ANNAMACHARYA INSTITUE OF TECHNOLOGY AND SCIENCES –TIRUPATI

AUTONOMOUS

UNIT-1 Laplace Transforms

Why use Laplace Transforms?

- Find solution to differential equation using algebra
- Relationship to Fourier Transform allows easy way to characterize systems
- No need for convolution of input and differential equation solution
- Useful with multiple processes in system

History of the Transform

- Euler began looking at integrals as solutions to differential equations in the mid 1700's:

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$$z = \int X(x) e^{ax} \, dx \qquad z(x;a) = \int_0^x e^{at} X(t) \, dt ,$$

- Lagrange took this a step further while working on probability density functions and looked at forms of the following equation:

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$$\int X(x) e^{-ax} a^x \, dx ,$$

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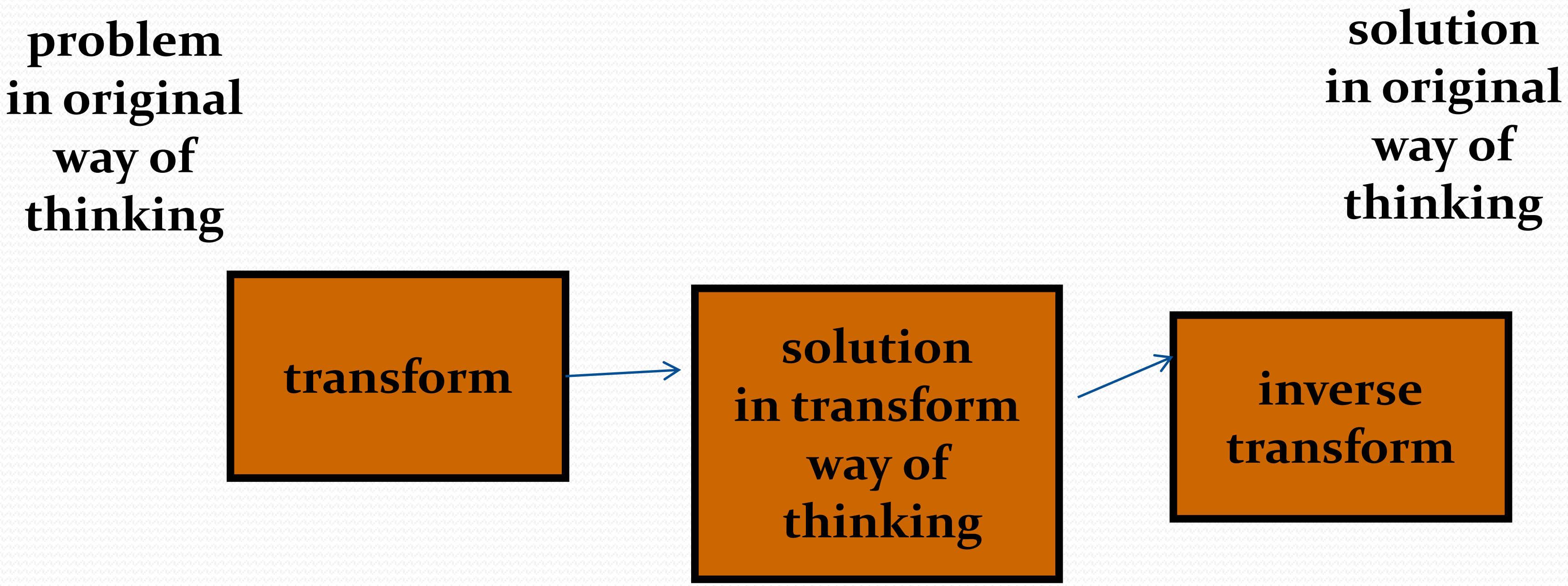
$$S = A y_s + B \Delta y_s + C \Delta^2 y_s + \ldots , \qquad y_s = \int e^{-sx} \phi(x) \, dx ,$$

Transforms -- a mathematical conversion from one way of thinking to another to make a problem easier to solve



Complex numbers

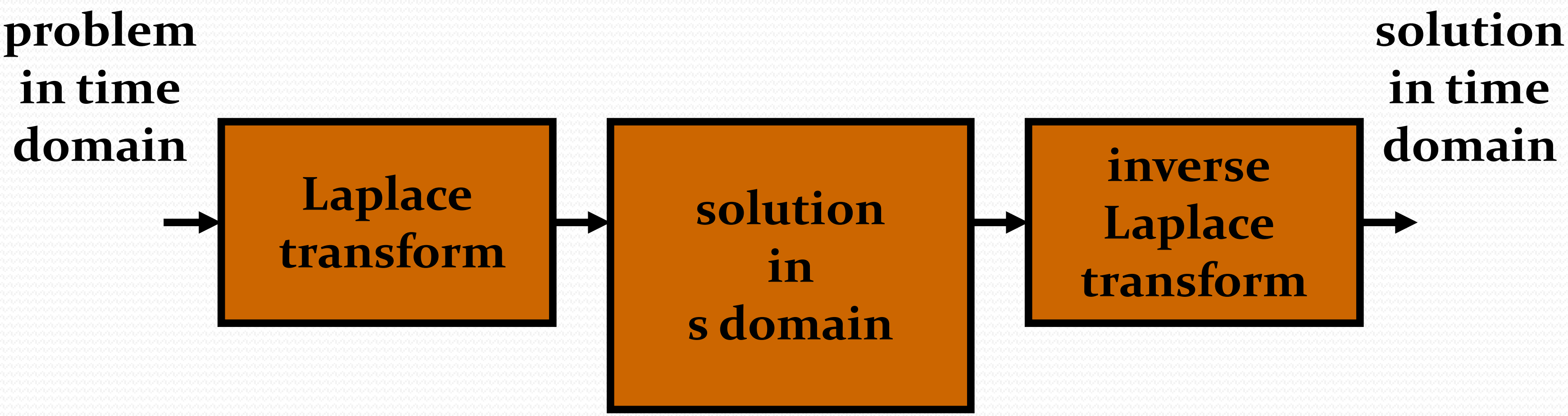
- complex number in Cartesian form: $z = x + jy$
- $x = Rz$, the Real part of z
- $y = \mathscr{I}z$, Imainary part of z
- $j = \sqrt{-1}$ (engineering notation)
- $i = \sqrt{-1}$ is polite term in mixed company

Complex numbers in polar form

- complex number in polar form: $z = r e \exp j\phi$
- r is the modulus or magnitude of z
- ϕ is the angle or phase of z
- $\exp(j\phi) = \cos\phi + j \sin\phi$

The Laplace transform

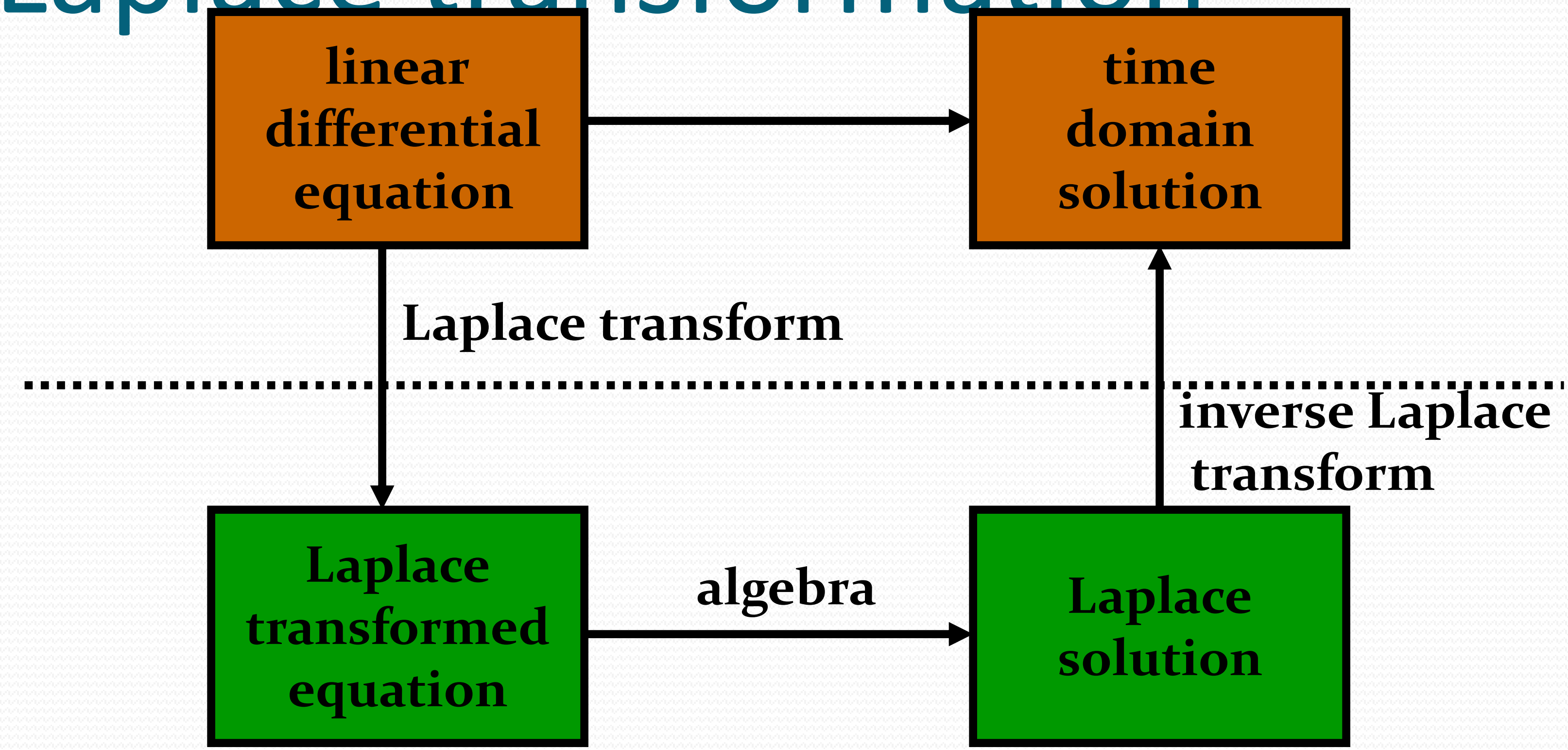
- we'll be interested in signals defined for $t \geq 0$ the Laplace transform of a signal (function) f is the function $F = L(f)$ defined by
- $F(s) = \int_0^\infty e^{-st} f(t)\, dt$ for those $s \in C$ for which the integral makes sense
- F is a complex-valued function of complex numbers •
- s is called the (complex) frequency variable, with units sec^{-1};
- t is called the time variable (in sec);
- st is unitless • for now, we assume f contains no impulses at $t = 0$



- Other transforms
  - Fourier
  - z-transform
  - wavelets

Laplace transformation

time domain



Laplace domain or
complex frequency domain

why to use Laplace Transform

- Find differential equations that describe system
- Obtain Laplace transform
- Perform algebra to solve for output or variable of interest
- Apply inverse transform to find solution

Definition of Laplace Transform

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

- t is real, s is complex!
- Inverse requires complex analysis to solve
- Note "transform": $f(t) \rightarrow F(s)$, where t is integrated and s is variable
- Conversely $F(s) \rightarrow f(t)$, t is variable and s is integrated

