AK23 Regulations

Year : I **Semester** : I **Branch of Study :** Common to All

Subject Code:23ABS9904	Subject Name:Linear Algebra and Calculus	L	T/CLC	P	Credits
		4	2	0	3

Course Outcomes (CO): Student will be able to

1. Analyze the matrix algebraic techniques for engineering applications.

2. Understand the concept of Eigen values, Eigen vectors and quadratic forms.

3. Analyzethe mean value theorems for real timeapplications.

4. Apply the concepts of partial differentiation to functions of several variables.

5. Apply the multivariable integral calculus for computation of Area and Volume.

CO	Action Verb	Knowledge Statement	Condition	Criteria	Blooms level
1	Analyze	the matrix algebraic techniques	for engineering applications.		L4
2	Understand	the concept of eigen values, eigen vectors and quadratic forms.	-		L2
3	Analyze	the mean value theorems	for real time applications.		L4
4	Apply	the concept of Maxima and Minima	to functions of several variables.		L3
5	Apply	the multivariable integral calculus	for computation of Area and volume.		L3

Unit I: Matrices **12hrs**

Rank of a matrix by Echelon form, Normal form, Cauchy-Binet formula (without proof).Inverse of Non-singular matrices by Gauss-Jordan method,system oflinear equations:solving system of Homogeneous and Non-homogeneous equations by Gauss Elimination method, Jacobi and Gauss Seidel Iteration methods.

Unit II:Eigen values, Eigen vectors and Orthogonal Transformation **9hrs**

Eigen values, Eigen vectors and their properties,Diagonalization of a matrix,Cayley-Hamilton theorem (without proof),finding inverse and power of a matrix by Cayley-Hamilton theorem, Quadratic forms and Nature of the Quadratic forms, Reduction of quadratic form to canonical forms by Orthogonal Transformation.

Unit III: Calculus **9hrs**

Mean Value Theorems:Rolle’s theorem, Lagrange’s mean value theorem with their geometrical interpretation, Cauchy’s mean value theorem, Taylor’s and Maclaurin's theorems with remainders (without proof), problems and applications on the above theorems.

Unit IV: Partial differentiation and Applications(Multi Variable Calculus) **10hrs**

Functions of several variables: Continuity and Differentiability, Partial derivatives, total derivatives, chain rule, Directional derivative, Taylor’s and Maclaurin's series expansion of functions of two variables,Jacobians, Functional dependence, Maxima and Minima of functions of two variables, method of Lagrange multipliers.

Unit V: Multiple Integrals **10hrs**

Double integrals, triple integrals change of order of integration, change of Variables to polar, Cylindrical and Spherical coordinates, Finding areas(by double integrals) and volumes (bydouble integrals and triple integrals).

Textbooks:

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

References:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass,Pearson Publishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar,Alpha Science International Ltd., 25th Edition(9th reprint).
3. Advanced Modern Engineering Mathematics, Glyn James, Pearsonpublishers, 2018, 5 th Edition.
4. Advanced Engineering Mathematics, Micheael Greenberg, ,Pearsonpublishers, 9 th edition.
5. Higher Engineering Mathematics, H. K Das, Er. RajnishVerma, S. ChandPublications,2014, Third Edition (Reprint 2021)

Mapping of COs to POs

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
1		3										
2		2										
3		3										
4	3											
5	3											

(Levels of Correlation, viz., 1-Low, 2-Moderate, 3 High)

CO-PO mapping justification:

CO	Percentage of contact hours over the total planned contact hours			CO		Program Outcome (PO)	PO(s): Action verb and BTL (for PO1 to PO5)	Level of Correlation (0-3)
	Lesson Plan (Hrs)	%	correlation	Verb	BTL			
1	**10**	14	2	**Analyze**	**L4**	PO2	Analyze	3
2	**15**	21.4	3	**Understand**	**L2**	PO2	Apply	2
3	**15**	21.4	**3**	Analyze	**L4**	PO2	Analyze	3
4	**16**	22.8	**3**	Apply	**L3**	PO1	Apply	3
5	**14**	20	**3**	Apply	**L3**	PO1	Apply	3

CO1: Analyze the matrix algebraic techniques that are needed for engineering applications.

Action Verb: Analyze(L4)

PO2 Verbs: Analyze (L4)

CO1 Action Verb is equal toPO2verb ; Therefore correlation is high (3).

CO2: Understand the concept of eigen values, eigen vectors and quadratic forms.

Action Verb: Understand (L2)

PO1 Verbs: Apply (L3)

CO2 Action Verb is low level to PO1 verb by one level; Therefore correlation is moderate (2).

CO3: Analyze the mean value theorems for real life problems.

Action Verb: Analyze **(L4)**

PO1 Verb: Analyze (L4)

CO3 Action Verb level is equal to PO2 verb; Therefore correlation is high (3).

CO4: Apply the concept of Maxima and Minima of functions of several variables.

Action Verb: Apply (L3)

PO2 Verb: Apply (L3)

CO4 Action Verb level is equal to PO1 verb; Therefore correlation is high (3).

CO5: Apply the multivariable integral calculus for computation of area and volume.

Action Verb: Apply(L3)

PO1 Verb: Apply (L3)

CO5 Action verb is high level to PO1 verb; therefore the correlation is high (3).

UNIT-I

MATRICES

Real Matrices:

REAL MATRIX: A Matrix A= (a_{ij}) is said to be a real matrix if every element a_{ij} of A is a real number. Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 8 & 0 & -1 \\ 5 & -6 & 9 \end{bmatrix}$

SYMMETRIC MATRIX: A Matrix A= (a_{ij}) is said to be a symmetric matrix if $a_{ij} = a_{ji}$ for i and j .Thus A is symmetric matrix if $A=A^T$.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 9 \\ 3 & 9 & 7 \end{bmatrix}$

SKEW-SYMMETRIC MATRIX: A real Matrix A= (a_{ij}) is said to be a skew-symmetric matrix if $a_{ij}= -a_{ji}$ for every i and j. Thus A is skew-symmetric if $A=-A^T$.

Note: Every diagonal element of a skew-symmetric matrix is necessarily zero.

Ex: $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$

ORTHOGONAL MATRIX: A real Matrix A= (a_{ij}) is said to be a orthogonal matrix If $A^{-1}=-A^T$.

Ex: $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

Complex Matrices:

Complex matrix: A matrix A is called a complex matrix if the elements of A are complex.

Ex: $\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$

Conjugate complex matrix: If the elements of the matrix of A are replaced by their conjugate complexes then the resulting matrix is defined as conjugate complex matrix.It is denoted by A bar

Ex: If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$, then conjugate of A = $\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Hermition: A complex square matrix A is said to be a Hermition if A= $A^{\theta} = \overline{A}^{T}$

$$A = \begin{pmatrix} 2 & 1+j & 2-j \\ 1-j & 1 & j \\ 2+j & -j & 1 \end{pmatrix}$$

Skew-Hermition: A complex square matrix A is said to be skew-Hermition if A= - $\overline{A}^{T}$

$$\begin{bmatrix} -i & 2+i \\ -(2-i) & 0 \end{bmatrix}$$

Unitary: A complex square matrix A is said to Unitary if A^{-1}=conjugate (A^T)

$$A = \begin{bmatrix} 2^{-1/2} & 2^{-1/2} & 0 \\ -2^{-1/2}\, i & 2^{-1/2}\, i & 0 \\ 0 & 0 & i \end{bmatrix}$$

Idempotent Matrix: A square matrix is said to be Idempotent if A^2=A

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Elementary row Transformation:

1) Interchange of two rows, if i^{th} row and j^{th} row are interchanged it is denoted as Ri→Rj

2) Multiplication of each element of a row with a non-zero scalar if ith row is multiplied with K then it is denoted as Ri→KRi

3) Multiplying every element of a row with a non-zero scalar and adding to the corresponding elements of another row. If all the elements of ith row are multiplied with k and added to the corresponding elements of jth row then it is denoted by Rj→Rj+kRi

Elementary column Transformation:

1) Interchange of two rows, if i^{th} column and j^{th} column are interchanged it is denoted as Ci→Cj

2) Multiplication of each element of a column with a non-zero scalar if ith column is multiplied with K then it is denoted as $C_i \to KC_i$

3) Multiplying every element of a column with a non-zero scalar and adding to the corresponding elements of another column. If all the elements of ith column are multiplied with k and added to the corresponding elements of jth column then it is denoted by $C_j \to C_j + kC_i$

Elementary Matrix:
A matrix which is obtained by applying elementary transformations in known as Elementary matrix.

Rank of a Matrix:

A matrix is said to be of rank r if

(i) It has atleast one non-zero minor of order r and

(ii) Every minor of order higher than r vanishes.

And it is denoted by $\rho(A)$.

Properties:

1) The rank of a null matrix is zero.

2) For a non-zero matrix A, $\rho(A) \geq 1$

3) The rank of every non-singular matrix of order n is n. The rank of a singular matrix of order n is $< n$.

4) The rank of a unit matrix of order n is n.

5) The rank of an $m \times n$ matrix $\leq \min(m,n)$.

6) The rank of a matrix every element of which is unity is unity.

Different methods to find the rank of a matrix:

Method 1:

Echelon form: A matrix is said to be Echelon form if

1) Zero rows, if any, are below any non-zero row

2) The first non-zero entry in each non-zero row is equal to one

3) The number of zeros before the first non-zero elements in a row is less than the number of such zeros in the next rows.

Ex: the rank of matrix which is in Echelon form $\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 3 since the no. of non-zero rows is

Note: Apply only row operations.

Method 2:

Normal Form: Every m×n matrix of rank r can be reduced to the form I_r or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

by a finite chain of elementary row or column operations.

Ex: the rank of matrix A= $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ after applying elementary operations reduced to normal form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I3 & 0 \\ 0 & 0 \end{bmatrix}$$

the rank of A is 3.

Finding the Inverse of a Nonsingular Matrix using Row/Column Transformations(Gauss-Jordan Method):

EXAMPLE: ***Find the inverse of the matrix*** $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ ***using the Gauss-Jordan method.***

Solution: Consider the matrix $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$ ***A sequence of steps in the Gauss-Jordan method are:***

1.
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1(1/2)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_{31}(-1)]{R_{21}(-1)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

3.
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \xrightarrow{R_2(2/3)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

4.
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \xrightarrow{R_{32}(-1/2)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$$

5.
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} \xrightarrow{R_3(3/4)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

6.
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix} \xrightarrow[R_{13}(-1/2)]{R_{23}(-1/3)} \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & \frac{-3}{8} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$$

7.
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & \frac{-3}{8} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix} \xrightarrow{R_{12}(-1/2)} \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix}.$$

8. Thus, the inverse of the given matrix is
$$\begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}.$$

EXERCISE : ***Find the inverse of the following matrices using the Gauss-Jordan method.***

$$(i) \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}, \quad (ii) \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}, \quad (iii) \begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -2 \\ 2 & 4 & 1 \end{bmatrix}.$$

Consistency of System of Linear equations (Homogeneous and NonHomogeneous) Using Rank of the Matrix:

Nature of solution:

1. **m ≠n** non-homogeneous with m equations and n unknowns

The system of equations AX=B is said to be

i) consistent if rank of A = rank of [AB]

ii) consistent and unique solution if rank of A = rank of [AB]=r=n
Where r is the rank of A and n is the no. of unknowns.

iii) Consistent and an infinite no. of solutions if rank of A < rank of [AB] i.e., r < n. In this case we have to give arbitrary values to n-r variables and the remaining variables can be expressed in terms of these arbitrary values.

iv) Inconsistent if rank of A ≠ rank of [AB]

2. **m = n** non-homogeneous equations with n equations and n unknowns

If A be an n-rowed non-singular matrix, the ranks of matrices A and [AB] are both n. Therefore the system of equations AX=B is consistent i.e., possesses a solution.

3. Method of finding the rank of A and [AB]:

Reduce the augmented matrix [A:B] to Echelon form by elementary row transformations.

Ex: Discuss for what values of λ and μ the simultaneous equations

x+y+z = 6 , x+2y+3z = 10 , x+2y+λz = μ have

(iv) no solution

(v) A unique solution

(vi) An infinite no. of solutions

Sol: The given equations can be written as AX=B

$$\text{i.e., } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

and we have the augmented matrix [AB] = $\begin{bmatrix} 1 & 1 & 1 & . & 6 \\ 1 & 2 & 3 & . & 10 \\ 1 & 2 & \lambda & . & \mu \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & . & 6 \\ 0 & 1 & 2 & . & 4 \\ 0 & 1 & \lambda-1 & . & \mu-6 \end{bmatrix} \; R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & . & 6 \\ 0 & 1 & 2 & . & 4 \\ 0 & 0 & \lambda-3 & . & \mu-10 \end{bmatrix} \; R_3 \to R_3 - R_2$$

Case-I: When $\lambda \neq 3$, then rank of A = 3 = rank of [AB]. So that the system of equations is consistent. And r=3=n, so the system has unique solution.

Case-II: When $\lambda = 3$ and $\mu \neq 10$, then rank of A = 2. And rank of [AB] = 3.

Therefore rank of A ≠ rank of [AB]. So the system is inconsistent.

Case-III: When $\lambda = 3$ and $\mu = 10$, then rank of A = rank of [AB] = 2.

Therefore the system is consistent and has an infinite no. of solutions.

Since the no. of unknowns = n = 3 > rank of A = 2.

Homogeneous linear equations

Consider the system of m homogeneous equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{m1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{mn}x_n &= 0 \end{aligned} \qquad (1)$$

(1) can be written as AX=O

Where A is the coefficient matrix formed by A = $\begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$

X= $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ and B= $\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$

Consistency: The matrix A and [AB] are same. So rank of A = rank of [AB]

Therefore the system (1) is always consistent.

Nature of solution:

Trivial solution: Obviously $x_1=x_2=x_3=$ -------- $=x_n=0$ is always a solution of the given system and this solution is called trivial solution.

Therefore trivial solution or zero solution always exists.

Non-Trivial solution: Let r be the rank of the matrix A and n be the no. of unknowns.

Case-I: If r=n, the equations AX=O will have n-n i.e., no linearly independent solutions. In this case, the zero solution will be the only solution.

Case-II: If r<n, we shall have n-r linearly independent solutions. Any linear combination of these n-r solutions will also be a solution of AX=O.

Case-III: If m<n then $r \le m < n$. Thus in this case n-r > 0.

Therefore when the no. of equations < No. of unknowns, the equations will have an infinite no. of solutions.

Ex: Show that the only real number λ for which the system

$X+2y+3z = \lambda x$, $3x+y+2z = \lambda y$, $2x+3y+z = \lambda z$ has non-zero solution is 6 and solve them when $\lambda=6$.

Sol: Given system of equations can be expressed as AX=O

Where $A=\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$; $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here the no. of variables = n = 3.

The given system of equations possesses a non-zero (non-zero) solution, if rank of A < number of unknowns i.e., rank of A < 3.

For this we must have detA = 0

$$\therefore \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\text{i.e.,} \quad \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0 \qquad R_1 \to R_1+R_2+R_3$$

$$\text{i.e., } (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } (6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0 \qquad C_2 \to C_2-C_1,\ C_3 \to C_3-C_1$$

i.e., $(6-\lambda)(\lambda^2+3\lambda+3) = 0$

i.e., $\lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_2 \to 5R_2+3R_1,\ R_3 \to 5R_3+2R_1$$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \to R_3+R_2$$

$\Rightarrow$ -5x+2y+3z = 0 and -19y+19z = 0

$\Rightarrow$ y=z

Since rank of A < No. of unknowns i.e., r < n (2 < 3)

Therefore, the given system has infinite no. of non-trivial solutions.

Let z=k $\Rightarrow$ y=k and -5x+2k+3k = 0 $\Rightarrow$ x=k

$\therefore$ x=k, y=k and z=k is the solutions

Solving mxn and nxm Linear System of Equations by Gauss Elimination:

<u>GAUSSIAN ELIMINATION METHOD</u>: Consider the system of linear equation s

$$a_1x_1+b_1x_2+c_1x_3=d_1$$
$$a_2x_1+b_2x_2+c_2x_3=d_2$$
$$a_3x_1+b_3x_2+c_3x_3=d_3$$

The augmented matrix is [A:B]= $\begin{bmatrix} a1 & b1 & c1 & d1 \\ a2 & b2 & c2 & d2 \\ a3 & b3 & c3 & d3 \end{bmatrix}$

After performing row operations or column operations we get

$$[A:B] = \begin{bmatrix} a1 & b1 & c1 & d1 \\ 0 & b2 & c2 & d2 \\ 0 & 0 & c3 & d3 \end{bmatrix}$$

Ex: solve the equations

2x1 + x2 + x3 = 10

3x1 + 2x2 +3x3 = 18

X1 + 4x2 + 9x3 = 16 using gauss elimination method

Sol: The augmented matrix of the given matrix is [A B] = $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$

[A B] ~ $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/4 & 11 \end{bmatrix}$ $R_2 \rightarrow 2R_3 - R_1$

[A B] ~ $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix}$ $R_3 \rightarrow R_3 - 7R_2$

This augmented matrix corresponds to the following upper triangular system. By backward substitution we get

2x1 + x2 + x3 = 10; $\frac{1}{2}$x2 + $\frac{3}{2}$x3 = 3; -2x3 = -10

x3 = 5 ; x2 = -9 ; x1 = 7

<u>CAYLEY-HAMILTON THEOREM</u>:

Every square matrix satisfies its own characteristics equation

Verification: Example 1: Cayley-Hamilton theorem

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Its characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2 = \lambda^2 - 2\lambda - 1$$

$p(A) = A^2 - 2A - I$

= 1 1 1 1 - 2 1 1 - 1 0

UNIT-II

Eigen Values,Eigen Vectors and Orthogonal Transformation

Introduction

In mathematics, a matrix is a rectangular array numbers.Matrices consisting of only one column or row are called vrctors,while higher – dimensional, arrays of numbers are called tensors.Matrix can also keep track of the coefficients in a system of linear equations.For a square matrix, the determinant and inverse matrix govern the behavior of solutions to the corresponding system of linear equations, and eigen values and eigen vectors provide insight into the geometry of the associated linear transformation.

Applications

1.Physics makes use of them in various domains, for example in geometrical optics and matrix mechanics.
2.Matrices encoding distances of knot points in a graph, such as cities connected by roads, are used in graph theory, and computer graphics use matrices to encode projections of three-dimensional spact onto a two-dimensional screen.
3. Serialism and dodecaphonism are musical movements of the 20^{th} century that utilize a square mathematical matrix to determine the pattern of music intervals.

Characteristic Equation

For a Linear transformation the characteristic equation [Latent equation] can be defined as $|A-\lambda I|=0$ where A is the given matrix from the linear equation and is the eigen constant or characteristic constant and I is the unit matrix with respect to the order of A.

Problems

- **Find the characteristic equation of $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.**

Solution:

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 =sum of the main diagonal elements=2+2=4

s_2=|A| $=\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4 + 1 = 5$

Hence the required characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$

- **Find the characteristic equation of $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$**

Solution:

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 =Sum of the main diagonal elements=1+2=3

s_2=|A| $=\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Hence the required characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

- **Find the characteristic equation of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.**

Solution:

Given matrix is a 3x3 matrix

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 =sum of the main diagonal elements=1+2+3=6

s_2 =sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= (6-4)+(3-1)+(2-1) = 2+2+1 = 5$

$s_3 = |A|$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$= 1(6-4) - 1(3-2) + 1(2-2)$

$= 2 - 1 = 1$

Hence the required characteristic equation is $\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$

- **Find the characteristic equation of** $\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$**.**

Solution:

Given matrix is a 3x3 matrix

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 =Sum of the main diagonal elements=2+2+2=6

s_2 =Sum of the minors of main diagonal element

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

=4+ $(4-1)$ +4=4+3+4=11

$s_3 = |A|$

$$= \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 2(4-0) - 0 + 1(0-2)$$

$$= 8 - 2 \quad = 6$$

Hence the required characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

EIGEN VALUES : Let A be a square matrix .The characteristic equation of A is $|A - \lambda I| = 0$ .The roots of the characteristic equation are called Eigen values of A.

EIGEN VECTOR: Let A be a square matrix .If there exists a non –zero column vector X such that AX=λX, then the vector X is called an Eigen vector of A corresponding to the Eigen value of λ.

Properties of Eigen values.

Solution:

i)The sum of the eigen values of a matrix is equal to the trace of the matrix and product of the eigen values is equal to the determinant of the matrix.

ii)A square matrix A and its transpose A^T have the same eigen values.

Problems

- **Prove that a square matrix and its transpose has the same eigen values.**

Solution:

Let A be a square matrix of order n.

The characteristic equation of A and A^T are $|A - \lambda I| = 0 \quad and$

$$|A^T - \lambda I| = 0$$

Since the determinant value is unaltered by the interchange of rows and columns.

(i.e) $|A| = |A^T|$

Hence characteristic equation of A and A^T are identical

Therefore the eigen values of A and A^T are the same.

