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Vector Algebra

Electromagnetics (EM):-

Electromagnetics is a branch of physics or electrical Engineering in which electric & magnetic phenomena are studied.

Maxwell's Equations:-

$$\nabla \cdot D = \rho_v$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

where,

∇ = Vector differential operator

D = The electric flux density

B = Magnetic " "

E = Electric field Intensity

H = Magnetic " "

ρ_v = Volume charge density

J = Current density.

Scalar:-

A "scalar" is a quantity that has only magnitude.

Eg:- Time, Mass, distance, temperature, Population, Electric potential ...

vector:-

A "vector" is a quantity that has both magnitude & direction.

Eg :- velocity, force, displacement, electric field Intensity

Field:-

A "field" is a function that specifies a particular quantity Everywhere in a region.

If the quantity is a scalar (or vector), the field is said to be a scalar (or vector).

Eg:- Scalar fields are temperature distribution in building

Sound Intensity in a Theater,

Electric potential in a region,

Refractive Index of a stratified medium.

The Gravitational force on a body in space & the velocity of raindrops in the atmosphere are Examples of vector fields.

Unit vector:-

- The magnitude of 'A' is a scalar written as A (or) |A|.
- A unit vector 'a_A' along A is defined as a vector whose magnitude is unity (i.e, 1) & its direction is along A, i.e.,

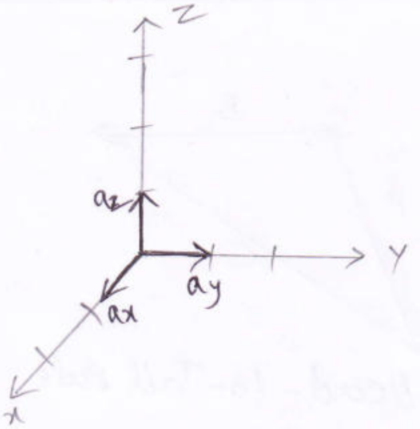
$$a_A = \frac{\bar{A}}{|\bar{A}|} = \frac{A}{A} = 1.$$

$$a_A = \frac{A}{|A|} \Rightarrow A = |A| \, a_A.$$

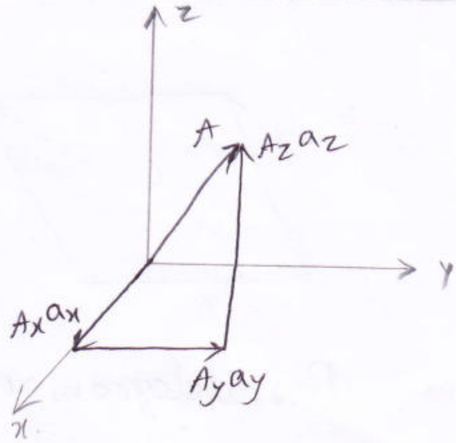
'A' in terms of its magnitude |A| & its direction 'a_A'.

'A' ⇒ is a vector, in Cartesian (or Rectangular) Coordinates may be represented as,

$$(A_x, A_y, A_z) \text{ (or) } A_x a_x + A_y a_y + A_z a_z.$$



Unit vectors a_x, a_y & a_z



Components of 'A' along a_x, a_y & a_z.

The magnitude of 'A' (vector) is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

& the unit vector along 'A' is,

$$a_A = \frac{A_x a_x + A_y a_y + A_z a_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Vector Addition & Subtraction:-

Two vectors A & B can be added together to give another vector 'C', i.e.,

$$C = \bar{A} + \bar{B}$$

if, $\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$

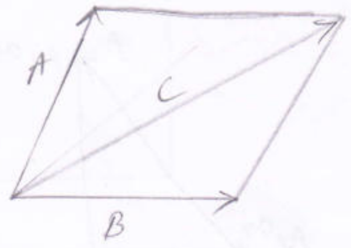
$$\bar{C} = (A_x + B_x) a_x + (A_y + B_y) a_y + (A_z + B_z) a_z$$

Vector subtraction,

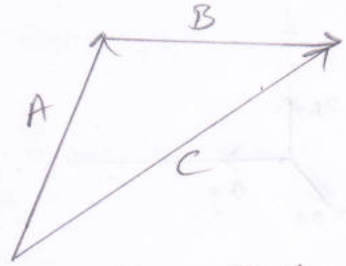
$$\bar{D} = \bar{A} - \bar{B} = \bar{A} + (-\bar{B})$$

$$= (A_x - B_x) a_x + (A_y - B_y) a_y + (A_z - B_z) a_z$$

$\bar{C} = \bar{A} + \bar{B}$

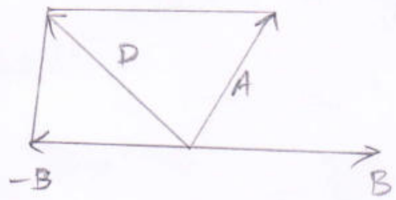


Parallelogram rule

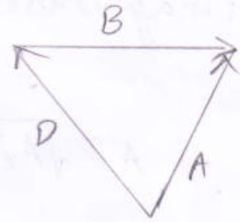


Head-to-Tail rule

$\bar{D} = \bar{A} - \bar{B}$



Parallelogram rule

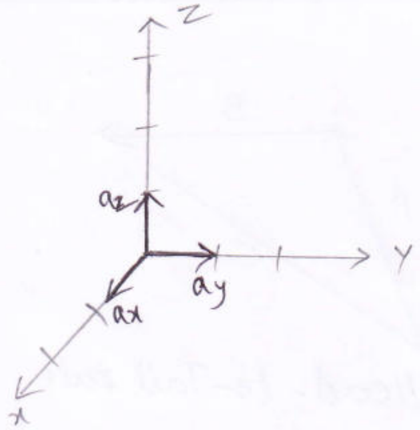


Head-to-Tail rule

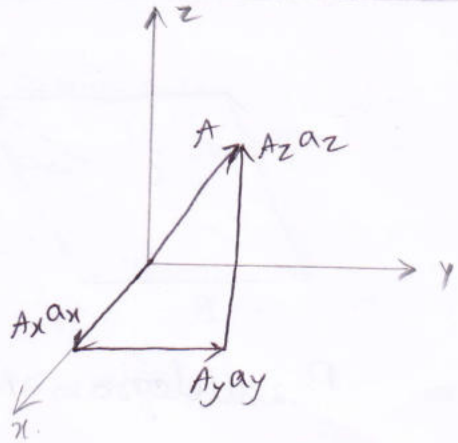
The three basic laws of algebra obeyed by any given vectors 'A', 'B', 'C' are summarized as,

Law	Addition	Multiplication
Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$K\bar{A} = \bar{A}K$
Associative	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$K(\lambda\bar{A}) = (K\lambda)\bar{A}$
Distributive	$K(A+B) = KA + KB$	—

K & $\lambda \Rightarrow$ Scalars.



Unit vectors a_x, a_y & a_z



Components of 'A' along a_x, a_y & a_z

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Vector Addition & Subtraction:-

Two vectors A & B can be added together to give another vector 'C', i.e.,

$$C = \bar{A} + \bar{B}$$

if, $\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$

$$\bar{C} = (A_x + B_x) a_x + (A_y + B_y) a_y + (A_z + B_z) a_z$$

vector subtraction,

$$\bar{D} = \bar{A} - \bar{B} = \bar{A} + (-\bar{B})$$

$$= (A_x - B_x) a_x + (A_y - B_y) a_y + (A_z - B_z) a_z$$

Position & Distance vectors:-

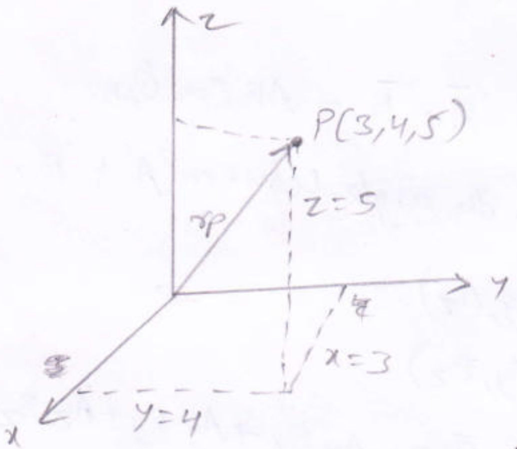
The position vector r_p (or radius vector) of point 'P' is as the directed distance from the origin 'O' to 'P' i.e.,

$$\bar{r}_p = OP = x a_x + y a_y + z a_z.$$

The position vector of point P is useful in defining its position in space,

Eg:- P(3, 4, 5)

$$\bar{r}_p = OP = 3a_x + 4a_y + 5a_z.$$



A "distance vector" (or Seperation vector) is the displacement from one point to another.

If two points P & Q are, (x_p, y_p, z_p) & (x_q, y_q, z_q).

$$\bar{r}_{pq} = r_Q - r_p$$
$$= (x_Q - x_p) a_x + (y_q - y_p) a_y + (z_q - z_p) a_z.$$

Vector Multiplication:-

When two vectors 'A' & 'B' are multiplied, the result is either a scalar or a vector depending on how they are multiplied

1. Scalar (or dot) Product $\Rightarrow \bar{A}\cdot\bar{B}$
2. Vector (or cross) Product $\Rightarrow \bar{A}\times\bar{B}$
3. Scalar triple product $\Rightarrow \bar{A}\cdot(\bar{B}\times\bar{C})$
4. Vector triple product $\Rightarrow \bar{A}\times(\bar{B}\times\bar{C})$

Dot product:-

The dot product of two vectors $\bar{A}$ & $\bar{B}$, as $\bar{A}\cdot\bar{B}$, is defined geometrically as the product of the magnitudes of A & B and the Cosine of the angle between them.

$$\bar{A}\cdot\bar{B} = AB\cos\theta_{AB}$$

$\Rightarrow \theta_{AB}$ is the angle between 'A' & 'B'.

If $\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$

then $\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z$

Two vectors 'A' & 'B' are said to be orthogonal (⊥er) with each other, if, $\bar{A}\cdot\bar{B} = AB\cos 90^\circ$

$\bar{A}\cdot\bar{B} = 0.$

Dot product obeys following laws:-

→ Commutative law: $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$.

→ Distributive law: $\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$

$\bar{A}\cdot\bar{A} = |\bar{A}|^2 = A^2$

→ $a_x\cdot a_y = a_y\cdot a_z = a_z\cdot a_x = 0.$

$a_x\cdot a_x = a_y\cdot a_y = a_z\cdot a_z = 1.$

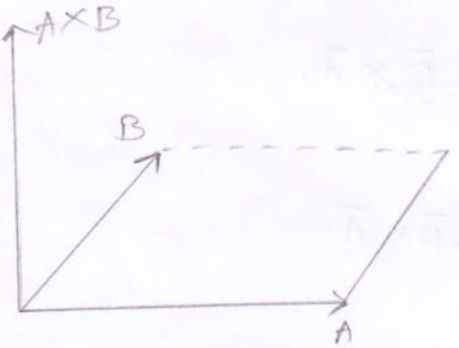
Cross product:-

The cross product of two vectors $\bar{A}$ & $\bar{B}$, as $\bar{A}\times\bar{B}$ is a vector quantity whose magnitude is the area of the parallelopiped formed by $\bar{A}$ & $\bar{B}$ & is in the direction of advance of a right-handed screw as $\bar{A}$ is turned into $\bar{B}$.

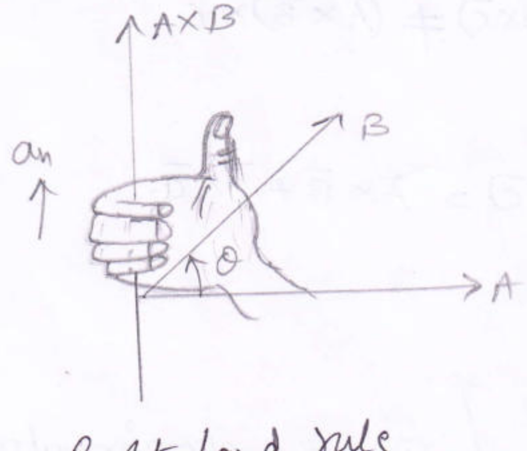
$$\bar{A}\times\bar{B} = AB\sin\theta_{AB}\; a_n.$$

$a_n \rightarrow$ Unit vector normal to the plane containing A & B.