Necessary and sufficient condition

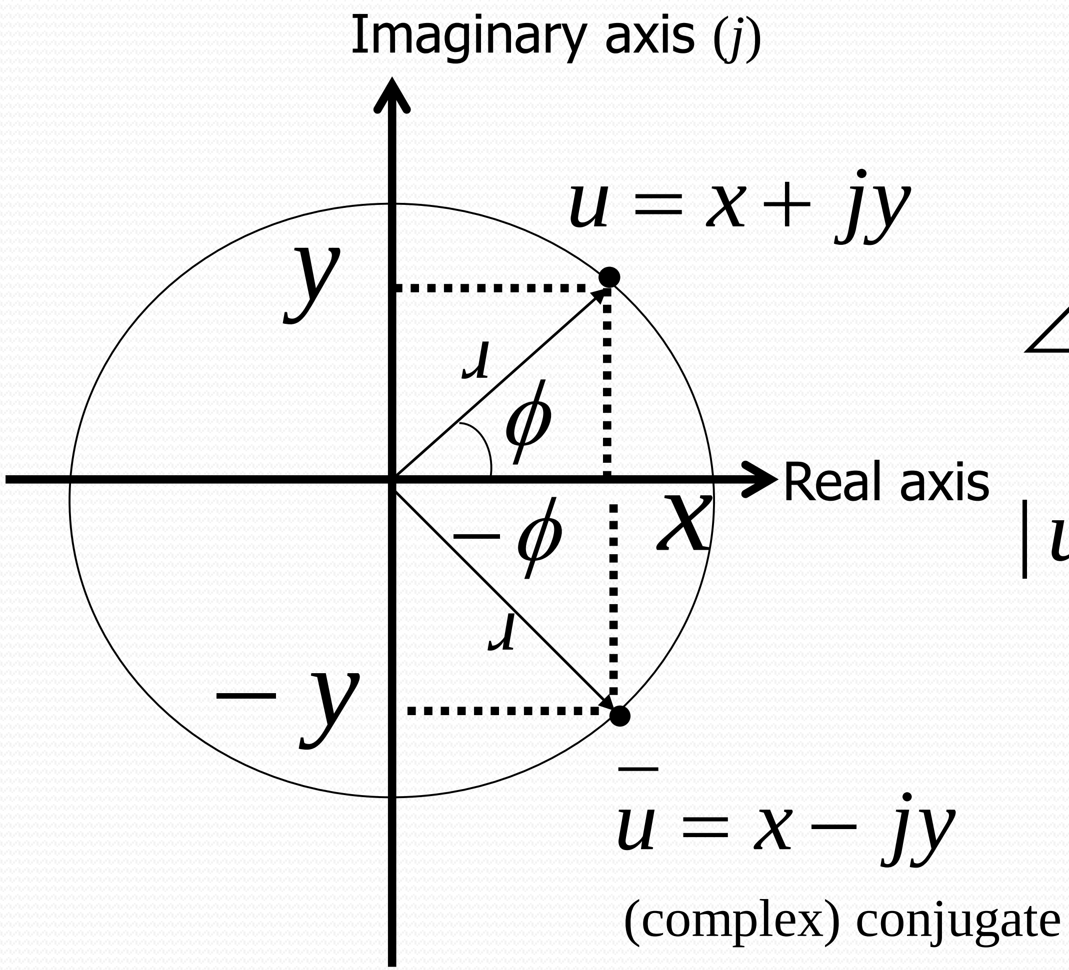
- There are two governing factors that determine whether Laplace transforms can be used:
  - $f(t)$ must be at least piecewise continuous for $t \geq 0$
  - $|f(t)| \leq Me^{\gamma t}$ where M and γ are constants

Basic Tool For Continuous Time: Laplace Transform

$$\mathbf{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

- Convert time-domain functions and operations into frequency-domain
  - $f(t) \to F(s) \quad (t \in R,\ s \in C)$
  - Linear differential equations (LDE) $\to$ algebraic expression in Complex plane
- Graphical solution for key LDE characteristics
- Discrete systems use the analogous z-transform

The Complex Plane (review)



$$\angle u \equiv \phi = \tan^{-1}\frac{y}{x}$$

$$|u| \equiv r \equiv |\bar{u}| = \sqrt{x^2 + y^2}$$

Continuity

- Since the general form of the Laplace transform is:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

it makes sense that f(t) must be at least piecewise continuous for $t \geq 0$.

- If f(t) were very nasty, the integral would not be computable.

Boundedness

- This criterion also follows directly from the general definition:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

- If f(t) is not bounded by $Me^{\gamma t}$ then the integral will not converge.

Laplace Transfor

- General Theory

$$F(s) = L(f(t)) = \int_0^{+\infty} e^{-st} f(t)\,dt = \lim_{\tau \to \infty} \int_0^{\tau} e^{-st} f(t)\,dt$$

- Example

$$f(t) = 1$$

$$L(f(t)) = \int_0^{+\infty} e^{-st} 1\,dt = \lim_{\tau \to \infty} \left(\frac{e^{-st}}{-s} \Bigg|_0^{\tau} \right)$$

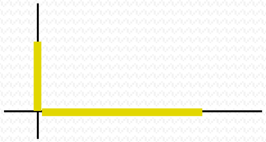
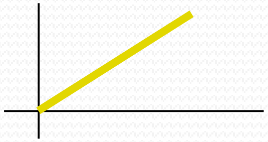
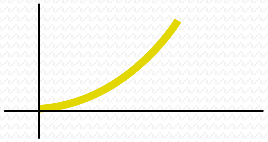
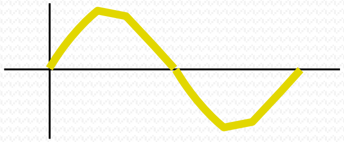
$$= \lim_{\tau \to \infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s}$$

- Convergence

$$f(t) = e^{t^2}$$

$$L(f(t)) = \lim_{\tau \to \infty} \int_0^{\tau} e^{-st} e^{t^2}\,dt = \lim_{\tau \to \infty} \int_0^{\tau} e^{t^2 - st}\,dt = \infty$$

Laplace Transforms of Common Functions

Name	$f(t)$		$F(s)$
Impulse	$f(t) = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases}$		1
Step	$f(t) = 1$		$\frac{1}{s}$
Ramp	$f(t) = t$		$\frac{1}{s^2}$
Exponential	$f(t) = e^{at}$		$\frac{1}{s-a}$
Sine	$f(t) = \sin(\omega t)$		$\frac{1}{\omega^2 + s^2}$

Some more Transforms

$$f(t) = t^n \Leftrightarrow F(s) = \frac{n!}{s^{n+1}}$$

$$\mathrm{n} = 0, \mathrm{f}(\mathrm{t}) = \mathrm{u}(\mathrm{t}) \Leftrightarrow \mathrm{F}(\mathrm{s}) = \frac{0!}{\mathrm{s}^1} = \frac{1}{\mathrm{s}}$$

$$\mathrm{n} = 1, \mathrm{f}(\mathrm{t}) = \mathrm{tu}(\mathrm{t}) \Leftrightarrow \mathrm{F}(\mathrm{s}) = \frac{1!}{\mathrm{s}^2}$$

$$\mathrm{n} = 5, \mathrm{f}(\mathrm{t}) = \mathrm{t}^5\mathrm{u}(\mathrm{t}) \Leftrightarrow \mathrm{F}(\mathrm{s}) = \frac{5!}{\mathrm{s}^6} = \frac{120}{\mathrm{s}^6}$$

Theorem 1

- **Linearity of the Laplace Transform**
- *The Laplace transform is a linear operation; that is, for any functions f*(t) *and* g(t) *whose transforms exist and any constants a and b the transform of* $af(t) + bg(t)$ *exists, and*

$$L\ \{ af(t) + bg(t) \} = aL\{f(t)\} + bL\{g(t)\}.$$

Laplace Transform

Table 6.1 **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a $(a \text{ positive})$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

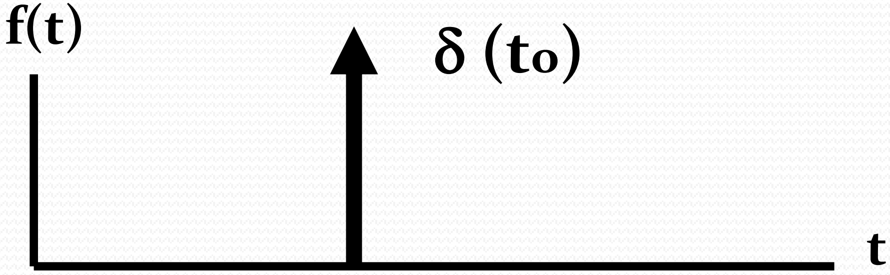
Laplace Transform Properties

Addition/S caling	$L[af_1(t) \pm bf_2(t)] = aF_1(s) \pm bF_2(s)$
Differenti ation	$L\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0\pm)$
Integratio n	$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]_{t=0\pm}$
Convolutio n	$\int_0^t f_1(t-\tau)f_2(\tau)d\tau = F_1(s)F_2(s)$
Initial -value theorem	$f(0+) = \lim_{s\to\infty} sF(s)$
Final -value theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

- SIMPLE TRANSFORMATIONS
- Impulse -- $\delta\ (t_o)$

$$F(s) = \int_{o}^{\infty} e^{-st}\, \delta\ (t_o)\ dt$$

$$= e^{-st_o}$$



- Step -- $u\,(t_o)$

$$F(s) = \int_0^\infty e^{-st}\, u\,(t_o)\; dt \quad = e^{-st_o}/s$$

Linearity	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
Constant multiplication	$a\ f(t)$	$a\ F(s)$
Complex shift	$e^{at}\ f(t)$	$F(s-a)$
Real shift	$f(t - T)$	$e^{Ts}\ F(as)$
Scaling	$f(t/a)$	$a\ F(as)$

First shifting Theorem

Theorem 2

- First Shifting Theorem, s-Shifting
- *If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,*

$$L\{e^{at}f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = L^{-1}\{F(s - a)\}$$

Properties: Multiplication by t^n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example :

$$L\{t^n u(t)\} =$$

$$(-1)^n \frac{d^n}{ds^n} (\frac{1}{s}) =$$

$$\frac{n!}{s^{n+1}}$$

Proof :

$$L\{t^n f(t)\} = \int_0^\infty t^n f(t) e^{-st} dt =$$

$$\int_0^\infty f(t) t^n e^{-st} dt =$$

$$(-1)^n \int_0^\infty f(t) \frac{\partial^n}{\partial s^n} e^{-st} dt =$$

$$(-1)^n \frac{\partial^n}{\partial s^n} \int_0^\infty f(t) e^{-st} dt = (-1)^n \frac{\partial^n}{\partial s^n} F(s)$$

The "D" Operator

1. Differentiation shorthand

$$Df(t) = \frac{df(t)}{dt}$$

$$D^2 f(t) = \frac{d^2}{dt^2} f(t)$$

2. Integration shorthand

if $g(t) = \int_{-\infty}^{t} f(t)dt$

then $Dg(t) = f(t)$

if $g(t) = \int_{a}^{t} f(t)dt$

then $g(t) = D_a^{-1} f(t)$

Properties: Integrals

$$L\{D_0^{-1} f(t)\} = \frac{F(s)}{s}$$

Proof : $g(t) = D_0^{-1} f(t)$

$$L\{\sin(t)\} = \int_0^{\infty} g(t) e^{-st} dt$$

let $u = g(t), du = f(t)dt$

$$dv = e^{-st}dt, v = -\frac{1}{s}e^{-st}$$

$$= [-\frac{1}{s} g(t) e^{-st}]_0^{\infty} + \frac{1}{s}\int f(t) e^{-st} dt = \frac{F(s)}{s}$$

$$g(t) = \int_0^t f(t)dt$$ If t=o, g(t)=o

for $(t = \infty) \Rightarrow \int_0^{\infty} f(t) e^{-st} dt < \infty$ so

$\int_0^{\infty} f(t)dt = g(t) \to \infty$ slower than $e^{-st} \to 0$

Example :

$$L\{D_0^{-1} \cos(t)\} =$$

$$(\frac{1}{s})(\frac{s}{s^2+1}) = \frac{1}{s^2+1}$$

$$L\{\sin(t)\}$$

Properties: Derivatives (this is the big one)

$$L\{Df(t)\} = sF(s) - f(0^+)$$

Example :

$$L\{D\cos(t)\} =$$

$$\frac{s^2}{s^2+1} - f(0^+) =$$

$$\frac{s^2}{s^2+1} - 1 =$$

$$\frac{s^2-(s^2+1)}{s^2+1}$$

$$\frac{-1}{s^2+1} = L\{-\sin(t)\}$$

Proof :

$$L\{Df(t)\} = \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$

let

$$u = e^{-st}, du = -se^{-st}$$

$$dv = \frac{d}{dt} f(t) dt, v = f(t)$$

$$[e^{-st} f(t)]_0^\infty + s\int_0^\infty f(t) e^{-st} dt =$$

$$-f(0^+) + sF(s)$$

Properties: Nth order derivatives

$$L\{D^2 f(t)\} = ?$$

let $g(t) = Df(t), g(0) = Df(0) = f'(0)$

$$= L\{D^2 g(t)\} = sG(s) - g(0) = sL\{Df(t)\} - f'(0)$$

$$= s(sF(s) - f(0)) - f'(0) = s^2 F(s) - sF(0) - f'(0)$$

$$L\{D^n f(t)\} = s^n F(s) - s^{(n-1)} f(0) - s^{(n-2)} f'(0) - \cdots - sf^{(n-2)'}(0) - f^{(n-1)'}(0)$$

NOTE: to take $L\{D^n f(t)\}$
you need the value @ t=0 for

$D^{n-1} f(t), D^{n-2} f(t), \ldots Df(t), f(t) \rightarrow$ called initial conditions!