- **If $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, find the eigen value of A^3.**

Solution:

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = 4 + 2 = 6$$

$$s_2 = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 8 - 3 = 5$$

The characteristic equation of A is $\lambda^2 - 6\lambda + 5 = 0$

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\lambda = 1,5$$

Eigen values of the given matrix A are 1,5

Eigen values of the matrix A^3 are 1,125

- **Find the eigen values of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.**

Solution:

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = 1 + 1 = 2$$

$$s_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 1 + 1 = 2$$

The characteristic equation of A is $\lambda^2 - 2\lambda + 2 = 0$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\lambda = \frac{2 \pm 2i}{2}$$

$$\lambda = 1 \pm i$$

$\therefore$ The eigen values are $1+i$ and $1-i$

- **Find the sum and product of the eigen values of** $\begin{pmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{pmatrix}$ **without finding the eigen values.**

  <u>Solution:</u>

  Sum of the eigen values of A = trace of he matrix A

  $$= -15 - 12 + 2$$

  $$= -25$$

  Product of the Eigen Values of A=$|A|$

  $$=-15(-24+24) - 4(20-120) + 3(-40+240)$$

  $$= -4(-100)+3(200)$$

  $$=400+600$$

  $$=1000$$

- **Find the sum and product of the eigen values of** $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ **without finding the eigen values.**

  <u>Solution:</u>

  Sum of the eigen values of A = trace of he matrix A

  $$= -2 + 1 + 0 = -1$$

  Product of the Eigen Values of A=$|A|$

$$= -2(0-12) - 2(0-6) - 3(-4+1)$$

$$=24+12+9$$

$$=45$$

- **Find the sum and product of the eigen values of** $\begin{pmatrix} -10 & -2 & -5 \\ 2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ **without finding the eigen values.**

<u>**Solution:**</u>

Sum of the eigen values of A = trace of the matrix A

$$= 10+2+5=17$$

Product of the Eigen Values of A=$|A|$

$$= 10(10-9) + 2(10+15) - 5(6+10)$$

$$=10+2(25) - 5(16)$$

$$=10 + 50 - 80 = -20$$

- **If the matrix** $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$**, find the eigen values of** A^{-1}**.**

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1+5+1 = 7$$

$$s_2 = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$=(5-1) + (5-1) + (1-9)$$

$$= 4+4-8 = 0$$

$$s_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$=1(5-1)-1(1-3)+3(1-15)$$

$$=4+2-42=-36$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

If $\lambda = -2, then\ (-2)^3 - 7(-2)^2 + 36 = 0$

$\therefore \lambda=-2$ is a root.

Using synthetic division,

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$$\therefore \lambda = -2 \text{ and } \lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\lambda = 3,6$$

Hence, The Eigen values of A are $-2,3,6$

The Eigen values of A^{-1} are $\frac{-1}{2}, \frac{1}{3}, \frac{1}{6}$

- **Find the eigen values of $2A^2$ if $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.**

Solution:

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = 4 + 2 = 6$$

$$s_2 = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$$

$$=8-3=5$$

The characteristic equation of A is $\lambda^2 - 6\lambda + 5 = 0$

$$\lambda = 1,5$$

Eigen values of A are 1,5

Eigen values of A^2 are 1,25

Eigen values of $2A^2$ are 2,50

- **Two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to 1(one) each. Find the eigen values of A^{-1}.**

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A.

Given that $\lambda_1 = \lambda_2 = 1$.

We know that,

Sum of the eigen values of A = trace of the matrix A

Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$

$$\Rightarrow 1 + 1 + \lambda_3 = 7$$

$$\lambda_3 = 5$$

$$\therefore \lambda_1 = 1, \ \lambda_2 = 1, \lambda_3 = 5$$

Therefore, Eigen values of A^{-1} are $1, 1, \frac{1}{5}$

- **If 3 and 15 are two eigen values of $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$, find $|A|$.**

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A.

Given $\lambda_1 = 3, \ \lambda_2 = 15, \ \lambda_3 = ?$

We know that,

Sum of the eigen values = sum of the main diagonal elements

$\Longrightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$\Longrightarrow \quad 3+15+\lambda_3=18$

$\Longrightarrow \quad \lambda_3 = 0$

$\therefore$ The eigen values are 3,15,0.

$|A| =$ Product of the eigen values

$=(3)(15)(0)$

$\Longrightarrow \quad |A| = 0$

- **Find the eigen values of the inverse of the matrix** $\boldsymbol{A} = \begin{pmatrix} \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{8} & \mathbf{4} & \mathbf{0} \\ \mathbf{2} & \mathbf{2} & \mathbf{5} \end{pmatrix}$**.**

<u>**Solution:**</u>

Given matrix A is a lower triangular matrix.

The eigen values of a triangular matrix are just the diagonal elements of the matrix

Hence eigen values of A are 3,4,5

$\therefore$ Eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

- **If 3 and 6 are the eigen values of** $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{pmatrix}$**. Write down the eigen values of** $\boldsymbol{A}^{-1}$ **and 3A.**

<u>**Solution:**</u>

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A

Given that $\lambda_1 = 3 \text{ and } \lambda_2 = 6$

We know that,

Sum of the eigen values =sum of the main diagonal elements

$\Longrightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$

$\Longrightarrow \quad 3+6+ \lambda_3=7$

$\Longrightarrow \quad \lambda_3 = 7 - 9$

$\Longrightarrow \quad \lambda_3 = -2$

Hence the eigen values of A are $-2,3,6$

Eigen values of 3A are $-6,9,18$

Eigen values of A^{-1} are $\frac{-1}{2}, \frac{1}{3}, \frac{1}{6}$

- **Find the eigen values of A^3 given $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix}$.**

<u>Solution:</u>

For a Triangular matrix, the diagonal elements are its Eigen values.

Therefore, The eigen values of A are 1,2,3

The eigen values of A^3 are 1,8,27

- **Two eigen values of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ are 2 and 8. Find the third eigen values.**

<u>Solution:</u>

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A

Given $\lambda_1 = 2, \ \lambda_2 = 8, \ \lambda_3 = ?$

We know that,

Sum of the eigen values = sum of the main diagonal elements

$\Longrightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 6 + 3 + 3$

$\Longrightarrow \quad 2+8+ \lambda_3=12$

$\Longrightarrow \quad \lambda_3 = 12 - 10 \Longrightarrow \lambda_3 = 2$

- **The product of the two eigen values of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value.**

Solution:

Let the eigen values of A be $\lambda_1, \ \lambda_2, \ \lambda_3$

Given $\lambda_1 \lambda_2 = 16$.

By the property, $|A|$= product of the eigen values

$$\Rightarrow \qquad |A| = \lambda_1 \lambda_2 \ \lambda_3$$

$$\lambda_1 \lambda_2 \ \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$16\lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$=6(8) + 2(-4) + 2(-4)$$

$$=48-8-8 = 32$$

$$\Rightarrow \quad \lambda_3 = \frac{32}{16} \quad \Rightarrow \lambda_3 = 2$$

- **Find the sum of the squares of the eigen values of** $\mathbf{A} = \begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}$.

Solution:

Given matrix A is a upper triangular matrix.
For a Triangular matrix, the diagonal elements are its Eigen values.

∴ Eigen values of A are 3,2,5
Sum of the squares of the eigen values of A=9+4+25=38

- **If the sum of two eigen values and trace of the matrix A are equal, find the value of |A|.**

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A.

By the property, Sum of the eigen values of A =Trace of the matrix A

$$(i.e)\ \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2$$

$$\Rightarrow \qquad \lambda_3 = 0$$

By the property, $|A|=$ product of the eigen values

$$(i.e) \qquad |A| = \lambda_1 \lambda_2 \lambda_3 = 0$$

- **Prove that the eigen values of $-3A^{-1}$ are the same as those of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.**

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = 1 + 1 = 2$$

$$s_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 4 = -3$$

The characteristic equation of A is $\lambda^2 - 2\lambda - 3 = 0$

$$\Rightarrow (\lambda + 1)(\lambda - 3) = 0 \Rightarrow \lambda = -1, 3$$

Eigen values of A are $-1, 3$

Eigen values of A^{-1} are $-1, \frac{1}{3}$

Eigen values of $-3A^{-1}$ are $3, -1$

$\therefore$ Eigen values of A = Eigen values of $-3A^{-1}$

- **Two eigen values of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and they are double the third. Find the eigen value of A^2.**

<u>**Solution:**</u>

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A.

Given $\lambda_1 = \lambda_2 = 2\lambda_3$

Sum of the eigen values = sum of the main diagonal elements

$$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 4+3-2$$

$$\Rightarrow \quad \lambda_1 + \lambda_1 + \frac{\lambda_1}{2} = 5$$

$$\Rightarrow \quad 2\lambda_1 + \frac{\lambda_1}{2} = 5, \quad \text{where } \lambda_3 = \frac{\lambda_1}{2}$$

$$\Rightarrow \quad \frac{5\lambda_1}{2} = 5$$

$$\Rightarrow \quad \frac{\lambda_1}{2} = 1$$

$$\lambda_1 = 2 = \lambda_2$$

$$2\lambda_3 = 2 \Rightarrow \lambda_3 = 1$$

Eigen values of A are 2,2,1

Eigen values of A^2 are 4,4,1.

Problems

- **Find the eigen values and eigen vectors of** $\boldsymbol{A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}}$

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 6 - 13 + 4$$

$$= 10 - 13 = -3$$

$$s_2 = \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix} + \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix}$$

$$= (-78 + 84) + (-52 + 60) + (24 - 35)$$

$$= 6+8 - 11 = 3$$

$$s_3 = |A|$$

$$=6(8) + 6(56 - 70) + 5(-84 + 91)$$

$$= 48 - 84 + 35 = -1$$

The characteristic equation of A is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

If $\lambda = -1$, then $(-1)^3 + 3 - 3 + 1 = 0$

$\therefore \lambda = -1$ is a root.

Using synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$\therefore \lambda = -1 \text{ and } \lambda^2 + 2\lambda + 1 = 0$$

$$\Longrightarrow \quad \lambda = -1, -1, -1$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i) : When $\lambda = -1 \; in \; (1)$,

$$\begin{pmatrix} 6+1 & -6 & 5 \\ 14 & -13+1 & 10 \\ 7 & -6 & 4+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \qquad 7x_1 - 6x_2 + 5x_3 = 0$$

$$14x_1 - 12x_2 + 10x_3 = 0$$

$$7x_1 - 6x_2 + 5x_3 = 0$$

The above equations represents the same equation $7x_1 - 6x_2 + 5x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$6x_2 = 5x_3$$

$$\frac{x_2}{5} = \frac{x_3}{6}$$

$$X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Choosing arbitrary values for x_2 , let $x_2 = 0$

$$7x_1 = -5x_3$$

$$\frac{x_1}{-5} = \frac{x_3}{7}$$

$$X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$$

Choosing arbitrary values for x_3 , let $x_3 = 0$

$$7x_1 = 6x_2$$

$$\frac{x_1}{6} = \frac{x_2}{7}$$

$$X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}, X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}, X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of** $\boldsymbol{A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}}$

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 6 + 3 + 3 = 12$$

$$s_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$$

=(18-4)+(9-1)+(18-4)

=14+8+14=36

$$s_3 = |A|$$

$$=6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8 = 32$$

The characteristic equation of A is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

If $\lambda = 2$, then $(2)^3 - 12(2)^2 + 36(2) - 32 = 0$

$\therefore \lambda = 2$ is a root.

Using synthetic division,

2	1	-12	36	-32

0	2	-20	32
1	-10	16	0

$\therefore\ \lambda = 2$ and $\lambda^2 - 10\lambda + 16 = 0$

$$\Longrightarrow \quad \lambda = 2,2,8$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When $\lambda = 2$ in (1), we get,

$$\begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

The above equations represents the same equation $2x_1 - x_2 + x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$x_2 = x_3$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Choosing arbitrary values for x_2, let $x_2 = 0$

$$2x_1 = -x_3$$

$$\frac{x_1}{-1} = \frac{x_3}{2}$$

$$X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Case (ii): When $\lambda = 8$ in (1),

$$\begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of A=** $\begin{pmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{pmatrix}$

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 2 + 1 - 1 = 2$$

$$s_2 = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}$$

$$=(2 + 2) + (-1 - 3) + (-2 - 2)$$

$$= 4 - 4 - 4 = -4$$

$$s_3 = |A|$$

$$= 2(-1 - 3) + 2(-1 - 1) + 2(3 - 1)$$

$$= -8 - 4 + 4 = -8$$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

If $\lambda = 2$, then $(2)^3 - 2(2)^2 - 4(2) + 8 = 0$

$\therefore \lambda = 2$ is a root.

Using synthetic division,

2	1	-2	-4	8
	0	2	0	-8

1 0 -4 0

$$\therefore\ \lambda = 2 \text{ and } \lambda^2 - 4 = 0$$

$$\Longrightarrow \lambda = 2,2,-2$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When $\lambda = 2\ in\ (1)$.

$$\begin{pmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0x_1 - 2x_2 + 2x_3 = 0$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + 3x_2 - 3x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-2+2} = \frac{x_2}{2-0} = \frac{x_3}{0+2}$$

$$\frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ as an eigen vector corresponding to $\lambda = 2$

Case (ii): When $\lambda = -2\ in\ (1)$

$$\begin{pmatrix} 2+2 & -2 & 2 \\ 1 & 1+2 & 1 \\ 1 & 3 & -1+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1 - 2x_2 + 2x_3 = 0$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-2-6} = \frac{x_2}{2-4} = \frac{x_3}{12+2}$$

$$\frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$$

$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

$$X_3 = \begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix}$$

∴ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of A=$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$.**

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 1 + 3 + 3=7$

$$s_2 = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$=3+(9-1)+3=14$

$s_3 = |A|$

$=1(9-1) - 0+0=8$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

If $\lambda = 1$, then 1-7+14-8=0

$\therefore \lambda = 1$ is a root.

Using synthetic division,

1	1	-7	14	-8
	0	1	-6	8
	1	-6	8	0

$\therefore\ \lambda = 1$ and $\lambda^2 - 6\lambda + 8 = 0$

$$\Longrightarrow \lambda = 1,2,4$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i): When $\lambda = 1$

$$\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 3-1 & -1 \\ 0 & -1 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0x_1 + 0x_2 + 0x_3 = 0$

$$0x_1 + 2x_2 - x_3 = 0$$

$$0x_1 - x_2 + 2x_3 = 0$$

Solving last two equations using cross rule method

$$\frac{x_1}{4-1} = \frac{-x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (ii) : when $\lambda = 2 \ in \ (1)$

$$\begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 3-2 & -1 \\ 0 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_1 + 0x_2 + 0x_3 = 0$

$$0x_1 + x_2 - x_3 = 0$$

$$0x_1 - x_2 + x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{0-0} = \frac{-x_2}{1-0} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Case (iii) : when $\lambda = 4\ in\ (1)$

$$\begin{pmatrix} 1-4 & 0 & 0 \\ 0 & 3-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{0-0} = \frac{-x_2}{3-0} = \frac{x_3}{3-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of** $\begin{pmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$**.**

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = -2 + 1 + 0 = -1$$

$$s_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= -12 - 3 - 6 = -21$$

$$s_3 = |A|$$

$$= (-2)(0 - 12) - 2(0 - 6) + (-3)(-4 + 1)$$

$$= 24 + 12 + 9 = 45$$

The characteristic equation of A is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

If $\lambda = -3$, then $(-3)^3 + (-3)^2 - 21(-3) - 45 = 0$

$\therefore \lambda = -3$ is a root.

Using synthetic division,

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$$\lambda = -3, \lambda^2 - 2\lambda - 15 = 0$$

$$\implies \lambda = -3, -3, 5$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When $\lambda = -3$ in (1),

$$\begin{pmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

The above equations represents the same equation $x_1 + 2x_2 - 3x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$2x_2 = 3x_3$$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

Choosing arbitrary values for x_2, let $x_2 = 0$

$$x_1 = 3x_3$$

$$\frac{x_1}{3} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Case(ii) : When $\lambda = 5$ in (1),

$$\begin{pmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-12-12} = \frac{-x_2}{42+6} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of A =** $\begin{pmatrix} \mathbf{7} & \mathbf{-2} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{1} & \mathbf{4} \\ \mathbf{-2} & \mathbf{4} & \mathbf{1} \end{pmatrix}$**.**

<u>Solution:</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 7 + 1 + 1 = 9$$

$$s_2 = \begin{vmatrix} 7 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 1 \end{vmatrix}$$

$$= (7 - 4) + (1 - 16) + (7 - 4)$$

$$= 3 - 15 + 3 = -9$$

$$s_3 = |A|$$

$$= 7(-15) + 2(-2 + 8) - 2(-8 + 2)$$

$$= -105 + 12 + 12 = -81$$

The characteristic equation of A is $\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$

If $\lambda = 3$, then $(3)^3 - 9(3)^2 - 9(3) + 81 = 0$

$\therefore \lambda = 3$ is a root.

Using synthetic division,

3	1	-9	-9	81
	0	3	-18	-81

1	-6	-27	0

$$\lambda = 3, \lambda^2 - 6\lambda - 27 = 0$$

$$\Longrightarrow \lambda = 3, 9, -3$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 7-\lambda & -2 & -2 \\ -2 & 1-\lambda & 4 \\ -2 & 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When λ =3 in (1),

$$\begin{pmatrix} 7-3 & -2 & -2 \\ -2 & 1-3 & 4 \\ -2 & 4 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ -2 & -2 & 4 \\ -2 & 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 - 2x_3 = 0$$

$$-2x_1 - 2x_2 + 4x_3 = 0$$

$$-2x_1 + 4x_2 - 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-8-4} = \frac{-x_2}{16-4} = \frac{x_3}{-8-4}$$

$$\frac{x_1}{-12} = \frac{x_2}{-12} = \frac{x_3}{-12}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii): When λ =9 in (1),

$$\begin{pmatrix} 7-9 & -2 & -2 \\ -2 & 1-9 & 4 \\ -2 & 4 & 1-9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 - 2x_3 = 0$$

$$-2x_1 - 8x_2 + 4x_3 = 0$$

$$-2x_1 + 4x_2 - 8x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-8-16} = \frac{-x_2}{-8-4} = \frac{x_3}{16-4}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii): When $\lambda = -3$ in (1)

$$\begin{pmatrix} 7+3 & -2 & -2 \\ -2 & 1+3 & 4 \\ -2 & 4 & 1+3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 2x_2 - 2x_3 = 0$$

$$-2x_1 + 4x_2 + 4x_3 = 0$$

$$-2x_1 + 4x_2 + 4x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-8+8} = \frac{-x_2}{40-4} = \frac{x_3}{40-4}$$

$$\frac{x_1}{0} = \frac{x_2}{-36} = \frac{x_3}{36}$$

$$X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

- **Find the eigen values and eigen vectors of $\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}$.**

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 0 + 0 + 0 = 0$$

$$s_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -1 - 1 - 1 = -3$$

$$s_3 = |A|$$

$$= 0 - 1(0 - 1) + 1(1) = 2$$

The characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

If $\lambda = 2$, then $(2)^3 - 0(2)^2 - 3(2) - 2 = 0$

$\therefore \lambda = 2$ is a root.

Using synthetic division,

2	1	0	-3	-2
	0	2	4	2
	1	2	1	0

$$\lambda = 2, \lambda^2 + 2\lambda + 1 = 0$$

$$\Longrightarrow \lambda = 2, -1, -1$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i): When $\lambda = 2$ in (1)

$$\begin{pmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + x_2 - 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii): when $\lambda = -1\ in\ (1)$

$$\begin{pmatrix} 0+1 & 1 & 1 \\ 1 & 0+1 & 1 \\ 1 & 1 & 0+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

The above equations represents the same equation $x_1 + x_2 + x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$x_2 = -x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Choosing arbitrary values for x_2, let $x_2 = 0$

$$x_1 = -x_3$$

$$\frac{x_1}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

∴ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Cayley-Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

Problems

- **Verify Cayley Hamilton theorem for the matrix** $A = \begin{pmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$**.**

Solution:

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

By Cayley Hamilton theorem, $A^2 - s_1 A + s_2 I = 0$

Here, $s_1 = 5 + 3 = 8$

$$s_2 = \begin{vmatrix} 5 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= 15 - 3 = 12$$

To prove $A^2 - 8A + 12I = 0$

$$A^2 = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 25+3 & 15+9 \\ 5+3 & 3+9 \end{pmatrix}$$

$$=\begin{pmatrix} 28 & 24 \\ 8 & 12 \end{pmatrix}$$

$$A^2 - 8A + 12I = \begin{pmatrix} 25+3 & 15+9 \\ 5+3 & 3+9 \end{pmatrix} + \begin{pmatrix} -40 & -24 \\ -8 & -24 \end{pmatrix} + \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

∴ Cayley Hamilton theorem is verified.