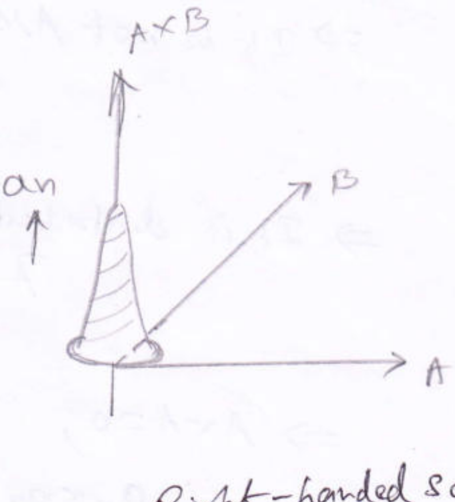
The direction of a_n is taken as the direction of the right thumb when the fingers of the right hand rotate from A to B. Alternatively, the direction of a_n is taken as that of the advance of a right-handed screw as A is turned into B.



⟹ The cross product of $\bar{A}$ & $\bar{B}$ is a vector with magnitude equal to the area of parallelogram & direction as indicated.



Right hand rule



Right-handed screw rule.

The vector multiplication is called cross product due to the cross sign, it is also called vector product because result is a vector.

If $\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$ then,

$$\bar{A} \times \bar{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y) a_x + (A_z B_x - A_x B_z) a_y + (A_x B_y - A_y B_x) a_z.$$

The cross product has the following properties:-

⇒ ~~not~~ Commutative:

$$\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}.$$

It is anti Commutative:

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}.$$

⇒ It is not Associative:

$$\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C}.$$

⇒ It is distributive:

$$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}.$$

⇒ $A \times A = 0$,

$$\left.\begin{aligned} a_x \times a_y &= a_z \\ a_y \times a_z &= a_x \\ a_z \times a_x &= a_y \end{aligned}\right\} \text{It is cyclic permutation.}$$

Cross product:-

The cross product of two vectors $\bar{A}$ & $\bar{B}$, as $\bar{A}\times\bar{B}$ is a vector quantity whose magnitude is the area of the parallelopiped formed by $\bar{A}$ & $\bar{B}$ & is in the direction of advance of a right-handed screw as $\bar{A}$ is turned into $\bar{B}$.

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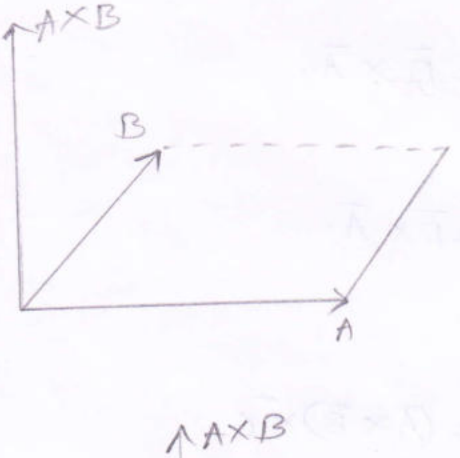
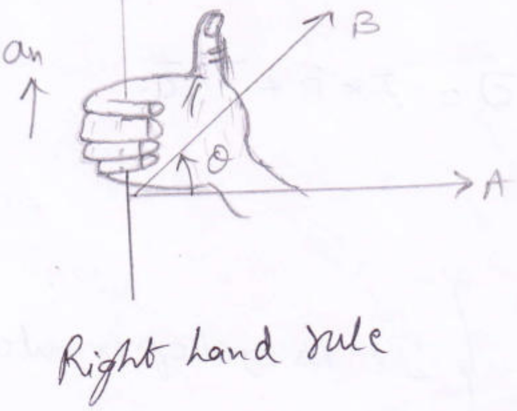
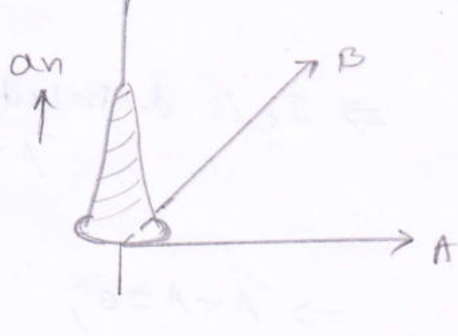


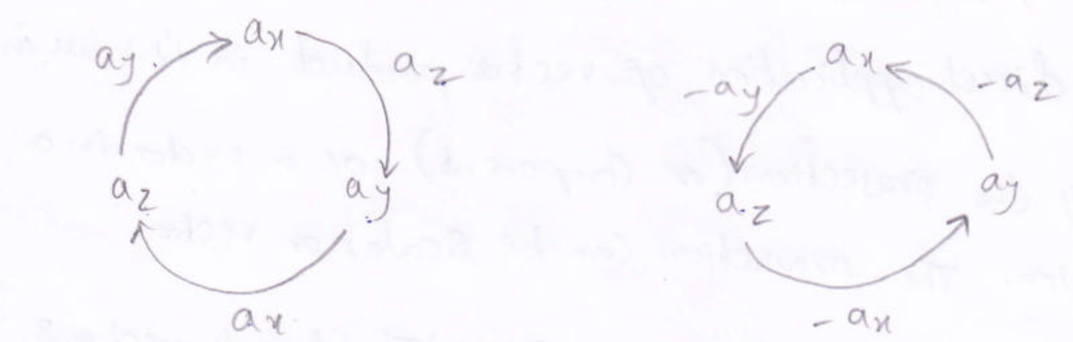
$\Rightarrow$ The cross product of $\bar{A}$ & $\bar{B}$ is a vector with magnitude equal to the area of parallelogram & direction as indicated.



Right hand rule



Right-handed screw rule.



Moving clockwise leads to Positive results.

Moving Counterclockwise leads to negative results.

Scalar Triple product:-

Let three vectors $\bar{A}$, $\bar{B}$ & $\bar{C}$, the scalar triple product as,

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{C}\times\bar{A}) = \bar{C}\cdot(\bar{A}\times\bar{B}).$$

If $\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$

$\bar{C} = (C_x, C_y, C_z)$ then,

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The result of this vector multiplication is scalar.

Vector Triple product:-

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

Obtained using 'bac - cab' rule.

$$(\bar{A}\cdot\bar{B})\bar{C} \neq \bar{A}(\bar{B}\cdot\bar{C})$$

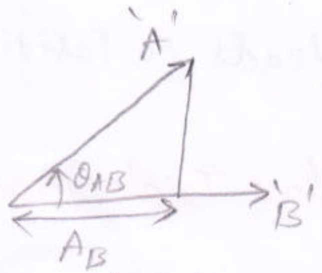
but, $(\bar{A}\cdot\bar{B})\bar{C} = \bar{C}(\bar{A}\cdot\bar{B})$

Components of a vector:-

A direct application of vector product is its use in determining the projection (or Component) of a vector in a given direction. The projection can be Scalar or vector.

Eg:- The scalar Component A_B of 'Ā' along the vector B,

$$A_B = A \cos \theta_{AB}$$
$$= |\bar{A}|\,|a_B| \cos \theta_{AB}.$$

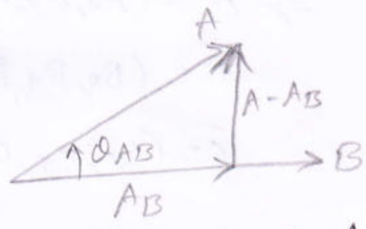


Scalar Component A_B

$$A_B = \bar{A} \cdot a_B.$$

The vector Component A_B of A along B is simply the scalar Component, multiplied by a unit vector along B,

i.e.,
$$\bar{A}_B = (A_B \cdot a_B)\, a_B.$$



Vector Component A_B.

CO-ordinate Systems & Transformation

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or non orthogonal.

⇒ An orthogonal system is one in which the coordinates are mutually perpendicular.

The coordinate systems are,

* The Cartesian (or) Rectangular
* Circular cylindrical
* Spherical
* Elliptic cylindrical
* Parabolic cylindrical
* Conical
* Prolate spheroidal
* Oblate spheroidal
* Ellipsoidal.

Cartesian Coordinates (x, y, z):-

A point P can be represented as (x, y, z). The ranges of the coordinate variables x, y, z are,

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

A vector 'A' in Cartesian, as,

$$(A_x, A_y, A_z) \text{ (or) } A_x a_x + A_y a_y + A_z a_z.$$

a_x, a_y, a_z } unit vectors. A_x, A_y, A_z } products. x, y, z } direction

Circular cylindrical Coordinates (ρ, ϕ, z):-

A point P in cylindrical coordinates is represented as (ρ, ϕ, z).

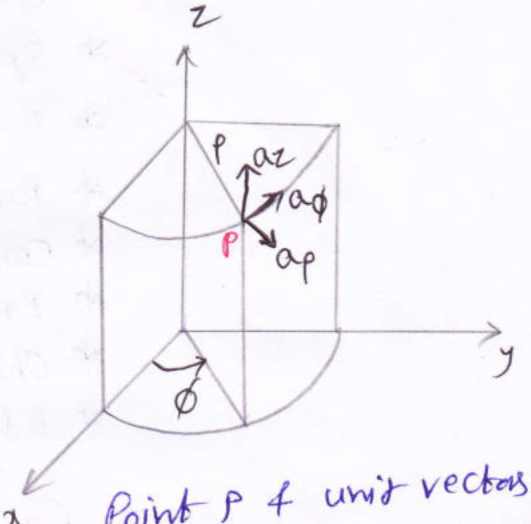
$\rho \rightarrow$ Radius of the Cylinder passing through P.
[Radial distance from the z-axis].

$\phi \rightarrow$ azimuthal angle, measured from x-axis in the xy-plane.

$z \rightarrow$ Same as in Cartesian System.

Ranges of the variables,

$$0 \leq \rho < \infty$$
$$0 \leq \phi < 2\pi$$
$$-\infty < z < \infty$$



Point P & unit vectors in the cylindrical Coordinate System

A vector $\bar{A}$ in cylindrical coordinates,

(A_ρ, A_ϕ, A_z) (or)

$$A_\rho a_\rho + A_\phi a_\phi + A_z a_z.$$

$$|\bar{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$$

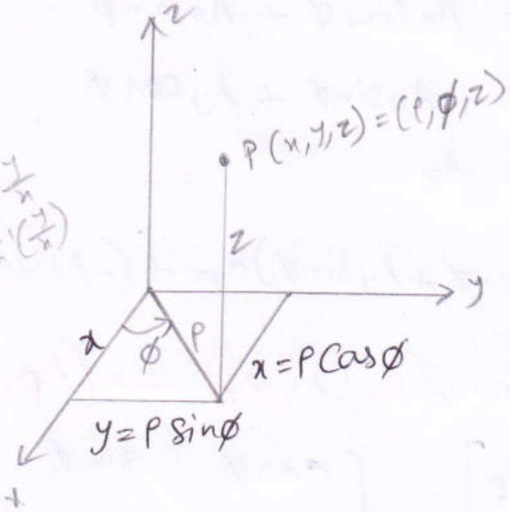
a_ρ, a_ϕ, a_z } mutually ⊥er

$$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1.$$
$$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0.$$

$$a_\rho \times a_\phi = a_z$$
$$a_\phi \times a_z = a_\rho$$
$$a_z \times a_\rho = a_\phi.$$

Transformation:-

Relation-Ship between (x, y, z) & (ρ, ϕ, z).