We will use this to solve differential equations!

Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

Unit Step Function(or) Second Shifting Theorem

- We shall introduce two auxiliary functions, the *unit step function or Heaviside function u(t – a) (following)* and *Dirac's delta δ(t – a)*

- These functions are suitable for solving ODEs with co mplicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (hammerblows, for example).

Second Shifting Theorem; Time Shifting

- *If $f(t)$ has the transform $F(s)$ then the "***shifted function***"*
- (3) $\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$
- has the transform $e^{-as}F(s)$. *That is, if* $\mathcal{L}\{f(t)\} = F(s)$, *then*
- **(4)** $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$.
- *Or, if we take the inverse on both sides, we can write*
- **(4*)** $f(t-a)u(t-a)\} = \mathcal{L}^{-1}\{e^{-as}F(s)\}$.

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geq 0). (1)$$

Definition

Unit Step Function (Heaviside Function) *u*(*t* − *a*) (continued)

Figure 1 shows the special case $u(t)$, which has its jump at zero, and Fig. 2 the general case $u(t - a)$ for an arbitrary positive a. (For Heaviside, see Sec. 6.1.)

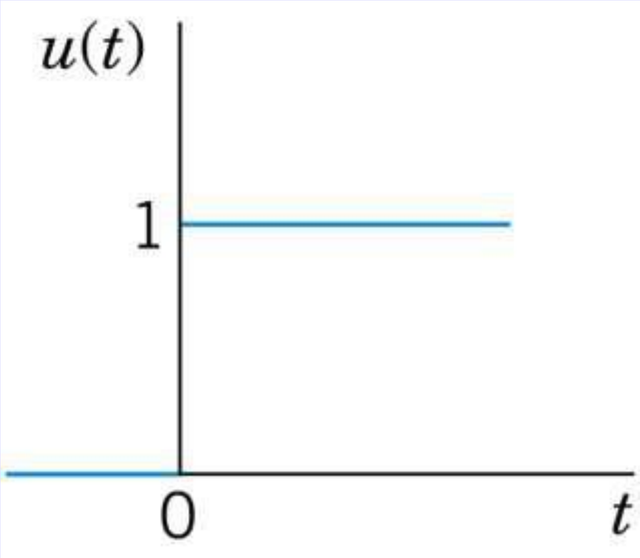


Fig. 1. Unit step function $u(t)$

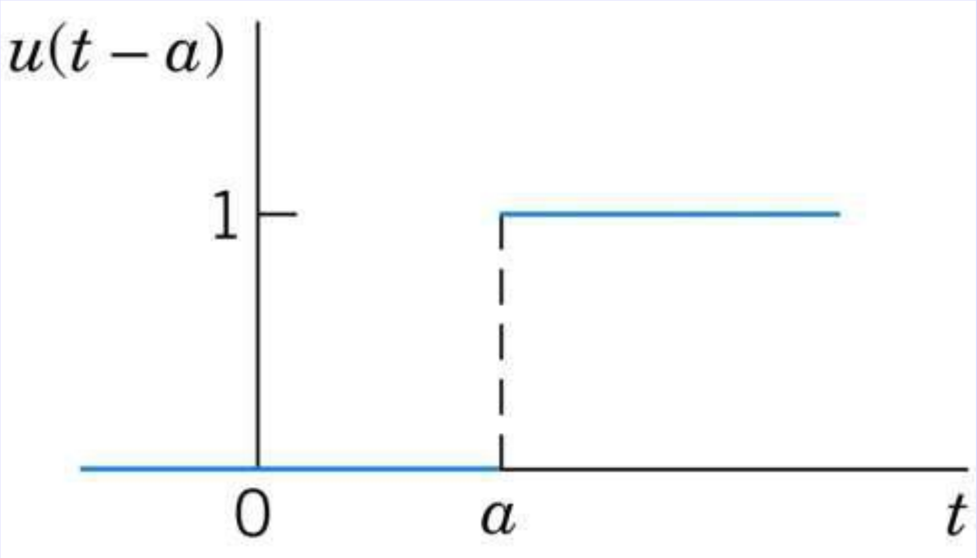


Fig. 2 Unit step function $u(t - a)$

Unit Step Function (Heaviside Function) $u(t - a)$ (continued)

- **(2)** $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s} \qquad (s > 0).$
- Multiplying functions $f(t)$ with $u(t - a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 3 In Fig. 120 the given function is shown in (A). In (B) it is switched off between $t = 0$ and $t = 2$ (because
- $u(t - 2) = 0$ when $t < 2$) and is switched on beginning
- at $t = 2$. In (C) it is shifted to the right by 2 units, say, for instance, by 2 sec, so that it begins 2 sec later in the same fashion as before.

- More generally we have the following.
- *Let $f(t) = 0$ for all negative t. Then $f(t - a)u(t - a)$ with $a > 0$ is*
- *$f(t)$* ***shifted*** *(translated) to the right by the amount a.*

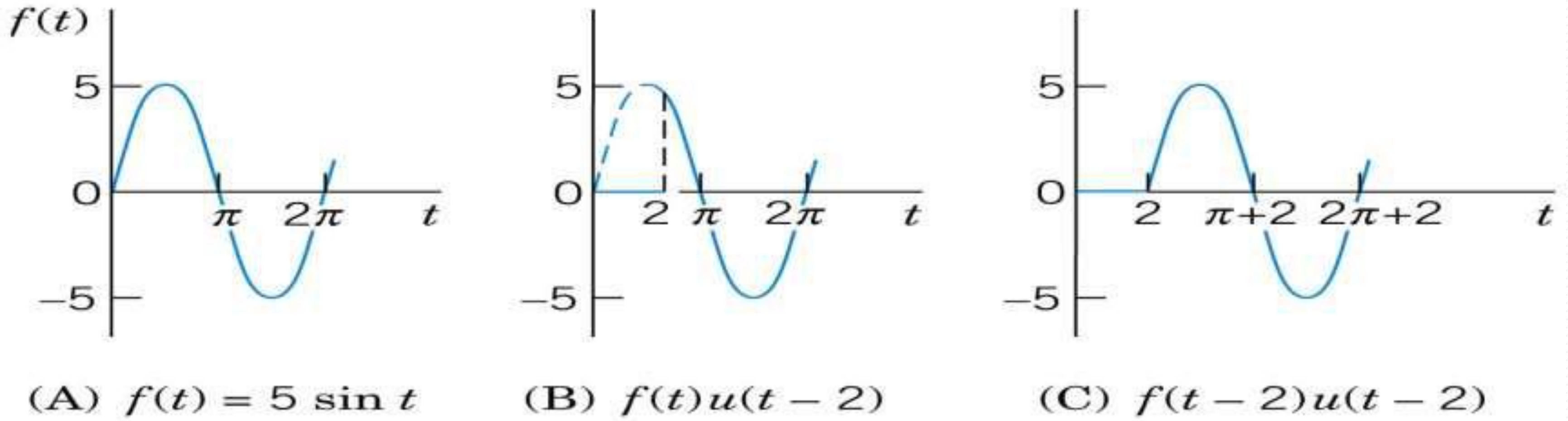
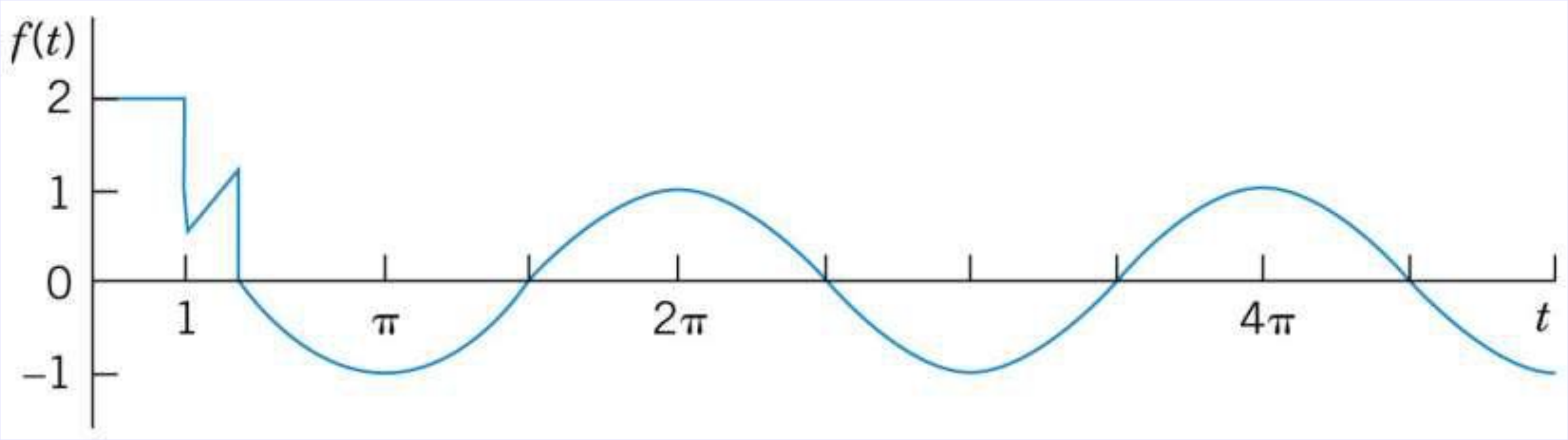


Fig. 3. Effects of the unit step function: (A) Given function.
(B) Switching off and on. (C) Shift

EXAMPLE 1

Application of Theorem 1. Use of Unit Step Functions

- Write the following function using unit step functions and find its transform.
-
$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \quad t > \frac{1}{2}\pi. \end{cases}$$
- (Fig. 122)



- **Fig. 122.** $f(t)$ in Example 1

Fig. 122. $f(t)$ in Example 1

Solution for example 1

- ***Step 1.***
- In terms of unit step functions
- $f(t) = 2(1 - u(t-1)) + \frac{1}{2}t^2(u(t-1) - u(t - \frac{1}{2}\pi)) + (\cos t)u(t - \frac{1}{2}\pi)$. Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so

***Solution** (continued)*

Step 2.

To apply Theorem 1, we must write each term in $f(t)$ in the form $(t - a)u(t - a)$. Thus, $2(1 - u(t-1))$ remains as it is and gives the transform $2(1 - e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left\{\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t - \frac{1}{2}\pi)\right\} = \mathcal{L}\left\{\left(\frac{1}{2}(t - \frac{1}{2}\pi)^2 + \frac{\pi}{2}(t - \frac{1}{2}\pi) + \frac{\pi^2}{8}\right)u(t - \frac{1}{2}\pi)\right\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

Solution (continued)

- ***Step 2.*** *(continued)*

$$\mathcal{L}\left\{(\cos t)u(t-\frac{1}{2}\pi)\right\} = \mathcal{L}\left\{-\left(\sin(t-\frac{1}{2}\pi)\right)u(t-\frac{1}{2}\pi)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

- Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

Short Impulses.

- **Dirac’s Delta Function.**

Short Impulses. Dirac's Delta Function.

Impulse examples:

*An airplane making a "hard" landing

•A mechanical system being hit by a hammer blow

•A ship being hit by a single high wave

• A tennis ball being hit by a racket, and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time.

We can model such phenomena and problems by "Dirac's delta function," and solve them very effectively by the Laplace transform.