- **Use Cayley Hamilton theorem to find A^3 of the given matrix $\boldsymbol{A} = \begin{pmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$**

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = 5 + 3$$

$$= 8$$

$$s_2 = \begin{vmatrix} 5 & 3 \\ 1 & 3 \end{vmatrix}$$

$$=15-3$$

$$=12$$

By Cayley Hamilton theorem,we have $A^2 - 8A + 12I = 0$

Premultiply by A on both sides , we get $A^3 - 8A^2 + 12A = 0$

$$\Rightarrow A^3 = 8A^2 - 12A$$

$$A^2 = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 28 & 24 \\ 8 & 12 \end{pmatrix}$$

$$\therefore \quad A^3 = 8\begin{pmatrix} 28 & 24 \\ 8 & 12 \end{pmatrix} - 12\begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 224 & 192 \\ 64 & 96 \end{pmatrix} - \begin{pmatrix} 60 & 36 \\ 12 & 36 \end{pmatrix}$$

$$=\begin{pmatrix}164 & 156\\ 52 & 60\end{pmatrix}$$

- **Use Cayley Hamilton theorem to find A^3 of the given matrix $A = \begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}$.**

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

$$s_1 = -1 + 4$$

$$=3$$

$$s_2 = \begin{vmatrix}-1 & 3\\ 2 & 4\end{vmatrix} = -4 - 6 = -10$$

By Cayley Hamilton theorem, we have $A^2 - 3A - 10I = 0$

Premultiply by A on both sides , we get $A^3 - 3A^2 - 10A = 0$

$$\Rightarrow A^3 = 3A^2 + 10A$$

$$A^2 = \begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}$$

$$=\begin{pmatrix}1+6 & -3+12\\ -2+8 & 6+16\end{pmatrix}$$

$$=\begin{pmatrix}7 & 9\\ 6 & 22\end{pmatrix}$$

$$\therefore\ A^3 = 3\begin{pmatrix}7 & 9\\ 6 & 22\end{pmatrix} + 10\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}$$

$$=\begin{pmatrix}21 & 27\\ 18 & 66\end{pmatrix} + \begin{pmatrix}-10 & 30\\ 20 & 40\end{pmatrix}$$

$$=\begin{pmatrix}11 & 57\\ 38 & 106\end{pmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find A^{-1} *and* A^4 ,** if$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$.

<u>Solution:</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 1 + 1 = 3$$

$$s_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$=1+(1-1) + (1 - 3)$$

$$=1+0-2 = -1$$

$$s_3 = |A|$$

$$= 1(1 - 1) - 0(2 + 1) + 3(-2 - 1)$$

$$= 0 - 0 + 3(-3) = -9$$

The characteristic equation of A is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley Hamilton Theorem,

$$A^3 - 3A^2 - A + 9I = 0 \qquad ---------(1)$$

Verification:

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix}$$

$$=\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = A.A^2$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+0 & -3+0-6 & 6+0+15 \\ 8+3+0 & -6+2+2 & 12+4-5 \\ 4-3+0 & -3-2-2 & 6-4+5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

$$A^3 - 3A^2 - A + 9I$$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^2 - 3A - I + 9 \ A^{-1} = 0$$

$$\Rightarrow A^{-1} = -\frac{1}{9}[A^2 - 3A - I]$$

$$A^{-1} = \frac{-1}{9}\left[\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right]$$

$$= \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To Find A^4:

Multiply both sides by A in (1), we get

$$A^4 - 3A^3 - A^2 + 9A = 0$$

$$\Longrightarrow \quad A^4 = 3A^3 + A^2 - 9A$$

$$= 3\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -27 & 63 \\ 33 & -6 & 33 \\ 3 & -21 & 21 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} -9 & 0 & -27 \\ -18 & -9 & 9 \\ -9 & 9 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 12+4-9 & -27-3+0 & 63+6-27 \\ 33+3-18 & -6+2-9 & 33+4+9 \\ 3+0-9 & -21-2+9 & 21+5-9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find $A^{-1} and\ A^4$,**

if$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 3 & 7 & -51 \end{bmatrix}$.

<u>Solution:</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 5 - 5 = 1$$

$$s_2 = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 5 & -4 \\ 7 & -5 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix}$$

$$= (5-4) + (-25+28) + (-5+6)$$

=1+3+1=5

$$s_3 = |A|$$

$$= 1(3) - 2(-10+12) - 2(14-15)$$

$$= 3 - 4 + 2 = 1$$

The characteristic equation of A is $\lambda^3 - \lambda^2 + 5\lambda - 1 = 0$

By Cayley Hamilton Theorem,

$$A^3 - A^2 + 5A - I = 0 ------(1)$$

Verification:

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix}$$

$$A^3 = A^2 . A$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{bmatrix}$$

$$A^3 - A^2 + 5A - I$$

$$= \begin{bmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{bmatrix} - \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix} + 5 \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1} = A^2 - A + 5I$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

To Find A^4: Multiply both sides by A in (1), we get

$$A^4 - A^3 + 5A^2 - A = 0$$

$$A^4 = A^3 - 5A^2 + A$$

$$A^4 = \begin{bmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{bmatrix} - 5 \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 0 \\ 0 & -5 & 20 \\ -10 & -30 & 45 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 8 \\ -8 & -23 & 32 \\ -20 & -52 & 57 \end{bmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find $A^{-1} and\ A^4$,**

  if$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$.

  Solution:

  The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

  $$s_1 = 1 - 1 - 1 = -1$$

$$s_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= (-1-4) + (1-4) + (-1-9)$$

$$= -5 - 3 - 10 = -18$$

$$s_3 = |A|$$

$$= 1(-3) - 2(-2-12) + 3(2+3)$$

$$= -3 + 28 + 15 = 40$$

The characteristic equation of A is $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

By Cayley Hamilton Theorem,

$$A^3 + A^2 - 18A - 40I = 0 - - - - - - -(1)$$

Verification:

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$A^3 + A^2 - 18A - 40I$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$- 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1} = \frac{1}{40}[A^2 + A - 18I]$$

$$A^{-1} = \frac{1}{40}\left[\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} + \begin{bmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{bmatrix}\right]$$

$$=\frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

To Find A^4: Multiply both sides by A in (1), we get

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = -A^3 + 18A^2 + 40A$$

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & -14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -44 & -33 & -46 \\ -24 & -13 & -74 \\ -52 & 14 & -8 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 36 & 72 & 252 \end{bmatrix} + \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 40 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 126 & 204 \end{bmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find A^{-1} *and* A^4 ,**

  if $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

  Solution:

  The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 2 + 2 + 2 = 6$$

$$s_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 1) + (4 - 2)$$

$$=3+3+2=8$$

$$s_3 = |A|$$

$$= 2(4 - 1) + 1(-2 + 1) + 2(1 - 2)$$

$$= 6 - 1 - 2 = 3$$

The characteristic equation of A is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

By Cayley Hamilton Theorem,

$$A^3 - 6A^2 + 8A - 3I = 0 - - - - - - -(1)$$

Verification:

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 14 + 5 + 10 & -12 - 6 - 10 & 18 + 6 + 14 \\ -7 - 10 - 5 & 6 + 12 + 5 & -9 - 12 - 7 \\ 7 + 5 + 10 & -6 - 6 - 10 & 9 + 6 + 14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$- 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1} = \frac{1}{3}[A^2 - 6A + 8I]$$

$$A^{-1} = \frac{1}{3}\left[\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$=\frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

To find A^4: Multiply both sides by A in (1), we get

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6\,[\,6A^2 - 8A + 3I]\text{-}8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I$$

$$A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\text{-}45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find A^{-1} *and* A^4 ,**

  if$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

  Solution:

  The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 2 + 2 = 5$$

$$s_2 = \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix}$$

=(2-0)+(4-0)+(2-0)

=2+4+2=8

$$s_3 = |A|$$

=1(4-0)+0-2(0-0) =4

The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

By Cayley Hamilton Theorem,

$$A^3 - 5A^2 + 8A - 4I = 0 - - - - - - -(1)$$

Verification:

$$A^2 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-4 \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$=\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-12 \\ 6+8+0 & 0+8+0 & -12+16+24 \\ 0+0+0 & 0+0+0 & 0+0+8 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A^3 - 5A^2 + 8A - 4I$$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1} = \frac{1}{4}[A^3 - 5A^2 + 8A]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}\right]$$

$$=\frac{1}{4}\left[\begin{bmatrix}1 & 0 & -14\\ 14 & 8 & 28\\ 0 & 0 & 8\end{bmatrix}+\begin{bmatrix}-5 & 0 & 30\\ -30 & -20 & -60\\ 0 & 0 & -20\end{bmatrix}+\begin{bmatrix}8 & 0 & -16\\ 16 & 16 & 32\\ 0 & 0 & 16\end{bmatrix}\right]$$

$$=\frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=I$$

To find A^4: Multiply both sides by A in (1), we get

$$A^4 - 5A^3 + 8A^2 - 4A = 0$$

$$A^4 = 5A^3 - 8A^2 + 4A$$

$$A^4 = \begin{bmatrix}5 & 0 & -70\\ 70 & 40 & 140\\ 0 & 0 & 40\end{bmatrix}+\begin{bmatrix}-8 & 0 & 48\\ -48 & -32 & -96\\ 0 & 0 & -32\end{bmatrix}+\begin{bmatrix}4 & 0 & -8\\ 8 & 8 & 16\\ 0 & 0 & 8\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & -30\\ 30 & 16 & 60\\ 0 & 0 & 16\end{bmatrix}$$

- **Verify Cayley –Hamilton theorem . Also find A^{-1} *and* A^4 ,**

if $A = \begin{bmatrix}-1 & 0 & 3\\ 8 & 1 & -7\\ -3 & 0 & 8\end{bmatrix}$.

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = -1 + 1 + 8 = 8$$

$$s_2 = \begin{vmatrix}-1 & 0\\ 8 & 1\end{vmatrix} + \begin{vmatrix}1 & -7\\ 0 & 8\end{vmatrix} + \begin{vmatrix}-1 & 3\\ -3 & 8\end{vmatrix}$$

$$=-1 + 8 + (-8 + 9)$$

$$=8 + 1 - 1 = 8$$

$$s_3 = |A|$$

$$= -1(8-0) + 0 + 3(0+3) = 1$$

The characteristic equation of A is $\lambda^3 - 8\lambda^2 + 8\lambda - 1 = 0$

By Cayley Hamilton Theorem,

$$A^3 - 8A^2 + 8A - I = 0 -------(1)$$

Verification:

$$A^2 = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-9 & 0+0+0 & -3+0+24 \\ -8+8+21 & 0+1+0 & 24-7-56 \\ 3+0-24 & 0+0+0 & -9+0+64 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix} \begin{bmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0-63 & 0+0+0 & -21+0+165 \\ -64+21+147 & 0+1+0 & 168-39-385 \\ 24+0-168 & 0+0+0 & -63+0+440 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 0 & 144 \\ 104 & 1 & -256 \\ -144 & 0 & 377 \end{bmatrix}$$

$$A^3 - 8A^2 + 8A - I = \begin{bmatrix} -55 & 0 & 144 \\ 104 & 1 & -256 \\ -144 & 0 & 377 \end{bmatrix} - 8\begin{bmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{bmatrix} + 8\begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1} = A^2 - 8A + 8I$$

$$A^{-1} = \begin{bmatrix}-8 & 0 & 21\\21 & 1 & -39\\-21 & 0 & 55\end{bmatrix} + \begin{bmatrix}8 & 0 & 24\\-64 & -8 & 56\\24 & 0 & -64\end{bmatrix} + \begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0 & -3\\-43 & 10 & 17\\3 & 0 & -1\end{bmatrix}$$

To find A^4: Multiply both sides by A in (1), we get

$$A^4 - 8A^3 + 8A^2 - A = 0$$

$$A^4 = 8A^3 - 8A^2 + A$$

$$= 8\ [8A^2 - 8A + I]\text{-}8A^2 + A$$

$$= 56A^2 - 63A + 8I$$

$$= 56\begin{bmatrix}-8 & 0 & 21\\21 & 1 & -39\\-21 & 0 & 55\end{bmatrix} - 63\begin{bmatrix}-1 & 0 & 3\\8 & 1 & -7\\-3 & 0 & 8\end{bmatrix} + \begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix}$$

$$= 56\begin{bmatrix}-8 & 0 & 21\\21 & 1 & -39\\-21 & 0 & 55\end{bmatrix} + \begin{bmatrix}63 & 0 & -189\\-504 & -63 & 441\\189 & 0 & -504\end{bmatrix} + \begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix}$$

$$=\begin{bmatrix}-377 & 0 & 987\\672 & 1 & 2625\\-987 & 0 & 2584\end{bmatrix}$$

- **Verify Cayley –Hamilton therorem . Also find A^{-1}*and* A^4 ,**

$$\text{if}\boldsymbol{A} = \begin{bmatrix} \mathbf{7} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-6} & \mathbf{-1} & \mathbf{2} \\ \mathbf{6} & \mathbf{2} & \mathbf{-1} \end{bmatrix}.$$

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 7 - 1 - 1 = 5$$

$$s_2 = \begin{vmatrix} 7 & 2 \\ -6 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ 6 & -1 \end{vmatrix}$$

$$= (-7 + 12) + (1 - 4) + (-7 + 12)$$

$$=5 - 3 + 5 = 7$$

$$s_3 = |A|$$

$$=7(-3) - 2(6 - 12) - 2(-12 + 6)$$

$$= -21 + 12 + 12 = 3$$

The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton Theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 - - - - - - -(1)$$

Verification:

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 49 - 12 - 12 & 14 - 2 - 4 & -14 + 4 + 2 \\ -42 + 6 + 12 & -12 + 1 + 4 & 12 - 2 - 2 \\ 42 - 12 - 6 & 12 - 2 - 2 & -12 + 4 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = A.A^2$$

$$=\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$=\begin{bmatrix} 175-48-48 & 56-14-16 & -56+16+14 \\ -150+24+48 & -48+7+16 & 48-8-14 \\ 150-48-24 & 48-14-8 & -48+16+7 \end{bmatrix}$$

$$=\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^3-5A^2+7A-3I=\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}-$$

$$5\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}+7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}-3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}: Multiply both sides by A^{-1} in (1), we get

$$A^{-1}=\frac{1}{3}[A^2-5A+7I]$$

$$A^{-1}=\frac{1}{3}\left[\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}+\begin{bmatrix} -35 & -10 & 10 \\ 30 & 5 & -10 \\ -30 & -10 & 5 \end{bmatrix}+\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right]$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

To Find A^4: Multiply both sides by A in (1), we get

$$A^4-5A^3+7A^2-3A=0$$

$$A^4=5A^3-7A^2+3A$$

$$A^4 = 5\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 7\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 3\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} + \begin{bmatrix} -175 & -56 & 56 \\ 168 & 49 & -56 \\ -168 & -56 & 49 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Orthogonal reduction of a symmetric matrix to Diagonal form.

Orthogonal matrix

A square matrix A is said to be an orthogonal matrix, if $A^T A = AA^T = I$

Diagonalisation of a matrix

The process of finding a matrix M such that $M^{-1}AM = D$, where D is the diagonal matrix, is called as Diagonalisation of A.

Problems

- **Show that A =** $\begin{pmatrix} \boldsymbol{cos\theta} & \boldsymbol{sin\theta} \\ \boldsymbol{-sin\theta} & \boldsymbol{cos\theta} \end{pmatrix}$ **is orthogonal.**

Solution:

$$(i.e)\ A^T A = AA^T = I$$

$$AA^T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} cos^2\theta + sin^2\theta & -cos\theta sin\theta + sin\theta cos\theta \\ -sin\theta cos\theta + cos\theta sin\theta & sin^2\theta + cos^2\theta \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=I$$

Similarly, $A^T A = I$

$$\therefore A^T A = AA^T = I$$

Hence A is orthogonal matrix.

- **Construct a Diagonalised matrix by an orthogonal transformation of** $\boldsymbol{A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}}$**.**

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 6 + 3 + 3 = 12$$

$$s_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= (18 - 4) + (9 - 1) + (18 - 4)$$

$$=14+8+14=36$$

$$s_3 = |A|$$

$$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$$

$$= 48 - 8 - 8=32$$

The characteristic equation of A is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

If $\lambda = 2$,then $(2)^3 - 12(2)^2 + 36(2) - 32 = 0$

$\therefore \lambda = 2$ is a root.

Using synthetic division,

2	1	-12	36	-32
	0	2	-20	32
	1	-10	16	0

$$\lambda = 2 \ \ and\ \lambda^2 - 10\lambda + 16 = 0$$

$$\lambda = 2{,}2{,}8$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When $\lambda = 8$ in (1),

$$\begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case(ii): When $\lambda = 2$ in (1),

$$\begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1 - 2x_2 + 2x_3 = 0$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

The above equations represents the same equation $2x_1 - x_2 + x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$x_2 = x_3$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$X_1{}^T X_3 = 0 \Rightarrow 2l - m + n = 0$ ……………….(1)

$X_2{}^T X_3 = 0 \Rightarrow 0l + m + n = 0$ …………………(2)

Solving (1) and (2) we get

$$\frac{l}{-1-1} = \frac{-m}{2-0} = \frac{n}{2-0}$$

$$\frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$X_3 = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1 {X_2}^T = X_2 {X_3}^T = X_{,3} {X_1}^T = 0$

$\therefore$ The Modal Matrix M=$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

To Prove : $N^T AN=D(8,2,2)$

To find Normalised matrix

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

To find AN

$$AN=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{12+2+2}{\sqrt{6}} & \frac{0-2+2}{\sqrt{2}} & \frac{6-2-2}{\sqrt{3}} \\ \frac{-4-3-1}{\sqrt{6}} & \frac{0+3-1}{\sqrt{2}} & \frac{-2+3+1}{\sqrt{3}} \\ \frac{4+1+3}{\sqrt{6}} & \frac{0-1+3}{\sqrt{2}} & \frac{2-1-3}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

Calculate $D = \mathrm{N}^T\ \mathrm{AN}$

$$D=\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{32+8+8}{6} & \frac{0-2+2}{\sqrt{12}} & \frac{4-2-2}{\sqrt{18}} \\ \frac{0-8+8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0+2-2}{\sqrt{6}} \\ \frac{16-8-8}{\sqrt{18}} & \frac{0+2-2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D(8,2,2)$$

- **Construct a Diagonalised matrix by an orthogonal transformation of**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{10} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{-2} & \mathbf{2} & \mathbf{3} \\ \mathbf{-5} & \mathbf{3} & \mathbf{5} \end{bmatrix}.$$

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 10 + 2 + 5 = 17$

$$s_2 = \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix}$$

$=(20 - 4) + (10 - 9) + (50 - 25)$

$=16+1+25=42$

$s_3 = |A|$

$=10(10 - 9) + 2(-10 + 15) - 5(-6+10)$

$=10(1)+2(5) - 5(4)$

$=10+10-20=0$

The characteristic equation of A is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$$\lambda(\lambda^2 - 17\,\lambda + 42) = 0$$

$$\lambda(\lambda - 3)(\lambda + 14) = 0$$

$$\lambda = 0,3,14$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad -------(1)$$

Case (i): When $\lambda = 0\ in\ (1)$,

$$\begin{pmatrix} 10-0 & -2 & -5 \\ -2 & 2-0 & 3 \\ -5 & 3 & 5-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 + 2x_2 + 3x_3 = 0$$

$$-5x_1 + 3 - 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-6+10} = \frac{-x_2}{30-10} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

Case (ii): When $\lambda = 3\ in\ (1)$,

$$\begin{pmatrix} 10-3 & -2 & -5 \\ -2 & 2-3 & 3 \\ -5 & 3 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 - x_2 + 3x_3 = 0$$

$$-5x_1 + 3 + 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-6-5} = \frac{x_2}{21-10} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii): When $\lambda = 14\ in\ (1)$,

$$\begin{pmatrix} 10-14 & -2 & -5 \\ -2 & 2-14 & 3 \\ -5 & 3 & 5-14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 - 12x_2 + 3x_3 = 0$$

$$-5x_1 + 3x_2 - 9x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-6-60} = \frac{-x_2}{-12-10} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1{X_2}^T = X_2{X_3}^T = X_3{X_1}^T = 0$

$\therefore$ The Modal Matrix M= $\begin{pmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

To Prove : $N^T AN = D(0,3,14)$

To find Normalised matrix

$$N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

To find AN

$$AN = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{10+10-20}{\sqrt{42}} & \frac{10-2-5}{\sqrt{3}} & \frac{-30-2-10}{\sqrt{14}} \\ \frac{-2-10+12}{\sqrt{42}} & \frac{-2+2+3}{\sqrt{3}} & \frac{6+2+6}{\sqrt{14}} \\ \frac{-5-15+20}{\sqrt{42}} & \frac{-5+3+5}{\sqrt{3}} & \frac{15+3+10}{\sqrt{14}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{\sqrt{3}} & \frac{-42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{bmatrix}$$

Calculate $D = \mathrm{N}^T \,\mathrm{AN}$

$$D = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{\sqrt{3}} & \frac{-42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{3-5+12}{\sqrt{126}} & \frac{-42-70+112}{\sqrt{588}} \\ 0 & \frac{3+3+3}{3} & \frac{-42+14+28}{\sqrt{42}} \\ 0 & \frac{-9+3+6}{\sqrt{42}} & \frac{126+14+56}{14} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix} = \mathrm{D}(0,3,14)$$

- **Construct a Diagonalised matrix by an orthogonal transformation of**

  $$\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}.$$

<u>Solution:</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 8 + 7 + 3 = 18$

$$s_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix}$$

$=(56-36)+(21-16)+(24-4)$

$=20+5+20=45$

$s_3 = |A|$

$=8(21-16)+6(-18+8)+2(24-14)$

$=8(5)+6(-10)+2(10)$

$=40-60+20=0$

The characteristic equation of A is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \quad \lambda(\lambda^2 - 18\,\lambda + 45) = 0$$

$$\Rightarrow \quad \lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = 0,3,15$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i): When $\lambda = 0$ *in* (1),

$$\begin{pmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 + 3x_2 + 3x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii): When $\lambda = 3\ in\ (1)$,

$$\begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -\ \ 4 \\ 2 & -4 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 + 3x_2 + 0x_3 = 0$$

Solving last two equations using cross rule method

$$\frac{x_1}{24-8} = \frac{-x_2}{-20+12} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii): When $\lambda = 15$,

$$\begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

solve last two equations using cross rule method

$$\frac{x_1}{24+16} = \frac{-x_2}{28+12} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1 {X_2}^T = X_2 {X_3}^T = X_3 {X_1}^T = 0$

$\therefore$ The Modal Matrix M= $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

To Prove : N^T AN=D(0,3,15)

To find Normalised matrix

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\mathrm{N}^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

To find AN

$$\mathrm{AN} = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+12 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

Calculate $D = \mathrm{N}^T\ \mathrm{AN}$

$$D = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

$$D = \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \mathrm{D}(0,3,15)$$

- **Construct a Diagonalised matrix by an orthogonal transformation of**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}.$$

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 1 + 1 \quad =3$$

$$s_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$=(1 - 1) + (1 - 1) + (1 - 1)=0$$

$$s_3 = |A|$$

$$=1(1 - 1) + 1(-1 - 1) - 1(1 + 1) = -4$$

The characteristic equation of A is $\lambda^3 - 3\lambda^2 + 4 = 0$

If $\lambda = -1$, then $-1-3+4=0$

$\therefore \lambda = -1$ is a root.