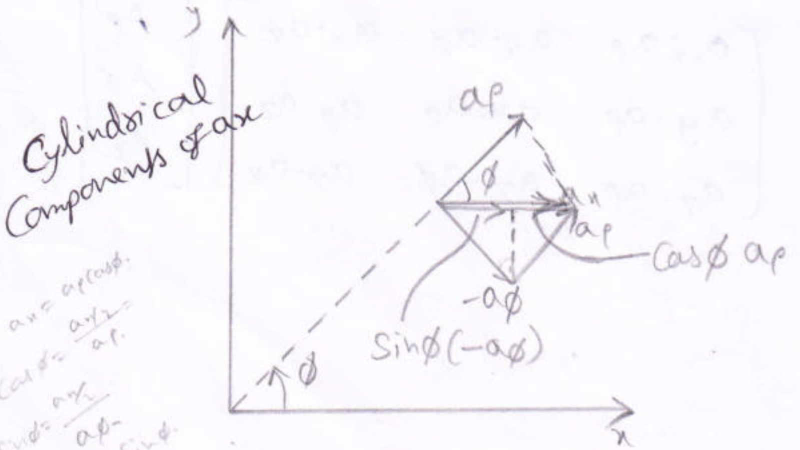


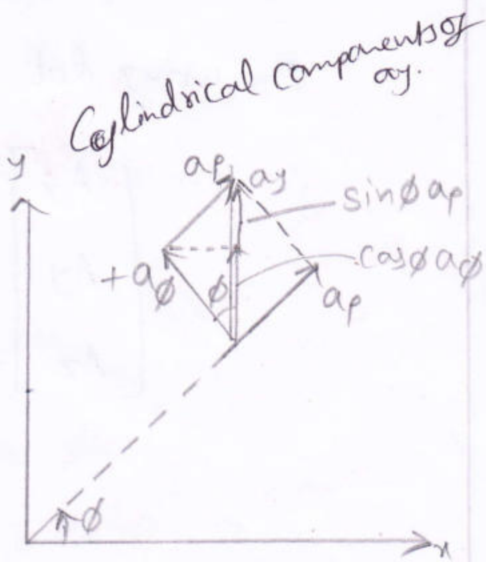
The relationship between the variables (x, y, z) of the Cartesian Coordinate system & those of the cylindrical system (ρ, ϕ, z) are,

$$\rho = \sqrt{x^2+y^2}\ ,\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)\ ,\quad z=z.$$

$$x = \rho\cos\phi\ ,\quad y = \rho\sin\phi\ ,\quad z=z.$$

The relationships between (a_x, a_y, a_z) & (a_ρ, a_ϕ, a_z) from unit vector transformation.





$$a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi$$
$$a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi$$
$$a_z = a_z$$

$$a_\rho = \cos\phi\, a_x + \sin\phi\, a_y$$
$$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$$
$$a_z = a_z$$

from the unit vectors,

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z.$$

$$\vec{A} = (A_x \cos\phi + A_y \sin\phi) a_\rho + (-A_x \sin\phi + A_y \cos\phi) + A_z a_z.$$

In matrix form, $(A_x, A_y, A_z) \rightarrow (A_\rho, A_\phi, A_z)$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

By using dot product,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} a_x \cdot a_\rho & a_x \cdot a_\phi & a_x \cdot a_z \\ a_y \cdot a_\rho & a_y \cdot a_\phi & a_y \cdot a_z \\ a_z \cdot a_\rho & a_z \cdot a_\phi & a_z \cdot a_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Spherical Coordinates:-

$\Rightarrow P(r, \theta, \phi)$.

Ranges, $\Rightarrow$ $0 \leq r < \infty$

$0 \leq \theta \leq \pi$

$0 \leq \phi < 2\pi$.

(A_r, A_θ, A_ϕ) , $A_r a_r + A_\theta a_\theta + A_\phi a_\phi$.

$|A| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$

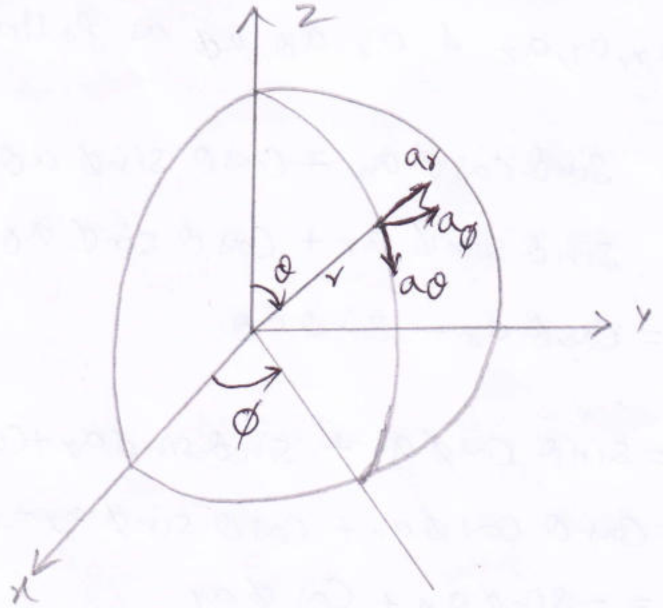
$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$

$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$

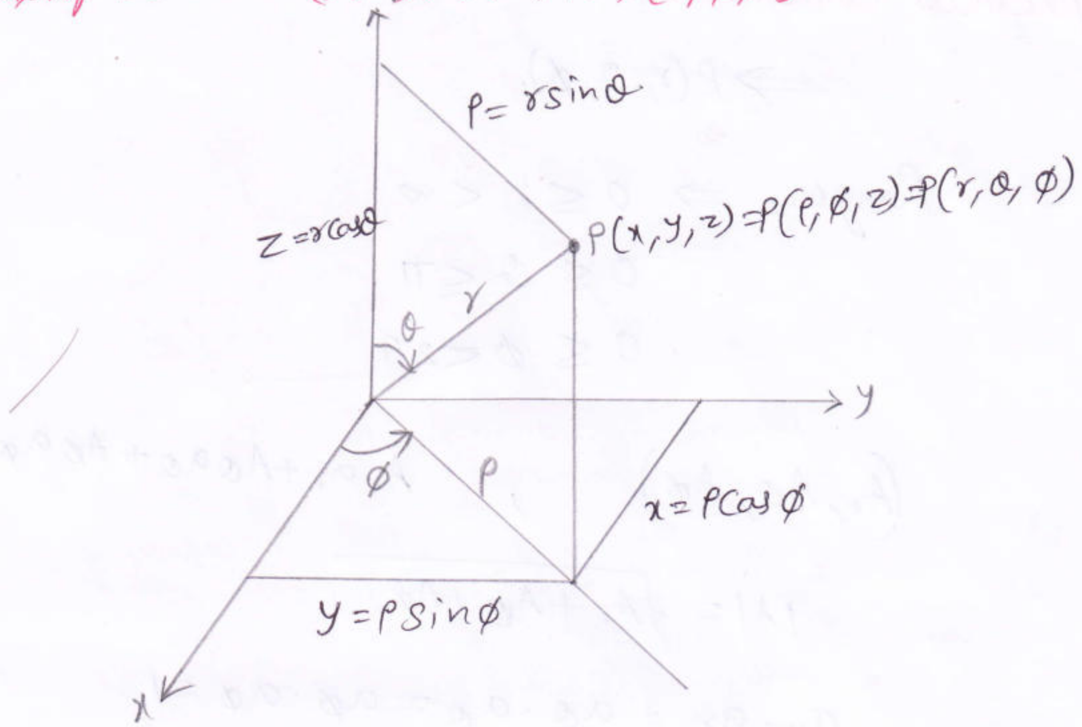
$a_r \times a_\theta = a_\phi$

$a_\theta \times a_\phi = a_r$

$a_\phi \times a_r = a_\theta$



Relationship between (x, y, z), (r, θ, ϕ), (ρ, ϕ, z):-



$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$z = r\cos\theta.$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

The unit vectors a_x, a_y, a_z & a_r, a_θ, a_ϕ as follows:

$$a_x = \sin\theta\cos\phi\, a_r + \cos\theta\sin\phi\, a_\theta - \sin\phi\, a_\phi$$

$$a_y = \sin\theta\sin\phi\, a_r + \cos\theta\sin\phi\, a_\theta + \cos\phi\, a_\phi$$

$$a_z = \cos\theta\, a_r - \sin\theta\, a_\theta.$$

$$a_r = \sin\theta\cos\phi\, a_x + \sin\theta\sin\phi\, a_y + \cos\theta\, a_z$$

$$a_\theta = \cos\theta\cos\phi\, a_x + \cos\theta\sin\phi\, a_y - \sin\theta\, a_z$$

$$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$$

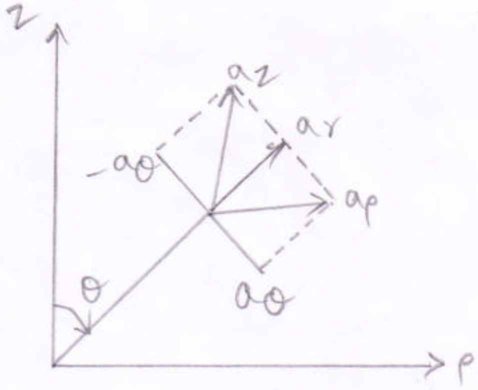
$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} a_r \cdot a_x & a_r \cdot a_y & a_r \cdot a_z \\ a_\theta \cdot a_x & a_\theta \cdot a_y & a_\theta \cdot a_y \\ a_\phi \cdot a_x & a_\phi \cdot a_y & a_\phi \cdot a_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

*The distance d between two points with position vectors r_1 & r_2 is,

$$d = |r_2 - r_1|$$



Unit vector transformations for Cylindrical & spherical coordinates

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \quad \{\text{Cartesian}\}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2-\phi_1) + (z_2-z_1)^2 \quad \{\text{Cylindrical}\}$$

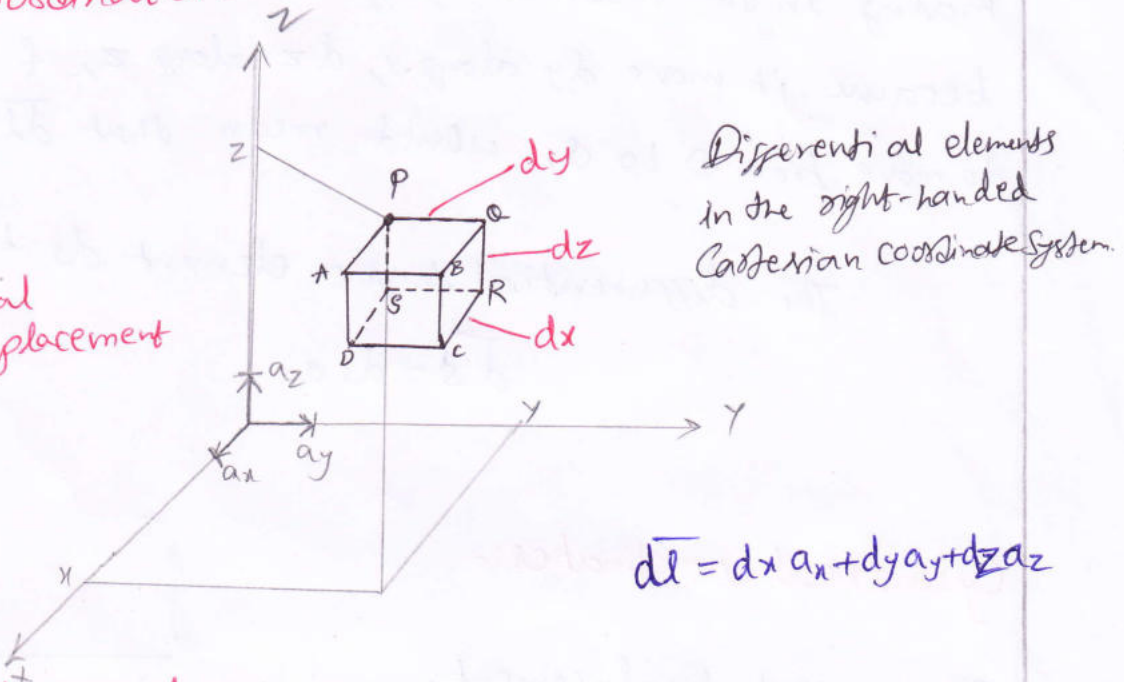
$$d^2 = r_2^2 + r_1^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\phi_2-\phi_1) \quad \{\text{Spherical}\}$$

Vector Calculus

Differential length, Area & Volume:-

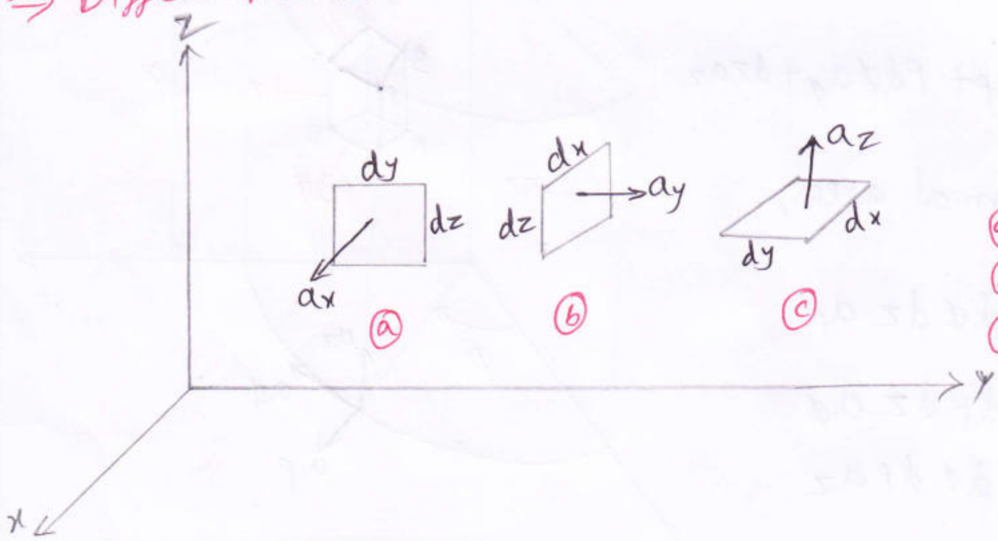
Cartesian Coordinates:-

⇒ Differential displacement



$\overline{dl} = dx\, a_x + dy\, a_y + dz\, a_z$

⇒ Differential normal areas



(a) $\overline{ds} = dy\, dz\, a_x$

(b) $\overline{ds} = dx\, dz\, a_y$

(c) $\overline{ds} = dx\, dy\, a_z$

⇒ Differential volume

$$dv = dx\, dy\, dz$$

⇒ $\overline{dl}$ & $\overline{ds}$ are vectors

⇒ dv scalar.