To model situations of that type, we consider the function

$$(1) \qquad f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig. 132)}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \le t \le a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 132 has area 1, the impulse of f_k in (1) is

$$(2) \qquad I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1.$$

Impulse function

- Phenomena of an impulsive nature: such as the action of forces or voltages over short intervals of time:
  - a mechanical system is hit by a hammerblow,
  - an airplane makes a “hard” landing,
  - a ship is hit by a single high wave, or
- Goal:
  - Dirac’s delta function.
  - solve the equation efficiently by the Laplace transform..

Laplace Transform of Periodic Function

- Definition: A function f(t) is said to be periodic function with period p(> 0) if f(t+p)=f(t) for all t>0.
- Theorem 1: Transform of Periodic Functions

– The Laplace transform of a piecewise continuous periodic function f(t) with period p is

$$L\{f(t)\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s>0)$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

Laplace Transform of Periodic Function

Laplace Transform of Periodic Function

- Put t=u+p in the second integral,

$$\therefore dt = du, \text{ when } t = p, u = 0 \text{ and when } t \to \infty, u \to \infty.$$

$$\therefore L\{f(t)\} = \int_0^p e^{-st} f(t)\, dt + \int_0^\infty e^{-s(u+p)} f(u+p)\, du \text{ since } f(p+u) = f(u)$$

$$= \int_0^\infty e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-su} f(u)\, du$$

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt + e^{-sp} \cdot \bar{f}(s)$$

Solving for $\bar{f}(s)$ the desired result follows.

$$\therefore L\{f(t)\} = \bar{f}(s) = \frac{1}{1-e^{-ps}} \int_0^{\infty} e^{-st} f(t)\, dt, \ (s > 0)$$

$\underline{\underline{Ex}}:$ *Find Laplace Transform of Half − wave Rectifier*

$$f(t) = \begin{cases} \sin wt, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{2} < t < \dfrac{2\pi}{\omega} \end{cases}$$

$\underline{\underline{Sol^n}}:$ Here $p = \dfrac{2\pi}{\omega} - 0$

$$= \frac{2\pi}{\omega}$$

– By definition of L.T. of periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} f(t)dt\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t dt$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{(-s)^2+(w)^2}(-s\cdot\sin\omega t-\omega\cdot\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\left\{\frac{e^{-st}}{s^2+\omega^2}(-s\cdot\sin\pi-\omega\cos\omega)\right\}-\right.$$

$$\left.\left\{\frac{e^{-st}}{s^2+\omega^2}(-s\cdot\sin 0-\omega\cos 0)\right\}\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{1}{s^2+\omega^2}\left(e^{\frac{-s\pi}{\omega}}\cdot\omega+\omega\right)\right]$$

Inverse Laplace transform

The Laplace transform is an expression involving variable s and can be denoted as such by $F(s)$. That is:

$$F(s) = L\{f(t)\}$$

It is said that $f(t)$ and $F(s)$ form a *transform pair*.

This means that if $F(s)$ is the *Laplace transform* of $f(t)$ then $f(t)$ is the *inverse Laplace transform* of $F(s)$.

That is:

$$f(t) = L^{-1}\{F(s)\}$$

INVERSE LAPLACE TRANSFORMS FORMULAE

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	
k	$\dfrac{k}{s}$	$s>0$
e^{-kt}	$\dfrac{1}{s+k}$	$s>-k$
te^{-kt}	$\dfrac{1}{(s+k)^2}$	$s>-k$

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$s^2+k^2>0$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$s^2+k^2>0$

- Definition :

- Partial fractions are several fractions whose sum equals a given fraction
- Purpose -- Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

Example : Determine the inverse transform of the function below.

$$F(s)=\frac{5}{s}+\frac{12}{s^{2}}+\frac{8}{s+3}$$

$$f(t)=5+12t+8e^{-3t}$$

Inverse Laplace Transforms

There are three cases to consider in doing the partial fraction expansion of F(s).

<u>Case 1</u>: F(s) has all non repeated simple roots.

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots + \frac{k_n}{s+p_n}$$

<u>Case 2</u>: F(s) has complex poles:

$$F(s) = \frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{k_1}{s+\alpha-j\beta} + \frac{k_1^*}{s+\alpha+j\beta} + \ldots +$$

<u>Case 3</u>: F(s) has repeated poles.

$$F(s) = \frac{P_1(s)}{Q_1(s)(s+p_1)^r} = \frac{k_{11}}{s+p_1} + \frac{k_{12}}{(s+p_1)^2} + \ldots + \frac{k_{1r}}{(s+p_1)^r} + \ldots + \frac{P_1(s)}{Q_1(s)}$$

Inverse Laplace Transforms

EXAMPLE 1:

$$F(s)=\frac{4(s+2)}{(s+1)(s+4)(s+10)}=\frac{A_1}{(s+1)}+\frac{A_2}{(s+4)}+\frac{A_3}{(s+10)}$$

$$A_1=\frac{(s+1)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-1}= 4/27$$

$$A_2=\frac{(s+4)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-4} = 4/9$$

$$A_3=\frac{(s+10)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-10} = -16/27$$

$$f(t)=\left[(4/27)e^{-t} + (4/9)e^{-4t} + (-16/27)e^{-10t}\right]u(t)$$

Complex Roots: An Example.

For the given F(s) find f(t)

$$F(s) = \frac{(s+1)}{s(s^2+4s+5)} = \frac{(s+1)}{s(s+2-j)(s+2+j)}$$

$$F(s) = \frac{A}{s} + \frac{K_1}{s+2-j} + \frac{K_1^*}{s+2+j}$$

$$A = \frac{(s+1)}{(s^2+4s+5)}\bigg|_{s=0} = \frac{1}{5}$$

$$K_1 = \frac{(s+1)}{s(s+2+j)}\bigg|_{s=-2+j} = \frac{-2+j+1}{(-2+j)(2j)} = 0.32\angle -108^o$$

Example-2. Determine exponential portion of inverse transform of function below.

$$F(s) = \frac{50(s+3)}{(s+1)(s+2)(s^2+2s+5)}$$

$$F_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = \left.\frac{50(s+3)}{(s+2)(s^2+2s+5)}\right]_{s=-1} = \frac{(50)(2)}{(1)(4)} = 25$$

$$A_2 = \left.\frac{50(s+3)}{(s+1)(s^2+2s+5)}\right]_{s=-2} = \frac{(50)(1)}{(-1)(5)} = -10$$

$$f_1(t) = 25e^{-t} - 10e^{-2t}$$

Example 3. Determine inverse transform of function below.

$$F(s) = \frac{60}{s(s+2)^2}$$

$$F(s) = \frac{60}{s(s+2)^2} = \frac{A}{s} + \frac{C_1}{(s+2)^2} + \frac{C_2}{(s+2)}$$

$$A = sF(s)\Big]_{s=0} = \frac{60}{(s+2)^2}\Bigg]_{s=0} = \frac{60}{(0+2)^2} = 15$$

$$C_1 = (s+2)^2 F(s)\Big]_{s=-2} = \frac{60}{s}\Bigg]_{s=-2} = \frac{60}{-2} = -30$$

CONT..NEXT PAGE

$$F(s)=\frac{60}{s(s+2)^2}=\frac{15}{s}-\frac{30}{(s+2)^2}+\frac{C_2}{s+2}$$

$$\frac{60}{(1)(1+2)^2}=\frac{15}{1}-\frac{30}{(1+2)^2}+\frac{C_2}{(1+2)}$$

$$F(s)=\frac{60}{s(s+2)^2}=\frac{15}{s}-\frac{30}{(s+2)^2}-\frac{15}{s+2}$$

$$f(t)=15-30te^{-2t}-15e^{-2t}=15-15e^{-2t}(1+2t)$$

Inverse Laplace transform of Derivatives,Integrals

$$If\ L^{-1}\{\bar{f}(s)\} = f(t), then$$

$$ILTD(INVERSELAPLACETRANSFORMOFDERIVATIVES)$$

$$L^{-1}\{f^{-(n)}(s)\} = \{-1\}^{n} t^{n} f(t)$$

$$ILTI(INVERSELAPLACETRANSFORMOFINTGRALS)$$

$$L^{-1}\{\int_{s}^{\infty} \overline{f}(s)ds\} = \frac{f(t)}{t}$$

Inverse laplace Transform of powers of ‘s’ & Division by ‘s’

$$If\ L^{-1}\{\bar{f}(s)\}=f(t), then$$

Inverse Laplace Transform of Powers of s states

$$L^{-1}\{s^n\bar{f}(s)\}=f^{(n)}(t), if\ f^{(n)}(t)=0\ for, n=1,2,3,\ldots\ldots,n-1.$$

Inverse Lapse Transform of Division by s states

$$L^{-1}\{\frac{\bar{f}(s)}{s}\}=\int_0^t f(u)du$$

Convolution Theorem

Convolution theorem

$$F(s)\cdot G(s)=F(s)\int_0^\infty e^{-st}g(t)dt=\int_0^\infty F(s)e^{-s\tau}g(\tau)d\tau$$

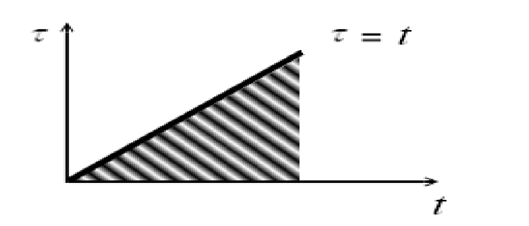
$$e^{-s\tau}F(s)=L[H(t-\tau)f(t-\tau)]$$

Proof:

$$F(s)\cdot G(s)=\int_0^\infty L[H(t-\tau)f(t-\tau)]g(\tau)d\tau$$

$$=\int_0^\infty\left[\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)dt\right]g(\tau)d\tau$$

$$=\int_0^\infty\int_0^\infty e^{-st}g(\tau)H(t-\tau)f(t-\tau)dtd$$



$$=\int_0^\infty\int_\tau^\infty e^{-st}g(\tau)f(t-\tau)dtd\tau$$

$$=\int_0^\infty\int_0^t e^{-st}g(\tau)f(t-\tau)d\tau dt=\int_0^\infty e^{-st}\left[\int_0^t g(\tau)f(t-\tau)d\tau\right]dt$$

$$=\int_0^\infty e^{-st}(f*g)(t)dt=L[f*g](s)$$

$$L^{-1}[FG] = f * g$$

$$L^{-1}\left[\frac{1}{s(s-4)^2}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s-4)^2}\right] = L^{-1}[F(s)\cdot G(s)]$$

$$L^{-1}\left[\frac{1}{s}\right] = 1 = f(t),\quad L^{-1}\left[\frac{1}{(s-4)^2}\right] = te^{4t} = g(t)$$

$$\therefore\ L^{-1}\left[\frac{1}{s(s-4)^2}\right] = f(t) * g(t) = 1 * te^{4t}$$

$$= \int_0^t \tau e^{4\tau} d\tau = te^{4t}/4 - e^{4t}/16 + 1/16$$

- Convolution has to do with the multiplication of transforms. The situation is as follows.
- *Addition* of transforms provides no problem;
- we know that $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$.
- Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product
- $\mathcal{L}(f)\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false. *The transform of a product is generally different from the product of the transforms of the factors,*
- $\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g)$ in general.

CONVOLUTION THEOREM

Example of Convolution Theorem

- If $f = e^t$ and $g = 1$.
- Then $fg = e^t$, $\mathcal{L}(fg) = 1/(s-1)$,
- but $\mathcal{L}(f) = 1/(s-1)$ and $\mathcal{L}(1) = 1/s$
- give $\mathcal{L}(f)\mathcal{L}(g) = 1/(s^2 - s)$.
- According to the next theorem, the correct answer is that
- $\mathcal{L}(f)\mathcal{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f*g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

APPLCATIONS OF LAPLACE TRANSFORMS

Working Rule to solve Differential Equations By Laplace Transform Method

- Step1: Take the Laplace transform of both sides of the given differential equation
- Step2:Use the formulas:

$$(i)L\{y'(t)\}=s\bar{y}(s)-y(0)$$

$$(ii)L\{y''(t)\}=s^2\bar{y}(s)-sy(0)-y'(0)$$

$$(iii)L\{y'''(t)\}=s^3\bar{y}(s)-s^2y(0)-sy'(0)-y''(0)$$

- $Step3: \text{Re}\ place \bullet y(0), y'(0) \,\&\, y''(0)$
- with the Initial conditions

Transform Method continuation

- step4:Transpose the terms with minus signs to the right.
- Step5:Divide by the coefficient of $\bar{y}$ getting $\bar{y}$ as a known function of x.
- Step6:Resolve this function into partial fractions.
- Step7:Take the Inverse of Laplace Transform of $\bar{y}$ obtained in step5.This gives y as a function of t which is the required solution of the given equation satisfying the given initial conditions.