Using synthetic division,

-1	1	-3	0	4
	0	-1	4	-4
	1	-4	4	0

$$\lambda = -1 \ and \ \lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = -1, (\lambda - 2)^2 = 0$$

$$\lambda = -1,2,2,$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad ------(1)$$

Case (i): When $\lambda = -1\ in\ (1)$,

$$\begin{pmatrix} 1+1 & -1 & -1 \\ -1 & 1+1 & -1 \\ -1 & -1 & 1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - x_2 - x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$-x_1 - x_2 + 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii): When $\lambda = 2\ in\ (1)$,

$$\begin{pmatrix} 1-2 & -1 & -1 \\ -1 & 1-2 & -1 \\ -1 & -1 & 1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 - x_2 - x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

The above equations represents the same equation $-x_1 - x_2 - x_3 = 0$

Choosing arbitrary values for x_1, let $x_1 = 0$

$$x_2 = -x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

To find the third eigen vector orthogonal to $X_1$and X_2,since the matrix A is symmetric.

Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$X_1{}^T X_3 = 0 \Longrightarrow l + m + n = 0$ ………………..(1)

$X_2{}^T X_3 = 0 \Longrightarrow 0l + m - n = 0$ …………………(2)

Solving (1) and (2) we get

$$\frac{x_1}{-1-1} = \frac{-x_2}{-1-0} = \frac{x_3}{1-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1{X_2}^T = X_2{X_3}^T = X_3{X_1}^T = 0$

$\therefore$ The Modal Matrix $M=\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

To Prove : $N^T AN=D(-1,2,2)$

To find Normalised matrix

$$N=\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

To find AN

$$AN=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{-4}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{2}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Calculate $D = N^T AN$

$$D = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{-4}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{2}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D(-1,2,2)$$

- **Construct a Diagonalised matrix by an orthogonal transformation of** $\boldsymbol{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}}$**.**

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 3 + 3 = 7$$

$$s_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$$=9 - 1 + 3+3=14$$

$$s_3 = |A|$$

$$= 1(9 - 1) - 0 + 0 = 8$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

If $\lambda = 1$,then1-7+14-8=0

$\therefore \lambda = 1$ is a root.

Using synthetic division

1	1	-7	14	-8
	0	1	-6	8
	1	-6	8	0

$$\lambda = 1, \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = 1, \lambda = 2{,}4$$

$$\lambda = 1{,}2{,}4$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i): When $\lambda = 1 \; in \; (1)$,

$$\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 3-1 & -1 \\ 0 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0x_1 + 0x_2 + 0x_3 = 0$

$$0x_1 + 2x_2 - x_3 = 0$$

$$0x_1 - x_2 + 2x_3 = 0$$

Solving last two equations using cross rule method

$$\frac{x_1}{4-1} = \frac{-x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (ii): When $\lambda = 2 \; in \; (1)$,

$$\begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 3-2 & -1 \\ 0 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + x_2 - x_3 = 0$$

$$0x_1 - x_2 + x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{0-0} = \frac{-x_2}{1-0} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) : When $\lambda = 4 \; in \; (1)$,

$$\begin{pmatrix} 1-4 & 0 & 0 \\ 0 & 3-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{0-0} = \frac{-x_2}{3-0} = \frac{x_3}{3-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1{X_2}^T = X_2{X_3}^T = X_3{X_1}^T = 0$

$\therefore$ The Modal Matrix M= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

To Prove : $N^T AN = D(1,2,4)$

To find Normalised matrix

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To Find AN

$$AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} & 0-\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} \\ 0+0+0 & 0-\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} & 0+\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & \sqrt{2} & 2\sqrt{2} \end{bmatrix}$$

Calculate $D = N^T AN$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & \sqrt{2} & 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D(1,2,4)$$

- **Construct a Diagonalised matrix by an orthogonal transformation of**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{7} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{6} & \mathbf{-2} \\ \mathbf{0} & \mathbf{-2} & \mathbf{5} \end{bmatrix}.$$

Solution:

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 7 + 6 + 5 = 18$$

$$s_2 = \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix}$$

$$=(42 - 4) + (30 - 4) + (35 - 0)$$

$$=38+26+35=99$$

$$s_3 = |A|$$

$$=7(30 - 4) + 2(-10 - 0) + 0$$

$$=7(26)+2(-10)=162$$

The characteristic equation of A is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

If $\lambda = 3$, then$(3)^3 - 18(3)^2 + 99(3) - 162 = 0$

$\therefore \lambda = 3$ is a root.

Using synthetic division

3	1	-18	99	-162
	0	3	-45	162
	1	-15	54	0

$$\lambda = 3, \lambda^2 - 15\lambda + 54 = 0$$

$$\lambda = 3, \lambda = 9,6$$

$$\lambda = 3,9,6$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad -------(1)$$

Case (i): When $\lambda = 3 \; in \; (1)$,

$$\begin{pmatrix} 7-3 & -2 & 0 \\ -2 & 6-3 & -2 \\ 0 & -2 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 3x_2 - 2x_3 = 0$$

$$0x_1 - 2x_2 + 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{4-0} = \frac{-x_2}{-8-0} = \frac{x_3}{12-4}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii): When $\lambda = 6 \; in \; (1)$,

$$\begin{pmatrix} 7-6 & -2 & 0 \\ -2 & 6-6 & -2 \\ 0 & -2 & 5-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - 2x_3 = 0$$

$$0x_1 - 2x_2 - x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{4-0} = \frac{-x_2}{-2-0} = \frac{x_3}{0-4}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii): When $\lambda = 9\ in\ (1)$,

$$\begin{pmatrix} 7-9 & -2 & 0 \\ -2 & 6-9 & -2 \\ 0 & -2 & 5-9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 - 3x_2 - 2x_3 = 0$$

$$0x_1 - 2x_2 - 4x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{4-0} = \frac{-x_2}{4-0} = \frac{x_3}{6-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Now clearly any two eigen vectors are pairwise orthogonal.

i.e $X_1{X_2}^T = X_2{X_3}^T = X_3{X_1}^T = 0$

$\therefore$ The Modal Matrix M= $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

To Prove : N^T AN=D(3,6,9)

To find Normalised matrix

$$N = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$N^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

To Find AN

$$AN=\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 7-4+0 & 14-2+0 & 14+4+0 \\ -2+12-4 & -4+6+4 & -4-12-2 \\ 0-4+10 & 0-2-10 & 0+4+5 \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 3 & 12 & 18 \\ 6 & 6 & -18 \\ 6 & -12 & 9 \end{bmatrix}=\begin{bmatrix} 1 & 4 & 6 \\ 2 & 2 & -6 \\ 2 & -4 & 3 \end{bmatrix}$$

Calculate $D = \mathrm{N}^T\ \mathrm{AN}$

$$D = \frac{1}{3}\begin{bmatrix} 1 & 4 & 6 \\ 2 & 2 & -6 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+4+4 & 4+4-8 & 6-12+6 \\ 2+2-4 & 8+2+8 & 12-6-6 \\ 2-4+2 & 8-4-4 & 12+12+3 \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 27 \end{bmatrix}=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \mathrm{D}(3,6,9)$$

- **Verify that the eigenvectors of the real symmetric matrix A=$\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$ are orthogonal in pairs.**

<u>**Solution:**</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 2 + 2 + 2 = 6$$

$$s_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$$=4 + 3 + 4 = 11$$

$$s_3 = |A|$$

$$=2(4-0) + 0(1) - 1(0+2)$$

$$=8-2=6$$

The characteristic equation of A is $\lambda^3-6\lambda^2+11\lambda-6=0$

If $\lambda=1$, then 1-6+11-6=0

$\therefore \lambda=1$ is a root.

Using synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda=1, \lambda^2-5\lambda+6=0$$

$$\lambda=1, \lambda=2,3$$

$$\lambda=1,2,3$$

To find eigen vectors solve $(A-\lambda I)X=0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ------(1)$$

Case (i): When $\lambda=1$ in (1),

$$\begin{pmatrix} 2-1 & 0 & -1 \\ 0 & 2-1 & 0 \\ -1 & 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1+0x_2-x_3=0$$

$$0x_1 + x_2 + 0x_3 = 0$$

$$-x_1 + 0x_2 + x_3 = 0$$

Solve first two equations using cross rule method

$$\frac{x_1}{0+1} = \frac{-x_2}{0-0} = \frac{x_3}{1-0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Case(ii): When $\lambda = 2 \; in \; (1)$,

$$\begin{pmatrix} 2-2 & 0 & -1 \\ 0 & 2-2 & 0 \\ -1 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 0x_2 - x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$-x_1 + 0x_2 + 0x_3 = 0$$

Solving first and last equations using cross rule method

$$\frac{x_1}{0+0} = \frac{-x_2}{0-1} = \frac{x_3}{0+0}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Case (iii) : When $\lambda = 3\ in\ (1)$,

$$\begin{pmatrix} 2-3 & 0 & -1 \\ 0 & 2-3 & 0 \\ -1 & 0 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 0x_2 - x_3 = 0$$

$$0x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + 0x_2 - x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{0-1} = \frac{-x_2}{0+0} = \frac{x_3}{1-0}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

∴ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

To prove the orthogonality condition :

$X_1X_2^T = 0$, $X_2X_3^T = 0$, $X_3X_1^T = 0$

i.e $X_1X_2^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}(0 \quad 1 \quad 0) = 0$

$$X_2X_3^T = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} = 0$$

$$X_3X_1^T = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = 0$$

i.e $X_1X_2^T = X_2X_3^T = X_3X_1^T = 0$

∴ The eigen vectors are pairwise orthogonal.

- **Verify that the eigenvectors of the real symmetric matrix A=$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are orthogonal in pairs.**

<u>Solution:</u>

The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 3 + 5 + 3 = 11$$

$$s_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 14 + 8 + 14 = 36$$

$$s_3 = |A|$$

$$= 3(14) + 1(-3 + 1) + 1(1 - 5)$$

$$= 42 - 2 - 4 = 36$$

The characteristic equation of A is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

If $\lambda = 2$, then $(2)^3 - 11(2)^2 + 36(2) - 36 = 0$

$\therefore \lambda = 2$ is a root.

Using synthetic division

2	1	-18	36	-36
	0	2	-18	36
	1	-9	18	0

$$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0$$

$$\lambda = 2, \lambda = 3,6$$

$$\lambda = 2,3,6$$

To find eigen vectors solve $(\mathrm{A} - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -------(1)$$

Case (i): When $\lambda = 2 \; in \; (1)$,

$$\begin{pmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{1-3} = \frac{-x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case (ii): when $\lambda = 3\ in\ (1)$,

$$\begin{pmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 0x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{1-2} = \frac{-x_2}{0+1} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii): When $\lambda = 6\ in\ (1)$,

$$\begin{pmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{1+1} = \frac{-x_2}{3+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

To prove the orthogonality condition :

$X_1{X_2}^T = 0$, $X_2{X_3}^T = 0$, $X_3{X_1}^T = 0$

$$\text{i.e } X_1{X_2}^T = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = 0$$

$$X_2{X_3}^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} = 0$$

$$X_3{X_1}^T = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}(1 \quad 0 \quad -1) = 0$$

i.e $X_1{X_2}^T = X_2{X_3}^T = X_3{X_1}^T = 0$

$\therefore$ The eigen vectors are pairwise orthogonal.

- **Verify that the eigenvectors of the real symmetric matrix A=$\begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$**

  are orthogonal in pairs.

  Solution:

  The characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 1 + 2 + 1 = 4$$

$$s_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (2 - 1) + (1 - 0) + (2 - 1)$$

$$= 1 + 1 + 1 = 3$$

$$s_3 = |A|$$

$$= 1(2 - 1) + 1(-1 - 0) + 0 = 0$$

The characteristic equation of A is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0, \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 0,1,3$$

To find eigen vectors solve $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad -------(1)$$

Case (i): When $\lambda = 0 \; in \; (1)$,

$$\begin{pmatrix} 1-0 & -1 & 0 \\ -1 & 2-0 & 1 \\ 0 & 1 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-1-0} = \frac{-x_2}{1-0} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Case (ii): When $\lambda = 1 \; in \; (1)$,

$$\begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-1-0} = \frac{-x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Case (iii): When $\lambda = 3 \; in \; (1)$,

$$\begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - x_2 + 0x_3 = 0$$

$$-x_1 - x_2 + x_3 = 0$$

$$0x_1 + x_2 - 2x_3 = 0$$

Solving first two equations using cross rule method

$$\frac{x_1}{-1-0} = \frac{-x_2}{-2-0} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vectors of A are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

To prove the orthogonality condition :

$$X_1 {X_2}^T = 0, \qquad X_2 {X_3}^T = 0, \qquad X_3 {X_1}^T = 0$$

$$\text{i.e } X_1 {X_2}^T = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (1 \quad 0 \quad 1) = 0$$

$$X_2 {X_3}^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} (-1 \quad 2 \quad 1) = 0$$

$$X_3 {X_1}^T = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} (1 \quad 1 \quad -1) = 0$$

$$\text{i.e } X_1 {X_2}^T = X_2 {X_3}^T = X_3 {X_1}^T = 0$$

$\therefore$ The eigen vectors are pairwise orthogonal.

<u>**UNIT II**</u>

<u>**CALCULUS**</u>

Rolle's Mean Value Theorem:

Suppose $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval [*a*,*b*].
2. $f(x)$ is differentiable on the open interval (*a*,*b*).
3. $f(a) = f(b)$

Then there is a number *c* such that $a < c < b$ and $f'(c) = 0$. Or, in other words $f(x)$ has a critical point in (*a*,*b*).

Let's take a look at a quick example that uses Rolle's Theorem.

Example 1 Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one real root.

Solution

From basic Algebra principles we know that since $f(x)$ is a 5^{th} degree polynomial it will have five roots. What we're being asked to prove here is that only one of those 5 is a real number and the other 4 must be complex roots.

First, we should show that it does have at least one real root. To do this note that $f(0) = -2$ and that $f(1) = 10$ and so we can see that $f(0) < 0 < f(1)$. Now, because $f(x)$ is a polynomial we know that it is continuous everywhere and so a number *c* such that $0 < c < 1$ and $f(c) = 0$. In other words $f(x)$ has at least one real root.

We now need to show that this is in fact the only real root. To do this we'll use an argument that is called contradiction proof. What we'll do is assume that $f(x)$ has at least two real roots. This means that we can find real numbers *a* and *b* (there might be more, but all we need for this particular argument is two) such that $f(a) = f(b) = 0$. But if we do this then we know from Rolle's

Theorem that there must then be another number c such that $f'(c)=0$ $f'(c)=0$. This is a problem however. The derivative of this function is,

$$f'(x)=20x^4+3x^2+7$$

Because the exponents on the first two terms are even we know that the first two terms will always be greater than or equal to zero and we are then going to add a positive number onto that and so we can see that the smallest the derivative will ever be is 7 and this contradicts the statement above that says we MUST have a number c such that $f'(c)=0$. We reached these contradictory statements by assuming that $f(x)$ has at least two roots. Since this assumption leads to a contradiction the assumption must be false and so we can only have a single real root.

Geometrical Interpretation of Rolle's Mean Value Theorem:

The proof of the mean value theorem is very simple and intuitive. We just need our intuition and a little of algebra. To prove it, we'll use a new theorem of its own: **Rolle's Theorem.**

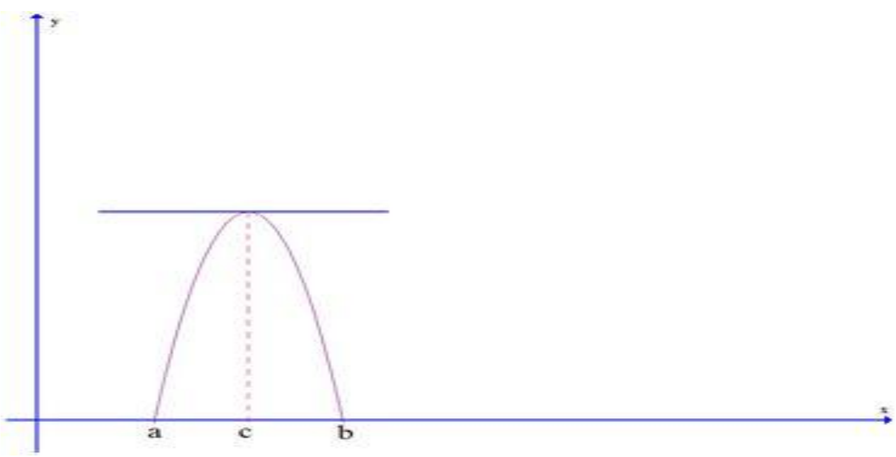
This theorem says that given a continuous function g on an interval [a,b], such that g(a)=g(b), then there is some c, such that:

$$a < c < b$$

And:

$$f'(c)=0$$

Graphically, this theorem says the following:



Given a function that looks like that, there is a point c, such that the derivative is zero at that point. That implies that the tangent line at that point is horizontal. Why? Because the derivative is the slope of the tangent line. Slope zero implies horizontal line.

Lagrange's Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

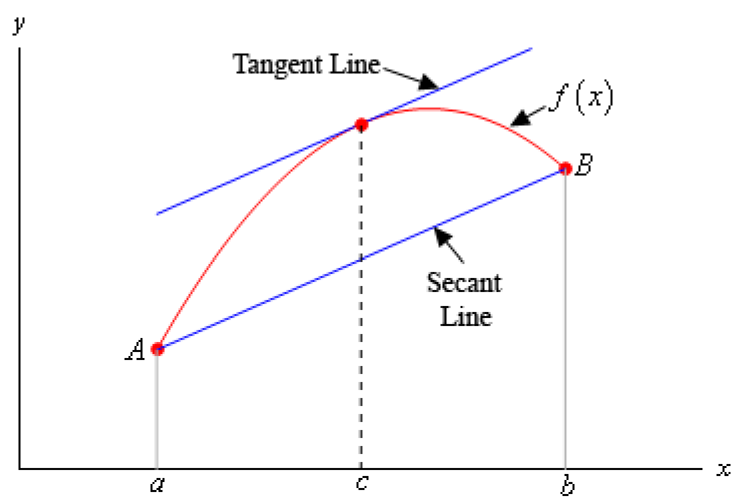
1. $f(x)$ is continuous on the closed interval [*a*,*b*].
2. $f(x)$ is differentiable on the open interval (*a*,*b*).

Then there is a number *c* such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$



Let's now take a look at a couple of examples using the Mean Value Theorem.

Example: Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1,2]$$

Solution

There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.First let's find the derivative.

$$f'(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$3c^2 + 4c - 1 = 4$$

$$3c^2 + 4c - 5 = 0$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of *c*.

$$c = \frac{-4+\sqrt{76}}{6} = 0.7863 \qquad\qquad c = \frac{-4-\sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c = 0.7863$$

Be careful to not assume that only one of the numbers will work. It is possible for both of them to work.

Cauchy's Mean Value Theorem:-

Statement:- **If two functions f(x) and g(x) are derivable in a closed interval [a,b] and g′(x) ≠ 0 for any value of x in [a,b] then there exists at least one value 'c' of x belonging to the open interval (a,b) such that**

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Fact

1. If $f'(x) > 0$ for every *x* on some interval *I*, then $f(x)$ is increasing on the interval.
2. If $f'(x) < 0$ for every *x* on some interval *I*, then $f(x)$ is decreasing on the interval.
3. If $f'(x) = 0$ for every *x* on some interval *I*, then $f(x)$ is constant on the interval.

First Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $x = c$ is a relative maximum.

2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $x = c$ is a relative minimum.

3. If $f'(x)$ is the same sign on both sides of $x = c$ then $x = c$ is neither a relative maximum nor a relative minimum.