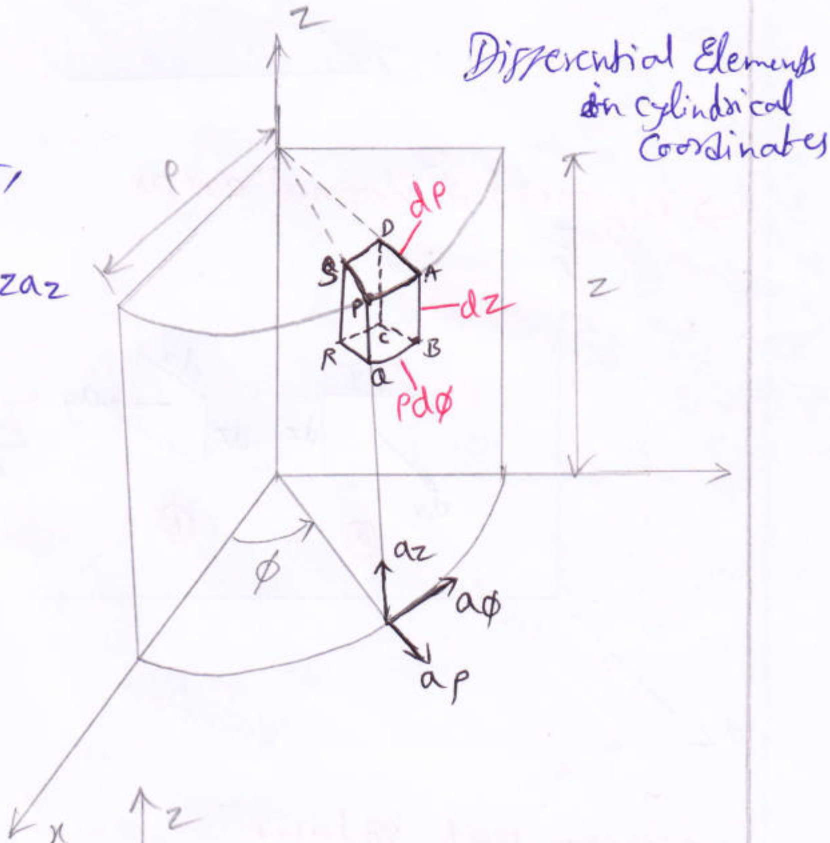
From figure, from point P to Q, $\overline{dl} = dy\, a_y$ because moving in the Y-direction, if it move from Q to S, $\overline{dl} = dy a_y + dz a_z$, because it move dy along y, dz along z, & $dx = 0$. Similarly, to move from D to Q, would mean that $\overline{dl} = dx a_x + dy a_y + dz a_z$.

The differential surface element ds is,

$$\overline{ds} = ds\, a_n$$

Cylindrical Coordinates:-

Differential Elements in cylindrical Coordinates



Differential displacement,

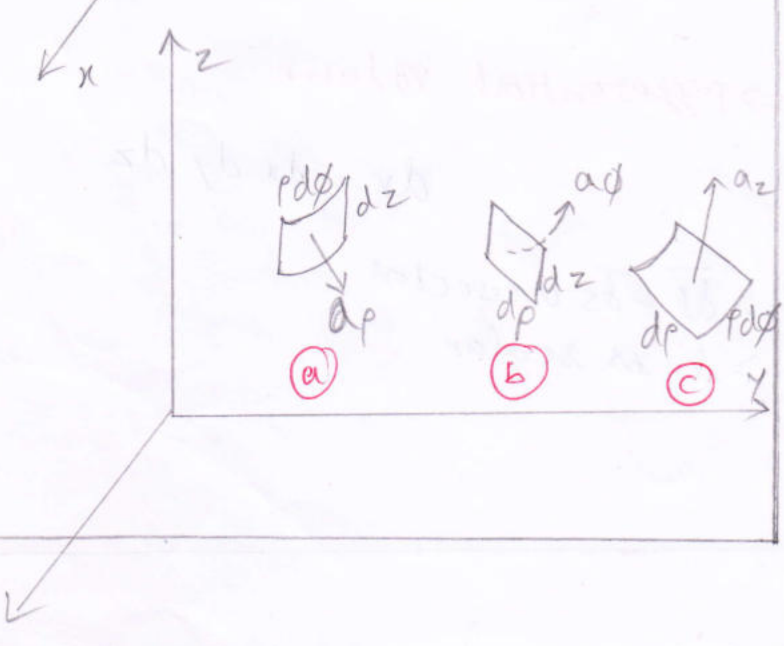
$$\overline{dl} = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$$

Differential normal area,

ⓐ $\overline{ds} = \rho\, d\phi\, dz\, a_\rho$

ⓑ $\overline{ds} = d\rho\, dz\, a_\phi$

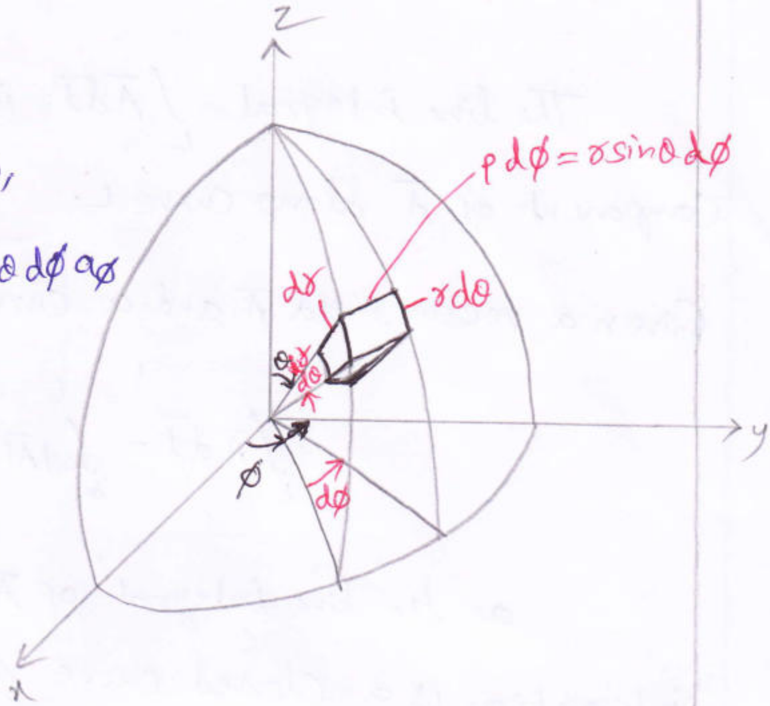
ⓒ $\overline{ds} = \rho\, d\phi\, d\rho\, a_z$



Differential volume is,

$$dv = \rho\, d\rho\, d\phi\, dz$$

Spherical Coordinates:-



⇒ Differential displacement is,

$$\overline{dl} = dr\, a_r + r\, d\theta\, a_\theta + r\sin\theta\, d\phi\, a_\phi$$

⇒ Differential normal area is,

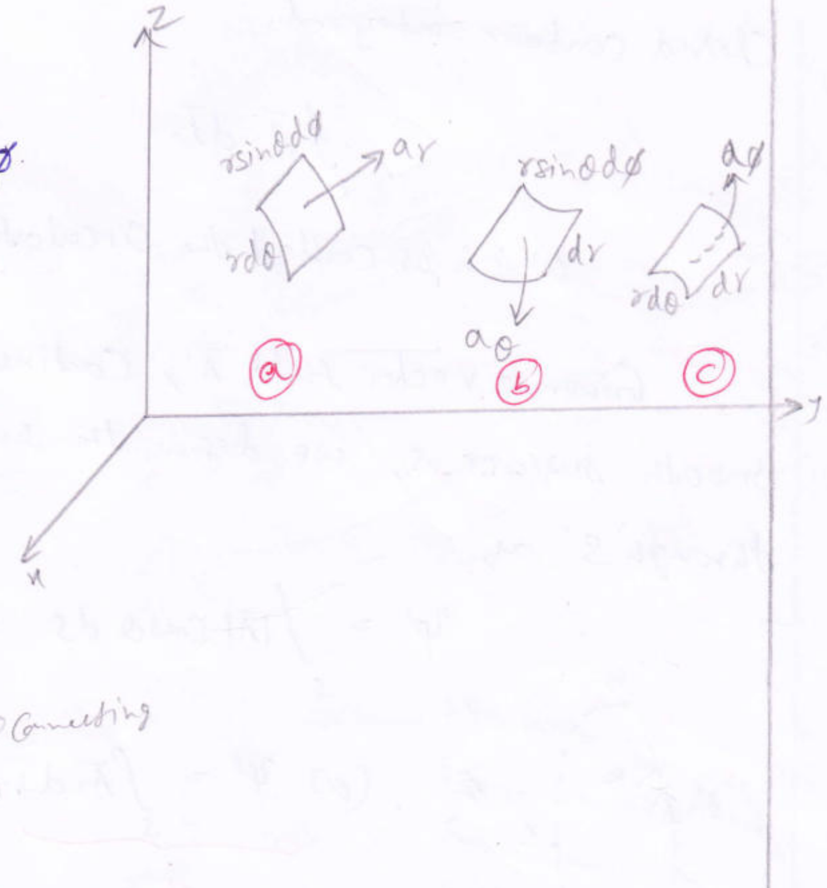
(a) $\overline{ds} = r^2 \sin\theta\, d\theta\, d\phi\, a_r$

(b) $\overline{ds} = r\sin\theta\, dr\, d\phi\, a_\theta$

(c) $\overline{ds} = r\, dr\, d\theta\, a_\phi$

⇒ Differential volume is,

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$



Contours: A line drawn on a map connecting points of equal points.

Line, Surface, & Volume Integrals:-

The line integral $\int_L \bar{A}\, d\bar{l}$. is the integral of the tangential Component of $\bar{A}$ along Curve L.

Given a vector field $\bar{A}$ and a curve L, the integral is,

$$\int_L \bar{A}\, d\bar{l} = \int_a^b |\bar{A}| \cos\theta\, dl$$

as the 'line integral' of $\bar{A}$ around L. If the path of integration is a closed curve such as abca, becomes a Closed contour integral.

$$\oint_L \bar{A} \cdot d\bar{l}.$$

Which is called the Circulation of '$\bar{A}$' around L.

Given a vector field '$\bar{A}$', Continuous in a region Containing the smooth surface S, we define the surface integral or the flux of $\bar{A}$ through 'S' as,

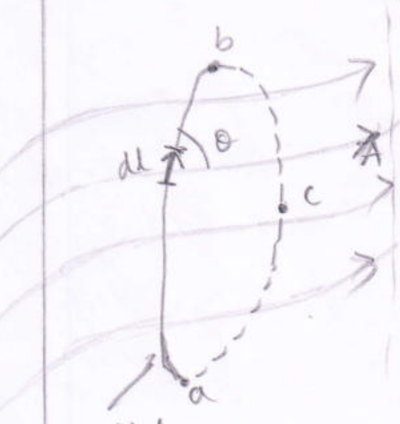
"$a_n \rightarrow$ unit normal to S."

$$\psi = \int_S |\bar{A}| \cos\theta\, ds = \int_S \bar{A} \cdot a_n \cdot d\bar{s}.$$

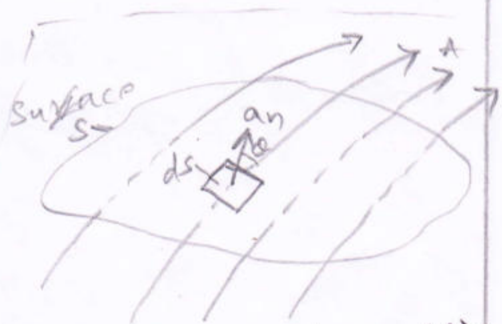
$$(\text{or})\ \psi = \int_S \bar{A} \cdot d\bar{s}.$$

⇓ at any point on 's'.

