 \frac{e^{-bx}}{-b} \Big|_0^{\infty} - \int_0^{\infty} 2x \frac{e^{-bx}}{-b} \cdot dx \right]$$

$$= b\left((0-0) + \frac{2}{b}\left(x \frac{e^{-bx}}{-b}\Big|_0^{\infty} - \frac{1}{b^2} e^{-bx}\Big|_0^{\infty} \right)\right)$$

$$E(X^2) = \cancel{b}\left(\frac{2}{\cancel{b}}\left(0 - 0 - \frac{1}{b^2}(0-1)\right)\right) = \frac{2}{b^2} \rightarrow (2)$$

$$E(X^3) = \int_{-\infty}^{\infty} x^3 \cdot f_X(x) \cdot dx.$$

$$= \int_0^{\infty} x^3 \cdot b e^{-bx} \cdot dx.$$

$$= b \cdot \left[x^3 \cdot \frac{e^{-bx}}{-b} \Big|_0^{\infty} - \int_0^{\infty} 3x^2 \frac{e^{-bx}}{-b} dx \right]$$

$$= b\left(0 + \frac{3}{b^2}\left(\int_{-\infty}^{\infty} x^2 \cdot b e^{-bx} u(x) \cdot dx\right)\right)$$

$$E(X^3) = \cancel{b} \cdot \frac{3}{b^{\cancel{2}}} E(X^2) = \frac{3}{b}\left(\frac{2}{b^2}\right) = \frac{6}{b^3} \rightarrow (3)$$

Coefficient of Skewness is given by, $\frac{\mu_3}{\sigma_X^3}$

where μ_3 is the third order central moment.

given by,

$$\mu_3 = E((X - \bar{X})^3)$$

$$= E(X^3 - \bar{X}^3 - 3X^2\bar{X} + 3X\bar{X}^2)$$

$$= E(X^3) - \bar{X}^3 - 3\bar{X}\,E(X^2) + 3\bar{X}^2 E(X)$$

$$= \frac{6}{b^3} - \left(\frac{1}{b}\right)^3 - 3\frac{1}{b}\left(\frac{2}{b^2}\right) + 3\left(\frac{1}{b^2}\right)\left(\frac{1}{b}\right)$$

$$\mu_3 = \cancel{\frac{6}{b^3}} - \frac{1}{b^3} - \cancel{\frac{6}{b^3}} + \frac{3}{b^3} = \frac{2}{b^3}$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{2}{b^2} - \left(\frac{1}{b}\right)^2 = \frac{1}{b^2}$$

∴ The Coefficient of Skewness is given as

$$\frac{\mu_3}{\sigma_x^3} = \frac{2/b^3}{1/b^3} = \underline{\underline{2}}$$

→ Kurtosis: The kurtosis is given by $\frac{\mu_4}{\sigma_x^4} = \frac{\mu_4}{\mu_2^2}$

$$E(X^4) = \int_{-\infty}^{\infty} x^4 f_X(x)\,dx$$

$$= b\int_0^{\infty} x^4 e^{-bx}\,dx$$

$$= b\left[x^4 \cdot \frac{e^{-bx}}{-b}\Big|_0^{\infty} - \int_0^{\infty} 4x^3 \frac{e^{-bx}}{-b}\,dx\right]$$

$$= b\left[0 + \frac{4}{b^2}\underbrace{\int_{-\infty}^{\infty} x^3 \cdot b e^{-bx} u(x)\,dx}\right]$$

$$= \cancel{b}\left[\frac{4}{b^{\cancel{2}}}\left(\frac{6}{b^3}\right)\right] = \frac{24}{b^4}$$

$$\therefore \mu_4 = E((X-\bar{X})^4)$$

$$= E\left((X-\bar{X})^2 (X-\bar{X})^2\right)$$

$$= E\left((X^2+\bar{X}^2-2\bar{X}X)(X^2+\bar{X}^2-2\bar{X}X)\right)$$

$$= E\big(X^4+\bar{X}^2X^2-2\bar{X}X^3+\bar{X}^2X^2+\bar{X}^4-2\bar{X}^3X$$
$$-2\bar{X}X^3-2\bar{X}^3X+4\bar{X}^2X^2\big)$$

$$= E\left(X^4+6\bar{X}^2X^2-4\bar{X}X^3-4\bar{X}^3X+\bar{X}^4\right)$$

$$= E(X^4)+6\bar{X}^2E(X^2)-4\bar{X}E(X^3)-4\bar{X}^3E(X)+\bar{X}^4$$

$$= \frac{24}{b^4}+6\frac{1}{b^2}\frac{2}{b^2}-4\frac{1}{b}\frac{6}{b^3}-4\frac{1}{b^3}\frac{1}{b}+\frac{1}{b^4}$$

$$= \cancel{\frac{24}{b^4}}+\frac{12}{b^4}-\cancel{\frac{24}{b^4}}-\frac{4}{b^4}+\frac{1}{b^4} = \frac{12-4+1}{b^4} = \frac{9}{b^4}$$

$$\therefore \text{Kurtosis} = \frac{\mu_4}{\sigma_x^4} = \frac{9/b^4}{1/b^4} = \underline{\underline{9}}$$

(5) a) State and prove the joint density function properties.

Ans: The properties of joint density function for two random variables X and Y are given as,

→ The joint density function is always non negative i.e $f_{XY}(x,y) \geq 0$

proof: By definition,

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \cdot \partial y} F_{XY}(x,y)$$

where $F_{XY}(x,y) = P(X \leq x, Y \leq y) = P(X \leq x \cap Y \leq y)$

Since $F_X(x) = P(X \leq x) \geq 0$ and $P(Y \leq y) = F_Y(y) \geq 0$

$\therefore$ $P(X \leq x \cap Y \leq y) \geq 0$ Hence

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) \geq 0.$$

→ The area under the joint density curve is always equal to 1 i.e

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1.$$

proof: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy$ gives the sum of all the joint probabilities for R.V's 'X and Y'. The sum of all probabilities must be equal to '1'.

$$\therefore \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1.$$

→ The joint distribution function is given by

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$$

proof:

$$\int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{x} \frac{\partial}{\partial x} \int_{-\infty}^{y} \frac{\partial}{\partial y} F_{XY}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{x} \frac{\partial}{\partial x} \cdot F_{XY}(x,y)\, dx$$

$$= F_{XY}(x,y).$$

$$\therefore F_{XY}(x,y) = \int_{-\infty}^{x} \cdot \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy.$$

→ The marginal distribution functions of 'X' and 'Y' are

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy \text{ and}$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

proof: We know that the marginal distribution function of 'X' is given by, $F_X(x) = F_{XY}(x,\infty) \longrightarrow ①$

we have,

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy.$$

$$y \to \infty, \quad F_{XY}(x,\infty) = F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

Similarly as $x \to \infty$ $F_{XY}(\infty,y) = F_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy.$

→ The marginal density functions of 'X' and 'Y' are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y).\,dy = \frac{\partial}{\partial x} F(x,\infty)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{\partial}{\partial y} F(\infty,y).$$

proof: We know that.

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

Similarly, $F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y).\,dx$$

→ The probability of the joint event $\{x_1 < X \le x_2, y_1 < Y \le y_2\}$ is given as

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int^{x_2} \int^{y_2} f_{XY}(x,y)\,dx\,dy.$$

(b) Verify that for what values of 'b' the given function is a valid density function.

$$g(x,y) = \begin{cases} b\,e^{-x}\cos y, & 0<x\leq 2,\ 0\leq y\leq \frac{\pi}{2} \\ 0, & \text{elsewhere.} \end{cases}$$ 'b' is positive const.

Soln:

→ Since 'b' is positive, and x, y takes positive values only, The function $g(x,y)$ is always non-negative i.e

$$g(x,y) \geq 0.$$

→ if $g(x,y)$ is a valid density, then area under this function is unity.

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy = 1$$

$$\int_0^2 \int_0^{\pi/2} b\,e^{-x}\cos y\,dx\,dy = 1$$

$$b \cdot \int_0^{\pi/2} \cos y \cdot \left(\frac{e^{-x}}{-1}\right)\Big|_0^2 dy = 1$$

$$b\int_0^{\pi/2} \cos y\,(1-e^{-2})\,dy = 1$$

$$b(1-e^{-2})\ \sin y\Big|_0^{\pi/2} = 1$$

$$b(1-e^{-2})(1-0) = 1$$

$$\boxed{b = \frac{1}{1-e^{-2}}}$$

⑥ a) Identify the value of moment μ_{22}, if statistically independent random variables X and Y have moments, $m_{10}=2$, $m_{20}=14$, $m_{02}=12$ and $m_{11}=-6$.

Soln:

Since 'x' and 'y' are independent random variables,

$$\mu_{22} = E([X-\bar{X}]^2[Y-\bar{Y}]^2)$$

$$= E([X-\bar{X}]^2)\cdot E([Y-\bar{Y}]^2)$$

$$\mu_{22} = \mu_{20}\cdot\mu_{02} \longrightarrow ①$$

given Data, $m_{10} = \bar{X} = 2$

$m_{20} = \overline{X^2} = 14$

$$m_{02} = 12 = \overline{Y^2}$$

$$m_{11} = \overline{XY} = \bar{X} \cdot \bar{Y} = -6$$

$$\bar{Y} = \frac{-6}{\bar{X}} = \frac{-6}{2} = -3$$

$$\therefore \mu_{02} = \overline{Y^2} - \bar{Y}^2$$

$$= 12 - (-3)^2 = 12 - 9 = 3$$

$$\therefore \mu_{22} = \mu_{20} \cdot \mu_{02} = 10(3) = \underline{\underline{30}}$$

(b) Two random variables 'X' and 'Y' have means $\bar{X} = 1$, and $\bar{Y} = 2$, variances $\sigma_X^2 = 4$ and $\sigma_Y^2 = 1$ and a correlation coefficient $\rho_{XY} = 0.4$. New random variables W and V are defined by $V = -X + 2Y$, $W = X + 3Y$. Find (i) The means (ii) The variances (iii) The correlations (iv) Correlation coefficient ρ_{VW} of V & W.

Soln:

Given Data:

$$E(X) = \bar{X} = 1, \quad \sigma_X^2 = 4$$

$$E(Y) = \bar{Y} = 2, \quad \sigma_Y^2 = 1$$

$$\therefore E(X^2) = \sigma_X^2 + \bar{X}^2 = 4 + 1 = 5$$

$$E(Y^2) = \sigma_Y^2 + \bar{Y}^2 = 1 + 4 = 5.$$

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y} = 0.4$$

$$\therefore C_{XY} = 0.4\sqrt{4} \cdot \sqrt{1} = 0.8$$

The new random variables W & V are defined as

$$V = -X + 2Y, \quad W = X + 3Y$$

(i) $E(V) = E(-X + 2Y) = -E(X) + 2E(Y) = -1 + 2(2) = 3$

$E(W) = E(X + 3Y) = E(X) + 3 \cdot E(Y) = 1 + 3(2) = 7$

(ii) $$\sigma_V^2 = E([V - \bar{V}]^2) = E([-X + 2Y - (-\bar{X} + 2\bar{Y})]^2)$$

$$= E([-(X - \bar{X}) + 2(Y - \bar{Y})]^2)$$

$$= E((X - \bar{X})^2 + 4(Y - \bar{Y})^2 - 4(X - \bar{X})(Y - \bar{Y}))$$

$$\sigma_V^2 = \sigma_X^2 + 4\sigma_Y^2 - 4C_{XY}$$
$$= 4 + 4(1) - 4(0.8) = 8 - 3.2 = \underline{\underline{0.8}}$$

$$\sigma_W^2 = E([W-\bar{W}]^2) = E([X+3Y-\bar{X}-3\bar{Y}]^2)$$
$$= E([(X-\bar{X}) + 3(Y-\bar{Y})]^2)$$
$$= E([X-\bar{X}]^2) + 9E([Y-\bar{Y}]^2) + 6E[(X-\bar{X})(Y-\bar{Y})]$$
$$= \sigma_X^2 + 9\sigma_Y^2 + 6C_{XY}$$
$$= 4 + 9(1) + 6(0.8) = 13 + 4.8 = \underline{\underline{17.8}}$$

(iii) $$R_{VW} = E(V.W) = E([X+2Y][X+3Y])$$
$$= E(-X^2 - 3XY + 2XY + 6Y^2)$$
$$= -E(X^2) - E(XY) + 6.E(Y^2)$$
$$= -(\sigma_X^2 + \bar{X}^2) - R_{XY} + 6(\sigma_Y^2 + \bar{Y}^2)$$
$$= -(4+1) - (C_{XY} + \bar{X}.\bar{Y}) + 6(1+4)$$
$$= -5 + 30 - (0.8+2) = 25 - 2.8$$
$$R_{VW} = \underline{\underline{22.2}}$$

(iv) $$\rho_{VW} = \frac{C_{VW}}{\sigma_V . \sigma_W} = \frac{R_{VW} - \bar{V}.\bar{W}}{\sigma_V . \sigma_W}$$
$$= \frac{22.2 - (3)(7)}{\sqrt{4.8}\sqrt{17.8}} \simeq \underline{\underline{0.1298}}$$

7 a) Define marginal density and distribution functions.

Ans: The distribution and density functions of one random variable can be obtained from the joint distribution or joint density functions and are known as marginal distribution and marginal density functions.

$F_X(x)$ can be obtained from $F_{XY}(x,y)$ by setting 'y' as '∞' i.e $F_X(x) = F_{XY}(x,\infty) = P(X \le x, Y \le \infty)$

Similarly we have

$$F_Y(y) = F_{XY}(\infty, y) = P(X \leq \infty, Y \leq y)$$
$$= P(S, Y \leq y)$$
$$= P(Y \leq y).$$

The marginal density function are given by,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \frac{d}{dx} F_X(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx. = \frac{d}{dy} F_Y(y).$$

(b) Let 'X' and 'Y' be jointly continuous random Variables with probability density function

$$f_{XY}(x,y) = x^2 + \frac{xy}{3}, \quad 0 \leq x \leq 1, 0 \leq y \leq 2$$
$$0 \quad , \text{ elsewhere.}$$

Find (i) $f_X(x)$. (ii) $f_Y(y)$ (iii) Are X and Y independent?

Soln: The joint density function of two random Variables 'X' and 'Y' is given by,

$$f_{XY}(x,y) = x^2 + \frac{xy}{3}, \quad 0 \leq x \leq 1, 0 \leq y \leq 2$$
$$0 \quad , \text{ elsewhere.}$$

(i) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y).\, dy$$

$$= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= x^2 \cdot y + \frac{x}{3}\frac{y^2}{2} \Big|_0^2$$

$$= 2x^2 + \frac{2}{3}x - 0 \cdot = 2\left(x^2 + \frac{x}{3}\right), \quad 0 \leq x \leq 1$$

(ii) $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y), dx$$

$$= \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \frac{x^3}{3} + \frac{y}{3}\frac{x^2}{2} \Big|_0^1$$

$$= \frac{1}{3} + \frac{y}{6} - 0 = \frac{1}{3}\left(1 + \frac{y}{2}\right)$$

(iii) $f_X(x) \cdot f_Y(y) = 2\left(x^2+\frac{x}{3}\right)\frac{1}{3}\left(1+\frac{y}{3}\right)$

$$= \frac{2}{3}\left(x^2+\frac{x^2y}{3}+\frac{x}{3}+\frac{xy}{9}\right) \neq f_{XY}(x,y)$$

∴ 'X' and 'Y' are not Statistically independent.

(8) a) Random variables 'X' and 'Y' have the joint density

$$f_{XY}(x,y) = \frac{1}{24}, \quad 0<x<6 \text{ and } 0<y<4$$
$$0, \quad \text{elsewhere.}$$

What is the expected value of the function $g(X,Y) = (XY)^2$?

Soln: The joint density of two random variables X & Y is given by,

$$f_{XY}(x,y) = \frac{1}{24}, \quad 0<x<6 \text{ and } 0<y<4$$
$$0, \quad \text{elsewhere,}$$

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^6\int_0^4 x^2y^2 \cdot \frac{1}{24}\,dx\,dy$$

$$= \frac{1}{24}\int_0^6 x^2 \cdot \frac{y^3}{3}\Big|_0^4 dx$$

$$= \frac{1}{24\times3}\int_0^6 x^2 \cdot (4^3)\,dx$$

$$= \frac{4^3}{24\times3} \cdot \frac{x^3}{3}\Big|_0^6$$

$$= \frac{4^3}{24\times9}(6^3-0)$$

$$= \frac{\cancel{4}\times4\times4\times\cancel{6}\times\cancel{6}\times\cancel{6}^{2}}{\cancel{6}\times\cancel{4}\times\cancel{3}\times\cancel{3}}$$

$$E(g(X,Y)) = 64$$

(b) How expectation is calculated for two random variables.

Ans: Whenever there exists multiple random variables, the expectation must be taken w.r.t. all the variables involved. For example if $g(X,Y)$ is some function of two random variables 'X' and 'Y', the expected value of

9) a) Show that the density of sum of two statistically independent random variables is equal to the convolution of the individual density functions.

Ans: Let 'w' be a random variable equals to the sum of two independent random variables, 'x' and 'y'

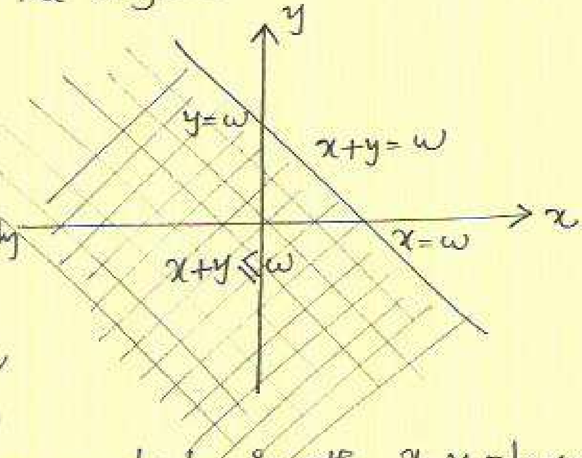
$$\therefore \quad W = X + Y \rightarrow ①$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \rightarrow ②$$

The distribution function of 'w' is then given by,

$$F_W(\omega) = P(W \leq \omega) = P(X+Y \leq \omega) \rightarrow ③$$

Below fig. shows the region in the xy-plane where, $x+y \leq \omega$.



We know that,

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy$$

From the above expression it is clear that the probability corresponding to an elemental area $dx\,dy$ in the x-y plane located at the point (x,y) is $f_{XY}(x,y)\,dx\,dy$. If we sum all such probabilities, over the region $x+y \leq \omega$ we obtain $F_W(\omega)$,

Thus,
$$F_W(\omega) = \int_{-\infty}^{\infty}\int_{x=-\infty}^{\omega-y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{x=-\infty}^{\omega-y} f_X(x) \cdot f_Y(y)\,dx\,dy \qquad \because f_{XY}(x,y) = f_X(x) \cdot f_Y(y).$$

$$F_W(\omega) = \int_{-\infty}^{\infty} f_Y(y) \cdot \int_{x=-\infty}^{\omega-y} f_X(x) \cdot dx\,dy$$

By differentiating above equation w.r.t. 'ω' and by using Leibniz's rule, we get the desired density function,

$$f_W(\omega) = \frac{d}{d\omega}(F_W(\omega)) = \int_{-\infty}^{\infty} f_Y(y) \cdot \frac{d}{d\omega}\int_{x=-\infty}^{\omega-y} f_X(x)\,dx \cdot dy.$$

$$\therefore f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\left(f_X(w-y) - 0\right) dy$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) . f_X(w-y)\, dy \longrightarrow ④$$

From the above equation it is clear that the density function of the sum of two statistically independent random variables is the convolution of their individual density functions.

(b) If statistically independent random variables 'X' and 'Y' having respective densities $f_X(x) = 5\,u(x)\,e^{-5x}$, $f_Y(y) = 2u(y)e^{-2y}$. then derive the density function of $W = X+Y$.

Soln: given the density functions of two independent random variables 'X' and 'Y' as,

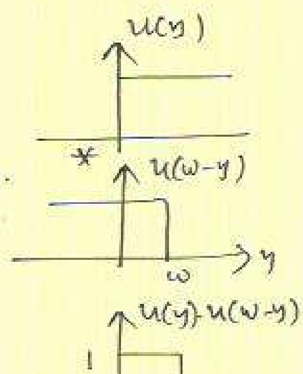
$$f_X(x) = 5\,u(x)\,e^{-5x}$$

$$f_Y(y) = 2\,u(y)\,e^{-2y}.$$

The density function of $W = X+Y$ is given by the convolution of densities of 'X' and 'Y'. as

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) . f_X(w-y)\, dy.$$

$$= \int_{-\infty}^{\infty} 2 . u(y) . e^{-2y} . 5 . u(w-y)\, e^{-5(w-y)}\, dy.$$



$$= 10 \cdot e^{-5w} \int_0^w e^{-2y} \cdot e^{5y}\, dy$$

$$= 10 e^{-5w} \int_0^w e^{3y}\, dy$$

$$= \frac{10}{3} e^{-5w}\left(e^{3y}\Big|_0^w \right)$$

$$f_W(w) = \frac{10}{3} e^{-5w}\left(e^{3w} - 1\right) = \frac{10}{3}\left(e^{-2w} - e^{-5w}\right)$$

(10) a) Show that $Var(ax+by) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$.