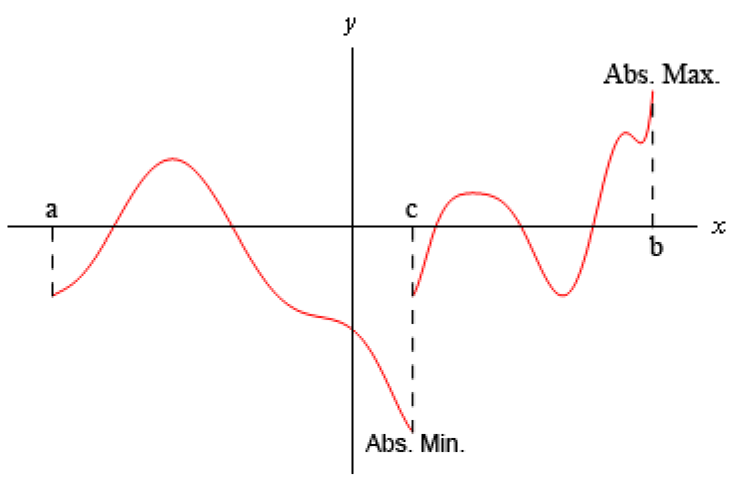
Definition

1. We say that *f(x)* has an **absolute (or global) maximum** at if for every *x* in the domain we are working on.

2. We say that *f(x)* has a **relative (or local) maximum** at if for every *x* in some open interval around .

3. We say that *f(x)* has an **absolute (or global) minimum** at if for every *x* in the domain we are working on.

4. We say that *f(x)* has a **relative (or local) minimum** at if for every *x* in some open interval around .



Example: **Verify Cauchy's mean value theorem for $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ in [a,b] when 0<a<b.**

Sol: -

Given $f(x)=\sqrt{x}.g(x)=\frac{1}{\sqrt{x}}$

$\because f(x)-g(x)$ are conts in [a,b] & f(x), g(x) are derivable in (a,b)

$\therefore f^1(x)=\frac{1}{2\sqrt{x}} \,\&\, g^1(x)=\frac{1}{2x\sqrt{x}}$ exists in (a,b)

Also $g^1(x)\neq 0 \forall x\in(a,b)\subseteq R^+$

$\therefore f(x)g(x)$ are satisfics all conditions of cauchy's mean value Theorem.

<u>VERIFICATION</u>: -

By Cauchy's mean value Theorem. Is al least one $c\in(a,b)$ such that $\frac{f^1(c)}{g(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

$$\therefore \frac{\frac{1}{2}\sqrt{c}}{-\frac{1}{2c\sqrt{c}}}=\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}\Rightarrow -c=\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}=-\sqrt{ab}\Rightarrow \therefore c=\sqrt{b}\text{E}(a,b)$$

Here c is Geometric mean of a & b

$\therefore$ Cauchy's mean value theorem verified

Generalised Mean Value Theorems

Taylor's theorem

If a function f(x) is such that f(x), $f^1(x)$,……. $f^{(n-1)}(x)$ are continuous in [a,a+h] and $f^{(n)}(x)$ exists for all x∈(a,a+h) then there exists θ∈(0,1) such that

$$f(a+h)=f(a)+hf^1(a)+\frac{h^2}{\angle 2}f''(a)+----\frac{h^{n-1}}{\angle(n-1)}f^{(n-1)}(a)+R_n$$

Here $R_n = \frac{h^n (1-\theta)^{n-p}}{(n-1)!\, p} f^{(n)}(a+\theta h)$ is called the Taylor's reminder after n terms

If p = 1 we get **Cauchy's form of Reminder** and

If p = n we get **Lagrange's form of reminder**

Maclaurin's theorem:

If a function f(x) is such that

(i) $f(x), f^1(x), \ldots\ldots f^{(n-1)}(x)$ are continuous in [0,x] and (ii) $f^{(n)}(x)$ exists for all $x \in (0,x)$ then there exists $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f^1(0) + \frac{x^2}{\angle 2} f''(0) + -----+ \frac{x^{n-1}}{\angle (n-1)} f^{(n-1)}(0) + R_n$$

where $R_n = \frac{x^n (1-\theta)^{n-p}}{(n-1)!\, p} f^{(n)}(\theta x)$ is called the Maclaurin's reminD.E.r after n terms

If p = 1 we get **Cauchy's form** and

If p = n we get **Lagrange's form of reminder**.

Verify Taylor's Theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder up to 2 terms in the interval [0,1]

Sol: -

Given $f(x) = (1-x)^{5/2} \Rightarrow f^1(x) = \frac{-5}{2}(1-x)^{3/2}$

$\because f^1(x)$ is polynomias in x

$\because f^1(x)$ is conts in [0,1] & $f^1(x)$ is derivable in (0,1)

They f(x) satisfies all condition of Taylor's theorem .

By Taylor's theorem with Lagrange's form of remainder is at least one $c \in (0,1)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f^1(a) + R_n$$

When $R_n = \frac{(b-a)^n}{n!} f^n(c)$

Here n = p = 2; a = 0, b = 1

$$f(1) = f(0) + f^1(0) + \frac{f^{11}(c)}{2!} \Rightarrow f(0) = 1;\ f(1) = 0$$

Now $f^1(x) = \frac{-5}{2}(1-x)^{3/2}$, $f^1(0) = \frac{-5}{2} \Rightarrow f^{11}(x) = \frac{15}{4}(1-x)^{\frac{1}{2}}$ $f^{11}(c) = \frac{15}{4}(1-c)^{\frac{1}{2}}$

$$\therefore 0 = 1 - \frac{5}{2} + \frac{15}{8}(1-c)^{1/2} \Rightarrow (1-c)^{1/2} = \frac{3}{2}.\frac{8}{15} = \frac{4}{5} \Rightarrow 1 - c = \frac{16}{25} \Rightarrow c = 1 - \frac{16}{25} = \frac{9}{25} = 0.36$$

Hence c = 0.36 $\in (0,1)$

$\therefore$ Taylor's Theorem is verified.

Show that $Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - -$

Sol : -wkt From Maclaurin's series

$$\text{f}(x) = f(0) + x.f^1(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + \frac{x^4}{4!} f^4(0) + - -$$

Given

$f(x) = \cos x$	$f(0) = 1$
$f^1(x) = -\sin x$	$f^1(0) = 0$
$f^{11}(x) = -Cosx$	$f^{11}(0) = -1$
$f^{111}(x) = \sin x$	$f^{111}(0) = 0$
$f^4(x) = Cosx$	$f^4(0) = 1$

$------$

Now $Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + - -$

UNIT-IV

Partial Differentiation and Application (Multivariable Calculus)

Partial Differential Co-Efficient

- Let Z be a function in two or more variables,it can be differentiated with respect to each of the variable by assuming that it varies only with that variable and others treated as constants.These differential Co-efficients are Known as Partial differential Co-efficient.They are denoted by $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial u}{\partial t}$ etc..

e.g.s

- **If u =$e^x siny$,find$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.**

$$\frac{\partial u}{\partial x} = e^x siny$$
$$\frac{\partial u}{\partial y} = e^x cosy$$

- $u = sin(x^2+y^2)$

$$\frac{\partial u}{\partial x} = 2x\, sin(x^2+y^2)$$

$$\frac{\partial u}{\partial y} = 2y\, sin(x^2+y^2)$$

Chain Rule For Partial Differentiation

- $u = f(x,y)$ and $x = f(s), y = f(s)$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

- $\phi = f(u,v,w)$and $u = f(x), \text{v} = f(x), w = f(x),$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}$$

Problems

- **If $\phi = f(y-z, z-x, x-y)$, show that $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0$**

Given $\phi = f(y-z, z-x, x-y)$

Let $u = y - z$

$v = z - x$

$w = x - y$

By chain rule,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial\phi}{\partial u}(0) + \frac{\partial\phi}{\partial v}(-1) + \frac{\partial\phi}{\partial w}(1)$$

$$= \frac{\partial\phi}{\partial w} - \frac{\partial\phi}{\partial v} \quad \text{-----------------------}(1)$$

Similarly,

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u} - \frac{\partial\phi}{\partial w} \quad \text{---------------------------}(2)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial v} - \frac{\partial\phi}{\partial u} \quad \text{------------------------- }(3)$$

Adding (1) ,(2) and (3), we get,

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0$$

- **If u $=e^x siny$ where $x = st^2$ and y=s²t .Find $\frac{\partial u}{\partial s}$ *and* $\frac{\partial u}{\partial t}$.**

Given that u $=e^x siny$, where $x = st^2$ and y =s²t

By chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x siny)(t^2) + (e^x cosy)(2st)$$

$$= e^x(t^2 siny + 2stcosy)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^x siny)(2st) + (e^x cosy)(s^2)$$

$$= e^x(2st\ siny + s^2 cosy).$$

- **If $v = (y - z)(z - x)(x - y)$, prove that $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$.**

$$v = (y - z)(z - x)(x - y)$$

$$= (yz - z^2 - xy + xz)(x - y)$$

$$= xyz - xz^2 - x^2y + x^2z - y^2z + yz^2 + xy^2 - xyz$$

$$= xy^2 + yz^2 + x^2z - xz^2 - x^2y - y^2z$$

$\frac{\partial v}{\partial x} = y^2 + 2xz - 2xy - z^2$ -------------(1)

$\frac{\partial v}{\partial y} = 2xy + z^2 - x^2 - 2yz$ --------------(2)

$\frac{\partial v}{\partial z} = 2yz + x^2 - 2xz - y^2$ ----------------(3)

Adding (1), (2) and (3)

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = y^2 + 2xz - 2xy - z^2 + 2xy + z^2 - x^2 - 2yz$$

$$+2yz + x^2 - 2xz - y^2 = 0$$

- **If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ (L6)**

Given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let $u = f(p, q, r)$, where $p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial u}{\partial p} - \frac{z}{x^2}\frac{\partial u}{\partial r} \quad ------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0)$$

$$= \frac{1}{z}\frac{\partial u}{\partial q} - \frac{x}{y^2}\frac{\partial u}{\partial p} \quad ------------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\ (0) + \frac{\partial u}{\partial q}\ \left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\ \left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{\partial u}{\partial r} - \frac{y}{z^2}\frac{\partial u}{\partial q} \qquad ------------------ (3)$$

Therefore, from (1), (2) and (3), we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} + \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q}$$

$$= 0 = \text{RHS}$$

- **If $u = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$.(L6)**

Given $u = f(x,y)$, $x = rcos\theta, y = rsin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\ \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\ \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\ (cos\theta) + \frac{\partial u}{\partial y}\ (sin\theta) \qquad --(1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\ \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\ \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-rsin\theta) + \frac{\partial u}{\partial y}\ (rcos\theta) \qquad -----------------------------------(2)$$

Therefore, from (1)and (2), we get,

$$\text{RHS} = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\ (cos\theta) + \frac{\partial u}{\partial y}\ (sin\theta)\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial x}(-rsin\theta) + \frac{\partial u}{\partial y}\ (rcos\theta)\right)^2$$

$$= \left[\left(\frac{\partial u}{\partial x}\right)^2 cos^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\ cos\theta sin\theta + sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{1}{r^2}\left[r^2sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\ r^2cos\theta sin\theta + r^2cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= cos^2\theta \left(\frac{\partial u}{\partial x}\right)^2 + sin^2\theta \left(\frac{\partial u}{\partial y}\right)^2 + sin^2\theta \left(\frac{\partial u}{\partial x}\right)^2 + cos^2\theta \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 (cos^2\theta + sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (sin^2\theta + cos^2\theta) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \text{LHS}$$

- **If $\boldsymbol{u = f(x - y, y - z, z - x)}$, prove that $\boldsymbol{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$**

Given $u = f(x - y, y - z, z - x)$

Let $u = f(p, q, r)$, where $p = x - y, q = y - z, r = z - x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad ----------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$= \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad ----------------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad ----------------------(3)$$

Therefore, From (1), (2) and (3), we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} = 0 = \text{RHS}$$

- **If $\boldsymbol{u = f(x^2+2yz, y^2+2zx)}$,prove that**

$$(\boldsymbol{y^2 - zx})\ \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + (\boldsymbol{x^2 - yz})\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}} + (\boldsymbol{z^2 - xy})\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{z}} = \boldsymbol{0}$$

Given $u = f(x^2 + 2yz, y^2 + 2zx)$

Let $u = f(p, q)$, where $p = x^2 + 2yz, q = y^2 + 2zx$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\ \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\ \frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p}\ (2x) + \frac{\partial u}{\partial q}\ (2z)$$

$$= 2x\frac{\partial u}{\partial p} + 2z\frac{\partial u}{\partial q} \qquad ----------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\ \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\ \frac{\partial q}{\partial y}$$

$$= \frac{\partial u}{\partial p}\ (2z) + \frac{\partial u}{\partial q}\ (2y)$$

$$= 2z\frac{\partial u}{\partial p} + 2y\frac{\partial u}{\partial q} \qquad ---------------------\ (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\ \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\ \frac{\partial q}{\partial z}$$

$$= \frac{\partial u}{\partial p}\ (2y) + \frac{\partial u}{\partial q}\ (2x)$$

$$= 2y\frac{\partial u}{\partial p} + 2x\frac{\partial u}{\partial q} \qquad ----------------------(3)$$

Therefore, From (1), (2) and (3), we get,

$$(y^2 - zx)\ \frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$$

$$= 2x\ (y^2 - zx)\frac{\partial u}{\partial p} + 2z\ (y^2 - zx)\frac{\partial u}{\partial q} + 2z(x^2 - yz)\frac{\partial u}{\partial p}$$

$$+ 2y(x^2 - yz)\frac{\partial u}{\partial q} + 2y(z^2 - xy)\frac{\partial u}{\partial p} + 2x(z^2 - xy)\frac{\partial u}{\partial q}$$

$$= \frac{\partial u}{\partial p}[2xy^2 - 2zx^2 + 2zx^2 - 2yz^2 + 2yz^2 - 2xy^2]$$

$$+ \frac{\partial u}{\partial q}[2zy^2 - 2xz^2 + 2yx^2 - 2zy^2 + 2xz^2 - 2yx^2]$$

$$= 0 = \text{RHS}$$

- **If $u = (x^2+y^2+z^2)^{1/2}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$**

Given that $u = (x^2+y^2+z^2)^{1/2}$

By chain rule,

$$\frac{\partial u}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2x)$$

$$= \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}$$

By quotient rule,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{(x^2 + y^2 + z^2)}\,(1) - x\,(\frac{1}{2})(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x)}{(\sqrt{(x^2 + y^2 + z^2)})^2}$$

$$= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad ---------------------(1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad ---------------------(2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad ---------------------(3)$$

Adding (1)(2)and (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2 + z^2 + y^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{2}{u}$$

- **If $z = f(u, v)$ where $u = x + y$ and $v = x - y$,**

  show that $2\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

  Given $z = f(u, v)$

  where $u = x + y, v = x - y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad ----------(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1)$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \qquad -------------(2)$$

  Therefore, From (1) and (2), we get,

$$\text{RHS} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= 2\frac{\partial z}{\partial u} = \text{LHS} .$$

- **If $z = f(x, y)$, where $x = u + v, y = uv$, prove that**

  $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$.(L6)

  Given $z = f(x, y)$

where $x = u + v, y = uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(v)$$

$$= \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \quad ----------------------(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(u)$$

$$= \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \quad -------------------(2)$$

Therefore, From (1) and (2), we get,

$$\text{LHS} = \mathrm{u}\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y}$$

$$= (u+v)\frac{\partial z}{\partial x} + (uv+uv)\frac{\partial z}{\partial y}$$

$$= (u+v)\frac{\partial z}{\partial x} + 2uv\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = \text{RHS}$$

- **Show that** $\boldsymbol{\frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 2logu}$, **where** $\boldsymbol{logu = \frac{(x^3+y^3)}{(3x+4y)}}$.

Differentiating partially w.r.to x by quotient rule,

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{(3x+4y)(3x^2) - (x^3+y^3)(3)}{(3x+4y)^2} = \frac{6x^3 + 12x^2y - 3y^3}{(3x+4y)^2} --(1)$$

Differentiating partially w.r.to y by quotient rule,

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{(3x+4y)(3y^2) - (x^3+y^3)(4)}{(3x+4y)^2} = \frac{8y^3 + 9xy^2 - 4x^3}{(3x+4y)^2} --(2)$$

Multiplying (1) by x and (2) by y we get,

$$\frac{x}{u}\frac{\partial u}{\partial x} = \frac{6x^4 + 12x^3y - 3xy^3}{(3x+4y)^2} \quad ----------------(3)$$

$$\frac{y}{u}\frac{\partial u}{\partial y} = \frac{8y^4 + 9xy^3 - 4x^3y}{(3x+4y)^2} ------------------(4)$$

Adding (4) and (5), we get,

$$\frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = \frac{6x^4 + 12x^3y - 3xy^3 + 8y^4 + 9xy^3 - 4x^3y}{(3x+4y)^2}$$

$$= \frac{6x^4 + 8x^3y + 6xy^3 + 8y^4}{(3x+4y)^2}$$

$$= \frac{6x(x^3+y^3) + 8y(x^3+y^3)}{(3x+4y)^2}$$

$$= \frac{(6x+8y)(x^3+y^3)}{(3x+4y)^2}$$

$$= \frac{2(3x+4y)(x^3+y^3)}{(3x+4y)^2}$$

$$= \frac{2(x^3+y^3)}{(3x+4y)}$$

$$\frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 2logu$$

- **If $z = f(u, v)$, where $u = x^2 - y^2$ and $v = 2xy$, prove that**

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Given $z = f(u, v)$

where $u = x^2 - y^2, v = 2xy$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y)$$

$$= 2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v} \qquad ---------------(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x)$$

$$= -2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v} \qquad ---------------(2)$$

Therefore, From (1) and (2), Squaring and addding, we get,

$$\text{LHS} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)^2 + \left(-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)^2$$

$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 + 4y^2\left(\frac{\partial z}{\partial u}\right)^2 - 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$= \left(\frac{\partial z}{\partial u}\right)^2[4x^2 + 4y^2] + \left(\frac{\partial z}{\partial v}\right)^2[4x^2 + 4y^2]$$

$$= [4x^2 + 4y^2]\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \text{RHS}$$

<u>**Total Differential Coefficient Of A Function**</u>

Let Z be a function in two variables x and y.If Z is continuous, then the total differential coefficient of Z is given by $dz = \frac{\partial Z}{\partial x}\,dx + \frac{\partial Z}{\partial y}\,dy$

e.g.

- **Find the total differential coefficient of the function $u = tan\,(3x - y + 2z)$.**

Given, $u = tan\,(3x - y + 2z)$.

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$ ------------------(1)

$$\frac{\partial u}{\partial x} = 3sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial y} = -sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial z} = 2sec^2(3x - y + 2z)$$

Substituting in (1)

$$du = 3sec^2(3x - y + 2z)dx - sec^2(3x - y + 2z)dy$$
$$+ 2sec^2(3x - y + 2z)dz$$

$$du = sec^2(3x - y + 2z)(3dx - dy + 2dz)$$

- **Find $\frac{du}{dt}$, if $u = log(x + y + z)$, where $x = e^{-t}, y = sint, z = cost$**

Given, $u = log(x + y + z)$,

where $x = e^{-t}$, $y = sint$, $z = cost$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= \frac{1}{x+y+z}(-e^{-t}) + \frac{1}{x+y+z}(cost) + \frac{1}{x+y+z}(-sint)$$

$$= \frac{cost - sint - e^{-t}}{e^{-t} + sint + cost}$$

- **Find $\frac{du}{dt}$, if $u = e^{xy}$, where $x = (a^2 - t^2)^{1/2}, y = sin^3t$**

Given, $u = e^{xy}$, where $x = (a^2 - t^2)^{1/2}, y = sin^3t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}\frac{1}{2}(a^2 - t^2)^{\frac{1}{2}-1}(-2t) + xe^{xy}3sin^2tcost$$

$$= e^{xy}\left[\frac{-yt}{\sqrt{a^2-t^2}} + 3xsin^2tcost\right]$$

- **Find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$ where $x = at^2, y = 2at$.**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (3x^2y^2 + 2xy^3)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= (3a^2t^4 4a^2t^2 + 2at^2 8a^3t^3)(2at) + (2a^3t^6 2at + 3a^2t^4 4a^2t^2)(2a)$$

$$=4a^4t^5(3t + 4)(2at) + 4a^4t^6(t + 3)(2a)$$

$$= 8a^5t^6(3t + 4) + 8a^5t^6(t + 3)$$

$$= 8a^5t^6(3t + 4 + t + 3)$$

$$= 8a^5t^6(4t + 7)$$

- **Find $\frac{du}{dt}$, if u $= \frac{x}{y}$, where $x = e^t$, and $y = \log t$. (L1)**

Given, $u = \frac{x}{y}$, where $x = e^t$, and $y = \log t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{y}e^t + \left(\frac{-x}{y^2}\right)\frac{1}{t}$$

$$= \frac{1}{\log t}\ e^t + \frac{-e^t}{(\log t)^2}\ \frac{1}{t}$$

$$= \frac{e^t}{\log t}\left(1 - \frac{1}{t \log t}\right)$$

- **If $u = \sin^{-1}(x - y)$, where $x = 3t$ and $y = 4t^3$. Show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.**

Given, $u = \sin^{-1}(x - y)$

where $x = 3t$ *and* $y = 4t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3) - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2) = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

Now $1 - (x - y)^2 = 1 - (3t - 4t^3)^2$

$$= 1 - t^2(3 - 4t^2)^2$$

$$= 1 - t^2(9 - 24t^2 + 16t^4)$$

$$= 1 - 9t^2 + 24t^4 - 16t^6$$

$$= 1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6$$

$$= (1 - t^2)(1 - 8t^2 + 16t^4)$$

$$= (1 - t^2)(1 - 4t^2)^2$$

$$\frac{du}{dt} = \frac{3(1 - 4t^2)}{\sqrt{(1 - t^2)(1 - 4t^2)^2}} = \frac{3}{\sqrt{1 - t^2}}$$

Implicit Function

A function of the form $f(x, y) = 0$ is called an implicit function.

e.g.1. $6x^3 + 12x^2y - 3y^3 = 0$

e.g.2. $x^3 + y^3 = 3ax^2y$

For an implicit function $f(x, y) = 0$,

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

- **Find $\frac{dy}{dx}$, when $x^3 + y^3 = 3ax^2y$**

Let $f(x, y) = x^3 + y^3 - 3ax^2y$.