Example of Solution of an ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

- ODE w/initial conditions

$$s^2 Y(s) + 6sY(s) + 8Y(s) = 2/s$$

- Apply Laplace transform to each term
- Solve for Y(s)

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

- Apply partial fraction expansion

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

- Apply inverse Laplace transform to each term

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

Example: Use of the Laplace Transform to solve the initial-value problem

- Solve $y''+4y=\sin t$, $y(0)=0$ & $y'(0)=1$.
- Solution: Taking the Laplace Transform of both sides i.e $L\{y''+4y\}=L\{\sin t\}$ $L\{y''\}+4L\{y\}=L\{\sin t\}$ using the derivative property

$$L\{y'\}=sL\{y\}-y(0)$$

$$L\{y''\}=s^2L\{y\}-sy(0)-y'(0)$$

$$s^2L\{y\}-sy(0)-y'(0)+4L\{y\}=L\{\sin t\}=\frac{1}{s^2+1}$$

- we obtain substituting the initial conditions $y(0)=0$ & $y'(0)=1$ and $Y(s)=L\{y\}$ using more suggestive , $s^2Y(s)-1+4Y(s)=\dfrac{1}{s^2+1}$

  we obtain

$$Y(s)(s^2+4)-1=\frac{1}{s^2+1}$$

i.e

$$Y(s)(s^2+4)=\frac{1}{s^2+1}+1=\frac{s^2+2}{s^2+1}$$

Solving

$$Y(s)=\frac{s^2+2}{(s^2+1)(s^2+4)}$$

- Solving partial fractions

$$Y(s)=\frac{s^2+2}{(s^2+1)(s^2+4)}=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2+4)}$$

we get

- =

$$=(As+B)(s^2+4)+(Cs+D)(s^2+1)$$

$$=(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)=s^2+2$$

$$A+C=0 \qquad B+D=1 \qquad 4A+C=0 \qquad 4B+D=2$$

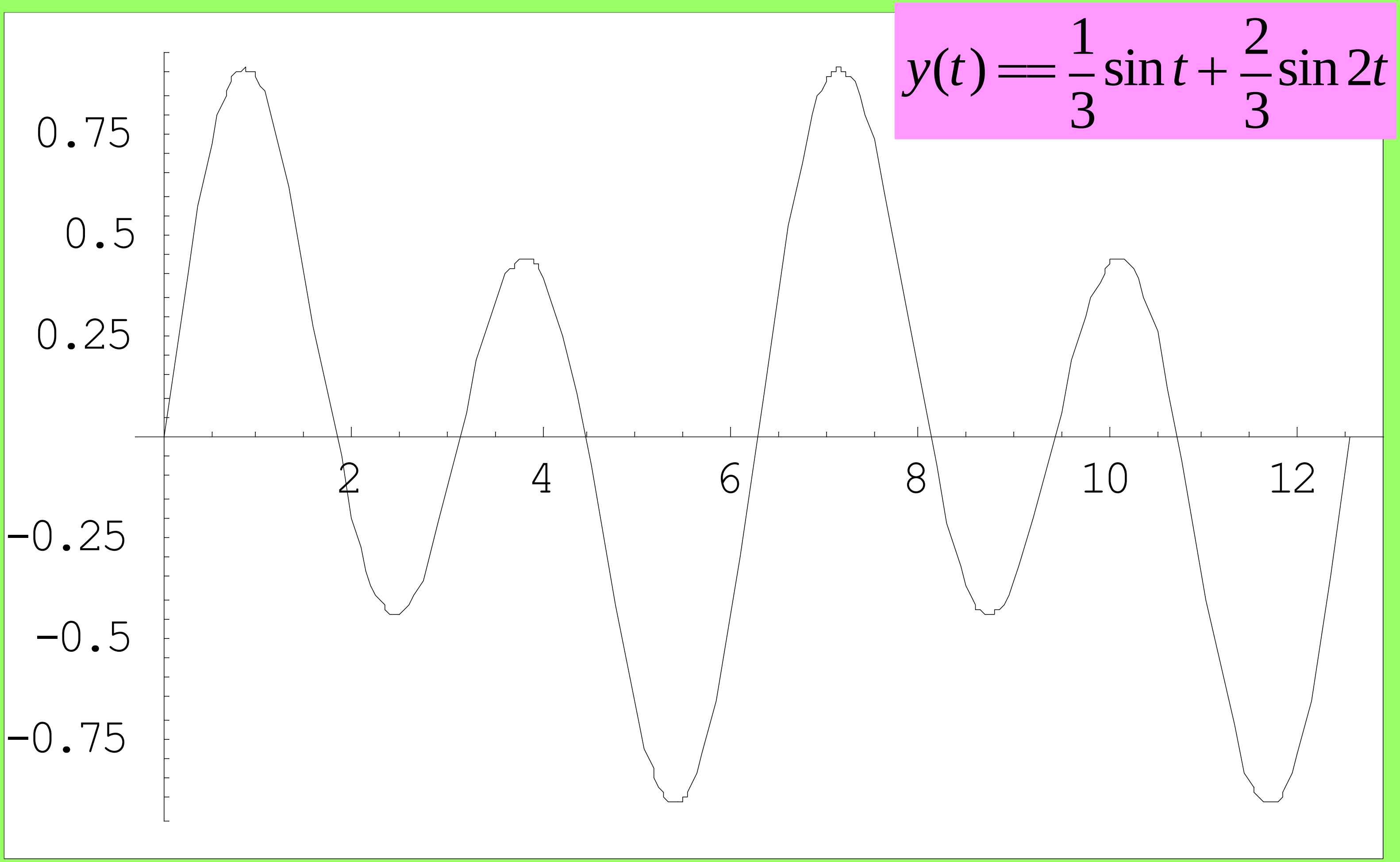
$$A=C=0 \qquad B=\frac{1}{3} \qquad D=\frac{2}{3}$$

$$Y(s)=\frac{1}{3(s^2+1)}+\frac{2}{3(s^2+4)}=\frac{1}{3}L\{\sin t\}+\frac{2}{3}L\{\sin 2t\}$$

- Hence the solution is $$y(t)==\frac{1}{3}\sin t+\frac{2}{3}\sin 2t$$

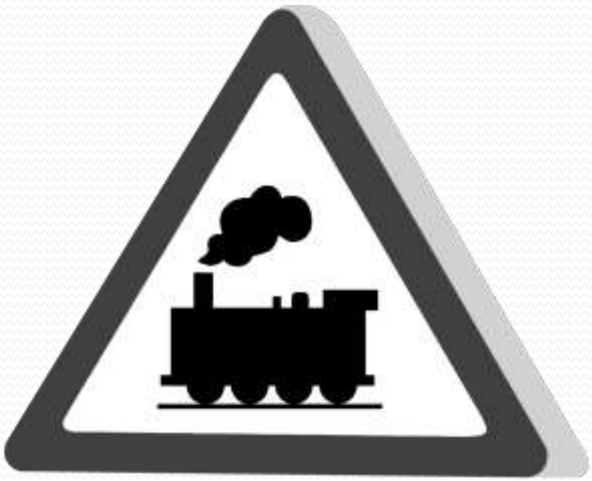
p4=Plot[((1/3)Sin[t]+(2/3)Sin[2t]),{t,0,4*Pi}] by Mathematica

$$y(t) = \frac{1}{3}\sin t + \frac{2}{3}\sin 2t$$



Real-Life Applications

- Semiconductor mobility
- Call completion in wireless networks
- Vehicle vibrations on compressed rails
- Behavior of magnetic and electric fields above the atmosphere



Diffusion Equation

$u_t = ku_{xx}$ in $(0,l)$

Initial Conditions:

$u(0,t) = u(l,t) = 1, \quad u(x,0) = 1 + \sin(\pi x/l)$

Using $\quad af(t) + bg(t) \rightarrow aF(s) + bG(s)$

and $\quad df/dt \rightarrow sF(s) - f(0)$

and noting that the partials with respect to x commute with the transforms with respect to t, the Laplace transform $U(x,s)$ satisfies

$sU(x,s) - u(x,0) = kU_{xx}(x,s)$

With $e^{at} \rightarrow 1/(s-a)$ and $a=0$,

the boundary conditions become $U(0,s) = U(l,s) = 1/s$.

So we have an ODE in the variable x together with some boundary conditions. The solution is then:

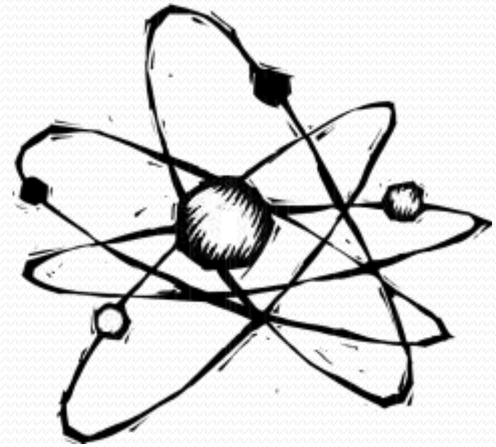
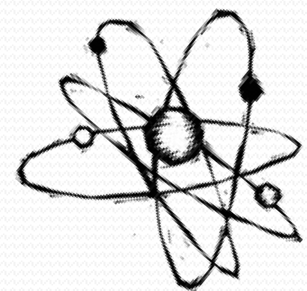
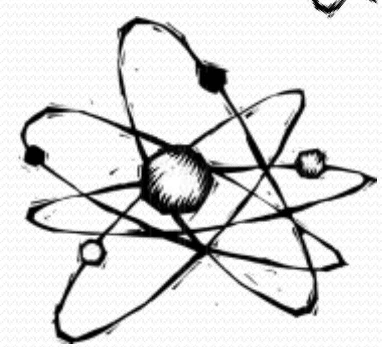
$U(x,s) = 1/s + (1/(s+k\pi^2/l^2))\sin(\pi x/l)$

Therefore, when we invert the transform, using the Laplace table:

$u(x,t) = 1 + e^{-k\pi^2 t/l^2}\sin(\pi x/l)$

Ex. Semiconductor Mobility

- Motivation
  - semiconductors are commonly made with superlattices having layers of differing compositions
  - need to determine properties of carriers in each layer
    - concentration of electrons and holes
    - mobility of electrons and holes
  - conductivity tensor can be related to Laplace transform of electron and hole densities

Real world Applications of Laplace Transform

- A simple Laplace Transform is conducted while sending signals over any two-way communication medium (FM/AM stereo, 2-way radio sets, cellular phones). When information is sent over medium such as cellular phones, they are first converted into time-varying wave, and then it is super-imposed on the medium. In this way, the information propagates. Now, at the receiving end, to decipher the information being sent, medium wave's time functions are converted to frequency functions.

Engineering Applications of Laplace Transform

- System Modeling
- Analysis of Electrical Circuits
- Analysis of Electronic Circuits
- Digital Signal Processing
- Nuclear Physics

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(AUTONOMOUS)

Transform Techniques and Complex Variables

Unit-2 : Fourier Series

Subject Code: 19ABS9912

Branch : ECE

By
Dr. K. Bhagya Lakshmi

FOURIER SERIES

Baron Jean Baptiste Joseph Fourier (1768−1830) introduced the idea that any periodic function can be represented by a series of Sines and cosines which are harmonically related.

Fourier series is an infinite series representation of a periodic function in terms of Sines and Cosines. Fourier series is useful to solve Ordinary and Partial differential equations particularly with periodic functions appearing as non-homogeneous terms.