Soln:

$$L.H.S = Var(ax+by) = E([ax+by]^2) - [E(ax+by)]^2$$

$$= E(a^2x^2 + b^2y^2 + 2abXY) - [a\bar{X} + b\bar{Y}]^2$$

$$= E(a^2x^2 + b^2y^2 + 2abXY) - a^2\bar{X}^2 - b^2\bar{Y}^2 - 2ab\bar{X}\bar{Y}$$

$$= a^2 E(X^2) + b^2 E(Y^2) + 2ab\, E(XY) - a^2\bar{X}^2 - b^2\bar{Y}^2 - 2ab\bar{X}\bar{Y}$$

$$= a^2[E(X^2) - \bar{X}^2] + b^2[E(Y^2) - \bar{Y}^2] + 2ab[R_{XY} - \bar{X}\bar{Y}]$$

$$Var(ax+by) = a^2 \cdot Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$$

(b) Let 'x' be a random variable with mean $\bar{X} = 3$ and variance $\sigma_X^2 = 2$. Determine the mean value of the new random variable $Y = -6X + 22$ and the correlation of 'x' and 'y'.

Soln:

given
$$\bar{X} = 3,$$
$$\sigma_X^2 = 2,$$
$$\therefore E(X^2) = \sigma_X^2 + \bar{X}^2 = 2 + 3^2 = 11$$

(i) $Y = -6X + 22$.

$$\bar{Y} = -6 \cdot \bar{X} + 22 = -6(3) + 22 = 4$$

(ii) The correlation of 'x' and 'y' is given by

$$R_{XY} = E(XY) = E(X(-6X+22))$$

$$= -6\, E(X^2) + 22\, E(X)$$

$$= -6(11) + 22(3)$$

$$R_{XY} = -66 + 66 = 0.$$

Since $R_{XY} = 0$, X & Y are orthogonal random variables.

Short Answer Questions:

① State Central limit theorem.

Ans: Central limit theorem broadly says that the distribution function of sum of 'N' no. of independent random variables approaches gaussian in the limit $N \to \infty$.

② Define Correlation Coefficient.

Ans: The second order joint moment of two standardised random variables $X' = \frac{X-\bar{X}}{\sigma_X}$ and $Y' = \frac{Y-\bar{Y}}{\sigma_Y}$ is known as correlation coefficient denoted by ρ_{XY} and is given by

$$\rho_{XY} = E\left(\frac{X-\bar{X}}{\sigma_X} \cdot \frac{Y-\bar{Y}}{\sigma_Y}\right)$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_X} = \frac{C_{XY}}{\sigma_X \cdot \sigma_Y}.$$

③ When 'N' random variables are said to be jointly gaussian.

Ans: 'N' random variables $X_1, X_2 \ldots X_N$ are said to be jointly gaussian if their N^{th} order joint density function is of the form,

$$f_{X_1X_2\ldots X_N}(x_1,x_2\ldots x_N) = \frac{\left|[C_X]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \exp\left(-\frac{[x-\bar{X}]^T[C_X]^{-1}[x-\bar{X}]}{2}\right)$$

Where we define the matrices

$$[x-\bar{X}] = \begin{bmatrix} x_1-\bar{X}_1 \\ x_2-\bar{X}_2 \\ \vdots \\ x_N-\bar{X}_N \end{bmatrix} \;\&\; [C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

④ Derive the expression of constant 'a' interms of moments of 'X' and Y if V & W are orthogonal, where $V = X+aY$ & $W = X-aY$.

Ans: Two R.V.'s V & W are orthogonal if their correlation becomes zero, $R_{VW} = 0 \Rightarrow E(V \cdot W) = 0$

$$E((X+aY)(X-aY)) = 0$$

$$E(X^2 - aXY + aYX - a^2Y^2) = 0$$

$$E(X^2) - a^2E(Y^2) = 0$$

$$a^2E(Y^2) = E(X^2)$$

⑤ The joint density function of two discrete random Variables 'X' and Y is $f_{XY}(x,y) = Kxy$, $0<x<4, 1<y<5$
0 , Otherwise. Find the Value of 'K'.

<u>Ans:</u> given $f_{XY}(x,y) = Kxy$, $0<x<4, 1<y<5$
0 , Otherwise

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$.

$$\int_0^4\int_1^5 Kxy\,dx\,dy = 1$$

$$K\int_0^4 x\cdot\frac{y^2}{2}\Big|_1^5 dx = 1$$

$$\frac{K}{2}(25-1)\ \frac{x^2}{2}\Big|_0^4 = 1$$

$$K\frac{24}{4}(4^2) = 1 \Rightarrow \boxed{K = \frac{1}{96}}$$

⑥ Define joint Characteristic function of two random Variables.

<u>Ans:</u> The joint Characteristic function of two random Variables 'X' and 'Y' denoted by $\phi_{XY}(\omega_1,\omega_2)$ and is defined as

$$\phi_{XY}(\omega_1,\omega_2) = E\left(e^{j\omega_1X+j\omega_2Y}\right)$$

where ω_1 & ω_2 are real numbers.

An alternative form of $\phi_{XY}(\omega_1,\omega_2)$ is given by

$$\phi_{XY}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,e^{j\omega_1x+j\omega_2y}\,dx\,dy.$$

⑦ Two random Variables X and Y have the following Values. $E(X) = E(Y) = \frac{7}{12}$, $E(XY) = \frac{1}{3}$ and $\sigma_X = \sigma_Y = \sqrt{\frac{11}{144}}$. Find the Correlation Coefficient.

<u>Ans:</u> $\bar{X} = \bar{Y} = \frac{7}{12}$, $R_{XY} = \frac{1}{3}$, $\sigma_X = \sigma_Y = \sqrt{\frac{11}{144}}$

$$\therefore\ \rho_{XY} = \frac{C_{XY}}{\sigma_X\cdot\sigma_Y} = \frac{R_{XY} - \bar{X}\bar{Y}}{\sigma_X\cdot\sigma_Y} = \frac{\frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}$$

$$\therefore\ \rho_{XY} = \frac{\frac{1}{3} - \frac{49}{144}}{\frac{11}{144}}$$

⑧ Define joint moments about the origin.

Ans: The joint moments about the origin are denoted by m_{nk} and are defined as

$$m_{nk} = E(X^n Y^k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\,dx\,dy.$$

Where $(n+k)$ are order of the moments,

The first order moments are, $m_{10} = E(X) = \bar{X}$, $m_{01} = \bar{Y} = E(Y)$

The Second order moments are, $m_{20} = E(X^2)$, $m_{02} = E(Y^2)$

and correlation $m_{11} = R_{XY} = E(X.Y)$.

⑨ Find the expected Value of the face Value while rolling fair die.

Ans: The distribution table of R.V 'X': face Value on the die is as shown.

$X = x_i$	1	2	3	4	5	6
$P(x_i)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$.

$$\therefore \quad E(X) = \sum_{i=1}^{6} x_i\, P(x_i)$$

$$= 1\cdot 1/6 + 2\cdot 1/6 + 3\cdot 1/6 + 4\cdot 1/6 + 5\cdot 1/6 + 6\cdot 1/6$$

$$E(X) = \frac{21}{6}$$

⑩ How interval conditioning is different from point conditioning.

Ans: The conditional distribution or density function of a random Variable 'X' given that another Variable 'Y' is fixed at a particular Value 'y_k' is known as point conditioning. defined as

$$F_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i,y_k)}{P(y_k)}\, u(x-x_i) \text{ and}$$

$$f_X(x/Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i,y_k)}{P(y_k)}\, \delta(x-x_i).$$

The conditional distribution & density functions of a random Variable 'X' given that another Variable 'Y' has an interval of Values $y_a < Y \le y_b$ is known as interval conditioning and is defined as

$$f_X(x/y_a<Y\le y_b) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

→ Conditioning Event is fixed

Unit - III

① What is ACF. State and explain any four properties of ACF. (or) List and Explain Various properties of Auto correlation function. (or) State and prove any four properties of ACF.

Ans: **Definition:** The Auto correlation function of a random process $X(t)$ is the correlation $E(X_1 \cdot X_2)$ of two random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ defined by the process $X(t)$ at $t = t_1$ and $t = t_2$ respectively. Mathematically,

$$R_{XX}(t_1, t_2) = E(X(t_1) \cdot X(t_2)) \longrightarrow ①$$

let $t_1 = t$ and $t_2 = t_1 + \tau$, then,

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau)) \longrightarrow ②$$

If $X(t)$ is atleast WSS process then $R_{XX}(t, t+\tau)$ is a function of only the time difference i.e $\tau = t_2 - t_1$. Thus for a WSS process,

$$R_{XX}(\tau) = E(X(t) \cdot X(t+\tau)) = \int_{-\infty}^{\infty} X(t) \cdot X(t+\tau) \cdot f_X(x)\, dx \longrightarrow ③$$

Properties of Auto Correlation function:

1. Auto correlation function is bounded by its Value at the origin. i.e $|R_{XX}(\tau)| \leq R_{XX}(0)$. This means that ACF has its maximum Value at $\tau = 0$.

proof. Since the Expectation of a non negative number is always non negative, Consider

$$E\left([X(t_1) \pm X(t_2)]^2\right) \geq 0$$

$$E\left(X^2(t_1) \pm X^2(t_2) \pm 2X(t_1) \cdot X(t_2)\right) \geq 0$$

$$E(X^2(t_1)) \pm E(X^2(t_2)) \pm 2E(X(t_1) \cdot X(t_2)) \geq 0$$

$$R_{XX}(0) \pm R_{XX}(0) \pm 2R_{XX}(\tau) \geq 0 \quad , t_1 = t, \; t_2 = t+\tau$$

$$\cancel{2} R_{XX}(\tau) \geq \cancel{2} R_{XX}(0) \text{ or}$$

$$\boxed{|R_{XX}(\tau)| \geq R_{XX}(0)}$$

2. Auto Correlation is an even function 'τ'. i.e ACF is Symmetric about the Vertical axis passing through origin

Thus

$$R_{XX}(-\tau) = R_{XX}(\tau).$$

proof: By definition, we have

$$R_{XX}(\tau) = E(X(t).X(t+\tau))$$

Replace τ by $-\tau$ we have,

$$R_{XX}(-\tau) = E(X(t).X(t-\tau))$$

let $t-\tau = t'$

$t = t'+\tau$

$$\therefore \quad R_{XX}(-\tau) = E(X[t'+\tau]X(t'))$$

$$= E(X(t').X(t'+\tau)) = R_{XX}(\tau).$$

$$\therefore \quad \boxed{R_{XX}(-\tau) = R_{XX}(\tau)}$$

3. The mean Squared Value of the random process can be obtained from $R_{XX}(\tau)$ by putting $\tau = 0$ and is known as the average power of the process. i.e

$$P_{avg} = E(X^2(t)) = R_{XX}(0).$$

proof: By definition, we have,

$$R_{XX}(\tau) = E(X(t).X(t+\tau))$$

$$\tau = 0 \Rightarrow R_{XX}(0) = E(X(t).X(t))$$

$$= E(X^2(t)) = P_{avg}.$$

4. If $X(t)$ is periodic, then its Auto correlation function $R_{XX}(\tau)$ is also periodic with the same period. i.e

if $X(t \pm T) = X(t)$ then $R_{XX}(\tau \pm T) = R_{XX}(\tau)$

proof: given

$$X(t \pm T) = X(t) \longrightarrow ①$$

From the definition,

$$R_{XX}(\tau) = E(X(t).X(t+\tau))$$

$$\tau \to \tau \pm T \Rightarrow R_{XX}(\tau \pm T) = E(X(t).X(t+\tau \pm T)) \longrightarrow ②$$

Since $X(t)$ is periodic $X(t+\tau)$ is also periodic

i.e $X(t+\tau \pm T) = X(t+\tau). \longrightarrow ③$

$$\therefore \quad R_{XX}(\tau \pm T) = E(X(t).X(t+\tau))$$

$$\boxed{R_{XX}(\tau \pm T) = R_{XX}(\tau)}$$

⑤ If $E(X(t)) = \bar{X} (\neq 0)$, and $X(t)$ is ergodic with no periodic components, then $\lim_{|\tau| \to \infty} R_{XX}(\tau) = \bar{X}^2$

⑥ The sum of two jointly WSS processes $X(t)$ and $Y(t)$ is also a stationary process. i.e if $Z(t) = X(t) + Y(t)$ then

$$E(Z(t)) = \text{Const. and}$$

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

If $X(t)$ is orthogonal to $Y(t)$ then $R_{XY}(\tau) = R_{YX}(\tau) = 0$

$$\therefore \quad R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau).$$

⑦ If a random process $X(t)$ has a D.C component, then its auto correlation function will also have D.C component.

② a) Two Random processes $X(t)$ and $Y(t)$ defined as below.

$$X(t) = A \cos\omega_0 t + B \sin\omega_0 t$$

$$Y(t) = B \cos\omega_0 t - A \sin\omega_0 t$$ where A and B are uncorrelated random variables with mean '0' and same variance and ω_0 is constant. Find whether $X(t)$ and $Y(t)$ are jointly wide sense stationary or not.

Soln: Given two random processes,

$$X(t) = A \cos\omega_0 t + B \sin\omega_0 t \longrightarrow ①$$

$$Y(t) = B \cos\omega_0 t - A \sin\omega_0 t \longrightarrow ②$$

$$\boxed{E(A^2) = \sigma_A^2 + [E(A)]^2 = \sigma^2 = E(B^2)}$$

$$E(A) = E(B) = 0, \quad E(AB) = 0 \text{ (Uncorrelated)}$$

$$\sigma_A^2 = \sigma_B^2 = \sigma^2 \text{ (let)} \Rightarrow E(A^2) = E(B^2) = \sigma^2$$

$$\left.\begin{aligned} E(X(t)) &= \cos\omega_0 t\, E(A) + \sin\omega_0 t\, E(B) = 0 \\ E(Y(t)) &= \cos\omega_0 t\, E(B) - \sin\omega_0 t\, E(A) = 0 \end{aligned}\right\} \rightarrow ③$$

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau))$$

$$= E\left([A \cos\omega_0 t + B \sin\omega_0 t][A \cos(\omega_0 t + \omega_0 \tau) + B \sin(\omega_0 t + \omega_0 \tau)]\right)$$

$$= E[A^2 \cos\omega_0 t \cos(\omega_0 t + \omega_0 \tau)] + E[AB \cos\omega_0 t \sin(\omega_0 t + \omega_0 \tau)]$$
$$+ E(AB \sin\omega_0 t \cos(\omega_0 t + \omega_0 \tau)) + E(B^2 \sin\omega_0 t \sin(\omega_0 t + \omega_0 \tau))$$

$$= E(A^2) \cos\omega_0 t \cos(\omega_0 t + \omega_0 \tau) + \overset{0}{\cancel{E(AB)}} \cos\omega_0 t \sin(\omega_0 t + \omega_0 \tau)$$
$$+ \overset{0}{\cancel{E(AB)}} \sin\omega_0 t \cos(\omega_0 t + \omega_0 \tau) + E(B^2) \sin\omega_0 t \sin(\omega_0 t + \omega_0 \tau)$$

$$R_{yy}(t,t+\tau) = E(Y(t)\cdot Y(t+\tau))$$

$$= E\left([B\cos\omega_0 t - A\sin\omega_0 t][B\cos(\omega_0 t+\omega_0\tau) - A\sin(\omega_0 t+\omega_0\tau)]\right)$$

$$= E(B^2)\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau) - E(\cancel{AB})^{0}\cos\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$

$$- E(\cancel{AB})^{0}\sin\omega_0 t\cos(\omega_0 t+\omega_0\tau) + E(A^2)\sin\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$

$$= \sigma^2\left[\cos(\cancel{\omega_0 t} - \cancel{\omega_0 t} - \omega_0\tau)\right) = \sigma^2\cos\omega_0\tau \rightarrow (5)$$

$$R_{xy}(t,t+\tau) = E(X(t)\cdot Y(t+\tau))$$

$$= E\left[(A\cos\omega_0 t + B\sin\omega_0 t)(\cancel{A}B\cos(\omega_0 t+\omega_0\tau) - A\sin(\omega_0 t+\omega_0\tau))\right]$$

$$= E(\cancel{AB})^{0}\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau) - E(A^2)\cos\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$

$$+ E(B^2)\cdot\sin\omega_0 t\cos(\omega_0 t+\omega_0\tau) - E(\cancel{AB})^{0}\sin\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$

$$= \sigma^2\left[\sin\omega_0 t\cos(\omega_0 t+\omega_0\tau) - \cos\omega_0 t\sin(\omega_0 t+\omega_0\tau)\right]$$

$$= \sigma^2\left(\sin(\cancel{\omega_0 t} - \cancel{\omega_0 t} - \omega_0\tau)\right) = -\sigma^2\sin\omega_0\tau \rightarrow (6)$$

from Equations (3), (4), (5) and (6) it is clear that the mean values of $X(t)$ and $Y(t)$ are constant and ACF's and CCF's are functions of time difference 'τ' only. Hence, $X(t)$ and $Y(t)$ are jointly wide sense processes.

(b) A Random process $X(t) = a\sin(\omega_0 t+\theta)$ where 'θ' is uniform over $(0, 2\pi)$. Find whether it is ergodic or not.

Soln:

Given the random process

$$X(t) = a\sin(\omega_0 t+\theta) \rightarrow (1)$$

θ is uniform R.V. over $(0, 2\pi)$ i.e

$$f(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi.$$

A Random process $X(t)$ is ergodic if its time averages becomes Statistical Averages. i.e

$$\bar{x} = \bar{X} \text{ and } \mathcal{R}_{xx}(\tau) = R_{xx}(\tau).$$

$$A[x(t)] = \bar{x} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} a\sin(\omega_0 t+\theta)\,dt$$

$$= \lim \frac{1}{2T}\, a\, \frac{(-)\cos[\omega_0 t+\theta]}{\omega_0}\Big|^{T}$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2T}\left(\frac{-a}{\omega_0}\right)\left[\cancel{\sin}\ \cos(\omega_0 T+\theta) - \cos(-\omega_0 T+\theta)\right]$$

$$= \lim_{T\to\infty} \frac{-a}{2T\omega_0}\left(\cos(2\pi+\theta) - \cos(2\pi-\theta)\right) \qquad \frac{2\pi}{\omega_0} = T$$

$$= \lim_{T\to\infty} \frac{-a}{2T\omega_0}\left(\cancel{\cos}\theta - \cos\cancel{\theta}\right) = 0$$

$$\therefore \boxed{\bar{x} = 0} \longrightarrow ①$$

$$\bar{X} = E(x(t)) = \int_{-\infty}^{\infty} x(t)\cdot f(\theta)\,d\theta$$

$$= \int_{\cancel{-\infty}\ 0}^{\cancel{\infty}\ 2\pi} a\sin(\omega_0 t+\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{a}{2\pi}(-)\cos(\omega_0 t+\theta)\Big|_0^{2\pi}$$

$$= \frac{-a}{2\pi}\left(\cos(2\pi+\omega_0 t) - \cos\omega_0 t\right)$$

$$= \frac{-a}{2\pi}\left(\cos\cancel{\omega_0 t} - \cos\cancel{\omega_0 t}\right) = 0$$

$$\therefore \boxed{\bar{X} = 0} \longrightarrow ②$$

The time auto correlation function is given by,

$$R_{xx}(\tau) = A[x(t)\,x(t+\tau)]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} a\sin(\omega_0 t+\theta)\,a\sin(\omega_0 t+\omega_0\tau+\theta)\,dt$$

$$= \lim_{T\to\infty}\frac{a^2}{4T}\int_{-T}^{T}\left[\cos(\omega_0\tau) - \cos(2\omega_0 t+2\omega_0\tau+2\theta)\right]dt$$

$\omega_0 T = 2\pi$

$$= \frac{a^2}{4}\lim_{T\to\infty}\frac{1}{T}\left[(\cos\omega_0\tau)\,t\Big|_{-T}^{T} - \frac{\sin(2\omega_0 t+2\omega_0\tau+2\theta)}{2\omega_0}\Big|_{-T}^{T}\right]$$

$2\omega_0 T = 4\pi$

$$\boxed{\begin{array}{l}\sin(4\pi+\theta) = \sin\theta \\ \sin(-(4\pi-\theta)) = -\sin\theta\end{array}}$$

$$= \frac{a^2}{4}\left(\lim_{T\to\infty}\frac{1}{T}\left(2T\cdot\cos\omega_0\tau - \frac{1}{2\omega_0}\left(\sin(2\omega_0 T+2\omega_0\tau+2\theta) - \sin(-2\omega_0 T+2\omega_0\tau+2\theta)\right)\right)\right)$$

$$= \frac{a^2}{4}\left(\lim \frac{1}{\cancel{T}}\left(2\cancel{T}\cos\omega_0\tau - \frac{1}{2\omega_0}\left(\sin(2\omega_0\tau+2\theta) - \sin(2\omega_0\tau+2\theta)\right)\right)\right)$$

The Autocorrelation function $R_{XX}(\tau)$ is given by

$$R_{XX}(\tau) = E(X(t).X(t+\tau))$$

$$= \int_{-\infty}^{\infty} X(t).X(t+\tau).f(\theta).d\theta$$

$$= \int_{0}^{2\pi} a\sin(\omega_0 t+\theta).a\sin(\omega_0 t+\omega_0\tau+\theta).\frac{1}{2\pi}d\theta$$

$$= \frac{a^2}{2\pi}\int_{0}^{2\pi}\frac{1}{2}\left[\cos\omega_0\tau - \cos(2\omega_0 t+2\theta+\omega_0\tau)\right].d\theta$$

$$= \frac{a^2}{4\pi}\left((\cos\omega_0\tau)\,\theta\Big|_0^{2\pi} - \frac{\sin(2\omega_0 t+2\theta+\omega_0\tau)}{2}\Big|_0^{2\pi}\right)$$

$$= \frac{a^2}{\cancel{4\pi}_2}\left(\cos\omega_0\tau\,(\cancel{2}\pi-0) - \tfrac{1}{2}\left(\cancel{\sin(2\omega_0 t+\omega_0\tau)} - \cancel{\sin(2\omega_0 t+\omega_0\tau)}\right)\right)$$

$$R_{XX}(\tau) = \frac{a^2}{2}\cos\omega_0\tau \longrightarrow ④$$

from ③ & ④ it is clear that

$$\boxed{\mathcal{R}_{XX}(\tau) = R_{XX}(\tau)} \quad \text{also} \quad \bar{x} = \bar{X}$$

$\therefore$ $X(t)$ is mean and correlated ergodic process.