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial x} = 3x^2 - 6axy$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax^2$$

$$\frac{dy}{dx} = \frac{-(3x^2 - 6axy)}{3y^2 - 3ax^2}$$

$$= -\frac{3x(x-6ay)}{3(y^2-ax^2)} = \frac{-x(x-6ay)}{(y^2-ax^2)}$$

- **Find $\frac{dy}{dx}$, when $x^y + y^x = c$**

Let $\mathrm{u}(x,y) = x^y + y^x - c$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} = yx^{y-1} + y^x logy$$

$$\frac{\partial u}{\partial y} = x^y logx + xy^{x-1}$$

$$\therefore \frac{dy}{dt} = \frac{-(yx^{y-1} + y^x logy)}{x^y logx + xy^{x-1}}$$

Taylor's Theorem For A Function Of Two Variables.

If $f(x,y)$ and all its partial derivatives are finite and continuous at all points , then the Taylor series of $f(x,y)$ about the point (a,b) is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

- **Write the Taylor's series expansion of x^y near the point (1, 1) up to the second degree terms**

Taylor's series expansion of x^y near the point (1, 1) is given by

$$x^y = f(1,1) + \frac{1}{1!}\left[(x-1)\frac{\partial f(1,1)}{\partial x} + (y-1)\frac{\partial f(1,1)}{\partial y}\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,1)}{\partial x^2}+2(x-1)(y-1)\frac{\partial^2 f(1,1)}{\partial x\partial y}+(y-1)^2\frac{\partial^2 f(1,1)}{\partial y^2}\right]$$

$$+\cdots$$

Function	Value at (1,1)
$f=x^y$	1
$f_x=yx^{y-1}$	1
$f_y=x^y logx$	0 $\quad [since\ log1=0]$
$f_{xx}=y(y-1)x^{y-2}$	0
$f_{xy}=yx^{y-1}logx+x^{y-1}$	1
$f_{yy}=x^y(logx)^2$	0

$$x^y=1+\frac{1}{1!}[(x-1)1+(y-1)0]$$

$$+\frac{1}{2!}[(x-1)^2(0)+2(x-1)(y-1)+(y-1)^2(0)]$$

$$x^y=1+\frac{1}{1!}[(x-1)]+\frac{1}{2!}[2(x-1)(y-1)]+\cdots$$

- **Write the Taylor series expansion of $e^x\log(1+y)$ in powers of x and y up to the terms of first degree.**

Taylor's series expansion of $e^x\log(1+y)$ near the point (0, 0) or Maclaurin's expansion is given by

$$e^x\log(1+y)=f(0,0)+\frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x}+(y-0)\frac{\partial f(0,0)}{\partial y}\right]+\cdots$$

Function	Value at (0,0)
$f=e^x\log(1+y)$	0 $\quad [since\ \log 1=0]$
$f_x=e^x\log(1+y)$	0
$f_y=e^x\frac{1}{1+y}$	1

$\therefore \quad e^x \log(1+y) = 0 + \frac{1}{1!}[(x)0 + (y)1] + \cdots = \mathrm{y}$

- **Expand x^2y+3y-2 in powers of $(x-1)$ and $(y+2)$ up to the third terms**

Taylor′s series about the point (a, b)is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Taylor′s series about the point (1, −2)is given by

$$x^2y + 3y - 2 = f(1,-2) + \frac{1}{1!}\left[(x-1)\frac{\partial f(1,-2)}{\partial x} + (y+2)\frac{\partial f(1,-2)}{\partial y}\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,-2)}{\partial x^2} + 2(x-1)(y+2)\frac{\partial^2 f(1,-2)}{\partial x\partial y} + (y+2)^2\frac{\partial^2 f(1,-2)}{\partial y^2}\right] + \cdots$$

Function	Value at (1,-2)
$f = x^2y+3y-2$	$(1)^2(-2) + 3(-2) - 2 = -2 - 6 - 2 = -10$
$f_x = 2xy$	$2(1)(-2) = -4$
$f_y = x^2 + 3$	$(1)^2 + 3 = 4$
$f_{xx} = 2y$	$2(-2) = -4$
$f_{xy} = 2x$	$2(1) = 2$
$f_{yy} = 0$	0
$f_{xxx} = 0$	0
$f_{xxy} = 2$	2
$f_{xyy} = 0$	0
$f_{yyy} = 0$	0

Using the table values

$x^2y+3y-2 = -10 + \frac{1}{1!}[(x-1)(-4) + (y+2)4] + \frac{1}{2!}[(x-1)^2(-4) +$
$2(x-1)(y+2)2 + (y+2)^2 0\,] + \frac{1}{3!}[(x-1)^3(0) + 3(x-1)^2(y+2)2 +$
$+3(x-1)(y+2)^2(0) + (y+2)^3 0] + \cdots$

$$= -10 + \frac{1}{1!}[-4(x-1) + 4(y+2)] + \frac{1}{2!}[-4(x-1)^2 + 4(x-1)(y+2)] + \frac{1}{3!}[6(x-1)^2(y+2)] + \cdots$$

$$= -10 - 4[(x-1) - (y+2)] - 2[(x-1)^2 - (x-1)(y+2)\,] + [(x-1)^2(y+2)] + \cdots$$

- **Expand $f(x,y) = x^2y + siny + e^x$ in Taylor's series about the point (1,π).(L2)**

Taylor's series about the point (a, b)is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Taylor series about the point $(1,\pi)$ is

$$x^2y + siny + e^x = f(1,\pi) + \frac{1}{1!}\left[(x-1)\frac{\partial f(1,\pi)}{\partial x} + (y-\pi)\frac{\partial f(1,\pi)}{\partial y}\right] + \frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,\pi)}{\partial x^2} + 2(x-1)(y-\pi)\frac{\partial^2 f(1,\pi)}{\partial x\partial y} + (y-\pi)^2\frac{\partial^2 f(1,\pi)}{\partial y^2}\right] + \frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f(1,\pi)}{\partial x^3} + 3(x-1)^2(y-\pi)\frac{\partial^3(1,\pi)}{\partial x^2\partial y} + 3(x-1)(y-\pi)^2\frac{\partial^3 f(1,\pi)}{\partial x\partial y^2} + (y-\pi)^3\frac{\partial^3 f(1,\pi)}{\partial y^3}\right] + \cdots$$

Function	$Value\ at\,(1,\pi)$
$f = x^2y + siny + e^x$	$f = \pi + e$
$f_x = 2xy + e^x$	$f_x = 2\pi + e$
$f_y = x^2 + cosy$	$f_y = 0$
$f_{xx} = 2y + e^x$	$f_{xx} = 2\pi + e$
$f_{xy} = 2x$	$f_{xy} = 2$
$f_{yy} = -siny$	$f_{yy} = 0$
$f_{xxx} = e^x$	$f_{xxx} = e$
$f_{xxy} = 2$	$f_{xxy} = 2$
$f_{xyy} = 0$	$f_{xyy} = 0$
$f_{yyy} = -cosy$	$f_{yyy} = 1$

$$x^2y + siny + e^x = \pi + e + \frac{1}{1!}[(x-1)(2\pi+e) + (y-\pi)(0)]$$
$$+\frac{1}{2!}[(x-1)^2(2\pi+e) + 2(x-1)(y-\pi)(2) + (y-\pi)^2(0)]$$
$$+\frac{1}{3!}[(x-1)^3e + 3(x-1)^2(y-\pi)(2) + 3(x-1)(y-\pi)^2(0)$$
$$+ (y-\pi)^3(1)] + \cdots$$

$$\therefore x^2y + siny + e^x = \pi + e + \frac{1}{1!}[(x-1)(2\pi+e)] +$$
$$\frac{1}{2!}[(x-1)^2(2\pi + e) + 4(x-1)(y-\pi)]$$
$$+\frac{1}{3!}[e(x-1)^3 + 6(x-1)^2(y-\pi) + (y-\pi)^3] + \cdots$$

- **Write the Taylor's series expansion of $e^x siny$ near the point $(-1, \pi/4)$ up to the third degree terms.**

Taylor's series about the point (a, b)is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)\right.$$
$$\left. + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Taylor's series about the point $\left(-1, \frac{\pi}{4}\right)$ is given by

$$e^x siny = f\left(-1,\frac{\pi}{4}\right) + \frac{1}{1!}\left[(x+1)\frac{\partial f\left(-1,\frac{\pi}{4}\right)}{\partial x} + \left(y-\frac{\pi}{4}\right)\frac{\partial f\left(-1,\frac{\pi}{4}\right)}{\partial y}\right]$$
$$+\frac{1}{2!}\left[(x+1)^2\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial x^2} + 2(x+1)\left(y-\frac{\pi}{4}\right)\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial x\partial y} + \left(y-\frac{\pi}{4}\right)^2\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial y^2}\right] + \frac{1}{3!}\left[(x+1)^3\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x^3} + 3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x^2\partial y} +\right.$$
$$\left. 3(x+1)\left(y-\frac{\pi}{4}\right)^2\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x\partial y^2} + \left(y-\frac{\pi}{4}\right)^3\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial y^3}\right] + \cdots \ldots \ldots$$

Function	Value at $\left(-1,\frac{\pi}{4}\right)$
$f = e^x siny$	$f = e^{-1} sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_x = e^x siny$	$f_x = e^{-1} sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_y = e^x cosy$	$f_y = e^{-1} cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xx} = e^x siny$	$f_{xx} = e^{-1} sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xy} = e^x cosy$	$f_{xy} = e^{-1} cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$

$f_{yy} = -e^x siny$	$f_{yy} = -e^{-1} sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xxx} = e^x siny$	$f_{xxx} = e^{-1} cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xxy} = e^x cosy$	$f_{xxy} = e^{-1} cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xyy} = -e^x siny$	$f_{xyy} = -e^{-1} sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{yyy} = -e^x cosy$	$f_{yyy} = -e^{-1} sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$

$$e^x siny = \frac{1}{e}(\frac{1}{\sqrt{2}}) + \frac{1}{1!}\left[(x+1)\frac{1}{e}(\frac{1}{\sqrt{2}}) + \left(y-\frac{\pi}{4}\right)\frac{1}{e}(\frac{1}{\sqrt{2}})\right]$$

$$+\frac{1}{2!}\left[(x+1)^2\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + 2(x+1)\left(y-\frac{\pi}{4}\right)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^2\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right)\right]$$

$$+\frac{1}{3!}\left[(x+1)^3\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + 3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\right.$$

$$\left. +\ 3(x+1)\left(y-\frac{\pi}{4}\right)^2\left(-\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\right) + \left(y-\frac{\pi}{4}\pi\right)^3\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right)\right] + \cdots$$

$$\Rightarrow\ e^x siny = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\left[1 + (x+1) + \left(y-\frac{\pi}{4}\right) + \frac{1}{2!}\left\{(x+1)^2 + \right.\right.$$

$$\left. 2(x+1)\left(y-\frac{\pi}{4}\right) - \left(y-\frac{\pi}{4}\right)^2\right\} + \frac{1}{3!}\left\{(x+1)^3 +\ 3(x+1)^2\left(y-\frac{\pi}{4}\right) - \right.$$

$$\left.\left. 3(x+1)\left(y-\frac{\pi}{4}\right)^2 - \left(y-\frac{\pi}{4}\right)^3\right\} + \cdots\right]$$

<u>Maclaurin's Expansion Of $f(x,y)$</u>

Taylor′s series about the point (0,0) is known as Maclaurin's Expansion

Maclaurin's expansion of $f(x,y)$is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x} + (y-0)\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x \partial y} + (y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\frac{\partial^3 f(0,0)}{\partial x^3} + 3(x-0)^2(y-0)\frac{\partial^3 f(0,0)}{\partial x^2 \partial y}\right.$$

$$\left. + 3(x-0)(y-0)^2\frac{\partial^3 f(0,0)}{\partial x \partial y^2} + (y-0)^3\frac{\partial^3 f(0,0)}{\partial y^3}\right]$$

- **Write down the Maclaurin's series for $\boldsymbol{sin(x + y)}$.**

  Maclaurin's expansion of $f(x,y)$is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x} + (y-0)\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x \partial y} + (y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\frac{\partial^3 f(0,0)}{\partial x^3} + 3(x-0)^2(y-0)\frac{\partial^3 f(0,0)}{\partial x^2 \partial y}\right.$$

$$\left. + 3(x-0)(y-0)^2\frac{\partial^3 f(0,0)}{\partial x \partial y^2} + (y-0)^3\frac{\partial^3 f(0,0)}{\partial y^3}\right] + \cdots$$

Function	Value at (0,0)
$f = \sin(x+y)$	0
$f_x = \cos(x+y)$	1
$f_y = \cos(x+y)$	1
$f_{xx} = -\sin(x+y)$	0
$f_{xy} = -\sin(x+y)$	0
$f_{yy} = -\sin(x+y)$	0
$f_{xxx} = -\cos(x+y)$	-1

$f_{xxy} = -\cos(x+y)$	-1
$f_{xyy} = -\cos(x+y)$	-1
$f_{yyy} = -\cos(x+y)$	-1

Substituting the table values,

$$\sin(x+y) = 0 + \tfrac{1}{1!}[(x-0)1 + (y-0)1] \; + \tfrac{1}{2!}[(x-0)^2 0 + 2(x-0)(y-0)0 + (y-0)^2 0] + \tfrac{1}{3!}[(x-0)^3(-1) + 3(x-0)^2(y-0)(-1) + 3(x-0)(y-0)^2(-1) + (y-0)^3(-1)] + \cdots$$

$$\sin(x+y) = (x+y) - \tfrac{1}{3!}(x+y)^3 + \cdots$$

- **Write down the Maclaurin's series for e^{x+y} .**

Maclaurin's expansion of e^{x+y} is given by

$$e^{x+y} = f(0,0) + \frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x} + (y-0)\frac{\partial f(0,0)}{\partial y}\right] + \frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x \partial y} + (y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2} + \cdots\right]$$

Function	Value at (0,0)
$f = e^{x+y}$	1
$f_x = e^{x+y}$	1
$f_y = e^{x+y}$	1
$f_{xx} = e^{x+y}$	1
$f_{xy} = e^{x+y}$	1
$f_{yy} = e^{x+y}$	1

$$\therefore \quad e^{x+y} = 1 + \frac{1}{1!}[(x)1 + (y)1] + \frac{1}{2!}[(x)^2 1 + 2(x)(y)1 + (y)^2 1] + \cdots$$

$$\Rightarrow \quad e^{x+y} = 1 + \frac{1}{1!}(x + y) + \frac{1}{2!}(x + y)^2 + \cdots$$

Jacobian.

If $u(x, y)$ and $v(x, y)$ are functions in two variables x and y,then the Jacobian of u and v w.r.t x and y is given by the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$.

Properties Of Jacobian

1. $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$
2. $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$
3. If $u(x, y)$ and $v(x, y)$ are functionally independent, then
$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

- **If $u = 2xy, v = x^2 - y^2, x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$.**

By the property of jacobian,

$$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)} \quad ------------(1)$$

Now,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2$$

$$= -4(x^2 + y^2)$$

$$= -4(r^2cos^2\theta + r^2sin^2\theta)$$

$$= -4\,r^2 \qquad ------------(2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= (rcos^2\theta + rsin^2\theta)$$

$$= r \qquad ------------(3)$$

substituting (2)and (3) in (1)

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4\,r^2\,(r) = -4r^3$$

- **If $\boldsymbol{x = rcos\theta, y = rsin\theta}$ and $\boldsymbol{z = \varphi}$. Find $\boldsymbol{\frac{\partial(x,y,z)}{\partial(r,\Theta,\varphi)}}$.**

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= cos\theta(rcos\theta - 0) + rsin\theta(sin\theta - 0) + 0$$

$$= rcos^2\theta + rsin^2\theta \; = r$$

- **If $x = u(1+v), y = v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.**

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv$$

$$= 1 + u + v + uv - uv$$

$$= 1 + u + v$$

- **If $x = e^r \sec\theta, y = e^r \tan\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$.**

$$Given,\ x = e^r \sec\theta \quad \Rightarrow \frac{\partial x}{\partial r} = e^r \sec\theta; \quad \frac{\partial x}{\partial \theta} = e^r \sec\theta \tan\theta;$$

$$y = e^r \tan\theta \quad \Rightarrow \frac{\partial y}{\partial r} = e^r \tan\theta; \quad \frac{\partial y}{\partial \theta} = e^r \sec^2\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} e^r \sec\theta & e^r \sec\theta \tan\theta \\ e^r \tan\theta & e^r \sec^2\theta \end{vmatrix} = e^{2r}(\sec^3\theta) - e^{2r}(\sec\theta \tan^2\theta)$$

$$= e^{2r}\sec\theta(\sec^2\theta - \tan^2\theta) = e^{2r}\sec\theta$$

- **If $u = x^2, v = y^2$, prove that $\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1$.**

If $u = x^2 \Rightarrow x = \sqrt{u}$ *and* $v = y^2 \Rightarrow y = \sqrt{v}$

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{vmatrix} \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix}$$

$$= \frac{1}{4\sqrt{uv}}.4xy = \frac{1}{4\sqrt{x^2y^2}}.4xy = \frac{1}{4xy}.4xy = 1$$

- **If $u = xyz, v + xy + yz + zx, w = x + y + z$. Find $\frac{\partial(u,v,w)}{\partial(x,y,w)}$**

$$\frac{\partial(u,v,w)}{\partial(x,y,w)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(x^2z + zxy - x^2y - xyz) - 1(xyz + y^2z - xy^2 - xyz) + 1(xyz + z^2y - xyz - xz^2)$$

$$= x^2(z-y) - y^2(z-x) + z^2(y-x)$$

- **If $x = uv$, $y = \frac{u}{v}$, show that $\frac{\partial(x,y)}{\partial(u,v)}.\frac{\partial(u,v)}{\partial(x,y)} = 1$**

Given, $x = uv$ --------------------(1)

$y = \frac{u}{v}$ -------------------(2)

(1) multiplied with (2)

$xy = uv.\frac{u}{v} = u^2$ ------------(3)

(1) divided with (2)

$\frac{x}{y} = \frac{uv}{\frac{u}{v}} = v^2$ -----------------(4)

From (1)&(2) , From (3)&(4)

$\frac{\partial x}{\partial u} = v$ $\qquad$ $2u\frac{\partial u}{\partial x} = y$

$\frac{\partial x}{\partial v} = u$ $\qquad$ $2u\frac{\partial u}{\partial y} = x$

$\frac{\partial y}{\partial u} = \frac{1}{v}$ $\qquad$ $2v\frac{\partial v}{\partial x} = \frac{1}{y}$

$\frac{\partial y}{\partial v} = \frac{-u}{v^2}$ $\qquad$ $2v\frac{\partial v}{\partial y} = -\frac{x}{y^2}$

$$\frac{\partial(x,y)}{\partial(u,v)}.\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-\mathrm{u}}{\mathrm{v}^2} \end{vmatrix} \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & \frac{-x}{2vy^2} \end{vmatrix}$$

$$= \left[\left(\mathrm{v}.\frac{-\mathrm{u}}{\mathrm{v}^2}\right) - (\mathrm{u}\,.\frac{1}{v})\right] \left[\left(\frac{y}{2u}.\frac{-x}{2vy^2}\right) - (\frac{x}{2u}.\frac{1}{2vy})\right]$$

$$= \left[\frac{-u}{v} - \frac{u}{v}\right]\left[-\frac{xy}{4uvy^2} - \frac{x}{4uvy}\right] = \left[\frac{-2u}{v}\right]\left[-\frac{2x}{4uvy}\right]$$

$$= \frac{x}{yv^2} = \frac{x}{y\left(\frac{x}{y}\right)} = 1$$

- **If we transform from three dimensional Cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ). Show that the Jacobian of x, y, z with respect to r, θ, ϕ is $r^2 sin\theta$**

Spherical polar co-ordinates are ,

$$x = r\sin\theta cos\varphi$$

$$y = rsin\ \theta sin\varphi$$

$$z = rcos\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta cos\varphi & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

$$= cos\theta[r^2 sin\theta cos\theta cos^2\varphi + r^2 sin\theta cos\theta sin^2\varphi]$$

$$+ rsin\theta[rsin^2\theta cos^2\varphi + rsin^2\theta sin^2\varphi]$$

$$= cos\theta r^2 sin\theta cos\theta[cos^2\varphi + sin^2\varphi] + rsin\theta rsin^2\theta[cos^2\varphi + sin^2\varphi]$$

$$= [cos^2\varphi + sin^2\varphi](r^2 sin\theta cos^2\theta + r^2 sin\theta sin^2\theta)$$

$$= r^2 sin\theta(cos^2\theta + sin^2\theta) = r^2 sin\theta$$

- **If $\boldsymbol{u = x + y + z, uv = y + z, uvw = z}$, evaluate.** $\boldsymbol{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Let $u = x + y + z$ ---------------(1)

$uv = y + z$ ----------------(2)

$uvw = z$ ----------------(3)

Put (2) in (1) we get,

$$u = x + uv \implies x = u - uv \implies x = u(1 - v)$$

Put (3) in (2) we get,

$$uv = y + uvw \implies y = uv - uvw \implies y = uv(1 - w)$$

From (3) we get, $z = uvw$.

$$\therefore x = u(1 - v) \implies \frac{\partial x}{\partial u} = 1 - v, \frac{\partial x}{\partial v} = -u, \frac{\partial x}{\partial w} = 0$$

$$y = uv - uvw \implies \frac{\partial y}{\partial u} = v(1 - w); \frac{\partial y}{\partial v} = u(1 - w); \frac{\partial y}{\partial w} = -uv$$

$$z = uvw \implies \frac{\partial z}{\partial u} = vw; \frac{\partial z}{\partial v} = uw; \frac{\partial z}{\partial v} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2w + uv^2w)$$

$$= u^2v - u^2v^2 + u^2v^2 = u^2v$$

Stationary Points.