Suppose that a given function *f(x)* defined by $[-\pi,\pi]$ or $[0,2\pi]$ or any other interval can be expressed as a trigonometric series as

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n Cosnx+b_n Sinnx)} \qquad \text{---------} \ (1)$$

Such series is known as the Fourier series for *f(x)* and the constants a_0, a_n & b_n ; $(n=1,2,3,.....)$ are called the Fourier Coefficients of *f(x)*.

Periodic Function

A function *f(x)* is said to be of period T or to be periodic with period T>0 if for all real *x*, *f(x+T)=f(T)* and T is the least of such values. (a function returning to the same value at regular intervals)

Example: Since $Sinx = Sin(x+\pi) = Sin(x+2\pi) = Sin(x+4\pi) = ----$ the function $Sinx$ is periodic with period 2π.

In a similar manner the period of *Cosx* is 2π

The period of *tanx* is π.

Euler's Formulae

The Fourier series for the function *f(x)* in the interval $C \le x \le C+2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n Cosnx + b_n Sinnx)}$$

$$\left.\begin{aligned} \text{where } a_0 &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)Cosnxdx \\ \& b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)Sinnxdx \end{aligned}\right\} \text{-------------------------------- (A)}$$

These values of $a_0, a_n \,\&\, b_n$ are called the Euler's formulae.

NOTE

- If *f(x)* is to be expanded as a Fourier series in the interval $0 \le x \le 2\pi$. Put C=0, then the formulae (A) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)Cosnxdx$$

$$\& b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)Sinnxdx$$

- If *f(x)* is to be expanded as a Fourier series in the interval $-\pi \le x \le \pi$. Put $C = -\pi$, then the formulae (A) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cosnxdx$$

$$\& b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinnxdx$$

SOLVED PROBLEMS

1. Expand $f(x)=x^2, 0<x<2\pi$ as a Fourier series.

Solution: Let $f(x)=x^2, 0<x<2\pi$

$$i.e, x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n Cosnx+b_n Sinnx) \text{---------}(1)$$

$$\text{Then } a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\int_0^{2\pi} x^2dx=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{8\pi^3-0}{3}\right]=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)Cosnxdx=\frac{1}{\pi}\int_0^{2\pi} x^2Cosnxdx=\frac{1}{\pi}\left[x^2\left(\frac{Sinnx}{n}\right)-2x\left(\frac{-Cosnx}{n^2}\right)+2\left(\frac{-Sinnx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[0+\left(\frac{4\pi Cos2n\pi}{n^2}\right)-0\right]=\frac{4}{n^2}\quad (\because Cos2n\pi=1)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)Sinnxdx=\frac{1}{\pi}\int_0^{2\pi} x^2Sinnxdx=\frac{1}{\pi}\left[x^2\left(\frac{-Cosnx}{n}\right)-2x\left(\frac{-Sinnx}{n^2}\right)+2\left(\frac{Cosnx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-\left(\frac{4\pi^2Cos2n\pi}{n}\right)+0+\left(\frac{2Cos2n\pi}{n^3}\right)-\left(0+0+\frac{2}{n^3}\right)\right]$$

$$=\frac{1}{\pi}\left[-\left(\frac{4\pi^2}{n}\right)+\left(\frac{2}{n^3}\right)-\left(\frac{2}{n^3}\right)\right]=\frac{-4\pi}{n}$$

Substituting the values of a_0, a_n & b_n in (1), we get the required Fourier series for *f(x)* as

$$x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}Cosnx + \frac{4\pi}{n}Sinnx\right)$$

$$= \frac{4}{3}\pi^2 + 4\left(Cosx + \frac{1}{2^2}Cos2x + \frac{1}{3^2}Cos3x + ----\right) - 4\pi\left(Sinx + \frac{1}{2}Sin2x + \frac{1}{3}Sin3x + ----\right)$$

2. Expand $f(x) = x, 0 < x < 2\pi$ as a Fourier series.

Solution: Let $f(x) = x, 0 < x < 2\pi$

$$i.e, x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cosnx + b_n Sinnx\right) \text{---------}(1)$$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$

$$= \frac{1}{\pi}\left[\frac{4\pi^2 - 0}{2}\right]$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)Cosnxdx = \frac{1}{\pi}\int_0^{2\pi} xCosnxdx = \frac{1}{\pi}\left[x\left(\frac{Sinnx}{n}\right) - \left(\frac{-Cosnx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \left(\frac{Cos2n\pi}{n^2} - \frac{1}{n^2}\right)\right] = \left(\frac{1}{n^2} - \frac{1}{n^2}\right) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)Sinnxdx = \frac{1}{\pi}\int_0^{2\pi} xSinnxdx = \frac{1}{\pi}\left[x\left(\frac{-Cosnx}{n}\right) - 1\left(\frac{-Sinnx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\left(\frac{2\pi Cos2n\pi}{n}\right) - (0+0)\right] = \frac{-2}{n}$$

Substituting the values of a_0, a_n & b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore x = \pi - 2\sum_{n=1}^{\infty}\left(\frac{1}{n}Sinnx\right) = \pi - 2\left(Sinx + \frac{1}{2}Sin2x + \frac{1}{3}Sin3x + ----\right)$$

3. Obtain the Fourier Series for the function $f(x) = e^x$ from $x = 0$ to $x = 2\pi$

Solution: Let $f(x) = e^x, 0 < x < 2\pi$

$$i.e, e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_nCosnx + b_nSinnx) ---------(1)$$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^x dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - 1\right]$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)Cosnxdx = \frac{1}{\pi}\int_0^{2\pi} e^xCosnxdx \ (\because \int e^{ax}Cosbxdx = \frac{e^{ax}}{a^2+b^2}(aCosbx + bSinbx)$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(Cosnx + nSinnx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}-1}{(1+n^2)}\right] \quad (\because Cos2n\pi = 1, Sin2n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) Sinnxdx = \frac{1}{\pi}\int_0^{2\pi} e^x Sinnxdx \ (\because \int e^{ax} Sinbxdx = \frac{e^{ax}}{a^2+b^2}(aSinbx - bCosbx)$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(Sinnx - nCosnx)\right]_0^{2\pi} = \frac{-n}{\pi}\left[\frac{e^{2\pi}-1}{(1+n^2)}\right]$$

Substituting the values of $a_0, a_n \ \& \ b_n$ in (1), we get the required Fourier series for *f(x)* as

$$\therefore e^x = \frac{1}{2\pi}\left(e^{2\pi}-1\right) + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}Cosnx + \sum_{n=1}^{\infty}\frac{(-n)\left(e^{2\pi}-1\right)}{\pi(1+n^2)}Sinnx$$

$$= \frac{1}{\pi}\left(e^{2\pi}-1\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{Cosnx}{(1+n^2)} - \sum_{n=1}^{\infty}\frac{Sinnx}{(1+n^2)}\right]$$

4. Expand $f(x) = xSinx, 0 < x < 2\pi$ as a Fourier series.

Solution: Let $f(x) = xSinx, 0 < x < 2\pi$

$$i.e, xSinx = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cosnx + b_n Sinnx\right) \text{---------}(1)$$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xSinxdx = \frac{1}{\pi}\left[x(-Cosx) - 1(-Sinx)\right]_0^{2\pi} = \frac{1}{\pi}\left[-2\pi - 0 - 0\right] = -2$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) Cosnxdx = \frac{1}{\pi}\int_0^{2\pi} xSinxCosnxdx = \frac{1}{2\pi}\int_0^{2\pi} x(2SinxCosnx)dx \quad ----(2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$=\frac{1}{2\pi}\left[x\left\{-\frac{Cos(n+1)x}{n+1}+\frac{Cos(n-1)x}{n-1}\right\}-1.\left\{-\frac{Sin(n+1)x}{(n+1)^2}+\frac{Sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi},(n\neq 1)$$

$$=\frac{1}{2\pi}\left[2\pi\left\{-\frac{Cos\,2(n+1)\pi}{n+1}+\frac{Cos\,2(n-1)\pi}{n-1}\right\}\right]=-\frac{1}{n+1}+\frac{1}{n-1}=\frac{2}{n^2-1},(n\neq 1)$$

If n=1, we have $\left[\text{Putting } n=1 \text{ in } (2)\right]$

$$a_1=\frac{1}{\pi}\int_0^{2\pi}x\left(Sin2x\right)dx=\frac{1}{2\pi}\left[x\left(\frac{-Cos2x}{2}\right)-1\left(\frac{-Sin2x}{4}\right)\right]_0^{2\pi}=\frac{1}{2\pi}\left[-\pi\right]=\frac{-1}{2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)Sinnxdx=\frac{1}{\pi}\int_0^{2\pi}xSinxSinnxdx=\frac{1}{2\pi}\int_0^{2\pi}x(2SinxSinnx)dx---------(3)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}x\left[Cos(n-1)x-Cos(n+1)x\right]dx$$

$$=\frac{1}{2\pi}\left[x\left\{\frac{Sin(n-1)x}{n-1}-\frac{Sin(n+1)x}{n+1}\right\}-1.\left\{-\frac{Cos(n-1)x}{(n-1)^2}+\frac{Cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi},(n\neq 1)$$

$$=\frac{1}{2\pi}\left[\left\{\frac{Cos\,2(n-1)\pi}{(n-1)^2}-\frac{Cos\,2(n+1)\pi}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right\}\right],(n\neq 1)$$

$$=\frac{1}{2\pi}\left[\left\{\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right\}\right]=0\ \text{ for } n\neq 1$$

If n=1, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\left(2Sin^2 x\right)dx \quad \left[\text{Putting } n = 1 \text{ in } (3)\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(1 - Cos2x)dx = \frac{1}{2\pi}\left[x\left(x - \frac{Sin2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{Cos2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi$$

Substituting the values of $a_0, a_n \ \& \ b_n$ in (1), we get the required Fourier series for *f(x)* as

$$\therefore\ xSinx = -1 - \frac{1}{2}Cosx + \sum_{n=2}^{\infty}\left(\frac{2}{n^2 - 1}Cosnx + \pi Sinx\right) = -1 + \pi Sinx - \frac{1}{2}Cosx + 2\sum_{n=2}^{\infty}\frac{Cosnx}{n^2 - 1}$$

This is the required Fourier series

EVEN AND ODD FUNCTIONS

A function *f(x)* is said to be even function if *f(-x)=f(x)* and odd function if *f(-x)=-f(x)*

Example:

$x^2, x^4+x^2+1, e^x+e^{-x}, Cosx, Secx$ are all even functions of *x*,

$x, x^3+2x^5+3, Sinx, Co\sec x, \tan x$ are odd functions of *x*

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

Case I. When *f(x)* is an even function in $(-\pi,\pi)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n Cosnx$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)Cosnxdx, n=0,1,2,.....$$

Case II. When *f(x)* is an odd function in $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n Sinnx$$

$$\text{Where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) Sinnx dx, n = 0, 1, 2, ...$$

SOLVED PROBLEMS

1. Express $f(x)=x$ as a Fourier series in $(-\pi,\pi)$.

Solution: Since $f(-x)=-x=-f(x)$

$\therefore f(x)$ is an odd function in $(-\pi,\pi)$.