~~for a closed~~



Path of integration of vector field 'A'.



The flux of a vector field 'A', through surface 'S'.

Flux: The rate of flow of energy (or) particles across a given surface

~~For a closed surface,~~

$$\Psi = \oint_S \bar{A} \cdot d\bar{s},$$

Which is referred to as the net outward flux of '$\bar{A}$' forms.

A closed path defines an open surface whereas a closed surface defines a volume. Then,

$$\int_v \rho_v \, dv.$$

The volume integral of the scalar ρ_v over the volume v.

The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by $\bar{A}$(or) ρ_v.

DEL Operator:-

The del operator, written ∇, is the vector differential operator. In Cartesian Coordinates,

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \quad -(1)$$

The vector differential operator, otherwise known as the gradient operator, is not a vector in itself, but when it operates on a scalar function,

Flux: The lines of force surrounding a permanent magnet.

The operator is useful in defining,

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector '$\bar{A}$', written as $\nabla \cdot \bar{A}$
3. The curl of a vector '$\bar{A}$', written as $\nabla \times \bar{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

The del operator ∇ in cylindrical & spherical coordinates,

To obtain ∇ in terms of ρ, ϕ, z,

from the conversions $A_x = \cos\phi A_\rho - \sin\phi A_\phi$, $A_y = \sin\phi A_\rho + \cos\phi A_\phi$

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\phi = \frac{y}{x}.$$

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \quad -(2)$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi} \quad -(3)$$

$$\nabla = \left(\cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\right) a_x + \left(\sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\right) a_y + \frac{\partial}{\partial z} a_z$$

$$= \frac{\partial}{\partial \rho}(\cos\phi\, a_x + a_y \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(-\sin\phi + \cos\phi) + \frac{\partial}{\partial z} a_z$$

$$\nabla = \frac{\partial}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi} a_\phi + \frac{\partial}{\partial z} a_z$$

Similarly, to obtain ∇ interms of r, θ, ϕ,

$$r = \sqrt{x^2 + y^2 + z^2}; \quad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \quad \tan\phi = \frac{y}{x}.$$

to obtain,

$$\frac{\partial}{\partial x} = \sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}$$

Substituting this in ∇ i.e., eq (1) then,

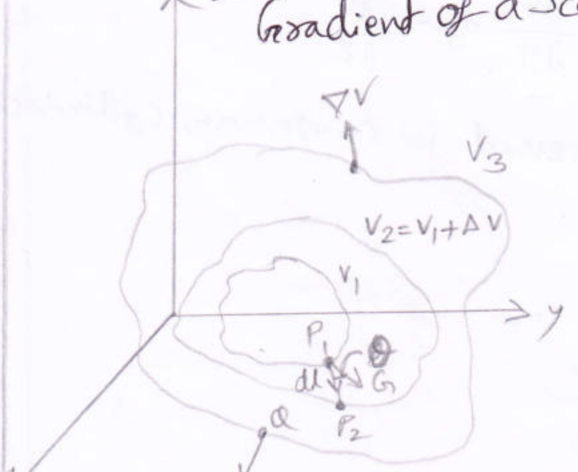
$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + a_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$

Gradient of a Scalar:-

The gradient of a scalar field 'V' is a vector that represents both the magnitude & the direction of the maximum space rate of increase of V.

The gradient expression is obtained by the difference in the field dV between points P_1 & P_2 of figure shown.

Gradient of a Scalar



$V_1, V_2, V_3 \rightarrow$ Contours

$V \rightarrow$ constant.

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right) \cdot \left(dx\, a_x + dy\, a_y + dz\, a_z\right)$$

Gradient: A graded change in the magnitude of some physical quantity of dimension

Let,

$$G = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

$$dV = \bar{G} \cdot d\bar{l} = G \cos\theta \, dl$$

$$\frac{dV}{dl} = G \cos\theta.$$

$dl \rightarrow$ Differential displacement from P_1 to P_2

$\theta \rightarrow$ Angle between G & dl.

From the above equation, that $\frac{dV}{dl}$ is maximum when $\theta = 0$, i.e., when dl is in the direction of G,

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$$

Where, $\frac{dV}{dn} \rightarrow$ Normal derivative.

G has its direction as those of the maximum rate of change of V. By definition, G is the gradient of V.

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z.$$

The gradient of V can be expressed in Cartesian, cylindrical, & spherical coordinates.

For Cartesian Coordinates,

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

For Cylindrical Coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$$

For Spherical Coordinates,

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi$$

Few Relations:- [Properties]:-

→ $\nabla (V+U) = \nabla V + \nabla U$

→ $\nabla (VU) = V\nabla U + U\nabla V$ where V & U are scalars

→ $\nabla \left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$ n is an integer.

→ $\nabla V^n = nV^{n-1}\nabla V$

Fundamental Properties of the gradient of a scalar field V:-

1. The magnitude of ∇V equals the maximum rate of change in 'V' per unit distance.
2. ∇V points in the direction of the maximum rate of change in V.
3. ∇V at any point is ⊥er to the Constant 'V' surface that passes through that point.

4. The projection of ∇V in the direction of a unit vector 'a' is $\nabla V \cdot a$ and is called the directional derivative of V along 'a'. This is the rate of change of 'V' in the direction of 'a'.

5. If $\bar{A} = \nabla V$, V is said to be scalar potential of $\bar{A}$.

Divergence of a Vector and Divergence Theorem:-

The net outflow of the flux of a vector field '$\bar{A}$' from a closed surface 'S' is obtained from the integral $\oint \bar{A} \cdot d\bar{S}$. Now the divergence of '$\bar{A}$' as the net outward flow of flux per unit volume over a closed incremental surface.

"The divergence of '$\bar{A}$' at a given point P is the outward flux per unit volume as the volume shrinks about P."

$$\text{div} \cdot \bar{A} = \nabla \cdot \bar{A} = \lim_{\Delta V \to 0} \frac{\oint \bar{A} \cdot d\bar{s}}{\Delta V}$$

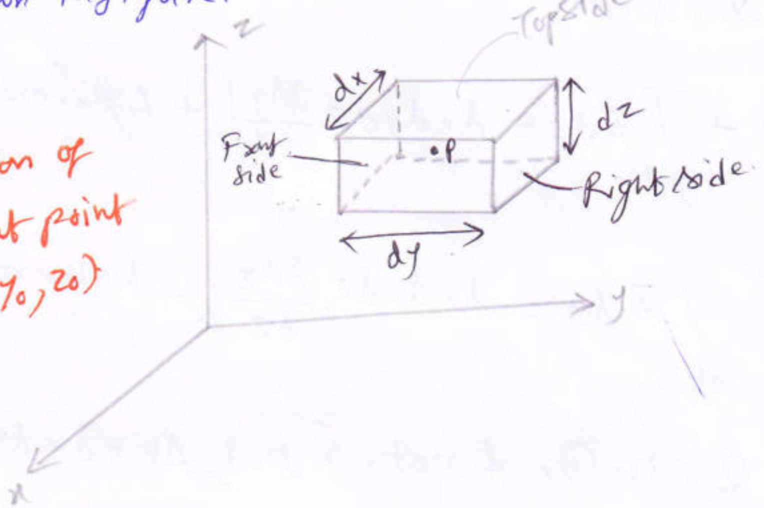
$\Delta V \rightarrow$ Volume enclosed by the closed surface S in which P is located.

Positive divergence — Negative divergence — Zero divergence

General Meaning [Divergence]: The act of moving away in different direction from a common point.

The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume. It is positive at a source point in the field, & negative at a sink point, or zero where there is neither sink nor source.

To evaluate the divergence of a vector field 'A' at point $P(x_0, y_0, z_0)$; Let point be enclosed by a differential volume as shown in figure.



Evaluation of $\nabla \cdot A$ at point $P(x_0, y_0, z_0)$

The surface integral is obtained from,

$$\oint_S A \cdot ds = \left(\int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom} \right) A \cdot ds \quad \text{--- (1)}$$

A three dimensional Taylor series expansion of A_x about P is,

$$A_x(x,y,z) = A_x(x_0, y_0, z_0) + (x - x_0)\frac{\partial A_x}{\partial x}\Big|_P + (y - y_0)\frac{\partial A_x}{\partial y}\Big|_P + (z - z_0)\frac{\partial A_x}{\partial z}\Big|_P + \text{higher order terms} \quad \text{--- (2)}$$

For the front side, $x = x_0 + dx/2$ & $ds = dy\,dz\,a_x$. then,

$$\int_{front} A \cdot ds = dy\,dz \left[A_x(x_0, y_0, z_0) + \frac{dx}{2}\frac{\partial A_x}{\partial x}\Big|_P \right] + \text{higher order terms} \quad \text{--- (3)}$$

For the back side, $x = x_0 - \frac{dx}{2}$, $ds = dy\,dz(-a_x)$ then,

$$\int_{back} \bar{A}\cdot ds = -dy\,dz\left[A_x(x_0,y_0,z_0) - \frac{dx}{2}\frac{\partial A_x}{\partial x}\bigg|_P\right] + \text{higher order terms} \quad -④$$

Hence,

$$\int_{front} \bar{A}\cdot ds + \int_{back} \bar{A}\cdot ds = dx\,dy\,dz\,\frac{\partial A_x}{\partial x}\bigg|_P + \text{higher order terms.} \quad -⑤$$

By taking similar steps,

$$\int_{left} \bar{A}\cdot ds + \int_{right} \bar{A}\cdot ds = dx\,dy\,dz\,\frac{\partial A_y}{\partial y}\bigg|_P + \text{higher order terms} \quad -⑥$$

$$\int_{Top} \bar{A}\cdot ds + \int_{bottom} \bar{A}\cdot ds = dx\,dy\,dz\,\frac{\partial A_z}{\partial z}\bigg|_P + \text{higher order terms} \quad -⑦$$

Sub. ⑤, ⑥, ⑦ in ①, & noting that $\Delta v = dx\,dy\,dz$,

$$\lim_{\Delta v \to 0} \frac{\oint \bar{A}\cdot ds}{\Delta v} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\bigg|_P \quad -⑧$$

Because the higher order terms will vanish as $\Delta v \to 0$, the divergence of 'A' at point $P(x_0, y_0, z_0)$ in Cartesian system is,

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad -⑨$$

Similarly, for Cylindrical,

$$\nabla\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad -⑩$$

The divergence of 'A' in spherical Coordinates,

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Properties of the divergence of a vector field :-

1. It produces a scalar field. [Because scalar product is involved].
2. The divergence of a Scalar V, div V, makes no sense.
3. $\nabla\cdot(\bar{A}+\bar{B}) = \nabla\cdot\bar{A} + \nabla\cdot\bar{B}$
4. $\nabla\cdot(V\bar{A}) = V\,\nabla\cdot\bar{A} + \bar{A}\cdot\Delta V$

From the divergence of $\bar{A}$',

$$\oint_S \bar{A}\cdot ds = \int_V \nabla\cdot\bar{A}\, dv.$$

This is called the divergence theorem, otherwise known as the Gauss - Ostrogoradsky theorem.

Statement :-

The divergence theorem states that the total outward flux of a vector field '$\bar{A}$' through the closed surface 'S' is the same as the volume integral of the divergence of $\bar{A}$'.

Proof :-

→ Subdivide volume V into a large number of small cells. If the K^{th} cell has volume ΔV_K & is bounded by surface S_K.