Let $f(x,y)$ be a function in x and y.Then the points at which $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$ are called stationary points. At these points the function takes an extreme value.

Maximum Value, Minimum Value And Extreme Value Of A Function Of Two Variables.

4. A function is said to have a maximum value at the point (a,b) if $f(a,b) > f(a+h, b+k)$ for all small values of h and k.
5. A function is said to have a minimum value at the point (a,b) if $f(a,b) < f(a+h, b+k)$ for all small values of h and k.
6. A function is said to have an extreme value at the point (a,b) if it is either maximum or minimum at (a,b).

Define Saddle Point Of A Function $f(x,y)$.

Let $f(x,y)$ be a function in x and y.The point (a,b) is said to be a saddle point ,if the function is neither maximum nor minimum at that point

Working Rule To Find Maximum/ Minimum Value

- Find the stationary points (a,b)
- Find the values A= $\frac{\partial^2 f}{\partial x^2}$, B= $\frac{\partial^2 f}{\partial x \partial y}$, C= $\frac{\partial^2 f}{\partial y^2}$ and $\Delta = AC - B^2$ at all the stationary points.
- If $\Delta > 0$ and A or B> 0 at (a,b),Then the function has a minima at (a,b)

- If Δ> 0 and A or B< 0 at (a,b),Then the function has a maxima at (a,b)
- If Δ< 0,Then (a,b) is a saddle point.
- If Δ=0,Then the nothing can be decided.
- **Examine the stationary points of the function**

  $\boldsymbol{f(x,y) = x^3 + y^3 - 3x - 12y + 20}$ **and also state their nature.**

Given that $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

To find stationary points

$$\frac{\partial f}{\partial x} = 0 \Longrightarrow 3x^2 - 3 = 0 \Longrightarrow 3(x^2 - 1) = 0$$

$$\Longrightarrow x^2 = 1$$

$$\Longrightarrow x = \pm 1 \text{ -----------------(1)}$$

$$\frac{\partial f}{\partial y} = 0 \Longrightarrow 3y^2 - 12 = 0 \Longrightarrow 3(y^2 - 4) = 0$$

$$\Longrightarrow y^2 = 4$$

$$\Longrightarrow y = \pm 2 \text{ -----------------(2))}$$

∴ The stationary points are $(-1,-2), (-1,2), (1,-2)$, and $(1,2)$

$$A = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$C = \frac{\partial^2 f}{\partial y^2} = 6y$$

Points	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	Δ= $AC - B^2$	Extremum
$(-1,-2)$	$-6 < 0$	0	$-12 < 0$	$72 > 0$	Maximum
$(-1,2)$	$-6 < 0$	0	12	$-72 < 0$	Saddle point
$(1,-2)$,	6	0	-12	$-72 < 0$	Saddle point

(1,2)	$6>0$	0	$12>0$	$72>0$	Minimum

The maximum value at $(-1,-2)$ is

$$f(x,y) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20 = 38$$

The minimum value at (1,2) is

$$f(x,y) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$
$$= 1 + 8 - 3 - 24 + 20 = 2$$

- **Examine $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$ for extreme values**

To find stationary points :

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$$

$$\Rightarrow x^2 + y^2 - 10x + 24 = 0 \quad \text{--------------------------}(1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 30y = 0$$

$$\Rightarrow y(6x - 30) = 0$$

$$\Rightarrow y = 0 \text{ or } 6x - 30 = 0$$

$$\Rightarrow y = 0 \text{ or } x = 5$$

Put $y = 0$ in (1)

$$\Rightarrow x^2 - 10x + 24 = 0$$

$$(x-6)(x-4) = 0 \Rightarrow x = 4,6$$

$\therefore$ For $y = 0$ *the points are* $(4,0)$ *and* $(6,0)$.

Let $x = 5$ in (1), we get, $25 + y^2 - 50 + 24 = 0$

$$\Rightarrow (y^2 - 1) = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

The points are $(5,1)$ $(5,-1)$

$\therefore$ The stationary points are $(4,0), (6,0), (5,1), (5,-1)$

$$A = \frac{\partial^2 f}{\partial x^2} = 6x - 30 \ ; \quad C = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

$B = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$\Delta = AC - B^2$

POINTS	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$\Delta = AC - B^2$	EXTREMUM
(4,0)	$-6<0$	0	$-6<0$	$36>0$	MAXIMA
(6,0)	$6>0$	0	$6>0$	$36>0$	MINIMA
(5,−1)	0	$-6<0$	0	$-36<0$	SADDLE POINT
(5,1)	0	$6>0$	0	$-36<0$	SADDLE POINT

- **Examine the function $f(x,y) = x^3y^2(12 - x - y)$ for extreme values**

Given that $f(x,y) = 12x^3y^2 - x^4y^2 - x^3y^3$

To find stationary points

$\frac{\partial f}{\partial x} = 0 \Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(36 - 4x - 3y) = 0$

$\Rightarrow x = 0, y = 0 \; or \; 36 - 4x - 3y = 0$

$\Rightarrow 4x + 3y = 36$ ------------------(1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 24x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(24 - 2x - 3y) = 0$

$\Rightarrow x = 0, y = 0 \; or \; 24 - 2x - 3y = 0$

$\Rightarrow 2x + 3y = 24$ ------------------(2)

Solving (1) and (2),

(2)x 2 $\Rightarrow 4x + 6y = 48$ ------------------(3)

(1)$\Rightarrow 4x + 3y = 36$ ------------------(4)

$(3) - (4) \Rightarrow 3y = 12 \Rightarrow y = 4 \; and \; x = 6$

$x = 0 \; in \; (1) \Longrightarrow y = 12 \Longrightarrow (0,12)$

$y = 0 \; in \; (1) \Longrightarrow x = 9 \Longrightarrow (9,0)$

$x = 0 \; in \; (2) \Longrightarrow y = 8 \Longrightarrow (0,8)$

$y = 0 \; in \; (2) \Longrightarrow x = 12 \Longrightarrow (12,0)$

The stationary points are (0,0), (0,12), (9,0), (0,8),(12,0) *and* (6,4)

$$A = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3$$

$$\mathrm{B} = \frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 8x^3y - 9x^2y^2$$

$$\mathrm{C} = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y$$

Points	$A = \frac{\partial^2 f}{\partial x^2}$	$\mathrm{B} = \frac{\partial^2 f}{\partial x \partial y}$	$\mathrm{C} = \frac{\partial^2 f}{\partial y^2}$	$\Delta = AC - B^2$	Extremum
(0,0)	0	0	0	0	Nothing can be decided
(0,12)	0	0	0	0	Nothing can be decided
(9,0)	0	0	4374	0	Nothing can be decided
(0,8)	0	0	0	0	Nothing can be decided

(12,0)	0	0	0	0	Nothing can be decided
(6,4)	-2304	-1728	-2592	> 0	Maximum

The maximum value at (6,4) is

$f(x,y)=(6)^3(4)^2(12-6-4) = 6912$

Lagrange's Method For Constrained Maxima And Minima

Let $f(x,y,z)$ be the function whose maximum/ minimum to be found subject to the constraint $\varphi(x,y,z) = 0$.

By Lagrange's Method,

- Form the auxiliary function $F = f + \lambda\,\varphi$,where λ is the Lagrangian multiplier.
- Solve for (x,y,z) from the equations

  $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $\varphi(x,y,z) = 0$ to find maximum/ minimum value of $f(x,y,z)$

- **Examine the minimum value of $x^2+y^2+z^2$, when $xyz = a^3$.**

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $\varphi(x,y,z) = xyz - a^3$

By Lagrange's method $F = f + \lambda\varphi$

$$(i.e)\ F = (x^2 + y^2 + z^2) + \lambda(xyz - a^3)$$

$$\frac{\partial F}{\partial x} = 0 \implies 2x + \lambda yz = 0$$

$$\implies \lambda = -\frac{2x}{yz} \quad \text{-----------------------------}(1)$$

$$\frac{\partial F}{\partial y} = 0 \implies 2y + \lambda xz = 0$$

$$\implies \lambda = -\frac{2y}{xz} \quad \text{-----------------------------}(2)$$

$$\frac{\partial F}{\partial z} = 0 \implies 2z + \lambda xy = 0$$

$$\implies \lambda = -\frac{2z}{xy} \quad \text{------------------------------(3)}$$

From (1) and (2), we get

$$-\frac{2x}{yz} = -\frac{2y}{xz}$$

$$\implies x^2 = y^2$$

$$\implies x = y \quad \text{------------------------------(4)}$$

From (2) and (3) We get

$$-\frac{2y}{xz} = -\frac{2z}{xy}$$

$$\implies y^2 = z^2$$

$$\implies y = z \quad \text{--------------------------------(5)}$$

From (4) and (5), we get

$$x = y = z$$

Using this in $xyz = a^3$, we get,

$$x(x)(x) = a^3$$

$$\implies x^3 = a^3$$

$$\implies x = a$$

$$\therefore x = y = z = a$$

$\therefore$ (a, a, a) *is a point of minima and* $f_{min} = a^2 + a^2 + a^2 = 3a^2$

- **The temperature at any point (x, y, z) in space is given by $T = kxyz^2$, where k is a constant. Determine the highest temperature on the surface of the sphere $x^2+y^2+z^2 = a^2$**

Given Temperature $T = kxyz^2$,

such that $x^2+y^2+z^2 = a^2$

$$f(x, y, z) = kxyz^2$$

$$\varphi(x, y, z) = x^2 + y^2 + z^2 - a^2$$

By Lagrange's Method,

Let $F = f + \lambda\varphi$, where λ is Lagrangian Multiplier.

$\Longrightarrow \quad F = kxyz^2 + \lambda(x^2 + y^2 + z^2 - a^2)$

$\therefore \quad \frac{\partial F}{\partial x} = 0 \quad \Longrightarrow kyz^2 + 2\lambda x = 0,$

$$\Longrightarrow \lambda = \frac{-kyz^2}{2x} \quad ----------------(1)$$

$\frac{\partial F}{\partial y} = 0 \quad \Longrightarrow kxz^2 + 2\lambda y = 0,$

$$\Longrightarrow \lambda = \frac{-kxz^2}{2y} \quad ---------------------(2)$$

$\frac{\partial F}{\partial z} = 0 \quad \Longrightarrow 2kxyz + 2\lambda z = 0,$

$$\Longrightarrow \lambda = \frac{-2kxyz}{2z} = -kxy \quad ------------- (3)$$

From (1) *and* (2), *we get*,

$$\frac{-kyz^2}{2x} = \frac{-kxz^2}{2y}$$

$\Longrightarrow \quad y^2z^2 = x^2z^2$

$\Longrightarrow \quad y^2 = x^2$

$\Longrightarrow \quad x = y \quad -------------(4)$

From (2) *and* (3), *we get*,

$$\frac{-kxz^2}{2y} = -kxy$$

$\Longrightarrow \quad xz^2 = 2xy^2$

$\Longrightarrow \quad z^2 = 2y^2$

$\Longrightarrow \quad \sqrt{2}\, y = z \quad ------------- (5)$

$From\ (4)\ and\ (5), we\ get, x = y\ \ \&\ z = \ \sqrt{2}\,y$

$Using\ this\ in\ x^2 + y^2 + z^2 - a^2 = 0, we\ get\ ,$

$$y^2 + y^2 + \left(\sqrt{2}\,y\right)^2 - a^2 = 0,$$

$$\Longrightarrow \qquad 2y^2 + 2y^2 - a^2 = 0,$$

$$\Longrightarrow \qquad 4y^2 - a^2 = 0$$

$$\Longrightarrow \qquad 4y^2 = a^2$$

$$\Longrightarrow \qquad y^2 = \frac{a^2}{4} \quad \Longrightarrow y = \frac{a}{2}$$

$$From\ (4), we\ get, \qquad x = \frac{a}{2}$$

$$From\ (5), we\ get\ , \qquad z = \sqrt{2}\,\frac{a}{2} = \frac{a}{\sqrt{2}}$$

Therefore, the maximum temperature on the given surface is

$$T = k\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right)^2 \ = k\frac{a^4}{8}$$

- **Determine the minimum value of $x^2 + y^2 + z^2$ when $x\ +\ y\ +\ z\ =\ 3a$.**

Let $\ f(x, y, z) = x^2 + y^2 + z^2$

and $\varphi(x, y, z) = x + y + z - 3a = 0$

By Lagrange's Method,

Let $F = f + \lambda\varphi$, where λ is Lagrangian Multiplier.

$F = (x^2 + y^2 + z^2) + \lambda(x + y + z - 3a)$

$$\frac{\partial F}{\partial x} = 0 \qquad \Longrightarrow 2x + \lambda = 0,$$

$$\Longrightarrow \ \lambda = -2x - - - - - - - - - - - - - - - (1)$$

$$\frac{\partial F}{\partial y} = 0 \qquad \Longrightarrow 2y + \lambda = 0,$$

$$\Longrightarrow \ \lambda = -2y - - - - - - - - - - - - - - - (2)$$

$$\frac{\partial F}{\partial z} = 0 \quad \Longrightarrow 2\text{z} + \lambda = 0,$$

$$\Longrightarrow \lambda = -2z ---------------(3)$$

$From\ (1),(2) and (3), we\ get,$

$$-2x = -2y = -2z$$

$$\Longrightarrow x = y = z$$

$Using\ this\ in\ x + y + z - 3\text{a} = 0,$ we get

$$3x - 3a = 0 \quad \Longrightarrow x = a$$

$$\Longrightarrow x = a = y = z$$

Therefore, Minimum value of $f(x, y, z) = \text{a}^2 + \text{a}^2 + \text{a}^2 = 3\text{a}^2$

- **Determine the volume of the largest rectangular solid which can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

Let the volume of the solid be xyz which is maximised in such a way that it can be inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let $f(x, y, z) = xyz$ and $\varphi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

By Lagrange's method $F = f + \lambda\,\varphi$

$$(i.e) \quad F = xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]$$

$$\therefore \frac{\partial F}{\partial x} = 0 \quad \Longrightarrow \quad yz + \frac{2x\lambda}{a^2} = 0$$

$$\Longrightarrow \lambda = \frac{-yza^2}{2x} ---------------(1)$$

$$\frac{\partial F}{\partial y} = 0 \quad \Longrightarrow xz + \frac{2y\lambda}{b^2} = 0$$

$$\Rightarrow \quad \lambda = \frac{-xzb^2}{2y} ---------------(2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \frac{2z\lambda}{c^2} = 0$$

$$\Rightarrow \quad \lambda = \frac{-yxc^2}{2z} ---------------(3)$$

From (1) &(2) we get

$$\frac{-yza^2}{2x} = \frac{-xzb^2}{2y} \quad \Rightarrow \quad \frac{ya^2}{x} = \frac{xb^2}{y}$$

$$\Rightarrow \quad y^2a^2 = x^2b^2$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} ---------------(4)$$

From (2) &(3) we get

$$\frac{-xzb^2}{2y} = \frac{-yxc^2}{2z} \Rightarrow \frac{zb^2}{y} = \frac{yc^2}{z}$$

$$\Rightarrow \frac{y}{b} = \frac{z}{c} \qquad ---------------(5)$$

From (4) &(5) we get

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \Rightarrow y = \frac{bx}{a}, z = \frac{cx}{a} \quad ------------(6)$$

We know that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{b^2x^2}{a^2b^2} + \frac{c^2x^2}{a^2c^2} = 1$$

$$\Rightarrow \ 3\frac{x^2}{a^2} = 1$$

$$\Rightarrow x^2 = \frac{a^2}{3} \qquad \Rightarrow \ x = \frac{a}{\sqrt{3}}$$

$\therefore$ Equation (6) $\Rightarrow$ $y = \frac{bx}{a} = \frac{b(\frac{a}{\sqrt{3}})}{a} = \frac{b}{\sqrt{3}}$

and $z = \frac{cx}{a} = \frac{c(\frac{a}{\sqrt{3}})}{a} = \frac{c}{\sqrt{3}}$

$\therefore$ The rectangular solid in a cube with dimensions are

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad z = \frac{c}{\sqrt{3}}$$

$$\therefore \text{ Volume} = xyz = \frac{a}{\sqrt{3}} \times \frac{b}{\sqrt{3}} \times \frac{c}{\sqrt{3}} = \frac{abc}{3\sqrt{3}}$$

- **Determine the minimum value of $x^m y^n z^p$ when $x + y + z = a$.**

Let $f(x,y,z) = x^m y^n z^p$ and $\varphi(x,y,z) = x + y + z - a$

By Lagrange's method $F = f + \lambda\,\varphi$

$(i.e)$ $F = x^m y^n z^p + \lambda(x + y + z - a)$

$$\therefore \frac{\partial F}{\partial x} = 0 \implies m x^{m-1} y^n z^p + \lambda = 0$$

$$\implies \lambda = -\, m x^{m-1} y^n z^p \quad \text{------------------(1)}$$

$$\frac{\partial F}{\partial y} = 0 \implies n x^m y^{n-1} z^p + \lambda = 0$$

$$\implies \lambda = -\, n x^m y^{n-1} z^p \quad \text{-------------------(2)}$$

$$\frac{\partial F}{\partial z} = 0 \implies p\, x^m y^n z^{p-1} + \lambda = 0$$

$$\implies \lambda = -\, p\, x^m y^n z^{p-1} \quad \text{------------------(3)}$$

From (1) &(2), we get

$$-\, m x^{m-1} y^n z^p = -\, n x^m y^{n-1} z^p$$

$$\implies m x^{m-1} y^n = n x^m y^{n-1}$$

$$\implies my = nx$$

$$\implies \frac{y}{n} = \frac{x}{m} \quad -------------------(4)$$

From (2) &(3), we get

$$-\, n x^m y^{n-1} z^p = -\, p\, x^m y^n z^{p-1}$$

$\Longrightarrow nz = py$

$$\Longrightarrow \quad \frac{y}{n} = \frac{z}{p} \text{ ------------------ } (5)$$

From (4) &(5), we get

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{p} \quad \Longrightarrow y = \frac{nx}{m}, z = \frac{px}{m}$$

Now consider , $\quad x + y + z = a$

$$\Longrightarrow x + \frac{nx}{m} + \frac{px}{m} = a$$

$$\Longrightarrow \quad mx + nx + px = am$$

$$\therefore \quad x = \frac{am}{m+n+p}$$

$$y = \frac{n\left(\frac{am}{m+n+p}\right)}{m} = \frac{na}{m+n+p}$$

$$z = \frac{p\left(\frac{am}{m+n+p}\right)}{m} = \frac{pa}{m+n+p}$$

$\therefore$ *The minimum value of* $f(x, y, z)$

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{na}{m+n+p}\right)^n \left(\frac{pa}{m+n+p}\right)^p$$

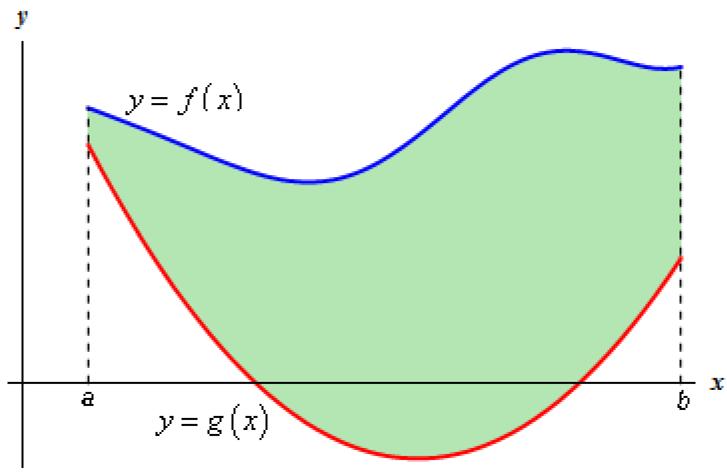
$$= a^{m+n+p} \frac{m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

Unit-V
Multiple Integrals

Area Between Curves

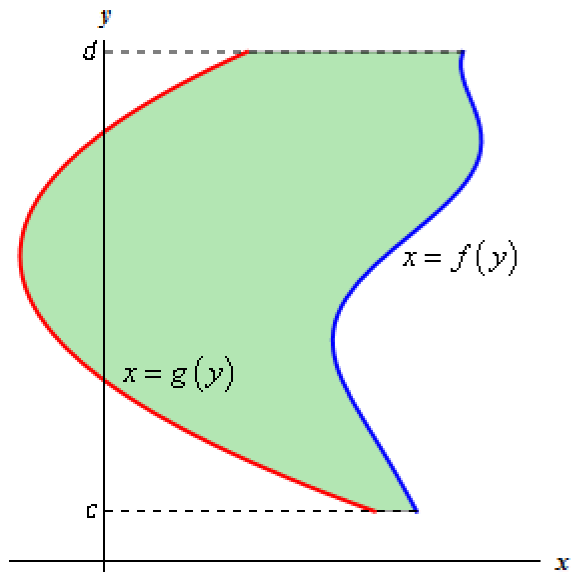
In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval [*a*,*b*]. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$A = \int_a^b f(x) - g(x)\,dx \qquad (1)$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval [*c*,*d*] with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\,dy \tag{2}$$

In the first case we will use,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \le x \le b$$

In the second case we will use,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \le y \le d$$

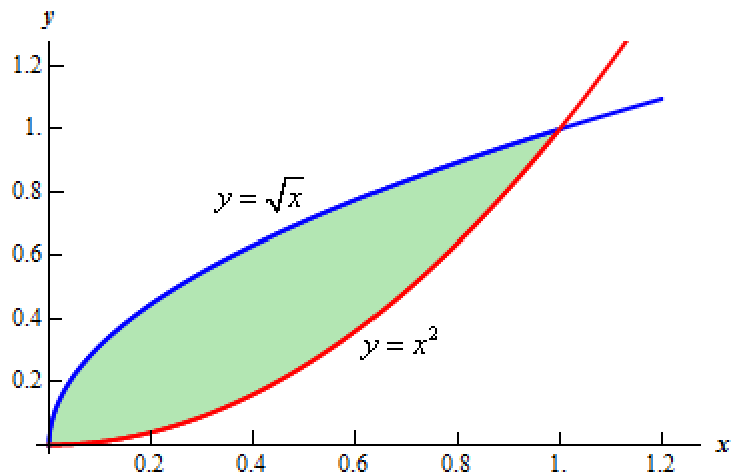
Using these formulas will always force us to think about what is going on with each problem and to make sure that we've got the correct order of functions when we go to use the formula.