Hence the Fourier series consists of **Sine** terms only.

$$i.e,\therefore x=\sum_{n=1}^{\infty} b_n Sinnx \text{---------}(1)$$

$$\text{Where } b_n=\frac{2}{\pi}\int_0^{\pi} f(x)Sinnxdx, n=0,1,2,.....$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)Sinnxdx=\frac{2}{\pi}\int_0^{\pi} xSinnxdx=\frac{2}{\pi}\left[x\left(\frac{-Cosnx}{n}\right)-1\left(\frac{-Sinnx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{-\pi Cosn\pi}{n}+0\right)-(0+0)\right]=(-1)^{n+1}\frac{2}{n}$$

Substituting the values of b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore x=2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}Sinnx\right)=2\left(Sinx-\frac{1}{2}Sin2x+\frac{1}{3}Sin3x-\frac{1}{4}Sin4x+---\right)$$

2. Expand the function $f(x)=x^2$ as a Fourier series in $(-\pi,\pi)$.

(OR) Prove that $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{Cosnx}{n^2}$

Hence deduce that $(i)\ \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$

$(ii)\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+----=\frac{\pi^2}{6}$

$(iii)\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+----=\frac{\pi^2}{8}$

Solution: Since $f(-x)=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is an even function in $(-\pi,\pi)$. Hence the Fourier series consists of **Cosine** terms only

$$i.e, x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nCosnx\text{---------}(1)$$

Then $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\pi^3-0}{3}\right]=\frac{2\pi^2}{3}$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)Cosnxdx=\frac{2}{\pi}\int_0^{\pi}x^2Cosnxdx=\frac{2}{\pi}\left[x^2\left(\frac{Sinnx}{n}\right)-2x\left(\frac{-Cosnx}{n^2}\right)+2\left(\frac{-Sinnx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[0+\left(\frac{2\pi Cosn\pi}{n^2}\right)-0\right]=\frac{4}{n^2}(-1)^n\ (\because Cosn\pi=(-1)^n)$$

Substituting the values of a_0 & a_n in (1), we get the required Fourier series for *f(x)* as

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n Cosnx = \frac{\pi^2}{3} - 4\left(Cosx - \frac{1}{2^2}Cos2x + \frac{1}{3^2}Cos3x + ----- \right) ....(2)$$

Deductions:

(i) Putting $x = 0$ in (2)

$$0 = \frac{\pi^2}{3} - 4\left(Cos0 - \frac{1}{2^2}Cos0 + \frac{1}{3^2}Cos0 - \frac{1}{4^2}Cos0 ---- \right)$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

(ii) Putting $x = \pi$ in (2)

$$\pi^2 = \frac{\pi^2}{3} - 4\left(Cos\pi - \frac{1}{2^2}Cos2\pi + \frac{1}{3^2}Cos3\pi + \frac{1}{4^2}Cos4\pi ---- \right)$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ----- \right)$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ----- \right)$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ----- = \frac{\pi^2}{6}$$

(iii) Adding (i) and (ii) and dividing it by 2, we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ----- = \frac{\pi^2}{8}$$

3. Find the Fourier series to represent the function $f(x)=|Sinx|, -\pi<x<\pi$ as a Fourier series

Solution: Since $|Sinx|$ is an even function in $(-\pi,\pi)$.

Hence the Fourier series consists of **Cosine** terms only

$$i.e, |Sinx| = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n Cosnx -----(1)\text{---------}(1)$$

$$\text{Then } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} Sinxdx = \frac{2}{\pi}\left[(-Cosx)\right]_0^{\pi} = \frac{2}{\pi}\left[-Cos(\pi)+Cos0\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)Cosnxdx$$

$$= \frac{1}{\pi}\int_0^{\pi}(2SinxCosnx)dx \qquad \left[\because 2SinAxCosBx = Sin(A+B)x - Sin(A-B)x\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x+\sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{Cos(1+n)x}{1+n}-\frac{Cos(1-n)x}{1-n}\right]_0^{\pi}, (n\neq 1)$$

$$= \frac{-1}{\pi}\left[\frac{Cos(1+n)\pi}{1+n}+\frac{Cos(1-n)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right], (n\neq 1)$$

$$= \frac{-1}{\pi}\left\{(-1)^{n+1}\left[\frac{1}{1+n}+\frac{1}{1-n}\right]-\left[\frac{1}{1+n}+\frac{1}{1-n}\right]\right\} = \frac{-2\left[(-1)^{n+1}+1\right]}{\pi\left(n^2-1\right)}, (n\neq 1)$$

$$\therefore a_n = \begin{cases} 0, \text{if } n \text{ is odd and } n \neq 1 \\ \dfrac{-4}{\pi(n^2-1)}, \text{if } n \text{ is even} \end{cases}$$

If n=1, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} Sinx Cosx dx = \frac{1}{\pi}\int_0^{\pi} Sin2x dx = \frac{1}{\pi}\left(\frac{-Cos2x}{2}\right)_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[Cos2\pi - 1\right] = 0$$

Substituting the values of   in (1), we get the required Fourier series for f(x) as

$$\therefore \left|Sinx\right| = \frac{2}{\pi} + \sum_{n=2,4,6,....}^{\infty}\left(\frac{-4}{\pi(n^2-1)} Cosnx\right)$$

$$\left|Sinx\right| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,....}^{\infty}\left(\frac{Cosnx}{(n^2-1)}\right)$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{Cos2nx}{(4n^2-1)}\right) \text{ (Replacing } n \text{ by } 2n\text{)}$$

Hence, $\left|Sinx\right| = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{Cos2x}{3} + \dfrac{Cos4x}{15} + ----\right]$

HALF RANGE FOURIER SERIES

It is often required to obtain Fourier series of a function *f(x)* in the interval $(0,\pi)$

The Sine Series:

The half range Sine series in $(0,\pi)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n Sinnx$$

$$\text{Where } b_n=\frac{2}{\pi}\int_0^{\pi} f(x) Sinnxdx, n=0,1,2,.....$$

The Cosine Series:

The half range Cosine series in $(0,\pi)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n Cosnx$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$ and

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)Cosnxdx, n=0,1,2,.....$$

1. Find the half-range cosine and sine series for the function $f(x)=x$ in the range $0<x<\pi$.

(OR)

Prove that the function $f(x)=x$ can be expanded in a series of cosines in $0\le x\le\pi$

as $x=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{Cosx}{1^2}+\frac{Cos3x}{3^2}+\frac{Cos5x}{5^2}+-----\right]$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+----=\frac{\pi^2}{8}$.

Solution:

The Cosine Series: The half range cosine series expansion of *f(x)* in $[0,\pi]$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nCosnx \quad \text{------------------(1)}$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$ and

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)Cosnxdx, n=0,1,2,.....$

Then $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}xdx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)Cosnxdx = \frac{2}{\pi}\int_0^\pi xCosnxdx = \frac{2}{\pi}\left[x\left(\frac{Sinnx}{n}\right) - \left(\frac{-Cosnx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 + \left(\frac{Cosn\pi}{n^2} - \frac{1}{n^2}\right)\right] = \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$\therefore a_n = \begin{cases} 0, \text{ for } n \text{ even} \\ \dfrac{-4}{\pi n^2}, \text{ for } n \text{ odd} \end{cases}$$

Substituting the values of a_0 & a_n in (1), we get the required Fourier series for *f(x)* as

$$i.e., \therefore x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,.....}^{\infty}\frac{1}{n^2}Cosnx \text{---------}(1)$$

$$\text{(or) } x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{Cosx}{1^2} + \frac{Cos3x}{3^2} + \frac{Cos5x}{5^2} + ----\right] \text{-------------}(2)$$

Deduction:

When *x=0, f(x)=0 i.e., f(0)=0*

Putting *x=0* in (2), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ----\right] \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ---- = \frac{\pi^2}{8}$$

The Sine Series:

$$f(x) = x = \sum_{n=1}^{\infty} b_n Sinnx \quad ----(3)$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) Sinnxdx, n = 0,1,2,.....$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) Sinnxdx = \frac{2}{\pi}\int_0^{\pi} xSinnxdx = \frac{2}{\pi}\left[x\left(\frac{-Cosnx}{n}\right) - 1\left(\frac{-Sinnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi Cosn\pi}{n} + 0\right) - (0+0)\right] = (-1)^{n+1}\frac{2}{n}$$

Substituting the values of b_n in (3), we get the required Fourier series for *f(x)* as

$$\therefore x = 2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n} Sinnx\right) = 2\left(Sinx - \frac{1}{2}Sin2x + \frac{1}{3}Sin3x - \frac{1}{4}Sin4x + ---\right)$$

2. Obtain the half-range sine series for the function $f(x)=e^x$ in $(0,\pi)$.

Solution: The half range sine series expansion of e^x in $(0,\pi)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n Sinnx,$$

Where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)Sinnxdx, n=0,1,2,.....$

$$i.e,\ e^x=\sum_{n=1}^{\infty} b_n Sinnx \quad \text{-------}(1)$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)Sinnxdx=\frac{2}{\pi}\int_0^{\pi} e^x Sinnxdx \ (\because \int e^{ax}Sinbxdx=\frac{e^{ax}}{a^2+b^2}\left(aSinbx-bCosbx\right)$$

$$=\frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(0-nCosn\pi\right)-\frac{1}{1+n^2}(0-n)\right]_0^{\pi}$$

$$=\frac{2n}{\pi(1+n^2)}\left[1+(-1)^{n+1}e^{\pi}\right]$$

Substituting the values of b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore e^x=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\left[1+(-1)^{n+1}e^{\pi}\right]}{(1+n^2)}Sinnx=\frac{2}{\pi}\left[\frac{(1+e^{\pi})}{1^2+1}Sinx+\frac{2(1-e^{\pi})}{2^2+1}Sin2x+\frac{3(1+e^{\pi})}{3^2+1}Sin3x+---\right]$$

3. Obtain the half-range sine series for the function $f(x) = Cosx$ in $(0, \pi)$.

Solution: The half range sine series expansion of $Cosx$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n Sinnx \ \ i.e, \ Cosx = \sum_{n=1}^{\infty} b_n Sinnx \quad \text{-------} \ (1)$$

Where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x) Sinnxdx, n = 0,1,2,.....$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) Sinnxdx = \frac{2}{\pi}\int_0^{\pi} CosxSinnxdx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2SinnxCosxdx$$

$$= \frac{1}{\pi}\left[\frac{-Cos(n+1)x}{n+1} - \frac{Cos(n-1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-Cos(n+1)\pi}{n+1} - \frac{Cos(n-1)\pi}{n+1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^n}{n+1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-1)^2}{n+1} + \frac{(-1)^n}{n+1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\left\{(-1)^n + 1\right\}\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right] = \frac{2n}{\pi}\left[\frac{(-1)^n + 1}{n^2 - 1}\right]$$

$$\therefore b_n = \begin{cases} 0, \text{when } n \text{ is odd} \\ \dfrac{4n}{\pi(n^2 - 1)}, \text{when } n \text{ is even} \end{cases}$$

If n=1, then

$$b_1 = \frac{2}{\pi}\int_0^{\pi} Cosx Sinx dx = \frac{1}{\pi}\int_0^{\pi} 2Cosx Sinx dx = \frac{1}{\pi}\int_0^{\pi} Sin2x dx$$

$$= \frac{1}{\pi}\left[\frac{-Cos2x}{2}\right]_0^{\pi} = 0$$

$$\therefore b_n = \begin{cases} 0 \text{ when } n \text{ is odd} \\ \dfrac{4n}{\pi(n^2 - 1)} \text{ when } n \text{ is even} \end{cases}$$

Substituting the values of b's in (1), we get

$$\therefore Cosx = \sum_{n=2,4,6,....}^{\infty} \frac{4n}{\pi(n^2-1)} Sinnx = \frac{4}{\pi}\sum_{n=2,4,6,....}^{\infty} \frac{n}{(n^2-1)} Sinnx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{2n}{(4n^2-1)} Sin2nx \quad (\because n \text{ is even, replace by } 2n)$$

$$= \frac{8}{\pi}\left(\frac{1}{3}Sin2x + \frac{2}{15}Sin4x + ----\right)$$

FOURIER SERIES IN AN ARBITRARY INTERVAL (CHANGE OF INTERVAL)

INTERVALS OTHER THAN $(-\pi,\pi)$ AND $(0,2\pi)$

(FOURIER SERIES FOR FUNCTIONS HAVING PERIOD 2ℓ):

The Fourier series for the function *f(x)* in the interval $C \le x \le C+2l$ is given by

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l}+b_n Sin\frac{n\pi x}{l}\right)}$$

$$\left.\begin{aligned}
\text{where } a_0 &= \frac{1}{l}\int_c^{c+2l} f(x)dx \\
a_n &= \frac{1}{l}\int_c^{c+2l} f(x)Cos\frac{n\pi x}{l}dx \\
\& b_n &= \frac{1}{l}\int_c^{c+2l} f(x)Sin\frac{n\pi x}{l}dx
\end{aligned}\right\} \text{-------------------------------- (A)}$$

NOTE

- If *f(x)* is to be expanded as a Fourier series in the interval $0 \le x \le 2l$. Put C=0, then the formulae (A) reduces to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)Cos\frac{n\pi x}{l}dx$$

$$\& b_n = \frac{1}{l}\int_0^{2l} f(x)Sin\frac{n\pi x}{l}dx$$

- If *f(x)* is to be expanded as a Fourier series in the interval $-l \le x \le l$. Put $C = -l$, then the formulae (A) reduces to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)Cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)Sin\frac{n\pi x}{l}dx$$

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that a function *f(x)* defined in the interval $(-l, l)$ can be represented by the Fourier series.