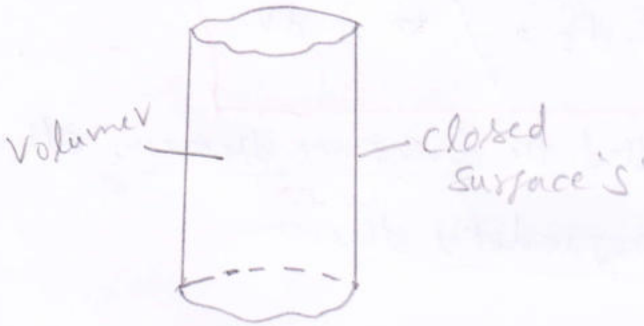
$$\oint_S A\cdot d\bar{s} = \sum_K \oint_{S_K} \bar{A}\cdot ds = \sum_K \frac{\oint_{S_K}\bar{A}\cdot ds}{\Delta V_K}\Delta V_K$$

Since the outward flux to one cell is inward to some neighbouring cells, there is Cancellation on every interior surface, so the sum of the surface integrals over S_k's is the same as the surface integral over the surface S.

Taking the limit of the right-hand side of the Eq.'s,

$$\oint_S \bar{A} \cdot d\bar{S} = \int_V \nabla \cdot A \, dv.$$

The theorem applies to any volume V bounded by the Closed surface 'S' such that as shown in figure, provided that $\bar{A}$ and $\nabla \cdot \bar{A}$ are Continuous in the region.



Volume 'V' enclosed by surface 'S'.

Curl of a vector & Stokes's Theorem:-

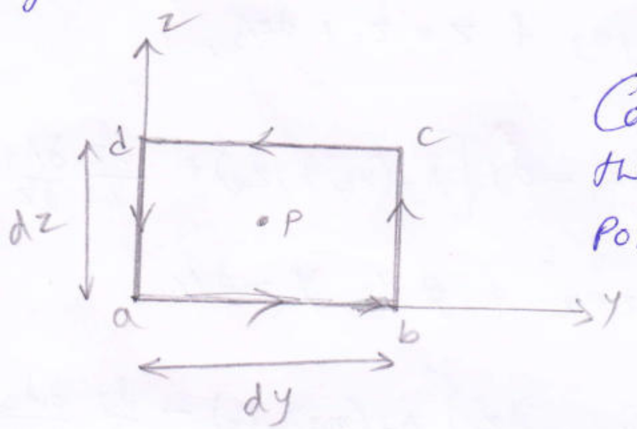
"The Curl of '$\bar{A}$' is an axial (or rational) vector whose magnitude is the maximum Circulation of '$\bar{A}$' per unit area as the area tends to zero & whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum."

[The circulation of a vector field '$\bar{A}$' around a closed path L as the integral $\oint_L \bar{A} \cdot d\bar{l}$.]

$$\text{Curl } \bar{A} = \nabla \times \bar{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \bar{A} \cdot dl}{\Delta S} \right) a_{n\,max} \quad -① $$

→ The area ΔS is bounded by the curve L and a_n is the unit vector normal to the surface ΔS & is determined using the right-hand rule.

Consider the differential area in the YZ-plane as shown in figure.



Contour used in evaluating the x-component of $\nabla \times \bar{A}$ at Point $P(x_0, y_0, z_0)$

The line integral in eq. ① is obtained as,

$$\oint_L \bar{A} \cdot dl = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \bar{A} \cdot dl$$

The field components in a Taylor series expansion about the center point $P(x_0, y_0, z_0)$ as in Eq. -

$$A_x(x, y, z) = A_x(x_0, y_0, z_0) + (x - x_0) \frac{\partial A_x}{\partial x}\Big|_P + (y - y_0) \frac{\partial A_x}{\partial y}\Big|_P + (z - z_0) \frac{\partial A_x}{\partial z}\Big|_P + \text{higher order terms}$$

On side ab,

$$d\bar{l} = a_y dy \quad \& \quad z = z_0 - dz/2,$$

$$\int_{ab} \bar{A}\cdot d\bar{l} = dy\left[A_y(x_0, y_0, z_0) - \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right]$$

on side bc, $d\bar{l} = dz\, a_z$ & $y = y_0 + dy/2$,

$$\int_{bc} \bar{A}\cdot d\bar{l} = dz\left[A_z(x_0, y_0, z_0) + \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right]$$

on side cd, $d\bar{l} = dy\, a_y$ & $z = z_0 + dz/2$,

$$\int_{cd} \bar{A}\cdot d\bar{l} = -dy\left[A_y(x_0, y_0, z_0) + \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right]$$

on side da, $d\bar{l} = dz\, a_z$ & $y = y_0 - dy/2$,

$$\int_{da} \bar{A}\cdot d\bar{l} = -dz\left[A_z(x_0, y_0, z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right]$$

Sub. all eqn's, then, $\Delta s = dy\, dz$,

$$\lim_{\Delta s \to 0} \oint_L \frac{\bar{A}\cdot d\bar{l}}{\Delta s} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

(or)

$$(\mathrm{Curl}\,\bar{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$\|^{ly}$,

$$(\mathrm{Curl}\,\bar{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$(\mathrm{Curl}\,\bar{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

The definition of $\nabla \times \bar{A}$ is independent of the coordinate system. In cartesian coordinates the curl of '$\bar{A}$' is easily found using,

$$\nabla \times \bar{A} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \bar{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] a_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] a_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] a_z$$

Curl of '$\bar{A}$' in cylindrical coordinates as,

$$\nabla \times \bar{A} = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \bar{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] a_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] a_\phi + \frac{1}{\rho}\left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right] a_z$$

Curl of '$\bar{A}$' in spherical coordinates as,

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$\nabla \times \bar{A} = \frac{1}{r\sin\theta}\left[\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] a_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r}\right] a_\theta + \frac{1}{r}\left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right] a_\phi$$

The properties of Curl :

1. The curl of a vector field is another vector field.

2. The curl of a scalar field V, $\nabla \times V$, makes no sense.

3. $\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$

4. $\nabla \times (\bar{A} \times \bar{B}) = \bar{A}(\nabla \cdot \bar{B}) - \bar{B}(\nabla \cdot \bar{A}) + (\bar{B} \cdot \nabla)\bar{A} - (\bar{A} \cdot \nabla)\bar{B}$

5. $\nabla \times (V\bar{A}) = V\nabla \times \bar{A} + \nabla V \times \bar{A}$

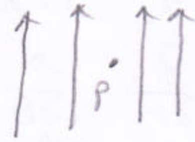
6. The divergence of the curl of a vector field vanishes,

$$\text{i.e., } \nabla \cdot (\nabla \times \bar{A}) = 0.$$

7. The curl of the gradient of a scalar field vanishes, $\nabla \times \nabla V = 0$.

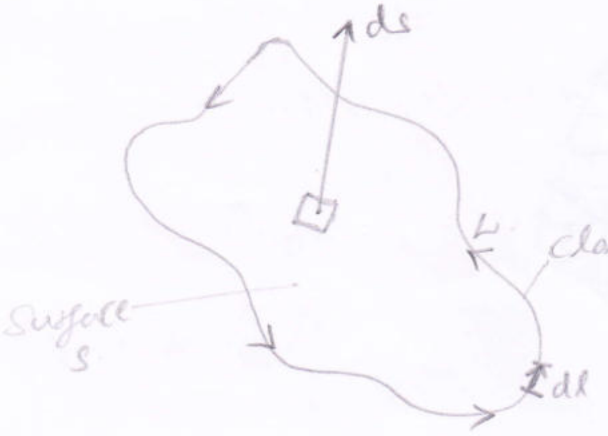
The curl provides the maximum value of the circulation of the field per unit area (or circulation density) & indicates the direction along which this maximum value occurs.

The curl of a vector field 'A' at a point P may be regarded as a measure of the circulation (or) how much the field curls around P.



Curl at P points out of the page

Curl at P is zero.



The sense of dl & ds involved in Stoke's theorem.

$$\oint_L \bar{A} \cdot d\bar{l} = \int_S (\nabla \times \bar{A}) \cdot d\bar{s}$$

This is called Stoke's theorem.

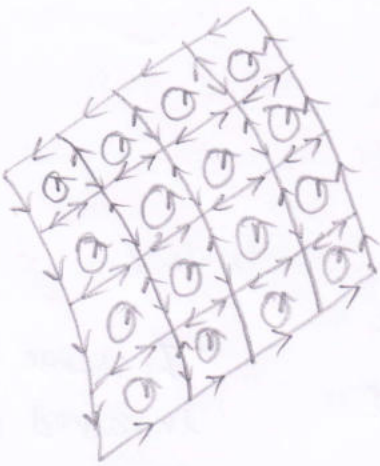
Statement:

The circulation of a vector field 'A' around a (closed) path L is equal to the surface integral of the curl of $\bar{A}$' over the open surface 'S' bounded by L, provided that $\bar{A}$' & $\nabla \times \bar{A}$ are continuous on S.

Proof:-

The surface of S is subdivided into a large number of cells as shown in figure. If the k^{th} cell has surface area ΔS_K and is bounded by path L_K.

$$\oint_L \bar{A} \cdot d\bar{l} = \sum_K \oint_{L_K} \bar{A} \cdot d\bar{l} = \sum_K \frac{\oint_{L_K} \bar{A} \cdot d\bar{l}}{\Delta S_K} \Delta S_K$$



There is a cancellation of every interior path, so the sum of the line integrals around L_k's is the same as the line integral around the bounding curve L. ∴ Taking the limit of the right-hand side of eq. as $\Delta s_k \to 0$,

$$\oint_L A \cdot dl = \int_S (\nabla \times A) \cdot ds.$$

The direction of dl and ds must be chosen using the right-handed screw rule. Let the fingers point in the direction of dl, the thumb will indicate the direction of ds. The divergence theorem relates a surface integral to a volume integral, Stoke's theorem relates a line integral (Circulation) to a surface integral.

Laplacian of a Scalar:-

For practical reasons, it is expedient to introduce a single operator which is the Composite of gradient and divergence operators. This operator is known as "Laplacian."

The Laplacian of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

$$\text{Laplacian } V = \nabla \cdot \nabla V = \nabla^2 V$$

$$= \left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] \cdot \left[\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right]$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

* The Laplacian of a scalar field is another scalar field.

Laplacian in Cylindrical Coordinates,

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian in Spherical Coordinates,

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$$

→ The scalar field is said to be harmonic in a given region if its Laplacian vanishes in that region.

$\nabla^2 V = 0$. ⇒ Laplace's Equation.

The Laplacian operator ∇^2 is a scalar operator, it is also possible to define the Laplacian of a vector A. $\nabla^2 A$ should not be viewed as the divergence of the gradient of A, which makes no sense.

* $\nabla^2 A$ is defined as the gradient of the divergence of 'A' minus the curl of the curl of A.

$$\nabla^2 \bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla\times\nabla\times\bar{A}.$$

$\nabla^2 \bar{A}$ in Cartesian system,

$$\nabla^2 \bar{A} = \nabla^2 A_x a_x + \nabla^2 A_y a_y + \nabla^2 A_z a_z.$$

Classification of vector fields:-

A vector field is uniquely characterized by its divergence & curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field.

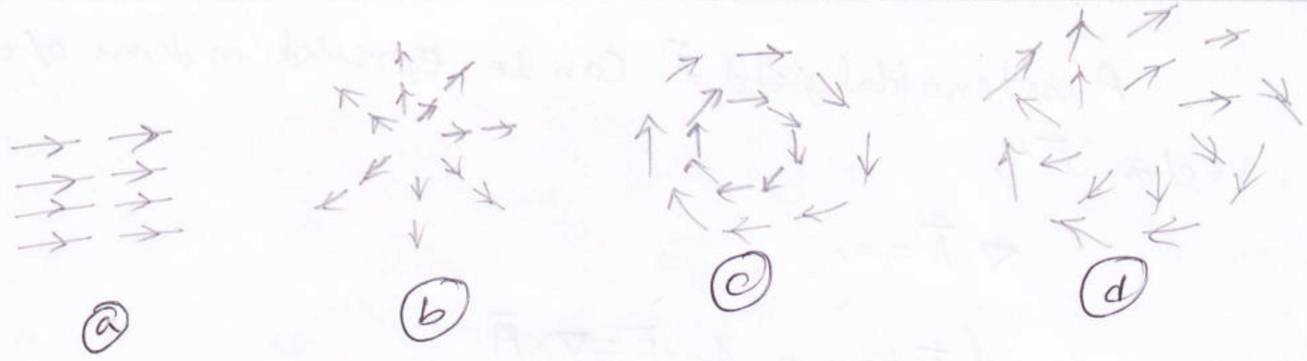
All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

$\nabla\cdot\bar{A} = 0$, $\nabla\times\bar{A} = 0$.