③ a) Find the mean, variance of the process $X(t)$, with ACF given as

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Solu:

given the ACF of the process,

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2} \longrightarrow ①$$

If $X(t)$ is ergodic and no, periodic components and $\bar{X} = E(X(t)) \neq 0$, we have,

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

$$\lim_{|\tau|\to\infty}\left(25+\frac{4}{1+6\tau^2}\right) = \bar{X}^2$$

$$\therefore \quad 25 = \bar{X}^2 \quad \text{or}$$

The mean value of the process is $\boxed{\bar{X} = 5}$

The mean square value of the process is given by

$$E(X^2(t)) = R_{XX}(0) = 25 + \frac{4}{1+6(0)}$$

$$= 25 + 4 = 29.$$

Hence, the variance of the process is given by

$$\sigma_X^2 = E(X^2(t)) - \overline{X}^2$$

$$\sigma_X^2 = 29 - 25 = \boxed{4 = \sigma_X^2}$$

(b) Evaluate the mean, average power and variance of the random process having $R_{XX}(\tau) = 36 + 25\exp(-|\tau|)$.

Soln: Given the ACF of the process,

$$R_{XX}(\tau) = 36 + 25\exp(-|\tau|).$$

For ergodic random process with no periodic components, we have.

$$\lim_{|\tau| \to \infty} R_{XX}(\tau) = \overline{X}^2$$

$$36 + 25e^{-\infty} = \overline{X}^2$$

$$\overline{X}^2 = 36$$

Thus the mean value is $\boxed{\overline{X} = 6}$

The Average power of the process is given as

$$P_{avg} = E(X^2(t)) = R_{XX}(0)$$

$$= 36 + 25e^{-0} = 36 + 25 = 61$$

$$\therefore \boxed{P_{avg} = E(X^2(t)) = 61}$$

The variance of the process is given as

$$\sigma_X^2 = E(X^2(t)) - \overline{X}^2$$

$$= 61 - 36$$

$$= 25$$

$$\therefore \boxed{\sigma_X^2 = 25}$$

④ Explain about first order, Second order, wide sense and Strict Sense Stationary processes.

Ans: Depending on the density functions of the random variables of the random process, there are several types of Stationarities. They are,

→ first order Stationary process

→ Second order and wide sense Stationary process

→ N^{th} order and Strict sense Stationary process.

first order Stationary process: A random process $X(t)$ is called Stationary to order one if its first order density function does not change with a shift in the time origin. i.e independent of time. In otherwords.

$f_X(x_1:t_1) = f_X(x_1:t_1+\tau)$ must be true for any t_1 and for any real number 'τ' if $X(t)$ is first order Stationary.

Thus the mean value of the process is const. Since $f_X(x:t_1)$ is independent of t_1 i.e

$$E(X(t_1)) = E(X(t+\tau)) \Rightarrow \boxed{E(X(t_1)) = \text{Constant}}$$

Second order Stationary Process: A Random process $X(t)$ is called Stationary to order two if its Second order density function Satisfies,

$$f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1+\tau, t_2+\tau) \;\forall\, t_1, t_2, \tau$$

where 'τ' is a real number.

A Second order Stationary process is also first order Stationary because the Second order densities determines the lower i.e first order densities.

The Auto correlation function of a Second order Stationary process is a function of only time differences and not on absolute time i.e if $\tau = t_2 - t_1$ then

$$\boxed{R_{XX}(t_1, t_1+\tau) = E(X(t_1).X(t_1+\tau)) = R_{XX}(\tau)}$$

Wide Sense Stationary process: A more relaxed form of Stationarity, called WSS process is defined as that for which the following conditions are true,

$$E(X(t)) = \bar{X} = \text{Constant and}$$

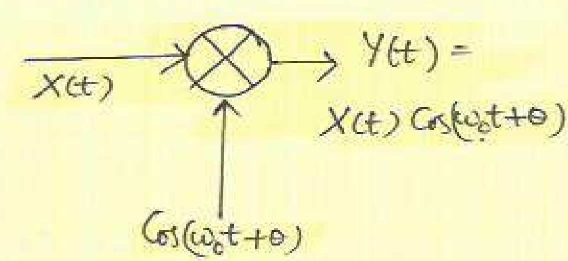
N^{th} order Stationary Process: A Random process $X(t)$ is Stationary to order 'N' if its N^{th} order density function is invariant to a time origin shift i.e

$$f_X(x_1, x_2 \dots x_N ; t_1, t_2 \dots t_N) = f_X(x_1, x_2, \dots x_N ; t_1+\tau, t_2+\tau, \dots t_N+\tau)$$
$$\forall, t_1, t_2 \dots t_N, \tau.$$

Stationarity to order 'N' implies that it is Stationary to all orders $k \le N$.

Strict Sense Stationary Process: A random process which is Stationary to all orders $N = 1, 2, 3 \dots$ is called Strict Sense Stationary process. So clearly an N^{th} order Stationary process is also Called as Strict Sense Stationary process.

(5) let $X(t)$ be a WSS process with Auto correlation function, $R_{XX}(\tau) = e^{-a|\tau|}$, where $a > 0$ is a Const. Assume $X(t)$ amplitude modulates a carrier $\cos(\omega_0 t + \theta)$ as Shown in fig. where ω_0 is const. and θ is a Random Variable uniform on $(-\pi, \pi)$ and is Statistically independent of $X(t)$. Determine the ACF of $Y(t)$.



Soln:

Given The ACF of a WSS process $X(t)$ is given by

$$R_{XX}(\tau) = e^{-a|\tau|}, \quad a > 0 \text{ is a Const.} \rightarrow (1)$$

$X(t)$, amplitude modulates a carrier $\cos(\omega_0 t + \theta)$, Thus we have,

$$Y(t) = X(t) \cdot \cos(\omega_0 t + \theta) \rightarrow (2)$$

Where 'θ' is uniform R.V. Over $(0, 2\pi)$

$$\therefore \quad f(\theta) = \frac{1}{2\pi}, \quad -\pi \le \theta \le \pi.$$

By definition,

$$\therefore R_{YY}(\tau) = E(Y(t) \cdot Y(t+\tau))$$
$$= E\left(X(t) \cos(\omega_0 t + \theta) \cdot X(t+\tau) \cos(\omega_0 t + \omega_0 \tau + \theta)\right)$$
$$= E(X(t) \cdot X(t+\tau)) \cdot \frac{1}{2} E\left(2 \cos(\omega_0 t + \theta) \cos(\omega_0 t + \omega_0 \tau + \theta)\right)$$

$$R_{yy}(\tau) = \frac{1}{2} \cdot R_{XX}(\tau) \cdot E\left[\cos(2\omega_0 t + \omega_0\tau + 2\theta) + \cos\omega_0\tau\right]$$

$$= \frac{1}{2} R_{XX}(\tau)\left[E(\cos(2\omega_0 t + \omega_0\tau + 2\theta)) + \cos\omega_0\tau\right]$$

$\because E(K) = K.$

$$E(\cos(2\omega_0 t + \omega_0\tau + 2\theta)) = \int_{-\pi}^{\pi} \cos(2\omega_0 t + \omega_0\tau + 2\theta) \cdot f(\theta) \cdot d\theta$$

$$= \frac{1}{2\pi} \left. \frac{\sin(2\omega_0 t + \omega_0\tau + 2\theta)}{2} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left(\sin(2\pi + 2\omega_0 t + 2\omega_0\tau) - \sin(-2\pi + 2\omega_0 t + \omega_0\tau)\right)$$

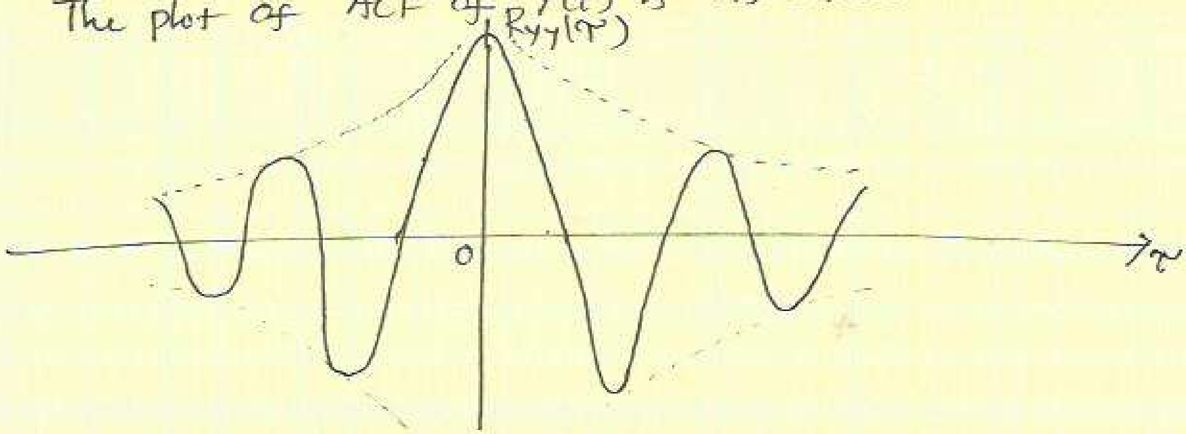
$$= \frac{1}{4\pi}\left(\cancel{\sin(2\omega_0 t + \omega_0\tau)} - \cancel{\sin(2\omega_0 t + \omega_0\tau)}\right)$$

$$= 0.$$

$$\therefore \quad R_{yy}(\tau) = \tfrac{1}{2} R_{XX}(\tau)\left(0 + \cos\omega_0\tau\right)$$

$$\boxed{R_{yy}(\tau) = \frac{1}{2} e^{-a|\tau|} \cos\omega_0\tau}$$

The plot of ACF of $Y(t)$ is as shown.



⑥ a) Compare cross correlation function with Autocorrelation function

Ans: → Cross Correlation function is defined as the correlation of two random variables $X(t_1)$ and $Y(t_2)$ which are derived from two random processes $X(t)$ and $Y(t)$ respectively where as Auto Correlation function is defined as the correlation of two R.V.'s $X(t_1)$ and $X(t_2)$ which

→ Cross correlation function is denoted by $R_{XY}(t_1,t_2)$ and is defined as $R_{XY}(t_1,t_2) = E(X(t_1).Y(t_2))$.

Auto correlation function is denoted by $R_{XX}(t_1,t_2)$ and is defined as $R_{XX}(t_1,t_2) = E(X(t_1).X(t_2))$

→ The two cross correlation functions are related by $R_{XY}(-\tau) = R_{YX}(\tau)$ where as the Auto correlation function is an even function of 'τ'

$$R_{XX}(-\tau) = R_{XX}(\tau).$$

→ $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0).R_{YY}(0)}$ But $|R_{XX}(\tau)| \leq R_{XX}(0)$.

b) Assume that an ergodic process $X(t)$ has an auto correlation function.

$$R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos 12\tau)$$

a) Find $\bar{X}$.

b) Does this process have a periodic Component.

c) What is the Average power in $X(t)$.

Soln: given, an ergodic random process $X(t)$ has an auto correlation function

$$R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos 12\tau) \longrightarrow (1)$$

(a) we know from the properties of ACF, if $X(t)$ is ergodic and have no periodic Components, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

$$\therefore \quad 18 + \frac{2}{\infty}(1+4\cos\infty) = \bar{X}^2$$

$$\bar{X}^2 = 18+0 \Rightarrow \bar{X} = 3\sqrt{2}$$

(b) No.

(c) The Average power in $X(t)$ is given by

$$P = R_{XX}(0) = 18 + \frac{2}{6+0}(1+4\cos 0)$$

(7) a) Define the terms (i) Random Process
(ii) Stationary Random Process
(iii) Wide Sense Stationary Process
(iv) Ergodic Random Process.

Ans: Random Process: The concept of random process is an extension of a random Variable which includes a time Variable 't'. Thus the random process is a function of 's' and 't'.

A random Variable as a function of time is Simply known as random process. In other words if we assign according to some rule a time function $x(t,s)$ to each outcome of 'S'. Then the family or ensemble of all such functions denoted by $X(t,s)$ or $X(t)$ is Called a random process.

Stationary Random Process: A random process is said to be Stationary if all its Statistical properties donot Change with time. Other processes are Called non Stationary. There are different levels of Stationarity all of which depends on the density functions of the random Variables of the process.

Wide Sense Stationary Process: A Random process $X(t)$ is Said to be Wide Sense Stationary process if it possesses the following Characteristics,

$$E(X(t)) = \text{Constant}$$

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau)) = R_{XX}(\tau).$$

Ergodic Random Process: The random processes that satisfy the ergodic theorem are known as ergodic random processes. The ergodic theorem allows the Validity of the following equations

$\bar{x} = \bar{X}$ and $\mathcal{R}_{XX}(\tau) = R_{XX}(\tau)$ where,

$\bar{x}$ and $\mathcal{R}_{XX}(\tau)$ are Time average mean & Time Auto correlation fn. and $\bar{X}$ and $R_{XX}(\tau)$ are Statistical mean and ACF's respectively.

Broadly says that if time averages

b) let $X(t)$ be a Stationary continuous random process that is differentiable, and denoted by $\dot{X}(t) = \frac{d}{dt}X(t)$. Determine

(i) $E(\dot{X}(t))$ (ii) Express the ACF of $\dot{X}(t)$ i.e $R_{\dot{X}\dot{X}}(\tau)$ interms of $R_{XX}(\tau)$

Ans:

given $X(t)$ be a WSS process, Thus we have

$$E(X(t)) = E(X(t+\varepsilon)) = \bar{X} = \text{Const.} \longrightarrow ①$$

The derivative of $X(t)$ is given by

$$\dot{X}(t) = \frac{d}{dt}X(t) = \lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon} \longrightarrow ②$$

(i) $$E(\dot{X}(t)) = E\left(\lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(E(X(t+\varepsilon)) - E(X(t))\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\cancel{\bar{X}} - \cancel{\bar{X}}) = 0$$

$$\therefore \boxed{E(\dot{X}(t)) = 0}$$

(ii) $$E(X(t).\dot{X}(t+\tau)) = R_{X\dot{X}}(\tau) = E\left(X(t). \lim_{\varepsilon \to 0} \frac{X(t+\tau+\varepsilon) - X(t+\tau)}{\varepsilon}\right)$$

$$= E\left(\lim_{\varepsilon \to 0} \frac{X(t).X(t+\tau+\varepsilon) - X(t).X(t+\tau)}{\varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(E(X(t).X(t+\tau+\varepsilon)) - E(X(t)X(t+\tau))\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(R_{XX}(\tau+\varepsilon) - R_{XX}(\tau)\right)$$

$$R_{X\dot{X}}(\tau) = \frac{d}{d\tau}\left(R_{XX}(\tau)\right) \longrightarrow ③$$

$$\therefore E(\dot{X}(t).\dot{X}(t+\tau)) = R_{\dot{X}\dot{X}}(\tau) = E\left[\lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon} . \dot{X}(t+\tau)\right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[E(X(t+\varepsilon).\dot{X}(t+\tau)) - E(X(t)\dot{X}(t+\tau))\right]$$

$$= \lim \frac{1}{\varepsilon}\left[R_{X\dot{X}}(\tau - \varepsilon) - R_{X\dot{X}}(\tau)\right]$$

$$\therefore \boxed{R_{\dot{X}\dot{X}}(\tau) = -\frac{d^2}{d\tau^2}\left(R_{XX}(\tau)\right)}$$

⑧ a) Distinguish b/w ensemble Average and time average of a random process.

Ans: Ensemble Average: The Expected Value of the random Variable $X_1 = X(t_1)$ is called the ensemble average as well as the mean or expected Value of the random process at $t = t_1$. Since t_1 may have Various Values the mean Value of a process may not be constant. It was denoted by $\overline{X} = E(X(t))$

Time Average: The time average of a random process $X(t)$ may be defined as

$$A[X(t)] = \overline{x} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt.$$

b) A random process is defined as $X(t) = A\sin(\omega t+\theta)$, where 'A' is a Constant and θ is a uniform random variable distributed over $(-\pi,\pi)$. Check $X(t)$ for Stationarity.

Soln: A Random process is given by

$$X(t) = A\sin(\omega t+\theta) \longrightarrow ①$$

where 'θ' is uniform R.V. over $(-\pi,\pi)$

$$\therefore \quad f(\theta) = \frac{1}{2\pi}, \quad -\pi \le \theta \le \pi.$$

$$\rightarrow E(X(t)) = \int_{-\infty}^{\infty} X(t)\cdot f(\theta)\cdot d\theta$$

$$= \int_{-\pi}^{\pi} A\sin(\omega t+\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A}{2\pi}(-)\cos(\omega t+\theta)\Big|_{-\pi}^{\pi}$$

$$= \frac{-A}{2\pi}\left(\cos(\pi+\omega t) - \cos(\omega t-\pi)\right)$$

$$= \frac{-A}{2\pi}\left(-\cancel{\cos\omega t} - (-\cancel{\cos\omega t})\right) = 0.$$

$$\rightarrow R_{XX}(t,t+\tau) = E(X(t).\,X(t+\tau))$$

$$= E\left(A\sin(\omega t+\theta)\, A\sin(\omega t+\omega\tau+\theta)\right)$$

$$= \frac{A^2}{2}E\left[\cos(\omega\tau) - \cos(2\omega t+\omega\tau+2\theta)\right]$$

$$= \frac{A^2}{2}\left[E(\cos\omega\tau) - E(\cos(2\omega t+\omega\tau+2\theta))\right]$$

$$= \frac{A^2}{2}\left(\cos\omega\tau - \int_{-\pi}^{\pi}\cos(2\omega t+\omega\tau+2\theta)\frac{1}{2\pi}\,d\theta\right)$$

$$= \frac{A^2}{2}\left(\cos\omega\tau - \frac{1}{2\pi}\,\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{A^2}{2}\left(\cos\omega\tau - \frac{1}{4\pi}\left(\sin(\cancel{2\omega t}+\omega\tau) - \sin(\cancel{2\omega t}+\omega\tau)\right)\right)$$

$$\boxed{R_{XX}(t,t+\tau) = \frac{A^2}{2}\cos\omega\tau = R_{XX}(\tau).} \longrightarrow (3)$$

from Eqn (2) & (3) we can say that $X(t)$ is WSS process

(9) a) Explain about mean ergodic process.

Ans: A process $X(t)$ with a constant mean value $\overline{X}$ is called mean ergodic or ergodic in mean if its statistical average $\overline{X}$ equals the time average $\overline{x}$ of any sample function $x(t)$ with probability '1' for all sample functions.

i.e

if $E(X(t)) = \overline{X} = A(x(t)) = \overline{x}$ with probability 1 for all $x(t)$.

b) If $X(t)$ is a WSS process having a mean value $E(X(t)) = 3$ and auto correlation function $R_{XX}(\tau) = 9 + 2e^{-|\tau|}$. find.

(a) The mean value

(b) The variance of the Random Variable $Y = \int_0^2 X(t)\,dt$.