Case I. When *f(x)* is an even function in $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos\frac{n\pi x}{l}$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx$

$$a_n = \frac{2}{l}\int_0^l f(x)Cos\frac{n\pi x}{l}dx, n = 0, 1, 2, \ldots.$$

Case II. When *f(x)* is an odd function in $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n Sin \frac{n\pi x}{l}$$

$$\text{Where } b_n = \frac{2}{l}\int_0^{\pi} f(x) Sin \frac{n\pi x}{l} dx, n = 0, 1, 2, ...$$

1. Find the Fourier series to represent $f(x)=x^2-2$, when $-2\leq x\leq 2$.

Solution: Since $f(-x)=(-x)^2-2=x^2-2=f(x)$

$\therefore f(x)$ is an even function in $(-2,2)$.

Hence the Fourier series consists of **Cosine** terms only

$$i.e, x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n Cos\frac{n\pi x}{2} \text{---------}(1) \text{ Then } a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{2}\int_0^2(x^2-2)dx=\left[\frac{x^3}{3}-2x\right]_0^2=\left[\frac{2^3}{3}-4\right]=\frac{-4}{3}$$

$$a_n=\frac{2}{l}\int_0^l f(x)Cos\frac{n\pi x}{l}dx=\frac{2}{2}\int_0^2(x^2-2)Cos\frac{n\pi x}{2}dx$$

$$=\left[(x^2-2)\left(\frac{Sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)-2x\left(\frac{-Cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)+2\left(\frac{-Sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3}\right)\right]_0^2$$

$$=\left[0+\left(\frac{16Cosn\pi}{n^2\pi^2}\right)-0\right]=\frac{16}{n^2\pi^2}(-1)^n \quad (\because Cosn\pi=(-1)^n)$$

Substituting the values of a_0 & a_n in (1), we get the required Fourier series for *f(x)* as

$$x^2=-\frac{2}{3}+\sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n Cos\frac{n\pi x}{2}=-\frac{2}{3}-\frac{16}{\pi^2}\left(\frac{Cos(\pi x)}{1^2}-\frac{Cos(2\pi x)}{2^2}+\frac{Cos(3\pi x)}{3^2}+----\right)$$

2. Find the Fourier series with period 3 to represent $f(x) = x + x^2$ in $(0,3)$

Solution:Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$ ------(1)

Here $2l = 3 \Rightarrow l = 3/2$

$i.e, x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{2n\pi x}{3} + b_n Sin\frac{2n\pi x}{3}\right)$---------(2)

Then $a_0 = \frac{2}{3}\int_0^{2l} f(x)dx = \frac{2}{3}\int_0^3 (x + x^2)dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)Cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (x + x^2)Cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(x + x^2)\left(\frac{Sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (1 + 2x)\left(\frac{-Cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right) + 2\left(\frac{-Sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) Sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{3}\int_0^{3}(x+x^2) Sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(x+x^2)\left(\frac{-Cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(1+2x)\left(\frac{-Sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)+2\left(\frac{Cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{-12}{n\pi}$$

Substituting the values of in (2), we get the required Fourier series for $f(x)$ as

$$x+x^2 = \frac{9}{2}+\sum_{n=1}^{\infty}\left\{\frac{9}{n^2\pi^2}Cos\left(\frac{2n\pi x}{3}\right)-\frac{12}{n\pi}Sin\left(\frac{2n\pi x}{3}\right)\right\}$$

3. Find the Fourier series to represent the function $f(x)=|x|, -2<x<2$ as a Fourier series

Solution: Since $|x|$ is an even function in $(-2,2)$.

Hence the Fourier series consists of **Cosine** terms only

$$i.e, |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos \frac{n\pi x}{2} \text{ ---------(1)}$$

$$\text{Then } a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \left[\frac{2^2}{2}\right] = 2 \quad (\because l = 2)$$

$$a_n = \frac{2}{l}\int_0^l f(x) Cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{2}\int_0^2 x Cos \frac{n\pi x}{2} dx$$

$$= \left[(x)\left(\frac{Sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - \left(\frac{-Cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) \right]_0^2$$

$$= \left[0 + \left(\frac{4}{n^2\pi^2}(Cos n\pi - 1) \right) \right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 & \text{when } n \text{ is even} \\ \dfrac{-8}{n^2\pi^2} & \text{when } n \text{ is odd} \end{cases}$$

Substituting the values of a_0 & a_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore |x| = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,....}^{\infty} \left(\frac{1}{n^2} Cos \frac{n\pi x}{2} \right)$$

$$|x| = 1 - \frac{8}{\pi^2} \left[Cos \frac{\pi x}{2} + \frac{1}{3^2} Cos \frac{3\pi x}{2} + ----- \right]$$

HALF RANGE FOURIER SERIES

It is often required to obtain Fourier series of a function *f(x)* in the interval $(0, l)$

The Sine Series:

The half range Sine series in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n Sin \frac{n\pi x}{l}$$

Where $b_n = \frac{2}{l} \int_0^l f(x) Sin \frac{n\pi x}{l} dx, n = 0, 1, 2, ...$

The Cosine Series:

The half range Cosine series in $(0,l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos \frac{n\pi x}{l}$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx$ and

$$a_n = \frac{2}{l}\int_0^l f(x) Cos \frac{n\pi x}{l} dx, n = 0,1,2,.....$$

SOLVED PROBLEMS

1. Obtain the half-range sine series for the function $f(x)=1$ in $[0,l]$.

Solution: The half range sine series expansion of $f(x)=1$ in $(0,l)$ is given by

$$f(x)=1=\sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{l}, \text{Where } b_n=\frac{2}{l}\int_0^l f(x)Sin\frac{n\pi x}{l}dx, n=0,1,2,..... ------(1)$$

$$i.e,\ 1=\sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{l} \quad -------(1)$$

$$b_n=\frac{2}{l}\int_0^l f(x)Sin\frac{n\pi x}{l}dx=\frac{2}{l}\int_0^l 1\cdot Sin\frac{n\pi x}{l}dx=\frac{2}{l}\left[\frac{-Cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l$$

$$=\frac{2}{n\pi}\left[-Cosn\pi+1\right]=\frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$$b_n=\begin{cases}0 & \text{when } n \text{ is even}\\ \dfrac{4}{n\pi} & \text{when } n \text{ is odd}\end{cases}$$

Substituting the values of b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore 1 = \frac{4}{\pi} \sum_{n=1,3,5,....}^{\infty} \frac{Sin \frac{n\pi x}{l}}{\pi} = \frac{4}{\pi}\left[\frac{Sin \frac{\pi x}{l}}{1} + \frac{Sin \frac{3\pi x}{l}}{3} + \frac{Sin \frac{5\pi x}{l}}{5} + ---\right]$$

2. Obtain the half-range sine series for the function $f(x) = ax + b$ in $0 < x < 1$.

Solution: The half range sine series expansion of $f(x) = ax + b$ in $(0,1)$ is given by

$$f(x) = ax + b = \sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{l},$$

Where $b_n = \frac{2}{l}\int_0^l f(x) Sin\frac{n\pi x}{l}dx, n = 0,1,2,.....$

$$i.e,\ ax + b = \sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{1} \quad \text{-------(1)}$$

$$b_n = \frac{2}{l}\int_0^l f(x) Sin\frac{n\pi x}{l}dx = \frac{2}{1}\int_0^1 (ax+b) Sin\frac{n\pi x}{1}dx = 2\left[(ax+b)\frac{-Cosn\pi}{n\pi} - a\frac{-Sinn\pi}{n\pi}\right]_0^1$$

$$= \frac{2}{n\pi}\left[(-1)(a+b)Cosn\pi + \frac{a}{n\pi}Sinn\pi + b\right]$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1}(a+b) + b\right]$$

Substituting the values of b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore ax + b = \frac{2}{\pi}(a+2b)Sin\pi x - \frac{2a}{2\pi}Sin2\pi x + \frac{2}{3\pi}(a+2b)Sin3\pi x ------$$

3. Find the half-range cosine series for the function $f(x) = x(2-x)$ in the range $0 \le x \le 2$ and hence find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - -$

Solution: The half range Cosine series expansion of $f(x) = x(2-x)$ in $[0,2]$

$$x(2-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos\frac{n\pi x}{2} \quad \text{--------(1)}$$

Then $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 (2x - x^2)dx = \frac{4}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)Cos\frac{n\pi x}{l}dx = \frac{2}{2}\int_0^2 (2x - x^2)Cos\frac{n\pi x}{2}dx$$

$$= \left[(2x - x^2)\left(\frac{Sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (2-2x)\left(\frac{-Cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right) + (2x)\left(\frac{Sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3} \right) \right]_0^{\pi}$$

$$= \frac{-8}{n^2\pi^2}Cosn\pi - \frac{8}{n^2\pi^2} = \frac{-8}{n^2\pi^2}\left(1 + (-1)^n\right)$$

$$\therefore a_n = \begin{cases} 0, \text{ for } n \text{ is odd} \\ \frac{-16}{n^2\pi^2}, \text{for } n \text{ is even} \end{cases}$$

Substituting the values of a_0 & a_n in (1), we get the required Fourier series for *f(x)* as

$$i.e, \therefore x(2-x) = \frac{2}{3} - \frac{16}{\pi^2} \sum_{n=2,4,6,.....}^{\infty} \frac{1}{n^2} Cos \frac{n\pi x}{2}$$

(or)

$$x(2-x) = \frac{2}{2} - \frac{4}{\pi^2}\left[Cos\pi x + \frac{Cos2\pi x}{2^2} + \frac{Cos3\pi x}{3^2} + ----\right] \text{-------------(2)}$$

Deduction:

Putting x=1 in (2), we get

$$2-1 = \frac{2}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} Cos\pi + \frac{1}{2^2} Cos2\pi + \frac{1}{3^2} Cos3\pi + ----\right]$$

$$\frac{1}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} Cos\pi + \frac{1}{2^2} Cos2\pi + \frac{1}{3^2} Cos3\pi + ----\right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + ---- = \frac{\pi^2}{12}$$

4. Obtain the half-range sine series for the function $f(x)=x^2$ in $[0,4]$.

Solution: The half range sine series expansion of $f(x)=x^2$ in $(0,4)$ is given by

Here $l=4$

$$f(x)=x^2=\sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{l} \text{ -----(1)}$$

Where $b_n=\frac{2}{l}\int_0^l f(x)Sin\frac{n\pi x}{l}dx$

$$b_n=\frac{2}{l}\int_0^l f(x)Sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{4}\int_0^4 x^2 Sin\frac{n\pi x}{4}dx=\frac{1}{2}\left[x^2\left(\frac{-Cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right)-2x\left(\frac{-Sin\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}\right)+2\left(\frac{Cos\frac{n\pi x}{4}}{\frac{n^3\pi^3}{64}}\right)\right]_0^4$$

$$=\frac{1}{2}\left[\frac{-4}{n\pi}16Cosn\pi+\frac{128}{n^2\pi^2}Sinn\pi+\frac{128}{n^3\pi^3}Cosn\pi-\frac{128}{n^3\pi^3}\right]$$

$$=32\left[\frac{2\left[(-1)^n-1\right]}{n^3\pi^3}+\frac{(-1)^{n+1}}{n\pi}\right]$$

Substituting the values of b_n in (1), we get the required Fourier series for *f(x)* as

$$\therefore x^2 = \sum_{n=1}^{\infty} 32\left[\frac{2\left[(-1)^n - 1\right]}{n^3\pi^3} + \frac{(-1)^{n+1}}{n\pi}\right] Sin\frac{n\pi x}{4}$$

THANK YOU