Let's work an example.

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution

First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of x's that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the x's. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$ so these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$$

$$= \int_0^1 \sqrt{x} - x^2 \, dx$$

$$= \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \Bigg|_0^1$$

$$= \frac{1}{3}$$

Before moving on to the next example, there are a couple of important things to note.

First, in almost all of these problems a graph is pretty much required. Often the bounding region, which will give the limits of integration, is difficult to determine without a graph.

Also, it can often be difficult to determine which of the functions is the upper function and which is the lower function without a graph. This is especially true in cases like the last example where the answer to that question actually depended upon the range of *x*'s that we were using.

Let's work some more examples.

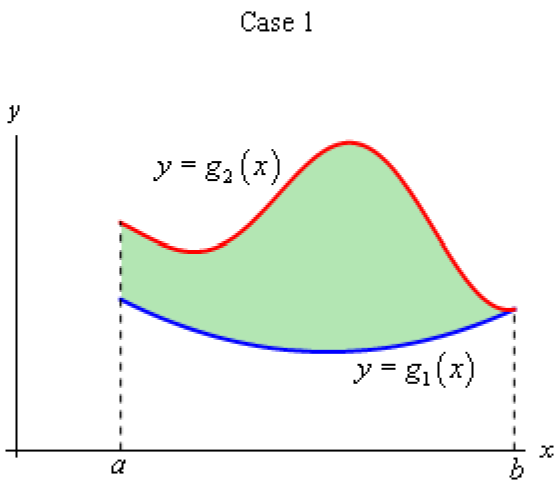
Double Integrals Over General Regions

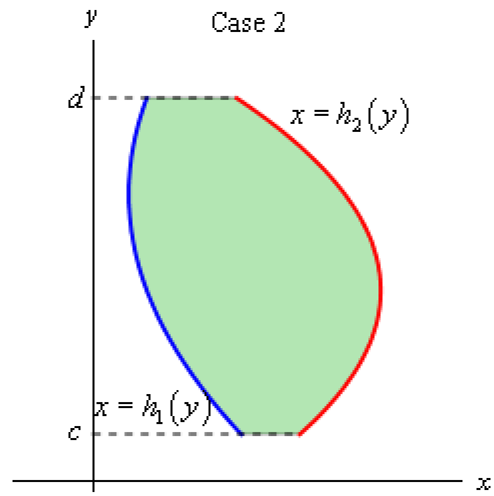
In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint_D f(x, y)\, dA$$

where *D* is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.





The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1

$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

In Case 2

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

Properties

1. $\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

2. $\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$, where *c* is any constant.

3. If the region *D* can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA$$

Example 1 Evaluate each of the following integrals over the given region *D*.

(a) $\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$

(b) $\iint\limits_D 4xy - y^3\,dA$, *D* is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

(c) $\iint\limits_D 6x^2 - 40\,y\,dA$, *D* is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Solution

(a) $\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$

Okay, this first one is set up to just use the formula above so let's do that.

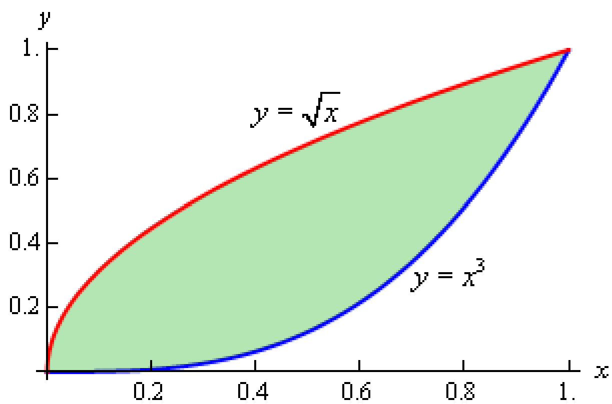
$$\iint_D \mathbf{e}^{\frac{x}{y}}\,dA = \int_1^2 \int_y^{y^3} \mathbf{e}^{\frac{x}{y}}\,dx\,dy = \int_1^2 y\mathbf{e}^{\frac{x}{y}}\Bigg|_y^{y^3}\,dy$$

$$= \int_1^2 y\mathbf{e}^{y^2} - y\mathbf{e}^1\,dy$$

$$= \left(\frac{1}{2}\mathbf{e}^{y^2} - \frac{1}{2}y^2\mathbf{e}^1\right)\Bigg|_1^2 = \frac{1}{2}\mathbf{e}^4 - 2\mathbf{e}^1$$

(b) $\iint_D 4xy - y^3\,dA$**,** ***D*** **is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$**.**

In this case we need to determine the two inequalities for x and y that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that that two inequalities are,

$$0 \le x \le 1 \qquad\qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral,

$$\iint\limits_D 4xy - y^3\, dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx$$

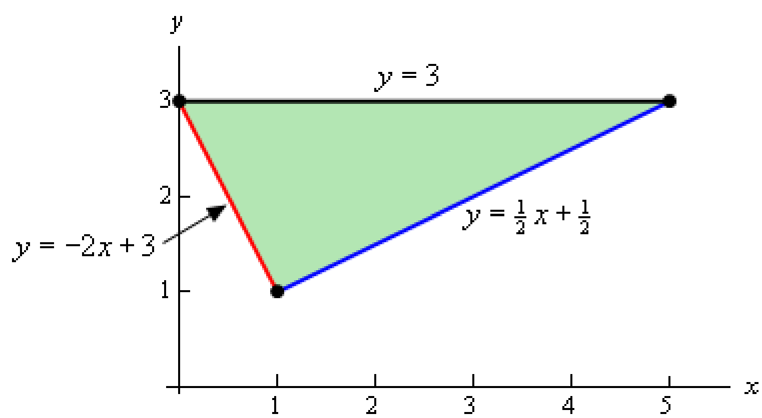
$$= \int_0^1 \left(2xy^2 - \frac{1}{4} y^4 \right) \Bigg|_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{7}{4} x^2 - 2x^7 + \frac{1}{4} x^{12}\, dx$$

$$= \left(\frac{7}{12} x^3 - \frac{1}{4} x^8 + \frac{1}{52} x^{13} \right) \Bigg|_0^1 = \frac{55}{156}$$

(c) $\iint\limits_D 6x^2 - 40y\, dA$, ***D* is the triangle with vertices** $(0,3)$**,** $(1,1)$**, and** $(5,3)$**.**

We got even less information about the region this time. Let's start this off by sketching the triangle.



Since we have two points on each edge it is easy to get the equations for each edge and so we'll leave it to you to verify the equations.

Now, there are two ways to describe this region. If we use functions of *x*, as shown in the image we will have to break the region up into two different pieces since the lower function is different depending upon the value of *x*. In this case the region would be given by $D = D_1 \cup D_2$ where,

$$D_1 = \left\{ (x, y) \,|\, 0 \le x \le 1, \; -2x + 3 \le y \le 3 \right\}$$

$$D_2 = \left\{ (x, y) \,|\, 1 \le x \le 5, \; \frac{1}{2}x + \frac{1}{2} \le y \le 3 \right\}$$

Note the $\cup$ is the "union" symbol and just means that *D* is the region we get by combing the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions.

To avoid this we could turn things around and solve the two equations for *x* to get,

$$y = -2x + 3 \qquad \Rightarrow \qquad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad \Rightarrow \qquad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \left\{ (x, y) \,|\, -\frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1, \; 1 \le y \le 3 \right\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Either way should give the same answer and so we can get an example in the notes of splitting a region up let's do both integrals.

Solution 1

$$\begin{aligned}\iint\limits_D 6x^2 - 40y\,dA &= \iint\limits_{D_1} 6x^2 - 40y\,dA + \iint\limits_{D_2} 6x^2 - 40y\,dA \\ &= \int_0^1 \int_{-2x+3}^3 6x^2 - 40y\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 6x^2 - 40y\,dy\,dx \\ &= \int_0^1 \left(6x^2y - 20y^2\right)\Big|_{-2x+3}^3 dx + \int_1^5 \left(6x^2y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^3 dx \\ &= \int_0^1 12x^3 - 180 + 20(3-2x)^2\,dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^2 dx \\ &= \left(3x^4 - 180x - \tfrac{10}{3}(3-2x)^3\right)\Big|_0^1 + \left(-\tfrac{3}{4}x^4 + 5x^3 - 180x + \tfrac{40}{3}\left(\tfrac{1}{2}x + \tfrac{1}{2}\right)^3\right)\Big|_1^5 \\ &= -\frac{935}{3}\end{aligned}$$

That was a lot of work. Notice however, that after we did the first substitution that we didn't multiply everything out. The two quadratic terms can be easily integrated with a basic Calc I substitution and so we didn't bother to multiply them out. We'll do that on occasion to make some of these integrals a little easier.

Solution 2

This solution will be a lot less work since we are only going to do a single integral.

$$\iint\limits_D 6x^2 - 40y\,dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} 6x^2 - 40y\,dx\,dy$$

$$= \int_1^3 \left(2x^3 - 40xy\right)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 100y - 100y^2 + 2(2y-1)^3 - 2\left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^3 dy$$

$$= \left(50y^2 - \tfrac{100}{3}y^3 + \tfrac{1}{4}(2y-1)^4 + \left(-\tfrac{1}{2}y + \tfrac{3}{2}\right)^4\right)\Big|_1^3$$

$$= -\frac{935}{3}$$

So, the numbers were a little messier, but other than that there was much less work for the same result

Triple Integrals

Now that we know how to integrate over a two-dimensional region we need to move on to integrating over a three-dimensional region. We used a double integral to integrate over a two-dimensional region and so it shouldn't be too surprising that we'll use a **triple integral** to integrate over a three dimensional region. The notation for the general triple integrals is,

$$\iiint\limits_E f(x,y,z)\,dV$$

Let's start simple by integrating over the box,

$$B = [a,b] \times [c,d] \times [r,s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is,

$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note that we integrated with respect to x first, then y, and finally z here, but in fact there is no reason to the integrals in this order. There are 6 different possible orders to do the integral in and which order you do the integral in will depend upon the function and the order that you feel will be the easiest. We will get the same answer regardless of the order however.

Let's do a quick example of this type of triple integral.

Example 1 Evaluate the following integral.

$$\iiint\limits_B 8xyz\,dV\;, \qquad B = [2,3]\times[1,2]\times[0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\iiint\limits_B 8xyz\,dV = \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy$$

$$= \int_1^2 \int_2^3 4xyz^2 \Big|_0^1 \,dx\,dy$$

$$= \int_1^2 \int_2^3 4xy\,dx\,dy$$

$$= \int_1^2 2x^2 y \Big|_2^3 \,dy$$

$$= \int_1^2 10y\,dy = 15$$

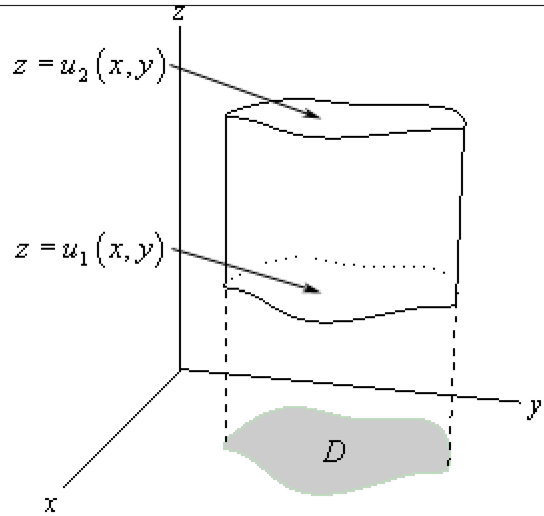
Before moving on to more general regions let's get a nice geometric interpretation about the triple integral out of the way so we can use it in some of the examples to follow.

Fact

The volume of the three-dimensional region *E* is given by the integral,

$$V = \iiint\limits_E dV$$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



In this case we define the region *E* as follows,

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

where $(x,y) \in D$ is the notation that means that the point (x,y) lies in the region *D* from the *xy*-plane. In this case we will evaluate the triple integral as follows,

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

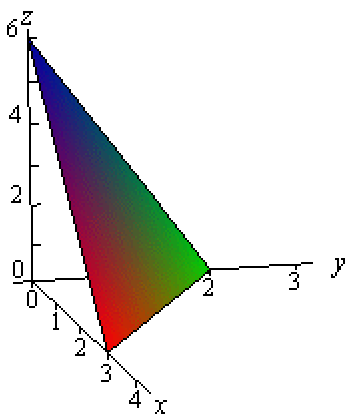
where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to *x*, we can integrate first with respect to *y*, or we can use polar coordinates as needed.

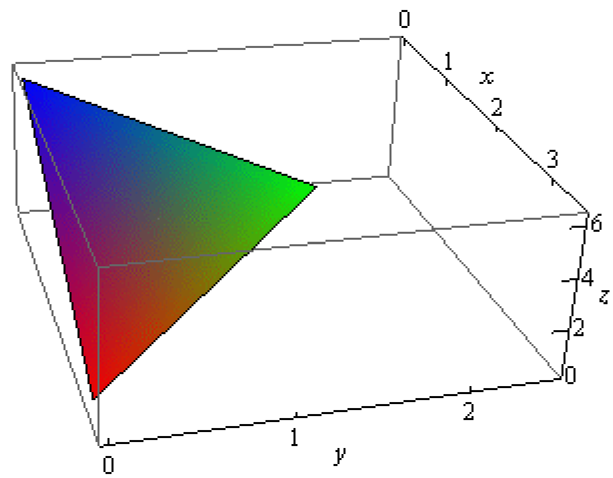
Example 2 Evaluate $\iiint\limits_E 2x\,dV$ where *E* is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

Solution

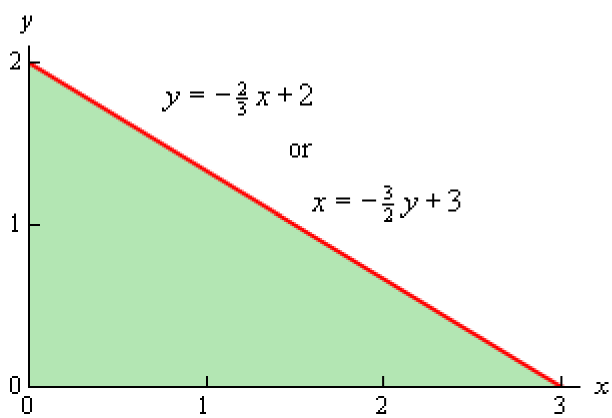
We should first define *octant*. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

Here is a sketch of the plane in the first octant.





We now need to determine the region D in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region D in the xy-plane. So D will be the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,2)$. Here is a sketch of D.



Now we need the limits of integration. Since we are under the plane and in the first octant (so we're above the plane $z = 0$) we have the following limits for *z*.

$$0 \le z \le 6 - 2x - 3y$$

We can integrate the double integral over *D* using either of the following two sets of inequalities.

$$0 \le x \le 3 \qquad\qquad 0 \le x \le -\frac{3}{2}y + 3$$

$$0 \le y \le -\frac{2}{3}x + 2 \qquad\qquad 0 \le y \le 2$$

Since neither really holds an advantage over the other we'll use the first one. The integral is then,

$$\begin{aligned}
\iiint_E 2x\,dV &= \iint_D \left[\int_0^{6-2x-3y} 2x\,dz \right] dA \\
&= \iint_D 2xz \Big|_0^{6-2x-3y} dA \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\left(6 - 2x - 3y\right) dy\,dx \\
&= \int_0^3 \left(12xy - 4x^2y - 3xy^2\right)\Big|_0^{-\frac{2}{3}x+2} dx \\
&= \int_0^3 \frac{4}{3}x^3 - 8x^2 + 12x\,dx \\
&= \left(\frac{1}{3}x^4 - \frac{8}{3}x^3 + 6x^2\right)\Big|_0^3 \\
&= 9
\end{aligned}$$

Change of Variables

$$\int_a^b f\left(g(x)\right)g'(x)\,dx = \int_c^d f(u)\,du \qquad \text{where} \quad u = g(x)$$

In essence this is taking an integral in terms of *x*'s and changing it into terms of *u*'s. We want to do something similar for double and triple integrals. In fact we've already done this to a certain extent when we converted double integrals to polar coordinates and when we converted triple integrals to cylindrical or spherical coordinates. The main difference is that we didn't actually go through the details of where the formulas came from. If you recall, in each of those cases we commented that we would justify the formulas for *dA* and *dV* eventually. Now is the time to do that justification.

While often the reason for changing variables is to get us an integral that we can do with the new variables, another reason for changing variables is to convert the region into a nicer region to work with. When we were converting the polar, cylindrical or spherical coordinates we didn't worry about this change since it was easy enough to determine the new limits based on the given region. That is not always the case however. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables.

First we need a little notation out of the way. We call the equations that define the change of variables a **transformation**. Also we will typically start out with a region, *R*, in *xy*-coordinates and transform it into a region in *uv*-coordinates.

Example 1 Determine the new region that we get by applying the given transformation to the region *R*.

(a) *R* is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.

(b) *R* is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

Solution

(a) ***R*** **is the ellipse** $x^2 + \frac{y^2}{36} = 1$ **and the transformation is** $x = \frac{u}{2}$, $y = 3v$.

There really isn't too much to do with this one other than to plug the transformation into the equation for the ellipse and see what we get.

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

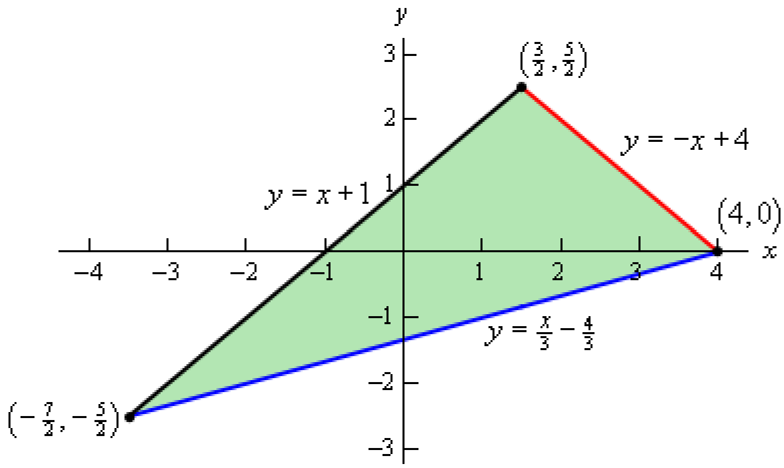
$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

So, we started out with an ellipse and after the transformation we had a disk of radius 2.

(b) ***R*** **is the region bounded by** $y = -x + 4$, $y = x + 1$, **and** $y = \frac{x}{3} - \frac{4}{3}$ **and the transformation is** $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

As with the first part we'll need to plug the transformation into the equation, however, in this case we will need to do it three times, once for each equation. Before we do that let's sketch the graph of the region and see what we've got.



So, we have a triangle. Now, let's go through the transformation. We will apply the transformation to each edge of the triangle and see where we get.

Let's do $y = -x + 4$ first. Plugging in the transformation gives,

$$\begin{aligned} \frac{1}{2}(u - v) &= -\frac{1}{2}(u + v) + 4 \\ u - v &= -u - v + 8 \\ 2u &= 8 \\ u &= 4 \end{aligned}$$

The first boundary transforms very nicely into a much simpler equation.

Now let's take a look at $y = x + 1$,