Soln: given Data, $X(t)$ is a WSS process with mean & ACF given as, $\overline{X} = 3$ and

A new Random Variable is defined as

$$Y = \int_0^2 X(t)\,dt.$$

(i) $$E(Y) = E\int_0^2 X(t)\,dt = \int_0^2 E(X(t))\,dt$$

$$= \int_0^2 3\cdot dt = 3\cdot t\Big|_0^2$$

$$\overline{Y} = 3(2-0) = 6$$

(ii) The variance of the Random Variable is given by

$$E(Y^2) - \overline{Y}^2 = \sigma_Y^2.$$

$$\therefore E(Y^2) = E(Y\cdot Y)$$

$$= E\left(\int_0^2 X(t)\,dt \cdot \int_0^2 X(u)\,du\right)$$

$$= \int_0^2\int_0^2 E(X(t).X(u))\,dt\,du$$

Since $X(t)$ is WSS, we have $E(X(t)\,X(u)) = R_{XX}(t-u)$

$$\therefore R_{XX}(t-u) = 9+2e^{-|\tau|}\Big|_{\tau=t-u}$$

$$\therefore E(Y^2) = \int_0^2\int_0^2 (9+2e^{-|t-u|})\,dt\,du$$

$$= \int_0^2\int_0^2 9\cdot dt\cdot du + 2\int_0^2\int_0^2 e^{-|t-u|}\,du\,dt$$

$$= \int_0^2 9\cdot t\Big|_0^2 du + 2\int_0^2 \frac{e^{-|t-u|}}{(-1)(-1)}\Big|_0^2 dt$$

$$= \int_0^2 9(2)\,du + 2\cdot\int_0^2 (e^{-|t-2|} - e^{-|t|})\,dt$$

$$= 18\cdot u\Big|_0^2 + 2\cdot\left[\frac{e^{-|t-2|}}{-1}\Big|_0^2 - \frac{e^{-|t|}}{-1}\Big|_0^2\right]$$

$$= 36 + 2\left[-(1-e^{-2}) + e^{-2} - 1\right]$$

$$= 36 + 2\left[2e^{-2} - 2\right] = 32 + 4e^{-2}$$

$$= 4(8+e^{-2})$$

$$\sigma_Y^2 = 4(8+e^{-2}) - 36$$

(10) a) When do you call two random processes to be jointly WSS.

Ans: Two random processes X(t) and Y(t) are jointly wide sense stationary if they possess the following properties.

$$E(X(t)) = \text{Const. and } E(Y(t)) = \text{Const.}$$

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau)) = R_{XX}(\tau)$$

$$R_{YY}(t, t+\tau) = E(Y(t) \cdot Y(t+\tau)) = R_{YY}(\tau) \text{ also}$$

$$R_{XY}(t, t+\tau) = E(X(t)\, Y(t+\tau)) = R_{XY}(\tau)$$

$$R_{YX}(t, t+\tau) = E(Y(t) \cdot X(t+\tau)) = R_{YX}(\tau)$$ i.e when two processes X(t) and Y(t) have const. mean's and their ACF's and CCF's are function of time difference 'τ' only.

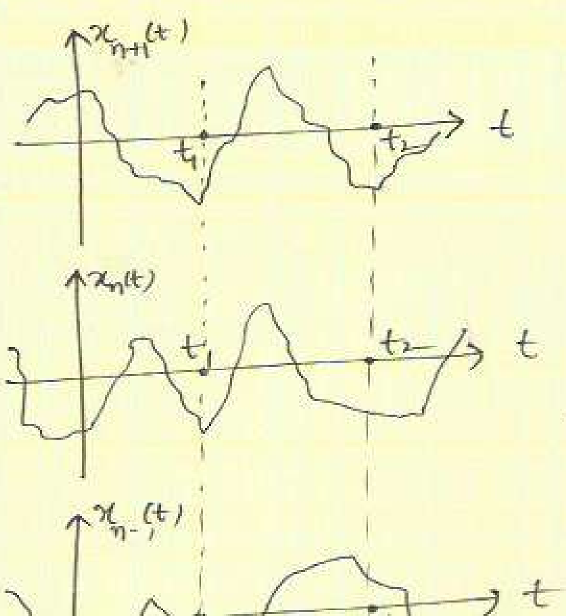
(b) Give the classification of Random Processes (or) Classify Random Processes.

Ans: Random processes are classified into four types according to the characteristics of the time variable 't' and the random variable $X = X(t)$ at time 't', They are,

→ Continuous Random process
→ Discrete Random process
→ Continuous Random Sequence
→ Discrete Random Sequence.

Continuous Random Process: If 'x' is continuous and 't' can have a continuous range of values then X(t) is called a continuous random process. Fig. shows few ensemble members of a continuous Random process.

The Thermal noise generated by any realizable n/w can be modelled as a sample function of this type of process.



→ Discrete Random Process: If 'X' is having only discrete values while 't' is continuous then X(t) is called as a discrete random process. This process is derived by heavily limiting the sampling functions to have either positive level or negative level.

Fig. Shows such type of process.

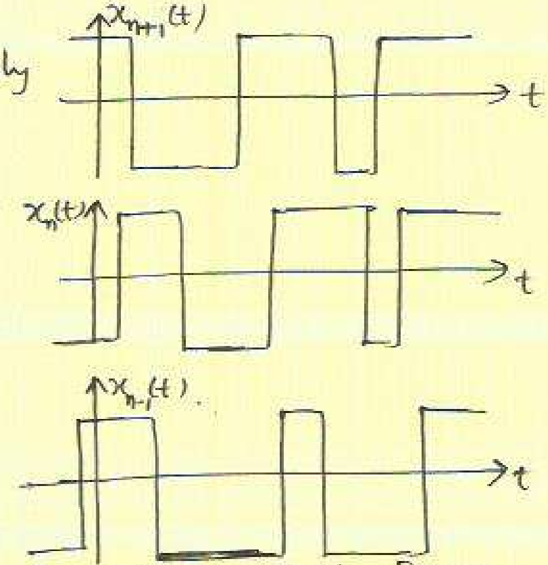


fig: Discrete Random Process.

→ Continuous random Sequence: A Random process for which 'X' is continuous and 't' has only discrete values is called as a continuous random sequence. This process can be formed by periodically sampling the ensemble members of continuous random process. The sample functions of this process are referred to as a discrete time signal.

Technically a continuous random sequence is a set of R.V's defined by $X(t_n)$ for $t_n = nT_s$, $n = 0, \pm1, \pm2 \ldots$ where T_s is the sampling interval.

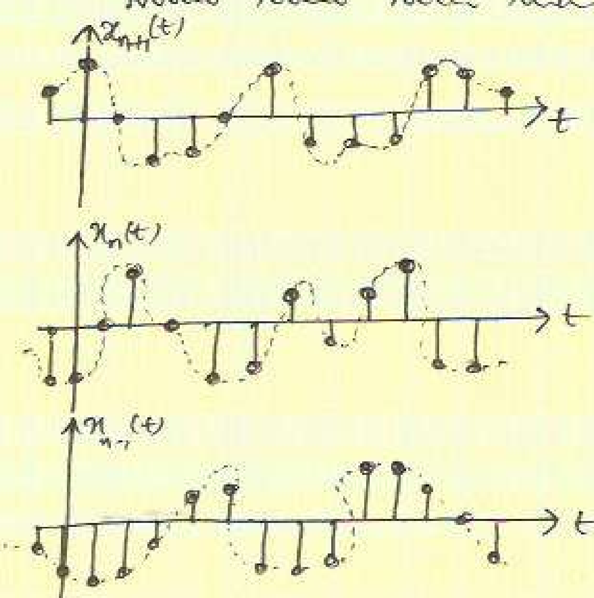
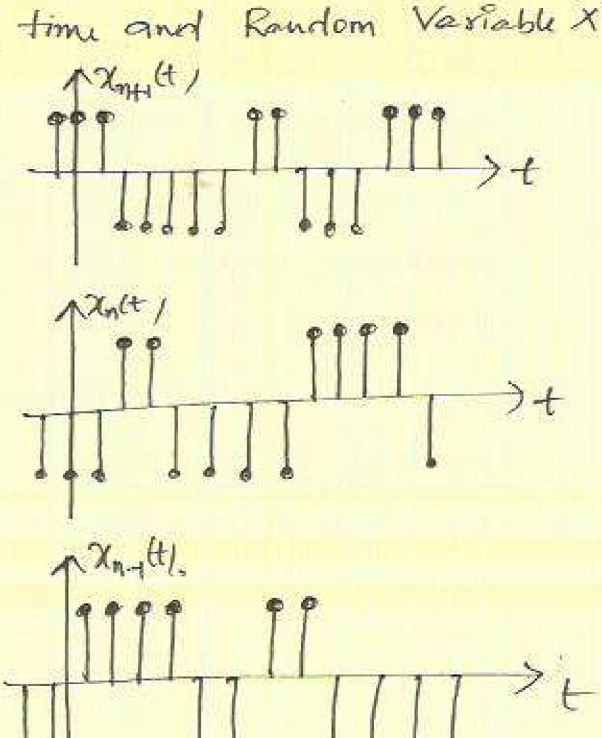


fig: Continuous Random Sequence.

→ Discrete Random Sequence: If both the time and Random Variable 'X' are having only discrete values then X(t) is called as Discrete Random Sequence. The Fig. shows a discrete Random Sequence which is developed by sampling the sample functions of Discrete random process.



Short Answer Questions:

① A Random process $X(t) = A \sin w_0 t$, where w_0 is Const. and 'A' is uniform R.V. over the interval $(0,1)$ Find whether $X(t)$ is Stationary process or not.

Ans:

$$X(t) = A \sin w_0 t, \quad f(A) = \frac{1}{1-0}, \quad 0 \leq A \leq 1$$

$$E(X(t)) = E(A \sin w_0 t)$$

$$= \int_0^1 A \sin w_0 t \cdot f(A) \cdot dA$$

$$E(X(t)) = \sin w_0 t \cdot \frac{A^2}{2}\Big|_0^1 = \frac{\sin w_0 t}{2}$$

Since the mean value of $X(t)$ is not Constant it is clear that $X(t)$ is not WSS process.

② State Auto correlation properties.

Ans:

→ $|R_{XX}(\tau)| \leq R_{XX}(0)$ i.e ACF has its max. Value at $\tau = 0$.

→ $R_{XX}(-\tau) = R_{XX}(\tau)$ i.e ACF is an even function of 'τ'

→ $P_{avg} = E(X^2(t)) = R_{XX}(0)$

→ Sum of two jointly Stationary processes is also Stationary process. i.e if $Z(t) = X(t) + Y(t)$, then $E(Z(t)) = $ Const. and $R_{ZZ}(t, t+\tau) = R_{ZZ}(\tau)$.

③ Explain about Strict Sense Stationary processes.

Ans: A Random process which is Stationary to all orders, $N = 1, 2, 3 \ldots$ is known as Strict Sense Stationary processes. Clearly an N^{th} order Stationary process is also Called Strict Sense Stationary process.

④ Where the poisson random is used? Explain. (or) Give any two examples of a poisson process.

Ans: Poisson random process is an example of discrete random process. It describes the no. of times that some event has occured as a function of time, where the events occur at random times.

Examples of such events are,

→ Arrival of a Customer at a bank or supermarket.

→ Failure of some Component in a System. etc. Thus.

poisson random processes are used where the Counting type of

⑤ List the Various Classification of random processes.

Ans: → Deterministic / Non Deterministic Processes

→ Stationary / Non- Stationary processes. Based on $x(t)$ and 's' processes also Classified as

- Continuous Time Random process
- Discrete Time Random process.
- Continuous Time random Sequence
- Discrete Time random Sequence.

⑥ prove the Statement $R_{XX}(-\tau) = R_{XX}(\tau)$.

Ans: By definition, we have,

$$R_{XX}(\tau) = E(X(t).X(t+\tau))$$

$$R_{XX}(-\tau) = E(X(t).X(t-\tau)), \quad \text{let } t-\tau = t'$$

$$t = t'+\tau$$

$$= E(X(t'+\tau)\cdot X(t'))$$

$$R_{XX}(-\tau) = R_{XX}(\tau).$$ i.e $R_{XX}(\tau)$ is an even function.

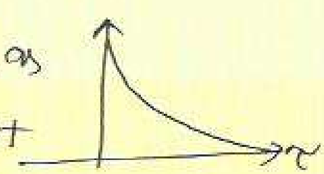
⑦ What is Stationary processes? Explain. (or) Distinguish b/w Stationary and non-Stationary processes.

Ans: A random process $X(t)$ is Said to be Stationary process if all its Statistical properties donot changes with time. There are different levels of Stationarities depending upon the density functions of the random Variables of the Process.

A random process $X(t)$ is Called non-Stationary if its Statistical properties are Variant to a shift of origin. i.e Statistical properties Change with time.

⑧ Test the function "$e^{-\tau}u(\tau)$" for a Valid ACF.

Ans: The function given $e^{-\tau}u(\tau)$ is plotted as



It is not a Valid ACF because it is not an even function 'τ' $R_{XX}(-\tau) \neq R_{XX}(\tau)$.

⑨ When two different random processes are said to be Statistically independent?

Ans: Two random processes $X(t)$ and $Y(t)$ are Statistically independent if the random Variable group $\{X(t_1), X(t_2) \cdots X(t_N)\}$ is independent of the group $\{Y(t'_1), Y(t'_2) \cdots Y(t'_N)\}$ for any

Choice of times $t_1, t_2 \ldots t_N, t_1', t_2' \ldots t_N'$ i.e

$$f_{XY}(x_1, x_2 \ldots x_N, y_1, y_2 \ldots y_N : t_1, t_2 \ldots t_N, t_1', t_2' \ldots t_N') = f_X(x_1, x_2 \ldots x_N : t_1, t_2 \ldots t_N)\, f_Y(y_1, y_2 \ldots y_N : t_1' \ldots t_N')$$

(10) State any two differences b/w Random Variable and random process.

Ans:

Random Variable	Random Process
1) Random Variable is a real valued function of the elements of the Sample Space 'S'	1) Random Variable as a function of time is called as random process.
2) It was denoted by $X(s)$ or X	2) It was denoted by $X(t,s)$ or $X(t)$
3) The mean value of a R.V is always const.	3) The mean value of a random process is a function of time.

(11) Define WSS process.

Ans: A Random process $X(t)$ is wide sense stationary process if it possesses the following properties.

$$E(X(t)) = \text{Constant and}$$

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau)) = R_{XX}(\tau).$$ i.e the mean value of the process is constant and the Auto correlation function is a function of the time difference 'τ' but not on absolute time 't'.

(12) Determine the mean square value of a random process with auto correlation function $R_{XX}(\tau) = e^{-|\tau|}$.

Ans: given

$$R_{XX}(\tau) = e^{-|\tau|}$$

The mean squared value of a random process is given by

$$E(X^2(t)) = R_{XX}(0) = e^{-|0|} = 1.$$

(13) Define Cross Covariance function.

Ans: The cross covariance function of two processes $X(t)$ and $Y(t)$ is defined as,

$$C_{XY}(t_1, t_2) = E([X(t_1) - E(X(t_1))][Y(t_2) - E(Y(t_2))])$$

if $t_1 = t$, $t_2 = t_1 + \tau$, we have,

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E(X(t))\, E(Y(t+\tau))$$

Unit – IV

① a) State the properties of power density Spectrum. (or) Define power Spectral density. List its properties. (or) Discuss the properties of Auto power Spectral Density in detail.

Ans: Definition : The power spectral density of a random process $X(t)$ is defined as,

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} \longrightarrow ①$$

where $X_T(\omega)$ is the fourier Transform of the truncated version of $x(t)$.

Properties: 1. The power Spectral density is always non negative. i.e $S_{XX}(\omega) \geq 0$. This property is valid Since the expected value of a non negative function is always non negative.

2. The power Spectral Density is a real Valued function Since $|X_T(\omega)|^2$ is real.

3. The mean Square Value of the Random process is given by.

$$E(X^2(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega.$$

4. The power density Spectrum and the time average of ACF form fourier transform pair. i.e.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]\, e^{-j\omega\tau}\, d\tau \text{ and}$$

$$A[R_{XX}(t,t+\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

if $X(t)$ is atleast WSS then $A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$

Therefore, PSD & ACF form fourier transform pair,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

5. If $X(t)$ is real, then $S_{XX}(\omega)$ is an even function of 'ω'

i.e $S_{XX}(-\omega) = S_{XX}(\omega)$.

6. If the power density Spectrum of $X(t)$ is $S_{XX}(\omega)$ then the power density Spectrum of $\frac{d}{dt}X(t)$ or $\dot{X}(t)$ is given by

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega), \quad \dot{X} = \frac{d}{dt}X(t).$$

(b) Find power Spectrum of WSS noise process $N(t)$ with ACF defined below. $R_{NN}(\tau) = P e^{-3|\tau|}$.

Soln:

Given Data:

The ACF of a WSS noise process $N(t)$ is given by,

$$R_{NN}(\tau) = P e^{-3|\tau|} \longrightarrow (1)$$

The power Spectrum of $N(t)$ is given by

$$S_{NN}(\omega) = \int_{-\infty}^{\infty} R_{NN}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} P e^{-3|\tau|} e^{-j\omega\tau} d\tau$$

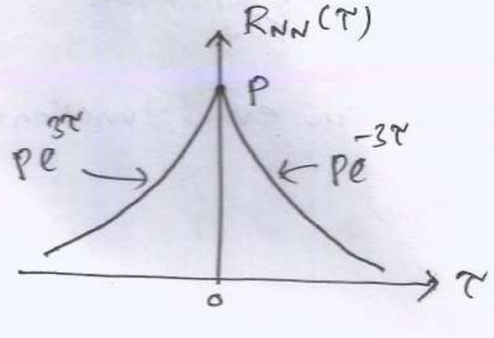
$$= P\left[\int_{-\infty}^{0} e^{3\tau} \cdot e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-3\tau} e^{-j\omega\tau} d\tau\right]$$

$$= P\left[\int_{-\infty}^{0} e^{(3-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(3+j\omega)\tau} d\tau\right]$$

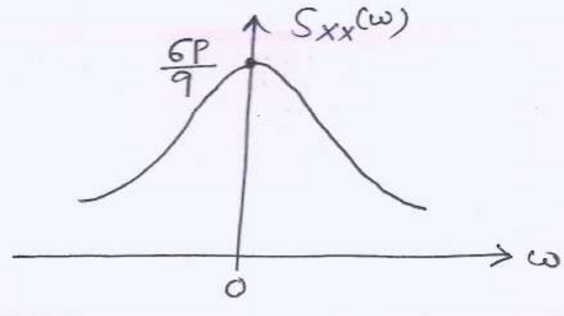
$$= P\left[\frac{e^{(3-j\omega)\tau}}{(3-j\omega)}\Bigg|_{-\infty}^{0} + \frac{e^{-(3+j\omega)\tau}}{-(3+j\omega)}\Bigg|_{0}^{\infty}\right]$$

$$= P\left[\frac{1}{3-j\omega}(1-0) - \frac{1}{3+j\omega}(0-1)\right]$$

$$S_{NN}(\omega) = P\left[\frac{1}{3-j\omega} + \frac{1}{3+j\omega}\right] = \frac{6P}{\omega^2+9}$$



$\longleftrightarrow$



② a) List the properties of Cross power density Spectrum. or Discuss the properties of cross power density Spectrum.

Ans: Definition: The cross power density Spectrum of two Random processes $X(t)$ and $Y(t)$ is defined as,

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega) \cdot Y_T(\omega)]}{2T} \longrightarrow ①$$

Similarly

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[Y_T^*(\omega) \cdot X_T(\omega)]}{2T} \longrightarrow ②$$

where $X_T(\omega)$ & $Y_T(\omega)$ are the fourier transforms of the truncated versions of $x(t)$ & $y(t)$ respectively.

Properties: for real Random processes $X(t)$ and $Y(t)$, the properties of cross power spectrum are as follows.

1. The two Cross power Spectrums $S_{XY}(\omega)$ and $S_{YX}(\omega)$ are related as

$$S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega).$$

2. The Real parts of $S_{XY}(\omega)$ and $S_{YX}(\omega)$ i.e $Re\{S_{XY}(\omega)\}$ and $Re\{S_{YX}(\omega)\}$ are even functions of 'ω'.

3. The Imaginary parts of $S_{XY}(\omega)$ and $S_{YX}(\omega)$ i.e $Im\{S_{XY}(\omega)\}$ and $Im\{S_{YX}(\omega)\}$ are odd functions of 'ω'.