$\nabla\cdot\bar{A} \neq 0$, $\nabla\times\bar{A} = 0$.

$\nabla\cdot\bar{A} = 0$, $\nabla\times\bar{A} \neq 0$.

$\nabla\cdot\bar{A} \neq 0$, $\nabla\times\bar{A} \neq 0$.



(a) $\bar{A} = k a_x$, $\longrightarrow \nabla \cdot \bar{A} = 0$, $\nabla \times \bar{A} = 0$

(b) $\bar{A} = k r$, $\longrightarrow \nabla \cdot \bar{A} = 3k$, $\nabla \times \bar{A} = 0$.

(c) $\bar{A} = k \times r$, $\longrightarrow \nabla \cdot \bar{A} = 0$, $\nabla \times \bar{A} = 2k$,

(d) $\bar{A} = k \times r + cr \longrightarrow \nabla \cdot \bar{A} = 3c$, $\nabla \times \bar{A} = 2k$.

"A vector field $\bar{A}$ is said to be 'Solenoidal' (or divergenceless) if $\nabla \cdot \bar{A} = 0$."

Such a field has neither source nor sink of flux.

From divergence theorem,

$$\oint_s \bar{A} \cdot ds = \int_v \nabla \cdot \bar{A} \, dv = 0.$$

Hence, flux lines of $\bar{A}$ entering any closed surface must also leave it.

Examples of Solenoidal fields:

* In-compressible fluids
* Magnetic fields
* Current density under steady state conditions.

The field of curl $\bar{F}$ is purely solenoidal because,

$$\nabla \cdot (\nabla \times F) = 0.$$

A solenoidal field 'A' can be epressed in terms of another vector 'F',

$$\nabla \cdot \bar{A} = 0,$$

$$\oint_S \bar{A} \cdot ds = 0 \quad \& \quad \bar{F} = \nabla \times \bar{A}.$$

A vector field 'A' is said to be irrotational (or potential) if $\nabla \times \bar{A} = 0$.

→ Curl-free vector is irrotational.

From stoke's theorem,

$$\int_S (\nabla \times \bar{A}) \cdot ds = \oint_L \bar{A} \cdot dl = 0.$$

Thus in an irrotational field 'A', the circulation of 'A' around a closed path is identically zero. This implies that the line integral of 'A' is independent of the chosen path.

∴, an irrotational field is known as conservative field.

Examples of irrotational fields,

→ The Electrostatic field

→ The gravitational field.

The irrotational field 'A' can always be expressed in terms of a scalar field 'V'.

$$\nabla \times \bar{A} = 0$$

$$\oint_L \bar{A} \cdot dl = 0 \quad \& \quad \bar{A} = -\nabla V$$

$\bar{A}$ → potential field

V → scalar potential of $\bar{A}$.

A vector '$\bar{A}$' is uniquely prescribed within a region by its divergence & its curl,

$$\nabla \cdot \bar{A} = \bar{\rho_v}$$

$$\nabla \times \bar{A} = \bar{\rho_s}.$$

ρ_v can be regarded as the source density of $\bar{A}$ & $\bar{\rho_s}$ is circulation density. Any vector '$\vec{A}$' satisfying the above eq.'s with both $\bar{\rho_v}$ & $\bar{\rho_s}$ vanishing at infinity can be written as the sum of two vectors: one irrotational (zero curl), the other solenoidal (zero divergence). This is called Helmholtz's theorem.

$$\bar{A} = -\nabla V + \nabla \times \bar{B}.$$

let $\bar{A_i} = -\nabla V$ &

$$A_s = \nabla \times \bar{B}$$

$\nabla \times \bar{A_i} = 0$ & $\quad \bar{A_i}$ is irrotational.

$\nabla \times \bar{A_s} = 0$. $\quad \bar{A_s}$ is solenoidal.

$$\therefore \nabla^2 A = \nabla \bar{\rho_v} - \nabla \times \bar{\rho_s}.$$

Electrostatic fields:

Electrostatic field is produced by a static charge distribution.

Coulomb's Law:-

Coulomb's law is an experimental law formulated by French colonel, Charles Augustin de Coloumb. It deals with the force a point charge exerts on another point charge. By a point charge we mean a charge i.e., located on a body whose dimensions are much small than other relavent dimensions.

Example:- A collection of electric charges on a pinhead may be regarded as point charge. Charges are measured in Coulombs (C). One coulomb is approximately equivalent to 6×10^{18} electrons; it is a very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.

Statement:-

Coulomb's law states that the force F between two point charges Q_1 & Q_2 is, along the line joining them,

, Directly proportional to the product Q_1Q_2 of the charges.

, Inversely proportional to the square of the distance R between them.

$$F = \frac{K \cdot Q_1 Q_2}{R^2} \quad -(1)$$

$K \rightarrow$ Proportionality constant.

Charges Q_1 & Q_2 are in coulombs (C),

R is distance in meters (m),

F is force in Newtons (N),

$$K = \frac{1}{4\pi\varepsilon_0}.$$

ε_0 is permittivity of free space (in farads permeter).

$$\varepsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-19}}{36\pi} \text{ F/m.}$$

$$K = \frac{1}{4\pi\varepsilon_0} \simeq 9 \times 10^{9} \text{ m/F.}$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \quad -(2)$$

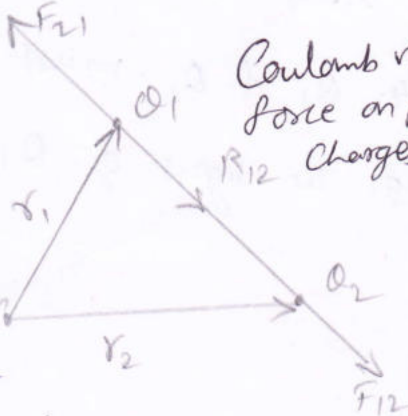
If point charges Q_1 & Q_2 are located at points having position vectors r_1 & r_2 then the force F_{12} on Q_2 due to Q_1, then,

$$\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} a_{R_{12}} \quad -(3)$$

where, $\bar{R}_{12} = \bar{r}_2 - \bar{r}_1$

$$R = |\bar{R}_{12}|$$

$$a_{R_{12}} = \frac{\bar{R}_{12}}{R}.$$



Coulomb vector force on point charges Q_1 & Q_2

Sub. in Eq-(3), $\bar{F}_{12} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^3} \bar{R}_{12}$

$$\bar{F}_{12} = \frac{Q_1 Q_2 (\bar{r}_2 - \bar{r}_1)}{4\pi\varepsilon_0 |\bar{r}_2 - \bar{r}_1|^3}$$

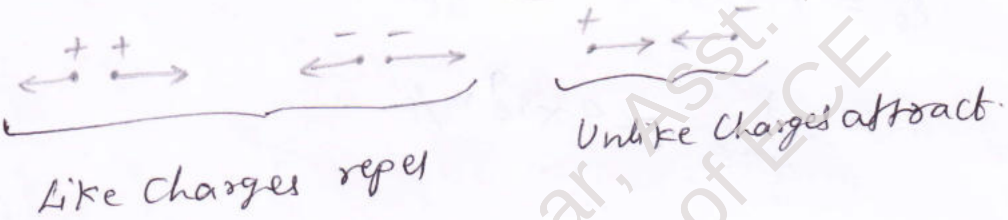
Note:-

1. As shown in figures, the force F_{21} on Q_1 due to Q_2 is,

$$\bar{F}_{21} = |\bar{F}_{12}| a_{R_{21}} = |\bar{F}_{12}| (-a_{R_{12}})$$

$$\bar{F}_{21} = -\bar{F}_{12}$$

$$a_{R_{21}} = -a_{R_{12}}$$



2. Like charges (charges of the same sign) repel each other while unlike charges attract

3. The distance R between the charged bodies Q_1 & Q_2 must be large compared with the linear dimensions of the bodies; i.e., Q_1 & Q_2 must be point charges.

4. Q_1 & Q_2 must be static (at rest).

5. The signs of Q_1 & Q_2 must be taken into account.

If more than two point charges, the principle of superposition to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2 \ldots Q_N$ located respectively, at points with position vectors $\bar{r}_1, \bar{r}_2 \ldots \bar{r}_N$, the resultant force $\bar{F}$ on a charge Q located at point 'r' is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2 \ldots Q_N$.

$$\bar{F} = \frac{Q Q_1 (\bar{r}-\bar{r}_1)}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_1|^3} + \frac{Q Q_2 (\bar{r}-\bar{r}_2)}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_2|^3} + \cdots + \frac{Q Q_N (\bar{r}-\bar{r}_n)}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_N|^3}$$

$$F = \frac{Q}{4\pi\varepsilon_0} \sum_{K=1}^{N} \frac{Q_K(\bar{r}-\bar{r}_K)}{|\bar{r}-\bar{r}_K|^3}$$

Electric Field Intensity :-

The electric field Intensity (or) Electric field strength $\bar{E}$ is the force per unit charge when placed in the electric field.

$$\bar{E} = \lim_{Q \to 0} \frac{\bar{F}}{Q}$$

$$\bar{E} = \frac{\bar{F}}{Q}$$

The electric field intensity $\bar{E}$ is obviously in the direction of the force '$\bar{F}$' and is measured in newtons/coulomb (or) volt/meter.

The electric field intensity at point 'r' due to a point Charge located at r' is readily obtained as,

$$\bar{E} = \frac{Q}{4\pi \varepsilon_0 R^2} a_R$$

$$\bar{E} = \frac{Q(\bar{r}-\bar{r}')}{4\pi \varepsilon_0 |\bar{r}-\bar{r}'|^3}$$

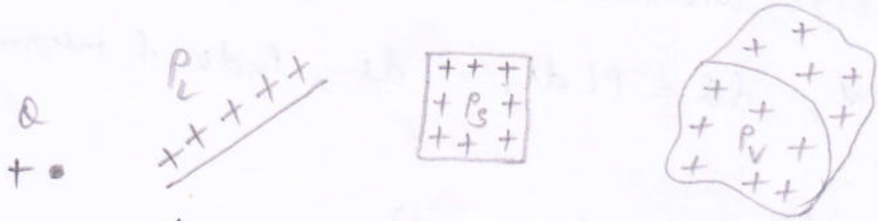
For N point charges $Q_1, Q_2 \ldots Q_N$ located at $r_1, r_2 \ldots r_N$ the electric field intensity at point 'r' is obtained from force eq.'n,

$$\bar{E} = \frac{Q_1(\bar{r}-\bar{r}_1)}{4\pi \varepsilon_0 |\bar{r}-\bar{r}_1|^3} + \frac{Q_2(\bar{r}-\bar{r}_2)}{4\pi \varepsilon_0 |\bar{r}-\bar{r}_2|^3} + \cdots + \frac{Q_N(\bar{r}-\bar{r}_N)}{4\pi \varepsilon_0 |\bar{r}-\bar{r}_N|^3}$$

$$\bar{E} = \frac{1}{4\pi \varepsilon_0} \sum_{K=1}^{N} \frac{Q_K(\bar{r}-\bar{r}_K)}{|\bar{r}-\bar{r}_K|^3}$$

Electric fields due to Continuous Charge Distributions:-

Various Charge distributions & Charge elements



Point Charge | Line Charge | Surface Charge | Volume Charge

It is also possible to have Continuous Charge distribution along a line, on a surface, or in a volume.