4. If $X(t)$ and $Y(t)$ are orthogonal then $S_{XY}(\omega) = S_{YX}(\omega) = 0$.

5. If $X(t)$ and $Y(t)$ are uncorrelated and have constant means $\bar{X}$ and $\bar{Y}$ respectively, then

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \bar{X} \cdot \bar{Y} \cdot \delta(\omega).$$

6. The Cross power density Spectrum and the time average of Cross correlation functions form a fourier transform pair. i.e.

$$S_{XY}(\omega) \xleftrightarrow{F.T} A[R_{XY}(t, t+\tau)]$$
$$S_{YX}(\omega) \xleftrightarrow{F.T} A[R_{YX}(t, t+\tau)]$$

In case if $X(t)$ and $Y(t)$ are jointly WSS processes, then

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau \quad \text{and} \quad R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau \quad \text{and} \quad R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega) e^{j\omega\tau} d\omega.$$

b) Find the cross correlation function for a cross power density Spectrum defined below.

$$S_{XY}(\omega) = \frac{8}{(\alpha+j\omega)^3}, \text{ where } \alpha>0 \text{ is a Const.}$$

Soln: given Data,

The cross power spectrum is given by

$$S_{XY}(\omega) = \frac{8}{(\alpha+j\omega)^3} \longrightarrow ①$$

∴ The cross correlation function is thus given as

$$R_{XY}(\tau) = \mathcal{F}^{-1}(S_{XY}(\omega))$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega).\, e^{j\omega\tau}\, d\omega \longrightarrow ②$$

we know that $\mathcal{F}^{-1}\left(\frac{1}{\alpha+j\omega}\right) = e^{-\alpha t}u(t)$ and

$$\mathcal{F}^{-1}\left(\frac{1}{(\alpha+j\omega)^2}\right) = t e^{-\alpha t}u(t) \text{ and}$$

$$\mathcal{F}^{-1}\left(\frac{1}{(\alpha+j\omega)^3}\right) = \frac{t^2}{2!}e^{-\alpha t}u(t) \longrightarrow ③$$

from Equations ② & ③ it is Clear that

$$R_{XY}(\tau) = \mathcal{F}^{-1}\left(\frac{8}{(\alpha+j\omega)^3}\right)$$

$$= 8 \cdot \frac{\tau^2}{2!} e^{-\alpha\tau} u(\tau)$$

$$R_{XY}(\tau) = 4\tau^2 e^{-\alpha\tau} u(\tau)$$

③ a) State and Explain the Weiner-Khintchine Relation. (or) Interpret the weiner–Khintchine relation for auto power Spectral density and auto correlation of a random process. (or) Bring out the relationship b/w power Spectral density & ACF of a R.P.

Ans: Statement: The power Spectral density and Time Average Of Auto Correlation function form fourier Transform pair i.e

$A[R_{XX}(t,t+\tau)] \leftrightarrow S_{XX}(\omega)$. if $X(t)$ is atleast Wide Sense Stationary then $A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$ i.e Power Spectral density & Auto Correlation function form fourier Transform pair

$$\therefore R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega).$$

$$\therefore \quad S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau} d\omega$$

proof: we know that from the definition of PSD,

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} \longrightarrow ①$$

where $$X_T(\omega) = \int_{-T}^{T} x(t).e^{-j\omega t} dt \longrightarrow ②$$

$$① \Rightarrow S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[X_T(\omega).\, X_T^*(\omega)\right]}{2T}$$

$$= \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} X(t_1) e^{+j\omega t_1} dt_1 . \int_{-T}^{T} X(t_2) e^{-j\omega t_2} dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E(X(t_1).X(t_2))\, e^{-j\omega(t_2-t_1)} dt_1 dt_2$$

$$E(X(t_1).X(t_2)) = R_{XX}(t_1,t_2), \quad -T\leq t_1, t_2 \leq T$$

$$\therefore \quad S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2).\, e^{-j\omega(t_2-t_1)} dt_1 dt_2 \longrightarrow ③$$

Now, Take inverse fourier Transform to Eqn ③ on b.s, we have,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\, e^{-j\omega(t_2-t_1)} dt_1 dt_2\, e^{j\omega\tau} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2).\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_2+t_1)} d\omega\, dt_1 dt_2$$

we know that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega x} d\omega = \delta(x)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2).\, \delta(\tau-t_2+t_1)\, dt_1 dt_2$$

Since impulse function is even function we have, $\delta(\tau-t_2+t_1) = \delta(t_2-(t_1+\tau))$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left\{\int_{-T}^{T} R_{XX}(t_1,t_2)\,\delta(t_2-(t_1+\tau))\, dt_2\right\} dt_1$$

from the Sampling property of impulse function we have,

$$\int_{-T}^{T} R_{XX}(t_1,t_2).\delta(t_2-(\tau+t_1))\,dt_2 = R_{XX}(t_1,t_1+\tau)$$

$$\because \int_{-\infty}^{\infty} f(x).\delta(x-x_0)dx = f(x_0).$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,e^{j\omega\tau}\,d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XX}(t_1,t_1+\tau).dt_1 \quad , t_1=t$$

$$= A[R_{XX}(t,t+\tau)] \longrightarrow (4)$$

i.e The inverse fourier Transform of power Spectrum is given by the time average of Auto correlation function. Hence direct transform also Valid, thus.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]\,e^{-j\omega\tau}\,d\tau \longrightarrow (5)$$

Equations (4) & (5) Shows that $S_{XX}(\omega)$ and $A[R_{XX}(t,t+\tau)]$ form fourier transform pair. For a special case if $X(t)$ is atleast WSS, we have, $A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$ Thus. we have,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,e^{-j\omega\tau}\,d\tau \longrightarrow (6)$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,e^{j\omega\tau}\,d\omega \longrightarrow (7) \text{ or}$$

$$R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega).$$

Equations (6) and (7) are usually called as Weiner- Khintchine relations which form the basic link b/w time domain description (correlation functions) of process and their description in the freq. domain (power Spectrum).

b) Find the Spectrum of random process whose auto correlation function is $R_{XX}(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau$. plot Correlation and its Spectrum.

Soln:

given Data,

The Auto correlation function of a Random process $X(t)$ is given by,

$$R_{XX}(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau \longrightarrow (1)$$

The power Spectrum of the process is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,e^{-j\omega\tau}\,d\tau \longrightarrow (2)$$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} \frac{A_0^2}{2} \cos\omega_0\tau \, e^{-j\omega\tau} d\tau$$

$$= \frac{A_0^2}{4} \int_{-\infty}^{\infty} \left(e^{j\omega_0\tau} + e^{-j\omega_0\tau}\right) e^{-j\omega\tau} d\tau$$

$$= \frac{A_0^2}{4}\left[\int_{-\infty}^{\infty} e^{j\omega_0\tau} \cdot e^{-j\omega\tau} d\tau + \int_{-\infty}^{\infty} e^{-j\omega_0\tau} \cdot e^{-j\omega\tau} d\tau\right]$$

$$= \frac{A_0^2}{4}\left[\mathcal{F}(e^{j\omega_0\tau}) + \mathcal{F}(e^{-j\omega_0\tau})\right] \longrightarrow ③$$

we know that $\mathcal{F}(1) = 2\pi\delta(\omega)$, and By using freq. shifting property of F.T. we have

$$\mathcal{F}(1 \cdot e^{j\omega_0\tau}) = 2\pi\delta(\omega-\omega_0)$$

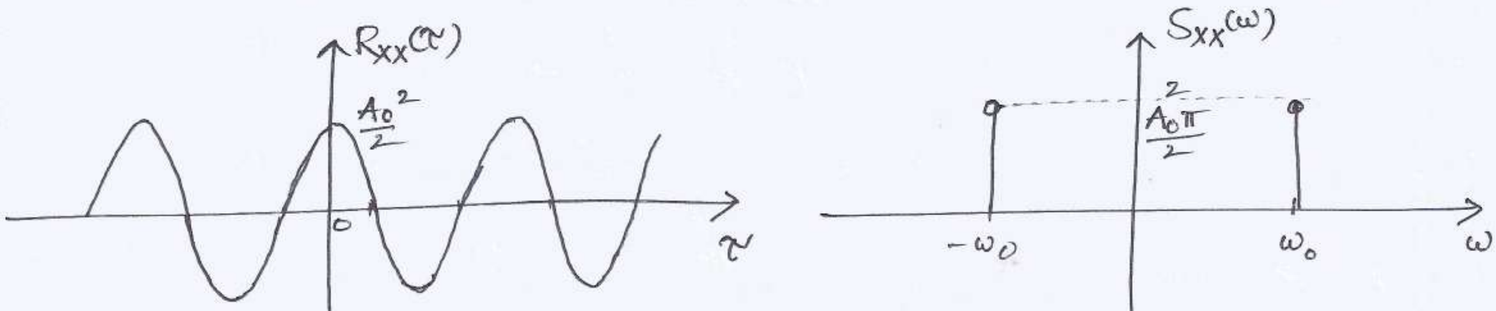
$$\mathcal{F}(1 \cdot e^{-j\omega_0\tau}) = 2\pi\delta(\omega+\omega_0).$$

$\therefore$ Equ ③ implies,

$$S_{XX}(\omega) = \frac{A_0^2}{4}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right)$$

$$S_{XX}(\omega) = \frac{A_0^2\pi}{2}\delta(\omega-\omega_0) + \frac{A_0^2\pi}{2}\delta(\omega+\omega_0) \longrightarrow ④$$

Equations ① & ④ Can be plotted as follows.



(a) Auto Correlation function (b) power Spectral Density.

④ a) Obtain the Auto correlation function corresponding to the power density Spectrum $S_{XX}(\omega) = \frac{8}{(9+\omega^2)^2}$

Soln:

Given the power Spectrum of a process

$$S_{XX}(\omega) = \frac{8}{(9+\omega^2)^2} \longrightarrow ①$$

$$R_{XX}(\tau) = R_1(\tau) * R_2(\tau) \xleftrightarrow{F.T} S_{XX}(\omega) = S_1(\omega) \cdot S_2(\omega).$$

Where $S_1(\omega) = S_2(\omega) = \frac{\sqrt{8}}{9+\omega^2}$ and

$$R_1(\tau) = R_2(\tau) = \mathcal{F}^{-1}\left(\frac{\sqrt{8}}{9+\omega^2}\right) = \frac{\sqrt{8}}{6} e^{-3|\tau|}$$

$$\therefore R_{XX}(\tau) = R_1(\tau) * R_2(\tau) = \int_{-\infty}^{\infty} \frac{\sqrt{8}}{6} e^{-3\xi} \cdot \frac{\sqrt{8}}{6} e^{-3|\tau-\xi|} d\xi$$

for the case when $\tau \geq 0$: Split the Above integral into three parts,

$$R_{XX}(\tau) = \int_{-\infty}^{0} \frac{\sqrt{8}}{6} e^{3\xi} \cdot \frac{\sqrt{8}}{6} e^{-3(\tau-\xi)} d\xi + \int_{0}^{\tau} \frac{\sqrt{8}}{6} e^{-3\xi} \frac{\sqrt{8}}{6} e^{-3(\tau-\xi)} d\xi + \int_{\tau}^{\infty} \frac{\sqrt{8}}{6} e^{-3\xi} \frac{\sqrt{8}}{6} e^{3(\tau-\xi)} d\xi$$

$$\therefore R_{XX}(\tau) = \frac{8}{36} \int_{-\infty}^{0} e^{-3\tau} e^{6\xi} d\xi + \frac{8}{36} e^{-3\tau} \int_{0}^{\tau} d\xi + \frac{8}{36} e^{3\tau} \int_{\tau}^{\infty} e^{-6\xi} d\xi$$

$$= \frac{8}{36} e^{-3\tau} \left.\frac{e^{6\xi}}{6}\right|_{-\infty}^{0} + \frac{8}{36} e^{-3\tau} (\xi)_0^{\tau} + \frac{8}{36} e^{3\tau} \left.\frac{e^{-6\xi}}{-6}\right|_{\tau}^{\infty}$$

$$= \frac{8}{36} e^{-3\tau} \left(\frac{1-0}{6}\right) + \frac{8}{36} e^{-3\tau} (\tau - 0) + \frac{8}{36} e^{3\tau} \left(\frac{0 - e^{-6\tau}}{-6}\right)$$

$$= \frac{8}{36} \frac{e^{-3\tau}}{6} + \frac{8\tau}{36} e^{-3\tau} + \frac{8}{36} \frac{e^{-3\tau}}{6}$$

$$R_{XX}(\tau) = \frac{8}{36} e^{-3\tau} \left(1/6 + 1/6 + \tau\right) = \frac{8}{36}(\tau + 1/3) e^{-3\tau}, \quad \tau \geq 0$$

Since $R_{XX}(-\tau) = R_{XX}(\tau)$, we have,

$$\boxed{R_{XX}(\tau) = \frac{8}{36} e^{-3|\tau|} (1/3 + |\tau|)}$$

(b) Compute the Average power of the process having power spectral density $\frac{6\omega^2}{1+\omega^4}$

Soln:

Given the power Spectral density

$$S_{XX}(\omega) = \frac{6\omega^2}{1+\omega^4}$$

The Avg. power of the process is thus given as

$$P_{XX} = E(X^2(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{6\omega^2}{1+\omega^4} d\omega \right).$$

we know that

$$\int \frac{x^2}{a^4+x^4}\,dx = \frac{-1}{4a\sqrt{2}} \ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) + \frac{1}{2a\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{ax\sqrt{2}}{a^2-x^2}\right)$$

By Using above integral we have,

$$P_{XX} = \frac{6}{2\pi}\left[\frac{-1}{4\sqrt{2}} \ln\left(\frac{\omega^2+\omega\sqrt{2}+1}{\omega^2-\omega\sqrt{2}+1}\right) + \frac{1}{2\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{\omega\sqrt{2}}{1-\omega^2}\right)\right]_{-\infty}^{\infty}$$

(or)

$$P_{XX} = \frac{6}{\pi}\int_0^\infty \frac{\omega^2}{1^4+\omega^4}\,d\omega$$

$$= \frac{6}{\pi}\left[\frac{-1}{4\sqrt{2}} \ln\left(\frac{\omega^2+\omega\sqrt{2}+1}{\omega^2-\omega\sqrt{2}+1}\right) + \frac{1}{2\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{\omega\sqrt{2}}{1-\omega^2}\right)\right]_0^\infty$$

$$= \frac{6}{\pi}\left(0+\frac{1}{2\sqrt{2}}(\pi)\right) = \frac{\sqrt{3}}{2} \text{ Watts.}$$

(5) Find the Auto Correlation function corresponding to the power density Spectrum

$$S_{XX}(\omega) = \frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} \longrightarrow (1)$$

Soln: Given the power density Spectrum.

$$S_{XX}(\omega) = \frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)}$$

$$\therefore\; R_{XX}(\tau) = \mathcal{F}^{-1}\left(\frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)}\right) \longrightarrow (2)$$

$$\frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} = \frac{A}{(16+\omega^2)} + \frac{B}{(9+\omega^2)}$$

where $A = (16+\omega^2).\,S_{XX}(\omega)\Big|_{\omega^2=-16} = \dfrac{157+12\omega^2}{9+\omega^2}\Big|_{\omega^2=-16}$

$$= \frac{157+12(-16)}{9-16}$$

$$= 5$$

$B = (9+\omega^2)\,S_{XX}(\omega)\Big|_{\omega^2=-9} = \dfrac{157+12\omega^2}{16+\omega^2}\Big|_{\omega^2=-9}$

$$= \frac{157+12(-9)}{16+(-9)} = 7$$

$$\therefore \quad S_{XX}(\omega) = \frac{5}{16+\omega^2} + \frac{7}{\omega^2+9}$$

$$\therefore \quad R_{XX}(\tau) = \mathcal{F}^{-1}\left(\frac{5}{16+\omega^2}\right) + \mathcal{F}^{-1}\left(\frac{7}{\omega^2+9}\right)$$

$$= \frac{5}{8}\mathcal{F}^{-1}\left(\frac{2\times 4}{\omega^2+4^2}\right) + \frac{7}{6}\mathcal{F}^{-1}\left(\frac{2\times 3}{\omega^2+3^2}\right)$$

$$\boxed{R_{XX}(\tau) = \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|}} \qquad \because \mathcal{F}\left(e^{-b|\tau|}\right) = \frac{2b}{\omega^2+b^2}$$

⑥ a) Discuss the relationship b/w Cross power Spectrum and Cross correlation function.

Ans: Statement: The cross power spectral density and the time Average of cross correlation function form fourier transform pair. i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega = A\left[R_{XY}(t,t+\tau)\right]$$

proof. From the definition of cross power spectrum, we have,

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E\left(X_T^*(\omega) \cdot Y_T(\omega)\right)}{2T} \longrightarrow ①$$

where, $X_T(\omega)$ and $Y_T(\omega)$ are fourier transforms of truncated versions of $x(t)$ and $y(t)$ respectively given by

$$X_T(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \rightarrow ②$$

$$Y_T(\omega) = \int_{-\infty}^{\infty} y(t_1)\, e^{-j\omega t_1}\, dt_1 \longrightarrow ③$$

Substitute Equations ② & ③ in Eqn ① we have,

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E\left[\int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt \cdot \int_{-\infty}^{\infty} y(t_1)\, e^{-j\omega t_1}\, dt_1\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left(x(t)\, y(t_1)\right) e^{-j\omega(t_1-t)}\, dt\, dt_1\right]$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XY}(t,t_1)\, e^{-j\omega(t_1-t)}\, dt\, dt_1\right] \rightarrow ④$$

Take inverse fourier transform to Equ ④ on b.s, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{T} R_{xy}(t,t_1)\, e^{-j\omega(t_1-t)}\, dt\,dt_1\right] e^{j\omega\tau} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xy}(t,t_1)\cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+j\omega(-t_1+t+\tau)} d\omega \cdot dt\,dt_1$$

we know that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega x} dx = \delta(x)$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xy}(t,t_1)\, \delta(\tau-t_1+t)\, dt\,dt_1$$

Since $\delta(-\omega) = \delta(\omega)$ we have $\delta(\tau-t_1+t) = \delta(t_1-(t+\tau))$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xy}(t,t_1)\,\delta(t_1-(t+\tau))\, dt_1\, dt.$$

from the Sampling property of impulse function, we have,

$$\int_{-T}^{T} R_{xy}(t,t_1)\,\delta(t_1-(t+\tau))\, dt_1 = R_{xy}(t,t+\tau).$$

$$\because \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t+\tau)\, dt = A\left[R_{xy}(t,t+\tau)\right] \rightarrow ⑤$$

from Equ ⑤ it is clear that inverse fourier transform of cross power spectrum is given by the Time average of cross correlation function. Hence direct transform also exists i.e

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} A\left[R_{xy}(t,t+\tau)\right] e^{-j\omega\tau} d\tau \rightarrow ⑥$$

for a special case if $X(t)$ and $Y(t)$ are jointly WSS, we have, $A[R_{xy}(t,t+\tau)] = R_{xy}(\tau)$, Thus Equ ⑥ & ⑤ reduced to

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau} d\tau \text{ and}$$

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau} d\omega. \text{ Similarly,}$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau} d\tau \quad \text{and} \quad R_{yx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yx}(\omega)\, e^{j\omega\tau} d\omega$$

b) For two jointly stationery Random processes, the cross correlation function is $R_{XY}(\tau) = 2e^{-2\tau}u(\tau)$. Find the two cross spectral density functions.

Solw: Given the cross correlation function of two processes $X(t)$ and $Y(t)$, as

$$R_{XY}(\tau) = 2e^{-2\tau}u(\tau) \longrightarrow (1)$$

The cross power spectrum $S_{XY}(\omega)$ is thus given by

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 2e^{-2\tau}u(\tau)\, e^{-j\omega\tau} d\tau$$

$$= 2 \cdot \int_{0}^{\infty} e^{-(2+j\omega)\tau} d\tau$$

$$= 2 \cdot \left. \frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)} \right|_{0}^{\infty}$$

$$S_{XY}(\omega) = \frac{-2}{2+j\omega}(0-1) = \frac{2}{2+j\omega}$$

we know that from the properties of cross PSD

$$S_{YX}(\omega) = S^{*}_{XY}(\omega) = \frac{2}{2-j\omega}.$$

(4) a) A wide sense stationary random process $X(t)$ has an auto correlation function $X(t)$, $R_{XX}(\tau) = \begin{cases} A_0\left[1 - \frac{|\tau|}{T}\right], & -T \le \tau \le T \\ 0, & \text{elsewhere.} \end{cases}$

Where $T > 0$ and A_0 are constants. Determine the power spectrum.

Solw: The ACF of a Random process $X(t)$ is given as

$$R_{XX}(\tau) = \begin{cases} A_0\left(1 - \frac{|\tau|}{T}\right), & -T \le \tau \le T \\ 0, & \text{elsewhere} \end{cases} \longrightarrow (1)$$

The power spectrum is thus given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \cdot e^{-j\omega\tau} d\tau \longrightarrow (2)$$

$$\therefore\ S_{xx}(\omega) = \int_{-\infty}^{\infty} A_0\left(1-\frac{|\tau|}{T}\right)e^{-j\omega\tau}\,d\tau$$

$$= \int_{-T}^{0} A_0\left(1+\frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau + \int_{0}^{T} A_0\left(1-\frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau$$

$$\therefore\ S_{xx}(\omega) = A_0\left[\left(1+\frac{\tau}{T}\right)\frac{e^{-j\omega\tau}}{(-j\omega)}\Big|_{-T}^{0} - \int_{-T}^{0}\frac{1}{T}\cdot\frac{e^{-j\omega\tau}}{(-j\omega)}d\tau\right] + A_0\left[\left(1-\frac{\tau}{T}\right)\frac{e^{-j\omega\tau}}{-j\omega}\Big|_{0}^{T} - \int_{0}^{T}\left(\frac{-1}{T}\right)\frac{e^{-j\omega\tau}}{-j\omega}d\tau\right]$$

$$= A_0\left[\frac{-1}{j\omega} - 0 + \frac{1}{Tj\omega}\,\frac{e^{-j\omega\tau}}{-j\omega}\Big|_{-T}^{0}\right] + A_0\left[0-\left(\frac{-1}{j\omega}\right) - \frac{1}{Tj\omega}\,\frac{e^{-j\omega\tau}}{-j\omega}\Big|_{0}^{T}\right]$$

$$= A_0\left[\frac{-1}{j\omega} + \frac{1}{\omega^2 T}\left(1-e^{j\omega T}\right)\right] + A_0\left[\frac{1}{j\omega} - \frac{1}{\omega^2 T}\left(e^{-j\omega T}-1\right)\right]$$

$$= A_0\left[\cancel{\frac{-1}{j\omega}} + \cancel{\frac{1}{j\omega}} + \frac{1}{\omega^2 T}\left(1-e^{j\omega T} - e^{-j\omega T} + 1\right)\right]$$

$$= \frac{A_0}{\omega^2 T}\left(2 - \left(e^{j\omega T} + e^{-j\omega T}\right)\right)$$

$$= \frac{A_0}{\omega^2 T}\left(2 - 2\cos\omega T\right)$$

$$\because\ 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \frac{2A_0}{\omega^2 T}\left(1-\cos\omega T\right)$$

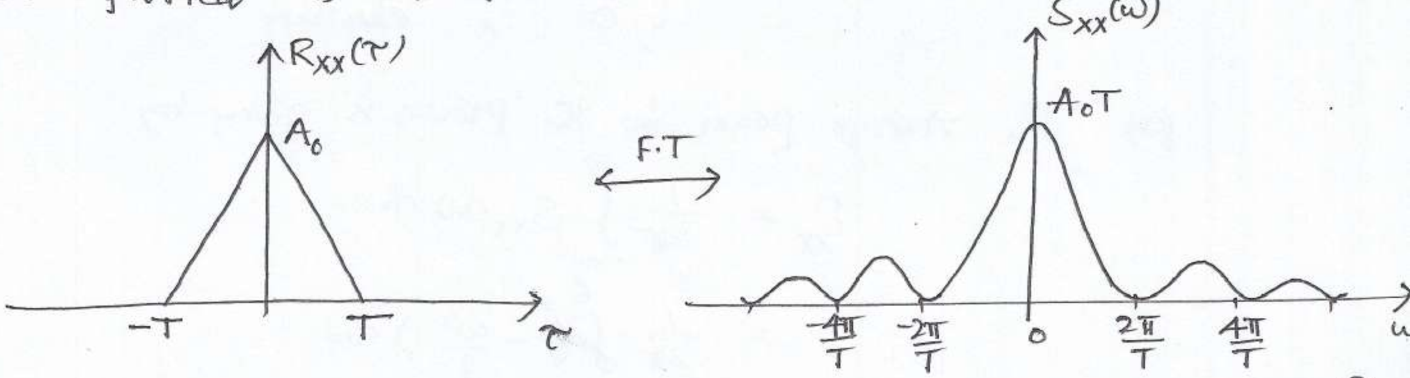
$$= \frac{4A_0}{\omega^2 T}\sin^2\left(\frac{\omega T}{2}\right)$$

$$= \frac{A_0 T}{\left(\frac{\omega T}{2}\right)^2}\sin^2\frac{\omega T}{2} = A_0 T\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2$$

$$= A_0 T\left(Sa\,\frac{\omega T}{2}\right)^2$$

$$S_{xx}(\omega) = A_0 T\, Sa^2\left(\frac{\omega T}{2}\right) \longrightarrow (3)$$

The Auto correlation function and the power spectrum are plotted as shown.



(b) The power spectral density of a Stationary Random process is given by,

$$S_{XX}(\omega) = \begin{cases} A, & -k \le \omega \le k \\ 0, & \text{otherwise.} \end{cases}$$

Find the ACF.

Soln: Given the power spectral density,

$$S_{XX}(\omega) = \begin{cases} A, & -k \le \omega \le k \\ 0, & \text{elsewhere,} \end{cases}$$

The Auto correlation function is thus given by

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-K}^{K} A\, e^{j\omega\tau}\, d\omega$$

$$= \frac{A}{2\pi} \left. \frac{e^{j\omega\tau}}{j\tau} \right|_{-K}^{K}$$

$$= \frac{A}{2\pi j\tau} \left(e^{j\tau k} - e^{-j\tau k} \right)$$

$$= \frac{A}{\pi\tau} \sin \tau k$$

$$R_{XX}(\tau) = \frac{AK}{\pi} \frac{\sin \tau k}{\tau K} = \frac{AK}{\pi} Sa(k\tau)$$

(8) A Random process has a power spectrum

$$S_{XX}(\omega) = \begin{cases} 4 - \frac{\omega^2}{9}, & |\omega| \le 6 \\ 0, & \text{elsewhere.} \end{cases}$$

Find (a) The Average power (b) The R.M.S. Bandwidth (c) The Auto correlation function of the process.

Soln: The power spectrum of a random process is given by

$$S_{XX}(\omega) = \begin{cases} 4 - \frac{\omega^2}{9}, & |\omega| \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

(a) The Average power in the process is given by

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-6}^{6} \left(4 - \frac{\omega^2}{9} \right) d\omega$$

$$P_{XX} = \frac{1}{2\pi}\left(4\omega - \frac{\omega^3}{27}\right)\Big|_{-6}^{6}$$

$$= \frac{1}{2\pi}\left(24 - \frac{216}{27} - \left(-24 - \left(\frac{-216}{27}\right)\right)\right)$$

$$P_{XX} = \frac{1}{2\pi}\left(48 - \frac{432}{27}\right) = \frac{1}{2\pi}(48-16) = \frac{32}{2\pi} = \frac{16}{\pi} \text{ watts}$$

(ii) The rms band width is given by.

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega} = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega} \longrightarrow ①$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-6}^{6}\left(4\omega^2 - \frac{\omega^4}{9}\right)d\omega$$

$$= \frac{1}{2\pi}\left[\left(\frac{4\omega^3}{3} - \frac{\omega^5}{45}\right)\Big|_{-6}^{6}\right]$$

$$= \frac{1}{2\pi}\left[\frac{216(4)}{3} - \frac{7776}{45} - \left(\frac{-216(4)}{3} - \frac{(-)7776}{45}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{864}{3} + \frac{864}{3} - \frac{7776}{45} - \frac{7776}{45}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1728}{3} - \frac{15,552}{45}\right]$$

$$= \frac{1}{2\pi}\left(576 - 345.6\right) = \frac{230.4}{2\pi}$$

$$\therefore \quad W_{rms}^2 = \frac{\frac{230.4}{2\cancel{\pi}}}{\frac{16}{\cancel{\pi}}} = \frac{230.4}{32} = 7.2.$$

$$\boxed{W_{rms} = \sqrt{7.2.}} \text{ rad/sec.}$$

(iii) The Auto correlation function is given by,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\,d\omega$$

$$\therefore\ R_{XX}(\tau) = \frac{1}{2\pi}\left[\int_{-6}^{6}\left(4-\frac{\omega^2}{9}\right)e^{j\omega\tau}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left(4-\frac{\omega^2}{9}\right)\frac{e^{j\omega\tau}}{j\tau}\Bigg|_{-6}^{6} - \int_{-6}^{6}\frac{-2\omega}{9}\,\frac{e^{j\omega\tau}}{j\tau}\,d\omega\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[(0-0) + \frac{\cancel{2}}{9j\tau}\left(\omega\cdot\frac{e^{j\omega\tau}}{j\tau}\Bigg|_{-6}^{6} - \int_{-6}^{6} 1\cdot\frac{e^{j\omega\tau}}{j\tau}\right)\right]$$

$$= \frac{1}{9\pi j\tau}\left(\frac{1}{j\tau}\left(6e^{6j\tau}+6e^{-6j\tau}\right) - \frac{1}{(j\tau)^2}\,e^{j\omega\tau}\Big|_{-6}^{6}\right)$$

$$= \frac{1}{9\pi j\tau}\left(\frac{6}{j\tau}(2\cos 6\tau) + \frac{1}{\tau^2}\left(e^{j6\tau}-e^{-j6\tau}\right)\right)$$

$$= \frac{1}{9\pi\tau}\left(\frac{-12\cos 6\tau}{\tau} + \frac{1}{\tau^2}\,2\sin 6\tau\right)$$

$$\frac{\sin 6\tau}{6\tau} = Sa\,6\tau$$

$$= \frac{1}{9\pi\tau}\left(\frac{-12\cos 6\tau}{\tau} + \frac{12}{\tau}\,Sa(6\tau)\right)$$

$$\boxed{R_{XX}(\tau) = \frac{4}{3\pi\tau^2}\left(Sa(6\tau) - \cos 6\tau\right)}$$

Short Answer Questions :

① Find the PSD if $R_{XX}(\tau) = e^{-2\lambda|\tau|}$.

Ans: $$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\lambda\tau}\, e^{-j\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-2\lambda\tau}\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{(2\lambda-j\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(2\lambda+j\omega)\tau}\, d\tau$$

$$= \left.\frac{e^{(2\lambda-j\omega)\tau}}{(2\lambda-j\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(2\lambda+j\omega)\tau}}{-(2\lambda+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{2\lambda-j\omega}(1-0) - \frac{1}{2\lambda+j\omega}(0-1)$$

$$S_{XX}(\omega) = \frac{1}{2\lambda-j\omega} + \frac{1}{2\lambda+j\omega} = \frac{4\lambda}{\omega^2+4\lambda^2}$$

② Examine the function $\dfrac{\omega^2}{\omega^6+3\omega^2+3}$ for Valid PSD.

Ans: $$S_{XX}(\omega) = \frac{\omega^2}{\omega^6+3\omega^2+3} \longrightarrow ①$$

- for any Value of ω, $S_{XX}(\omega) \geq 0$
- It is a real Valued function
- $S_{XX}(-\omega) = S_{XX}(\omega)$ i.e it is an even function of 'ω'

$\therefore$ Equ ① is a Valid power Spectral density.

③ ~~Analyze the power~~: Correlate CPSD and CCF.

Ans: The Cross PSD and Time Average of Cross Correlation function form fourier Transform pair. i.e

$$A[R_{XY}(t,t+\tau)] \xleftrightarrow{F.T} S_{XY}(\omega).$$

if $X(t)$ and $Y(t)$ are atleast jointly WSS. processes then, $A[R_{XY}(t,t+\tau)] = R_{XY}(\tau)$ $\therefore$ $R_{XY}(\tau) \xleftrightarrow{FT} S_{XY}(\omega)$

$$\therefore \quad S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau \text{ and}$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega.$$

④ List any two properties of Cross PSD.

Ans:

1) $S_{XY}(-\omega) = S_{YX}(\omega) = S^{*}_{XY}(\omega)$

2) if $X(t)$ and $Y(t)$ are orthogonal then $S_{XY}(\omega) = 0 = S_{YX}(\omega)$

3) $A[R_{XY}(t,t+\tau)] \xleftrightarrow{F.T} S_{XY}(\omega)$.

⑤ What is the Average power in $X(t)$ if the $R_{XX}(\tau) = 3 + 2e^{-4\tau^2}$

Ans:

$$P_{XX} = E(X^2(t)) = R_{XX}(0) = 3 + 2e^{-0} = 3 + 2 = 5 \text{ Watts.}$$

⑥ Define power Spectral density.

Ans: The power spectral density of a random process $X(t)$ is denoted by $S_{XX}(\omega)$ and is defined as

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right]$$

Where $X_T(\omega)$ is the fourier transform of the truncated version of $x(t)$ i.e $X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t} dt$.

⑦ If the PSD of $x(t)$ is $S_{XX}(\omega)$. Find the PSD of $\frac{d}{dt}x(t)$.

Ans: By definition we have,

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right] \longrightarrow ①$$

$\therefore$ The PSD of $\frac{d}{dt}x(t)$ or $\dot{X}$ is given by,

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{1}{2T}\left[E(|\dot{X}_T(\omega)|^2)\right] \longrightarrow ②$$

By definition we have,

$$\frac{d}{dt}X_T(t) = \lim_{\varepsilon \to 0} \frac{X_T(t+\varepsilon) - X_T(t)}{\varepsilon}$$

$$\therefore \dot{X}_T(\omega) = \mathcal{F}\left(\frac{d}{dt}X(t)\right) = \mathcal{F}\left(\lim_{\varepsilon \to 0} \frac{X_T(t+\varepsilon) - X_T(t)}{\varepsilon}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{e^{j\omega\varepsilon} \cdot X_T(\omega) - X_T(\omega)}{\varepsilon}$$

$$= X_T(\omega) \cdot \lim_{\varepsilon \to 0} \frac{e^{j\omega\varepsilon} - 1}{\varepsilon}$$

$$\dot{X}_T(\omega) = X_T(\omega) \cdot \lim_{\varepsilon \to 0} e^{j\omega\varepsilon/2} \frac{\left(e^{j\omega\varepsilon/2} - e^{-j\omega\varepsilon/2}\right)}{\varepsilon}$$

$$= X_T(\omega) \lim_{\varepsilon \to 0} e^{j\omega\varepsilon/2} \, 2j \, \frac{\sin(\omega\varepsilon/2)}{\varepsilon}$$

$$= X_T(\omega) \lim_{\varepsilon \to 0} (j\omega) \, e^{j\omega\varepsilon/2} \cdot \frac{\sin \omega\varepsilon/2}{\omega\varepsilon/2}$$

$$\dot{X}_T(\omega) = j\omega \cdot X_T(\omega) \longrightarrow (3)$$

$$\dot{X}_T^*(\omega) = -j\omega \, X_T^*(\omega) \longrightarrow (4)$$

$\therefore$ Eqn (2) $\Rightarrow$ $$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{1}{2T} E\left(\dot{X}_T^*(\omega) \, \dot{X}_T(\omega)\right)$$

$$= \lim_{T \to \infty} \frac{1}{2T} E\left(j\omega X_T(\omega) \cdot (-j\omega) X_T^*(\omega)\right)$$

$$= \omega^2 \lim_{T \to \infty} \frac{1}{2T} E\left(|X_T(\omega)|^2\right)$$

$$\boxed{S_{\dot{X}\dot{X}}(\omega) = \omega^2 \cdot S_{XX}(\omega)}$$ from Eqn (1)

(8) Define rms band width of the power spectrum.

Ans: The rms band width is a measure of spread in the normalized power spectrum and is given by

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) \, d\omega}.$$

Unit - V

① A Random Process $X(t)$ is applied to a n/w with impulse response $h(t) = u(t).t \cdot e^{-bt}$. where $b>0$ is a Constant.

The Cross Correlation of $X(t)$ with the o/p $Y(t)$ is known to have the Same form $R_{XY}(\tau) = u(\tau)\,\tau\, e^{-b\tau}$

(a) Find the Auto correlation of $Y(t)$

(b) What is the Average power in $Y(t)$. ?

Soln: Given data,

The impulse response of a n/w is

$$h(t) = u(t)\, t\, e^{-bt} \longrightarrow ① , \quad b>0 \text{ is a Const.}$$

The Cross Correlation function of o/p with input is given by,

$$R_{XY}(\tau) = u(\tau)\tau\, e^{-b\tau} \longrightarrow ②$$

(a) we know that,

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau+\xi)\, h(\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} u(\tau+\xi)(\tau+\xi)\, e^{-b(\tau+\xi)} \cdot u(\xi)\,\xi\, e^{-b\xi}\, d\xi$$

$$\therefore R_{YY}(\tau) = e^{-b\tau}\int_{-\infty}^{\infty} (\xi^2 + b\tau\xi)\, e^{-2b\xi}\, u(\xi)\, u(\tau+\xi)\, d\xi$$

Here we Consider two Cases $\tau \geq 0$ and $\tau < 0$ but however $R_{YY}(\tau) = R_{YY}(-\tau)$. Consider $\tau \geq 0$,

$$R_{YY}(\tau) = e^{-b\tau}\int_{0}^{\infty} (\xi^2 + \tau\xi)\, e^{-2b\xi}\, d\xi \qquad u(\xi)\,u(\tau+\xi) = 1 \text{ for } 0<\xi\leq\infty$$

$$= e^{-b\tau}\left[(\xi^2+\tau\xi)\frac{e^{-2b\xi}}{(-2b)}\Big|_0^\infty - \int_0^\infty (2\xi+\tau)\frac{e^{-2b\xi}}{-2b}\, d\xi \right]$$

$$= e^{-b\tau}\left[(0-0) + \frac{1}{2b}\left[(2\xi+\tau)\frac{e^{-2b\xi}}{-2b}\Big|_0^\infty - \int_0^\infty 2 \cdot \frac{e^{-2b\xi}}{-2b}\, d\xi \right] \right]$$

$$= e^{-b\tau}\left[\frac{1}{2b}\left(0 + \frac{\tau}{2b} \right] - 2\frac{1}{(2b)^2}\left(e^{-2b\xi}\right)_0^\infty \right) \Big]$$

$$R_{yy}(\tau) = e^{-b\tau}\left(\frac{1}{2b}\left(\frac{\tau}{2b} + \frac{\cancel{2}}{4b^2}\right)\right)$$

$$= e^{-b\tau}\left(\frac{1}{\cancel{2}b}\left(\frac{\cancel{2}b\tau + \cancel{2}}{4b^2}\right)\right)$$

$$R_{yy}(\tau) = \frac{1}{4b^3}(1+b\tau)\,e^{-b\tau} \quad , \quad \tau \geq 0$$

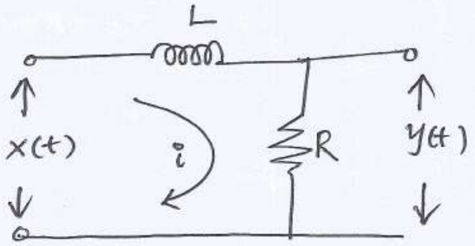
$$\therefore \boxed{R_{yy}(\tau) = \frac{1}{4b^3}\left(1+b|\tau|\right)e^{-b|\tau|}} \quad \because R_{yy}(-\tau) = R_{yy}(\tau).$$

(b) The Average power in $y(t)$ is given by,

$$P_{yy} = E(Y^2(t)) = R_{yy}(0) = \frac{1}{4b^3}(1+0)\,e^{-0}$$

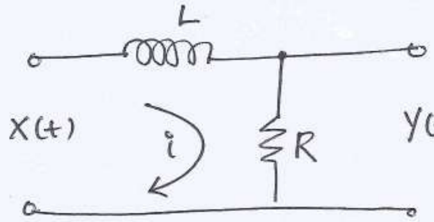
$$P_{yy} = \frac{1}{4b^3} \text{ watts}$$

(2) Determine the power Spectrum and Average power of the response of the network shown in fig. when $X(t)$ is white noise for which $S_{XX}(\omega) = \frac{N_0}{2}$.



Solⁿ: Given the network,

$S_{XX}(\omega) = \frac{N_0}{2}$



$S_{yy}(\omega) = ?$

$P_{yy} = ?$

we have

$$x(t) = L \cdot \frac{di}{dt} + y(t) \longrightarrow (1)$$

$$y(t) = i \cdot R \Rightarrow i = \frac{1}{R}\, y(t)$$

$$\frac{di}{dt} = \frac{1}{R}\frac{dy(t)}{dt} \rightarrow (2)$$

$$\therefore (1) \Rightarrow \quad x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t).$$

Take F.T. on b.s.

$$X(\omega) = \frac{L}{R} j\omega\, Y(\omega) + Y(\omega), = \left(1 + \frac{j\omega L}{R}\right)Y(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + \frac{j\omega L}{R}} \rightarrow (3)$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}$$

we know that the PSD of response of any n/w is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{1}{1+\frac{\omega^2 L^2}{R^2}} \cdot \frac{N_0}{2}$$

$$\boxed{S_{YY}(\omega) = \frac{N_0 R^2}{2(R^2+\omega^2 L^2)}}$$

$\therefore$ power in the response is given by

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0 R^2}{2} \frac{1}{R^2+\omega^2 L^2}\, d\omega$$

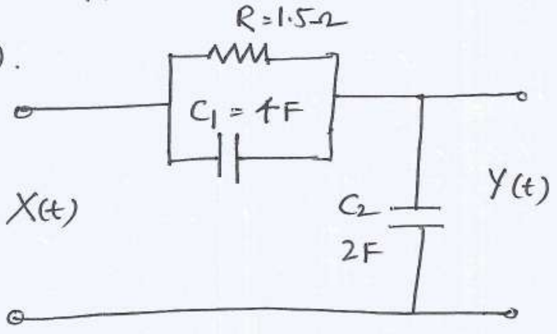
$$= \frac{N_0 R^{\cancel{2}}}{4\pi} \frac{1}{\cancel{R}L} \operatorname{Tan}^{-1} \frac{\omega L}{R}\Bigg|_{-\infty}^{\infty}$$

$$= \frac{N_0 R}{4\cancel{\pi} L} \left(\cancel{\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)}\right)$$

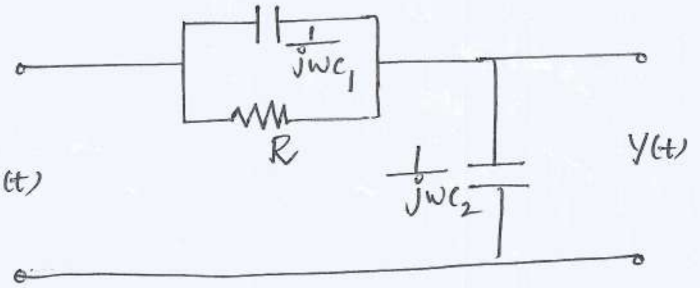
$$\therefore \quad \boxed{P_{YY} = \frac{N_0 R}{4L}}$$

③ A Stationary Random Process $X(t)$ having an ACF $R_{XX}(\tau) = 2e^{-4|\tau|}$ is applied to a n/w as shown.

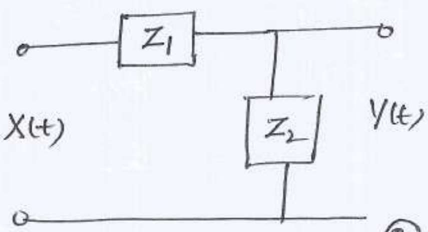
Find (a) $S_{XX}(\omega)$ (b) $|H(\omega)|^2$ (c) $S_{YY}(\omega)$.



Soln: (b) Transform the given n/w into frequency domain we have.



$\cong$



$$Z_1 = \frac{R \cdot \frac{1}{j\omega C_1}}{R + \frac{1}{j\omega C_1}}, \quad Z_2 = \frac{1}{j\omega C_2}$$

$$\therefore \quad Y(\omega) = \frac{X(\omega) Z_2}{Z_1 + Z_2} \Rightarrow \frac{Y(\omega)}{X(\omega)} = \frac{Z_2}{Z_1 + Z_2}$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{j\omega C_2}}{\frac{R \frac{1}{j\omega C_1}}{R + \frac{1}{j\omega C_1}} + \frac{1}{j\omega C_2}}$$

$$H(\omega) = \frac{\frac{1}{j\omega C_2}}{\frac{\frac{R}{\cancel{j\omega C_1}}}{\frac{j\omega R C_1 + 1}{\cancel{j\omega C_1}}} + \frac{1}{j\omega C_2}}$$

$$= \frac{1/\cancel{j\omega C_2}}{\frac{j\omega R C_2 + 1 + j\omega R C_1}{\cancel{j\omega C_2}(1 + j\omega R C_1)}}$$

$$= \frac{1 + j\omega R C_1}{1 + j\omega(RC_1 + RC_2)}$$

$R = 1.5\,\Omega$
$C_1 = 4F$
$C_2 = 2F$

$$= \frac{1 + j6\omega}{1 + j\omega(6+3)} = \frac{1 + j6\omega}{1 + j9\omega}$$

$$\therefore \quad |H(\omega)| = \frac{\sqrt{1 + 36\omega^2}}{\sqrt{1 + 81\omega^2}} \Rightarrow \boxed{|H(\omega)|^2 = \frac{1 + 36\omega^2}{1 + 81\omega^2}}$$

(a) We know that

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

given, $R_{XX}(\tau) = 2e^{-4|\tau|}$

$$\therefore \quad S_{XX}(\omega) = \int_{-\infty}^{\infty} 2e^{-4|\tau|} e^{-j\omega\tau} d\tau$$

$$= 2\left[\int_{-\infty}^{0} e^{4\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-4\tau} e^{-j\omega\tau} d\tau\right]$$

$$\therefore \quad S_{XX}(\omega) = 2\left[\int_{-\infty}^{0} e^{(4-j\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(4+j\omega)\tau}\, d\tau\right]$$

$$= 2\cdot\left[\frac{e^{(4-j\omega)\tau}}{(4-j\omega)}\Bigg|_{-\infty}^{0} + \frac{e^{-(4+j\omega)\tau}}{-(4+j\omega)}\Bigg|_{0}^{\infty}\right]$$

$$= 2\left[\frac{1}{4-j\omega}(1-0) - \frac{1}{4+j\omega}(0-1)\right]$$

$$= 2\left[\frac{1}{4-j\omega} + \frac{1}{4+j\omega}\right] = 2\left[\frac{4+\cancel{j\omega}+4-\cancel{j\omega}}{\omega^2+16}\right]$$

$$\boxed{S_{XX}(\omega) = \frac{16}{\omega^2+16}}$$

(c) we know that

$$S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega)$$

$$= \frac{1+36\omega^2}{1+81\omega^2} \cdot \frac{16}{\omega^2+16}$$

$$\boxed{S_{YY}(\omega) = \frac{16(1+36\omega^2)}{(1+81\omega^2)(\omega^2+16)}}$$

4. A random noise $X(t)$ having power spectrum $S_{XX}(\omega) = \frac{3}{\omega^2+49}$ is applied to a n/w for which $h(t) = u(t).\, t^2 e^{-7t}$. The network response is denoted by $Y(t)$.

(a) what is the Avg. power of $X(t)$?

(b) Find the power spectrum of $Y(t)$

(c) Find the Avg. power of $Y(t)$.

Soln: Given Data,

A Random Noise $X(t)$ has its power spectrum as

$$S_{XX}(\omega) = \frac{3}{\omega^2+49} \longrightarrow (1)$$

The impulse response of a n/w is given as

$$h(t) = u(t).\, t^2 e^{-7t} \longrightarrow (2)$$

(a) The Avg. power in $X(t)$ is given by

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{3}{\omega^2+49}\, d\omega$$

$$= \frac{3}{7\times 2\pi}\int_{-\infty}^{\infty} \frac{7}{\omega^2+7^2}\, d\omega$$

$$= \frac{3}{14\pi}\, \mathrm{Tan}^{-1}\left(\frac{\omega}{7}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{3}{14\cancel{\pi}}\left(+\frac{\cancel{\pi}}{2} - \left(-\frac{\cancel{\pi}}{2}\right)\right) = \frac{3}{14} \text{ watts.}$$

(b) We know that

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$\therefore\ H(\omega) = \mathcal{F}(h(t)) = \mathcal{F}(t^2 e^{-7t} u(t))$$

We know that $\mathcal{F}(e^{-7t}u(t)) = \dfrac{1}{j\omega+7}$

$$\therefore\ \mathcal{F}(t^2 e^{-7t} u(t)) = \frac{d^2}{d\omega^2}\left(\frac{1}{j\omega+7}\right)$$

$$= \frac{d}{d\omega}\left(\frac{-1}{(j\omega+7)^2}\right)$$

$$= \frac{2}{(j\omega+7)^3}$$

$$\therefore\ H(\omega) = \frac{2}{(j\omega+7)^3}$$

$$|H(\omega)| = \frac{2}{(\omega^2+49)^{3/2}} \Rightarrow |H(\omega)|^2 = \frac{2\times 2}{(\omega^2+49)^3}$$

$$\therefore\ S_{YY}(\omega) = \frac{4}{(\omega^2+49)^3}\cdot\frac{3}{(\omega^2+49)}$$

$$= \frac{12}{(\omega^2+49)^4}$$

(C) The Avg. Power in $Y(t)$ is given by,

$$P_{yy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(\omega^2+49)^4}\, d\omega$$

let $\omega = 7\tan\theta$

$d\omega = 7\sec^2\theta\, d\theta$

$\theta = \tan^{-1}(\omega/7)$

U.L = $\pi/2$

L.L = $-\pi/2$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{12}{(49\tan^2\theta+49)^4}\, 7\sec^2\theta\, d\theta$$

$$= \frac{\cancel{12}^{6}\times 7}{\cancel{2}\pi(49)^4}\int_{-\pi/2}^{\pi/2} \frac{\cancel{\sec^2\theta}}{\sec^{\cancel{8}6}\theta}\, d\theta$$

$$= \frac{42}{\pi(49)^4}\int_{-\pi/2}^{\pi/2} \cos^6\theta\, d\theta$$

$$= \frac{42}{\pi(49)^4}\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta$$

$$= \frac{\cancel{42}^{21}}{4\cancel{8}(49)^4\pi}\int_{-\pi/2}^{\pi/2} (1+\cos^3 2\theta + 3\cos 2\theta + 3\cos^2 2\theta)\, d\theta$$

$$P_{yy} = \frac{21}{4\pi(49)^4}\int_{-\pi/2}^{\pi/2}\left[1 + \frac{3\cos 2\theta - \cos 6\theta}{4} + 3\cos 2\theta + 3\left(\frac{1+\cos 4\theta}{2}\right)\right]d\theta$$

$$= \frac{21}{4\pi(49)^4}\left[\int_{-\pi/2}^{\pi/2}\left(\frac{5}{2} + \frac{15}{4}\cos 2\theta - \frac{1}{4}\cos 6\theta + \frac{3}{2}\cos 4\theta\right)d\theta\right]$$

$$= \frac{21}{4\pi(49)^4}\left[\frac{5}{2}\theta\Big|_{-\pi/2}^{\pi/2} + \frac{15}{8}\sin 2\theta\Big|_{-\pi/2}^{\pi/2} - \frac{1}{24}\sin 6\theta\Big|_{-\pi/2}^{\pi/2} + \frac{3}{8}\sin 4\theta\Big|_{-\pi/2}^{\pi/2}\right.$$

$$= \frac{21}{4\cancel{\pi}(49)^4}\left[\frac{5}{2}(\cancel{\pi}) + 0 - 0 + 0\right]$$

$$P_{yy} = \frac{105}{8\cdot(49)^4}\ \text{Watts}$$

(5) a) What is an LTI System? How the response can be obtained from LTI System.

A System is said to be Linear if its response to a weighted sum of inputs $\left\{ \sum_{n=1}^{N} \alpha_n x_n(t) \right\}$, $n = 1,2,3 \dots N$, is equal to corresponding weighted sum of responses taken separately. Thus if $x_n(t)$ causes a response $y_n(t)$, $n = 1,2 \cdot N$, then for a Linear System,

$$y(t) = T\left[\sum_{n=1}^{N} \alpha_n x_n(t)\right] = \sum_{n=1}^{N} \alpha_n T[x_n(t)]$$

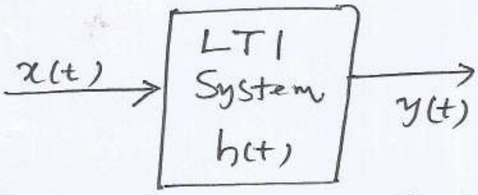
$$= \sum_{n=1}^{N} \alpha_n y_n(t) \text{ must hold.}$$

Where α_n are arbitrary constants & N may be finite.

A Linear System is also Time invariant if its response do not changes w.r.t. time, i.e if $y(t,\xi)$ is the response due to the delayed i/p $x(t-\xi)$ then if $y(t,\xi) = y(t-\xi)$ then we can say that System is Time inverient

$\therefore$ A System which obeys both Linearity and time invariance properties is known as LTI System.

Consider an LTI System whose impulse response is given by $h(t)$ and let $x(t)$ be the i/p & $y(t)$ be the response as shown.



We know that

$$y(t) = T[x(t)] \rightarrow (1)$$

From the properties of impulse function we may write

$$x(t) = \int_{-\infty}^{\infty} x(\xi)\, \delta(t-\xi)\, d\xi \rightarrow (2)$$

from (1) & (2) we have,

$$T[\delta(t)] = h(t)$$

$$\because T[\delta(t-\xi)] = h(t-\xi)$$

$$y(t) = T\left[\int_{-\infty}^{\infty} x(\xi)\, \delta(t-\xi)\, d\xi\right]$$

$$= \int_{-\infty}^{\infty} x(\xi)\, T[\delta(t-\xi)]\, d\xi = \int_{-\infty}^{\infty} x(\xi)\, h(t-\xi)\, d\xi \rightarrow (3)$$

$$\therefore \boxed{y(t) = x(t) * h(t)}$$

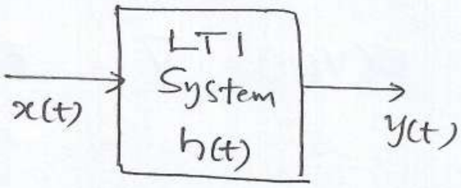
5) (b) Find the System response when a Signal $x(t) = u(t)e^{-2t}$ is applied to a n/w having an impulse response $h(t) = 3u(t)e^{-3t}$.

Soln:

given Data:

$x(t) = e^{-2t}u(t)$

$h(t) = 3e^{-3t}u(t)$



$\therefore \quad y(t) = x(t) * h(t).$

$$= \int_{-\infty}^{\infty} x(\xi)\, h(t-\xi)\, d\xi$$

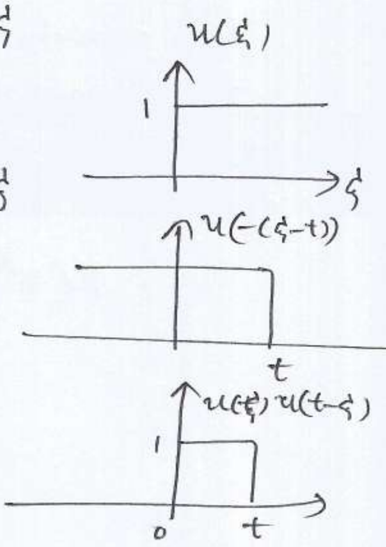
$$= \int_{-\infty}^{\infty} e^{-2\xi} u(\xi)\, 3\, e^{-3(t-\xi)} u(t-\xi)\, d\xi$$

$$= e^{-3t} \int_{-\infty}^{\infty} e^{-2\xi} \cdot e^{3\xi}\, u(\xi)\, u(t-\xi)\, d\xi$$

$$= e^{-3t} \int_{0}^{t} e^{\xi}\, d\xi$$

$$= e^{-3t}\, e^{\xi}\Big|_0^t$$

$$= e^{-3t}(e^{t}-1) = e^{-2t} - e^{-3t}, \quad t \geq 0$$



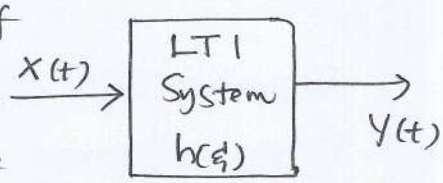
$\therefore \quad y(t) = (e^{-2t} - e^{-3t})\, u(t).$

6 a) Explain About mean and mean Square Value of System response

Ans:

Let us Consider an LTI System as shown.

The o/p process Y(t) interms of the input random Process X(t) and impulse response h(t) can be written as



$$Y(t) = \int_{-\infty}^{\infty} X(\xi)\, h(t-\xi)\, d\xi = \int_{-\infty}^{\infty} h(\xi)\, X(t-\xi)\, d\xi \rightarrow ①$$

Let us assume that $X(t)$ is WSS process. Thus we have,

$$E(X(t)) = E(X(t-\xi)) = \overline{X} = \text{Const.} \rightarrow (2)$$

On taking Expectation to Eqn (1) on b.s, we have,

$$E(Y(t)) = \overline{Y} = E\left[\int_{-\infty}^{\infty} h(\xi).X(t-\xi)\,d\xi\right]$$

$$= \int_{-\infty}^{\infty} h(\xi).\,E(X(t-\xi))\,d\xi$$

$$\overline{Y} = \int_{-\infty}^{\infty} h(\xi).\,\overline{X}.\,d\xi = \overline{X}.\int_{-\infty}^{\infty} h(\xi).\,d\xi \rightarrow (3)$$

from Eqn (3) it is clear that, the mean Value of $Y(t)$ equals the mean Value of $X(t)$ times the area under the impulse function.

The mean Squared Value of $Y(t)$ is given by

$$E(Y^2(t)) = \overline{Y^2(t)} = E\left[\left[\int_{-\infty}^{\infty} h(\xi).\,X(t-\xi)\right]^2\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\xi_1)\,X(t-\xi_1)\,d\xi_1 \cdot \int_{-\infty}^{\infty} h(\xi_2)X(t-\xi_2)\,d\xi_2\right]$$

$$\overline{Y^2(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1).h(\xi_2)\,E[X(t-\xi_1).X(t_2-\xi_2)]\,d\xi_1 d\xi_2$$

Since $X(t)$ is WSS process we have,

$$E[X(t-\xi_1).\,X(t-\xi_2)] = R_{XX}(\xi_1-\xi_2)$$

$$\therefore \quad \overline{Y^2(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1-\xi_2).\,h(\xi_1).h(\xi_2)\,d\xi_1 d\xi_2 \rightarrow (4)$$

Equations (3) & (4) gives the mean and mean Squared Value of System responses when $X(t)$ is WSS process.

⑥ b) A random Process $X(t)$ is applied to a network with impulse response $h(t) = u(t) \cdot t \cdot e^{-3t}$. The Cross Correlation of $X(t)$ with the output $Y(t)$ is known to have the same form

$R_{xy}(\tau) = u(\tau) \cdot \tau \cdot e^{-3\tau}$.

(i) Find the ACF of $Y(t)$

(ii) What is the Average power in $Y(t)$.

Soln:

Given Data:

The impulse response of a n/w is given by

$$h(t) = t e^{-3t} u(t) \rightarrow ①$$

The cross correlation of $X(t)$ with the o/p $Y(t)$ is given by

$$R_{xy}(\tau) = \tau e^{-3\tau} u(\tau) \rightarrow ②$$

(i) we know that the ACF of $Y(t)$ is given by

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau+\xi)\, h(\xi)\, d\xi$$

$$\therefore R_{yy}(\tau) = \int_{-\infty}^{\infty} (\tau+\xi)\, e^{-3(\tau+\xi)} u(\tau+\xi) \cdot \xi e^{-3\xi} \cdot u(\xi)\, d\xi$$

$$= e^{-3\tau} \int_{-\infty}^{\infty} (\xi^2+\tau\xi)\, e^{-6\xi}\, u(\xi) \cdot u(\tau+\xi)\, d\xi$$

Consider

$\underline{\tau \geq 0}$

$$= e^{-3\tau} \int_{0}^{\infty} (\xi^2+\tau\xi)\, e^{-6\xi}\, d\xi. \qquad \because u(\xi) \cdot u(\xi+\tau) = 1 \text{ for } 0 \leq \xi \leq \infty$$

$$= e^{-3\tau} \left[(\xi^2+\tau\xi) \frac{e^{-6\xi}}{-6} \Big|_0^{\infty} - \int_0^{\infty} (2\xi+\tau) \frac{e^{-6\xi}}{-6}\, d\xi \right]$$

$$= e^{-3\tau} \left[(0-0) + \frac{1}{6} \left\{ (2\xi+\tau) \frac{e^{-6\xi}}{-6} \Big|_0^{\infty} - \int_0^{\infty} 2 \cdot \frac{e^{-6\xi}}{-6}\, d\xi \right\} \right]$$

$$= \frac{e^{-3\tau}}{6} \left[0 - \left(\frac{-\tau}{6}\right) + \frac{2}{6} \frac{e^{-6\xi}}{-6} \Big|_0^{\infty} \right]$$

$$= \frac{e^{-3\tau}}{6} \left[\frac{\tau}{6} - \frac{2}{6^2}(0-1) \right]$$

$$= \frac{e^{-3\tau}}{6} \left(\frac{\tau}{6} + \frac{2}{6^2} \right)$$

$$R_{yy}(\tau) = \frac{e^{-3\tau}}{6}\left(\frac{6\tau+2}{6^2}\right) = \frac{e^{-3\tau}}{3}\left(\frac{1+3\tau}{36}\right) \qquad \tau \geq 0$$

$$\therefore \quad R_{yy}(\tau) = \frac{e^{-3|\tau|}}{108}(1+3|\tau|).$$

(ii) The power in $y(t)$ is given by

$$P_{yy} = R_{yy}(0) = \frac{1}{108} \text{ Watts}$$

⑦ $X(t)$ is a Stationary Random Process with Zero mean and auto Correlation function $R_{xx}(\tau) = e^{-2|\tau|}$ is applied to a System with $H(\omega) = \frac{1}{2+j\omega}$. Find the power Spectral density of its output.

Soln:

Given Data,

The ACF of input Random Process $X(t)$ is given by

$$R_{XX}(\tau) = e^{-2|\tau|} \longrightarrow ①$$

The System function is given by

$$H(\omega) = \frac{1}{2+j\omega} \longrightarrow ②$$

$$S_{yy}(\omega) = ?$$

we know that $S_{yy}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \longrightarrow ③$

Take F.T to Eqn ① on b.s we have

$$S_{XX}(\omega) = \mathcal{F}(R_{XX}(\tau)) = \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{+2\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2+j\omega)\tau} d\tau.$$

$$S_{XX}(\omega) = \left.\frac{e^{(2-j\omega)\tau}}{(2-j\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(2+j\omega)\tau}}{-(2+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{2-j\omega}(1-0) - \frac{1}{(2+j\omega)}(0-1)$$

$$= \frac{1}{2-j\omega} + \frac{1}{2+j\omega}$$

$$S_{XX}(\omega) = \frac{2+j\omega+2-j\omega}{(2-j\omega)(2+j\omega)} = \frac{4}{\omega^2+4} \longrightarrow (4)$$

$$(2) \Rightarrow \quad H(\omega) = \frac{1}{\sqrt{\omega^2+4}} \Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2+4} \longrightarrow (5)$$

$\therefore$ from (3), (4) & (5) we have, The power Spectral density of O/p process,

$$S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega)$$

$$= \frac{1}{\omega^2+4} \cdot \frac{4}{\omega^2+4}$$

$$\boxed{S_{YY}(\omega) = \frac{4}{(\omega^2+4)^2}}$$

(8) (a) Consider a Linear System as shown in fig. Where, $X(t)$ is the input and $Y(t)$ is the output of the System.

$$X(t) \longrightarrow \boxed{\frac{1}{6+j\omega}} \longrightarrow Y(t)$$

The ACF of $X(t)$ is $R_{XX}(\tau) = 5\,\delta(\tau)$. Determine the PSD & ACF of the output.

Soln:

Given Data:

The ACF of input process is

$$R_{XX}(\tau) = 5\,\delta(\tau) \longrightarrow (1)$$

The System function is given by

$$H(\omega) = \frac{1}{6+j\omega} \longrightarrow (2)$$

We know that the PSD of the O/p process is given by

$$S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega) \longrightarrow (3)$$

$$(1) \Rightarrow \quad S_{XX}(\omega) = \mathcal{F}(R_{XX}(\tau)) = \mathcal{F}(5\,\delta(\tau)) = 5$$

$$(2) \Rightarrow \quad |H(\omega)| = \frac{1}{\sqrt{\omega^2+36}} \Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2+36}$$

$$\therefore \quad S_{YY}(\omega) = \frac{1}{\omega^2+36} \cdot 5 = \frac{5}{\omega^2+36}$$

$$\therefore \quad \boxed{S_{YY}(\omega) = \frac{5}{\omega^2+36}}$$

The ACF of O/p is thus given by

$$R_{YY}(\tau) = \mathcal{F}^{-1}(S_{YY}(\omega)) = \mathcal{F}^{-1}\left(\frac{5}{\omega^2+36}\right)$$

$$R_{YY}(\tau) = \frac{5}{12}\,\mathcal{F}^{-1}\left(\frac{2(6)}{\omega^2+6^2}\right) \qquad \mathcal{F}^{-1}(e^{-a|\tau|}) = \frac{2a}{\omega^2+a^2}$$

$$\boxed{R_{YY}(\tau) = \frac{5}{12}\,e^{-6|\tau|}} \qquad \therefore \mathcal{F}^{-1}\left(\frac{2a}{\omega^2+a^2}\right) = e^{-a|\tau|}$$

(b) Write a Short notes on Band Limited, Band Pass and narrow band Processes.

Ans: In practice Random Processes Can be categorized into different types depending on their freq. Spectrums.

Some important types of random processes are,

(1) Band Pass Random Processes

(2) Band Limited Random Processes

(3) Narrow Band Random Processes

→ Band Pass Random Processes: A Random Process $X(t)$ is called a band Pass Process if its power Spectral density $S_{XX}(\omega)$ has its significant Components within a band width 'W' that does not include $\omega = 0$. But in practice the Spectrum may have a Small amount of power Spectrum at $\omega = 0$ as shown in fig. The Spectral Components outside the band 'W' are very small and can be neglected.

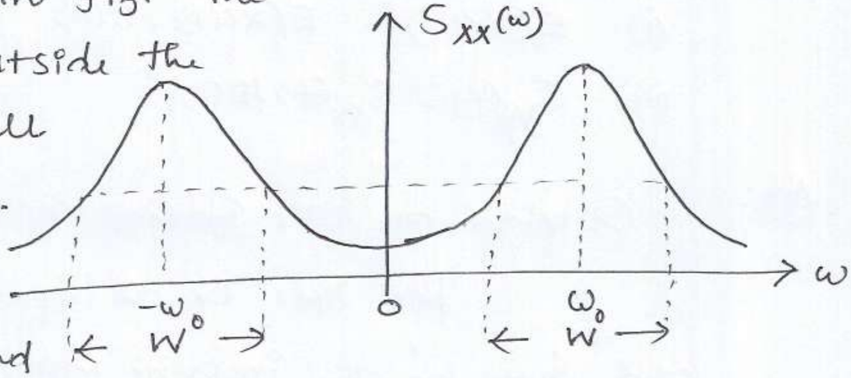


fig. Power Spectrum of a Band Pass Process.

Ex: Modulated Signals with Carrier freq. ω_0 and band width 'W' and noise transmitting over a channel.

Band Limited Random Process: A band Pass Random Process is Said to be band Limited if its power Spectrum Components are zero outside the freq. band of 'W' that does not include $\omega = 0$

The Power density Spectrum of the band Limited band pass process is as shown in fig.

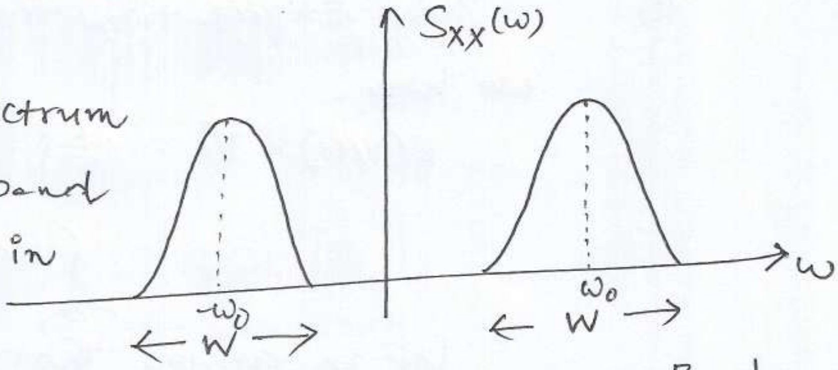


fig. Power Spectrum of a Band Limited Process.

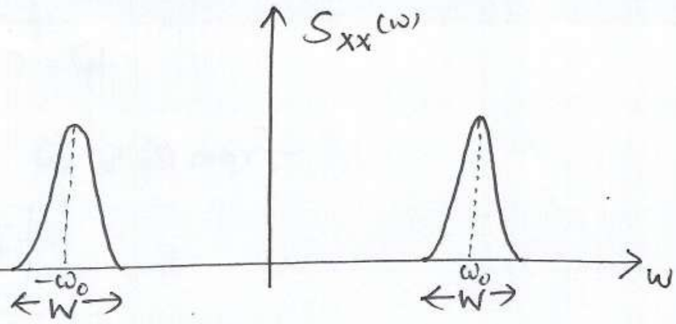
Narrow Band Random Process:

A band Limited Random Process $X(t)$ is Said to be narrow band random Process if the band width 'W' is very Small compared to the band Centre freq. ω_0 i.e $W \ll \omega_0$. where,

$W \rightarrow$ Bandwidth and

$\omega_0 \rightarrow$ freq. at which Spectrum is maximum.

The power density Spectrum of a narrow band Process $X(t)$ is as Shown in fig.



power Spectrum of a narrow band Process.

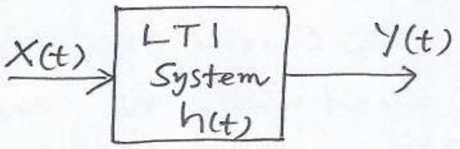
⑨ For an LTI System with impulse response $h(t)$, input $X(t)$ and output $Y(t)$, prove the following.

(i) $E(Y(t)) = E(X(t)) \cdot H(0)$ (ii) $R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$

(iii) $S_{YY}(f) = S_{XX}(f) |H(f)|^2$ (IV) $S_{XY}(f) = S_{XX}(f) \cdot H(f)$.

Ans: Consider an LTI System as shown.

Let $X(t)$ be the input and $Y(t)$ be the output and $h(t)$ be the impulse response of the System.



we know that the output process $Y(t)$ is given by the convolution of input process and impulse response. i.e

$$Y(t) = \int_{-\infty}^{\infty} h(\xi) \cdot X(t-\xi)\, d\xi \longrightarrow ①$$

(i). Take Expectation to Eqn ① on both sides we have,

$$E(Y(t)) = \bar{Y} = E\left(\int_{-\infty}^{\infty} h(\xi) \cdot X(t-\xi)\, d\xi\right)$$

$$= \int_{-\infty}^{\infty} h(\xi) \cdot E(X(t-\xi))\, d\xi$$

Let us assume that $X(t)$ is WSS process i.e

$$E(X(t)) = E(X(t-\xi)) = \bar{X} = \text{Constant}.$$

$$\therefore \quad \bar{Y} = \bar{X} \cdot \int_{-\infty}^{\infty} h(\xi) \cdot d\xi \longrightarrow ②$$

The Fourier Transform of impulse response $h(t)$ is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(\xi) \cdot e^{-j\omega\xi}\, d\xi$$

$$\omega = 0 \Rightarrow \quad H(0) = \int_{-\infty}^{\infty} h(\xi) \cdot d\xi \longrightarrow ③$$

from ② & ③ we have,

$$\boxed{\bar{Y} = \bar{X} \cdot H(0)} \quad (\text{or})$$

$$\boxed{E(Y(t)) = E(X(t)) \cdot H(0)}$$

(ii) The Auto Correlation function of the System response can be given by

$$R_{YY}(t,t+\tau) = E(Y(t).Y(t+\tau))$$

$$= E\left[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)d\xi_1 . \int_{-\infty}^{\infty} h(\xi_2)X(t+\tau-\xi_2)d\xi_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1).h(\xi_2).E(X(t-\xi_1)X(t+\tau-\xi_2))d\xi_1 d\xi_2$$

Since $X(t)$ is WSS process, we have,

$$E(X(t-\xi_1).X(t+\tau-\xi_2)) = R_{XX}(\tau-\xi_2+\xi_1)$$

$$\therefore R_{YY}(t,t+\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1).h(\xi_2).R_{XX}(\tau-\xi_2+\xi_1)d\xi_1 d\xi_2$$

Since the Above integral independent of 't'

we have,

$$R_{YY}(t,t+\tau) = R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)h(\xi_2)R_{XX}(\tau-\xi_2+\xi_1)d\xi_1 d\xi_2$$

Above equation may be viewed as two fold Convolution of $R_{XX}(\tau)$ with the impulse response $h(t)$ as,

$$\therefore R_{YY}(\tau) = \int_{-\infty}^{\infty} h(\xi_1).\int_{-\infty}^{\infty} h(\xi_2).R_{XX}(\tau+\xi_1-\xi_2)d\xi_2\, d\xi_1$$

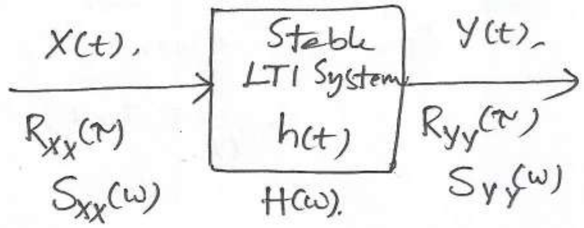
$$= \int_{-\infty}^{\infty} h(\xi_1).(h(\tau) * R_{XX}(\tau+\xi_1))d\xi_1$$

$$= h(\tau) * \int_{-\infty}^{\infty} h(\xi_1).R_{XX}(\tau+\xi_1)d\xi_1$$

$$= h(\tau) * R_{XX}(\tau) * h(-\tau)$$

$$\therefore \boxed{R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)}$$

(iii) Consider a Causal Stable LTI System with impulse response $h(t)$ as shown.



Let $R_{XX}(\tau)$ be input ACF

$R_{YY}(\tau)$ be output ACF.

$S_{XX}(\omega)$ be input PSD

$S_{YY}(\omega)$ be output PSD.

$H(\omega) \longrightarrow$ Transfer function of the System.

From, weiner- Khintchine Relation, we have,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) \cdot e^{-j\omega\tau} d\tau \longrightarrow ①$$

We know that the Auto correlation function of System response is

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)\, h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2 \longrightarrow ②$$

On Substituting Eqn ② in Eqn ① we have,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)\, h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2\; e^{-j\omega\tau} d\tau$$

Let $\tau + \xi_1 - \xi_2 = \xi \Rightarrow \tau = \xi - \xi_1 + \xi_2$

$d\tau = d\xi$

$$\therefore S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi) \cdot h(\xi_1)\, h(\xi_2) \cdot e^{-j\omega(\xi-\xi_1+\xi_2)}\, d\xi_1\, d\xi_2\, d\xi$$

$$= \int_{-\infty}^{\infty} R_{XX}(\xi) \cdot e^{-j\omega\xi} d\xi \cdot \int_{-\infty}^{\infty} h(\xi_1) \cdot e^{j\omega\xi_1} d\xi_1 \cdot \int_{-\infty}^{\infty} h(\xi_2)\, e^{-j\omega\xi_2} d\xi_2$$

$$= S_{XX}(\omega) \cdot H^*(\omega) \cdot H(\omega)$$

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot |H(\omega)|^2$$ also

$\omega = 2\pi f$

$$\boxed{S_{YY}(f) = |H(f)|^2\, S_{XX}(f)},$$

(IV) we know that the Cross power Spectrum $S_{XY}(\omega)$ is given by the fourier transform of the Cross Correlation function $R_{XY}(\tau)$ i.e

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau \longrightarrow ①$$

if $X(t)$ is atleast WSS then the cross correlation function of input $X(t)$ and output $Y(t)$ is given by

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi) h(\xi) d\xi \longrightarrow ②$$

Substitute Eqn ② in Eqn ① we have,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\tau-\xi) h(\xi) d\xi . e^{-j\omega\tau} d\tau$$

let $\tau - \xi = \xi_1 \Rightarrow \tau = \xi + \xi_1$

$d\tau = d\xi_1$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(\xi_1) h(\xi) e^{-j\omega(\xi+\xi_1)} d\xi \, d\xi_1$$

$$= \int_{-\infty}^{\infty} R_{XX}(\xi_1) e^{-j\omega\xi_1} d\xi_1 . \int_{-\infty}^{\infty} h(\xi) e^{-j\omega\xi} d\xi$$

$$S_{XY}(\omega) = S_{XX}(\omega) . H(\omega) \quad \text{also}$$

$\omega = 2\pi f.$

$$\boxed{S_{XY}(f) = H(f) . S_{XX}(f)}$$

Short Answer Questions

① Calculate the Noise equivalent band width of the filter defined with transfer function $H(f) = \frac{1}{1+j2\pi fRC}$

Ans: Given the Transfer function of a filter

$$H(f) = \frac{1}{1+j2\pi fRC} \Rightarrow H(\omega) = \frac{1}{1+j\omega RC}$$

The Noise equivalent B.W. of a filter is given by

$$W_N = \frac{\int_0^\infty |H(\omega)|^2 d\omega}{|H(0)|^2}$$

$$= \frac{\int_0^\infty \frac{1}{1+\omega^2R^2C^2} d\omega}{1}$$

let $\omega_c = \frac{1}{RC}$

$\omega_c^2 = \frac{1}{R^2C^2}$

$$= \int_0^\infty \frac{1}{1+\frac{\omega^2}{\omega_c^2}} d\omega$$

$$= \omega_c \cdot \int_0^\infty \frac{\omega_c}{\omega^2+\omega_c^2} d\omega = \omega_c . \operatorname{Tan}^{-1}\frac{\omega}{\omega_c}\Big|_0^\infty$$

$$= \omega_c\left(\pi/2 - 0\right)$$

$$\therefore W_N = \frac{\pi}{2}\omega_c \text{ or } \frac{\pi}{2}\frac{1}{RC} \text{ rad/sec.}$$

② State Weiner-Khintchine Theorem.

Ans: For a wide Sense Stationery Process $X(t)$, The power density Spectrum and Auto correlation function form fourier Transform pair. i.e

$$S_{XX}(\omega) \xleftrightarrow{F.T} R_{XX}(\tau). \text{ i.e.}$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau). e^{-j\omega\tau} d\tau \text{ and}$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau} d\omega.$$

③ Analyze the Power Density Spectrum of response:

Ans: The power density Spectrum of response of an LTI System is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \longrightarrow ①$$

where $S_{XX}(\omega) \rightarrow$ power Density Spectrum of input

$|H(\omega)|^2 \longrightarrow$ power Transfer function of the System.

④ List the properties of band Limited Processes.

Ans: Let $N(t)$ be any band Limited WSS process with zero mean and PSD $S_{NN}(\omega)$, If $N(t)$ is represented by

$N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$ then Some properties of $X(t)$ and $Y(t)$ are given below.

→ if $N(t)$ is WSS, then $X(t)$ & $Y(t)$ are jointly WSS processes

→ if $N(t)$ has zero mean i.e $E(N(t))=0$ then $E(X(t)) = E(Y(t)) = 0$.

→ The mean Square Values of the processes are equal i.e

$$E(N^2(t)) = E(X^2(t)) = E(Y^2(t)).$$

→ Both $X(t)$ and $Y(t)$ have Same Autocorrelation functions. i.e

$R_{XX}(\tau) = R_{YY}(\tau)$. also $S_{XX}(\omega) = S_{YY}(\omega)$.

⑤ Describe the Condition for a Stable System.

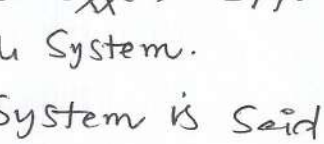
Ans: A Linear Time invariant System is Said to be Stable if its response to any bounded input is bounded. i.e if $|x(t)| < M$ where M is Some Constant, then $|y(t)| < MI$ for a Stable System where 'I' is another Const. independent of input. and it can be shown that

$$I = \int_{-\infty}^{\infty} |h(t)|\,dt < \infty \text{ for a Stable System}$$

whose impulse response is given by $h(t)$.

⑥ Explain About Mean Square Value of System Response.

The response of an LTI System is given by $Y(t) = \int_{-\infty}^{\infty} h(\xi).X(t-\xi)\,d\xi \rightarrow ①$



The mean Square Value of System response is given by

$$E(Y^2(t)) = \overline{Y^2(t)} = E\left[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)d\xi_1 \cdot \int_{-\infty}^{\infty} h(\xi_2)X(t-\xi_2)d\xi_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1).h(\xi_2).E\big(X(t-\xi_1)X(t-\xi_2)\big)d\xi_1 d\xi_2$$

Assume $X(t)$ is WSS, we have

$$E(X(t-\xi_1).X(t-\xi_2)) = R_{XX}(\xi_1-\xi_2)$$

$$\therefore \overline{Y^2(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1-\xi_2).h(\xi_1).h(\xi_2)\,d\xi_1\,d\xi_2$$ which

is independent of 't'.

(7) A wide Sense Stationary R.P $X(t)$ is applied to the input Of an LTI System whose impulse response is $5te^{-2t}u(t)$. The mean of $X(t)$ is 3. Find the mean output of the System.

Ans: given Data,

$$h(t) = 5te^{-2t}u(t).$$

$$\overline{X} = 3.$$

$$\overline{Y} = \overline{X}.\int_{-\infty}^{\infty} h(\xi)\,d\xi$$

$$= 3.\int_{-\infty}^{\infty} 5\xi e^{-2\xi}u(\xi)\,d\xi$$

$$= 15\int_{0}^{\infty} \xi e^{-2\xi}\,d\xi$$

$$= 15\left[\xi.\frac{e^{-2\xi}}{-2}\Big|_0^\infty - \frac{1}{4}e^{-2\xi}\Big|_0^\infty\right]$$

$$\overline{Y} = 15\left(0-0-\frac{1}{4}(0-1)\right) = \underline{\underline{\frac{15}{4}}}$$

(8) A Stationary Random Process with a mean 2 is passed through an LTI System with $h(t) = 2e^{-2t}u(t)$. Determine the mean value of the output process.

Ans: given $h(t) = 2e^{-2t}u(t)$

$$\overline{X} = 2.$$

$$\therefore \overline{Y} = \overline{X}.\int_{-\infty}^{\infty} h(\xi)\,d\xi = \overline{X}\int_{-\infty}^{\infty} 2e^{-2\xi}u(\xi)\,d\xi$$

$$= 2.2\int_{0}^{\infty} e^{-2\xi}\,d\xi$$

$$\overline{Y} = 4\,\frac{e^{-2\xi}}{-2}\Big|_0^\infty = -2(0-1) = 2$$