Line Charge density $\rightarrow \rho_L$ (in C/m)

Surface charge density $\rightarrow \rho_s$ (in C/m²)

Volume Charge density $\rightarrow \rho_v$ (in C/m³)

The Charge element dQ and the total Charge Q due to these charge distributions are obtained as,

$$dQ = \rho_L \, dl$$

$$Q = \int_L \rho_L \, dl \quad \text{(line Charge)}$$

$$dQ = \rho_s \, ds$$

$$Q = \int_s \rho_s \, ds. \quad \text{(surface Charge)}$$

$$dQ = \rho_v \, dv$$

$$Q = \int_v \rho_v \, dv \quad \text{(volume Charge)}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_s & ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution. Thus by replacing Q in with charge element, $dQ = \rho_L dl$, $\rho_s ds$, $\rho_v dv$ & integrating,

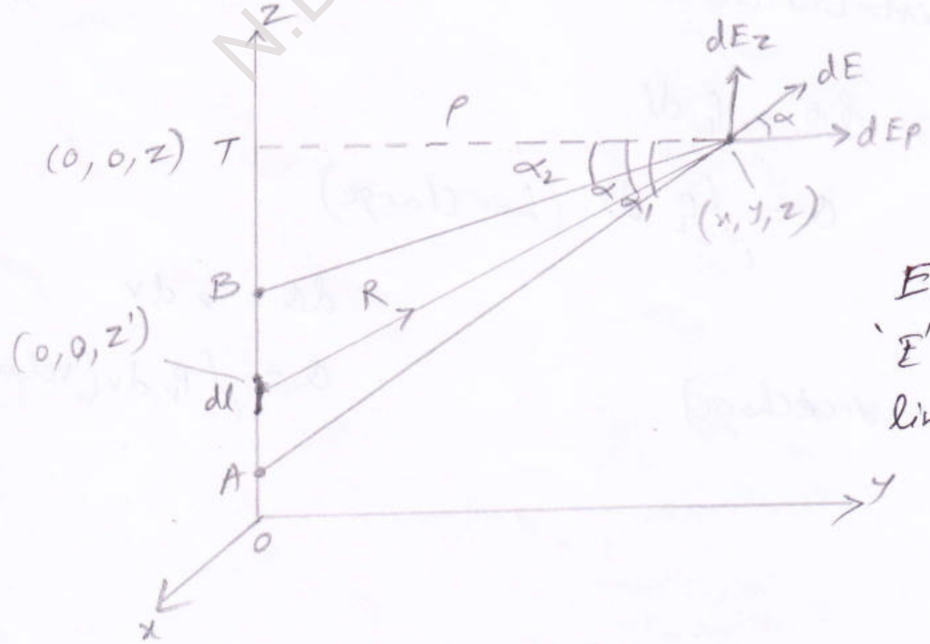
$$\bar{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \bar{a}_R \quad \text{(line charge)}$$

$$\bar{E} = \int \frac{\rho_s \, ds}{4\pi\varepsilon_0 R^2} \bar{a}_R \quad \text{(surface charge)}$$

$$\bar{E} = \int \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} \bar{a}_R \quad \text{(volume charge)}$$

Line Charge:-

A line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in figure.



Evaluation of the 'E' field due to a line charge.

The charge element dQ associated with element $dl = dz$ of the line is,

$$dQ = \rho_L \, dl = \rho_L \, dz$$

Hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz$$

The electric field intensity E at an arbitrary point $P(x, y, z)$ can be found using 'E' (line charge).

field point (x, y, z)
Source point (x', y', z')

$$dl = dz'$$

$$\bar{R} = (x, y, z) - (0, 0, z') = x a_x + y a_y + (z - z') a_z.$$

$$\bar{R} = \rho a_\rho + (z - z') a_z \quad \text{(addition vector)}$$

$$|\bar{R}| = \sqrt{(x^2 + y^2) + (z - z')^2}$$

$$|\bar{R}| = \sqrt{\rho^2 + (z - z')^2}$$

$$\bar{R} = \rho a_\rho + (z - z') a_z$$

$$E = \int \frac{\rho_L}{4\pi\varepsilon_0} \cdot \frac{dQ \, dl}{R^2}$$

$$R^2 = |\bar{R}|^2 = x^2 + y^2 + (z - z')^2$$
$$= \rho^2 + (z - z')^2$$

$$\frac{a_R}{R^2} = \frac{\bar{R}}{|\bar{R}|^3} = \frac{\rho a_\rho + (z - z') a_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Sub., in $\bar{E}$, then,

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho \bar{a}_\rho + (z - z') \bar{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} \, dz'$$

$\sec\alpha = \frac{R}{\rho}$, $R = \rho\sec\alpha$

$\sin\alpha = \frac{z-z'}{R}$

$\cos\alpha = \frac{\rho}{R}$

Change the differential term to α,

$$R^2 = \rho^2 + (z-z')^2$$

$$R = [\rho^2 + (z-z')^2]^{1/2} = \rho\sec\alpha$$

$$z' = OT - \rho\tan\alpha, \quad dz' = -\rho\sec^2\alpha\, d\alpha$$

Now,

$$\bar{E} = \frac{-\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\,[\cos\alpha\, a_\rho + \sin\alpha\, a_z]\, d\alpha}{\rho^2\sec^2\alpha}$$

$$= -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2} [\cos\alpha\, a_\rho + \sin\alpha\, a_z]\, d\alpha$$

Thus for a finite line charge,

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\, a_\rho + (\cos\alpha_2 - \cos\alpha_1)\, a_z\right]$$

As a special case, for an infinite line charge, point B is at $(0,0,\infty)$ & A at $(0,0,-\infty)$ so, that, $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$, the z-component vanishes, then,

$$E = \frac{\rho_L}{2\pi\varepsilon_0\rho}\, a_\rho.$$

If the line is not along the z-axis, ρ is the $\perp$er distance from the line to the point of interest and a_ρ is a unit vector along that distance directed from the line charge to the field point.

Surface Charge:-

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_s. The charge associated with an elemental area ds is,

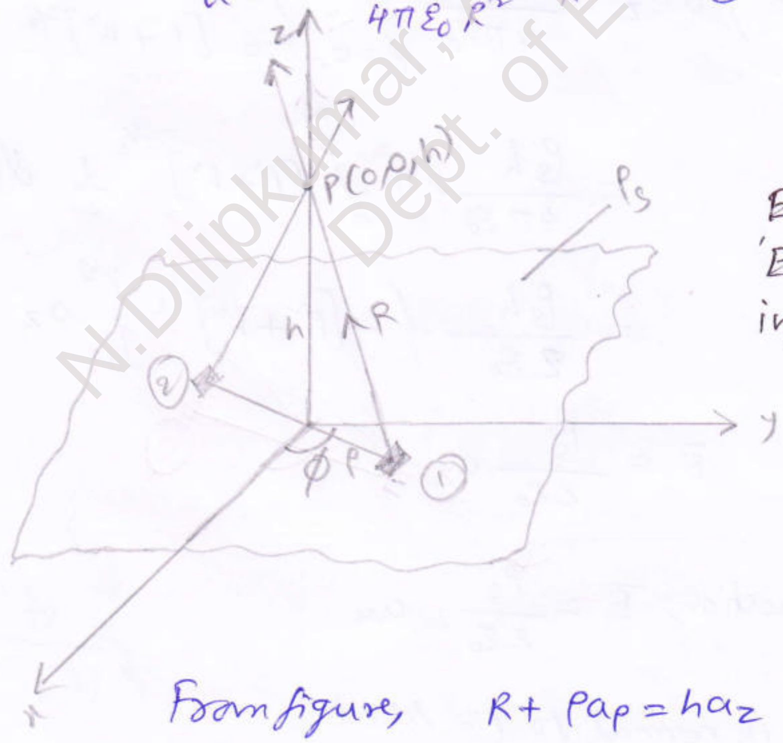
$$dQ = \rho_s \, ds.$$

& hence the total charge is,

$$Q = \int \rho_s \, ds.$$

The Contribution of 'E' field at point $P(0,0,h)$ by the elemental surface ① as shown in figure is,

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2} a_R \quad \text{—} ①$$



Evaluation of the 'E' field due to an infinite sheet of Charge.

From figure, $R + \rho a_\rho = h a_z$

$$\bar{R} = \rho(-a_\rho) + h a_z,$$

$$R = |R| = [\rho^2 + h^2]^{1/2}$$

$$a_R = \frac{\bar{R}}{R}, \quad dQ = \rho_s \, ds \Rightarrow dQ = \rho_s \rho \, d\phi \, d\rho$$

Substitute the terms in eq. ①,

$$d\bar{E} = \frac{\rho_s \, \rho \, d\phi \, d\rho \left[-\rho a_\rho + h a_z\right]}{4\pi\varepsilon_0 \left[\rho^2 + h^2\right]^{3/2}}$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along a_ρ cancels that of element 1, as shown in figure. Thus the contributions to E_ρ add upto zero. So that $\bar{E}$ has only z-component. This can also be shown mathematically by replacing a_ρ with $\cos\phi \, a_x + \sin\phi \, a_y$. Integration of $\cos\phi$ (or) $\sin\phi$ over $0 < \phi < 2\pi$ gives zero.

$$\bar{E} = \int dE_z = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\rho \, d\rho \, d\phi}{\left[\rho^2 + h^2\right]^{3/2}} \, a_z$$

$$= \frac{\rho_s h}{4\pi\varepsilon_0} 2\pi \int_0^{\infty} \left[\rho^2 + h^2\right]^{-3/2} \frac{1}{2} \, d(\rho^2) \, a_z$$

$$= \frac{\rho_s h}{2\varepsilon_0} \left\{ -\left[\rho^2 + h^2\right]^{-1/2} \right\}_0^{\infty} a_z$$

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0} a_z$$

⇒ General Equation, $\bar{E} = \dfrac{\rho_s}{2\varepsilon_0} a_n$

a_n is normal to the sheet.

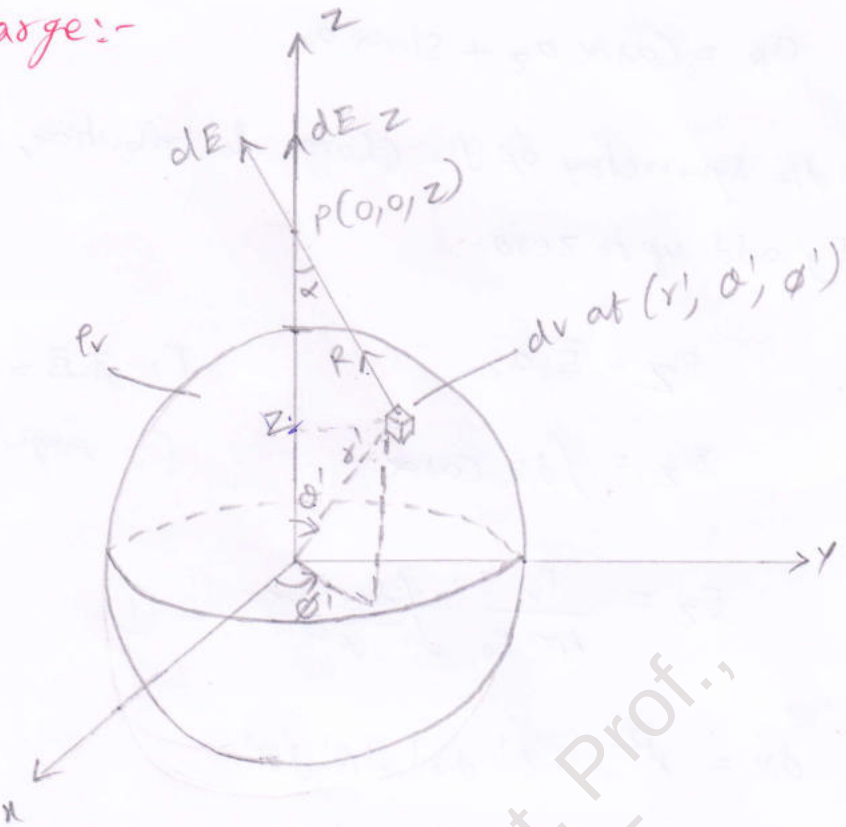
$$\int_0^{\infty} \frac{\left(\frac{1}{2}\right)}{\left(\rho^2 + h^2\right)^{3/2}} d(\rho^2)$$

$$\frac{1}{2} \int_0^{\infty} \frac{1}{\left(\rho^2 + h^2\right)^{3/2}} d(\rho^2)$$

$$\int \frac{1}{a^n} da = \frac{a^{-n+1}}{(-n+1)}$$

$$= \frac{a^{-3/2+1}}{\left(-\frac{3}{2}+1\right)}$$

Volume Charge:-



Volume Charge distribution with uniform Charge density ρ_v.

$$dQ = \rho_v \, dv.$$

Elemental Charge $dQ = \rho_v \, dv$.

Total charge in a sphere of radius a is,

$$Q = \int \rho_v \, dv$$

$$= \rho_v \int dv$$

$$= \rho_v \frac{4\pi a^3}{3}$$

Sphere $\Rightarrow \frac{4}{3}\pi \cdot a^3$

The electric field dE at P(0,0,z) due to the elementary volume charge is,

$$d\bar{E} = \frac{\rho_v \, dv}{4\pi \varepsilon_0 R^2} a_R$$

Where, $a_R = \cos\alpha\, a_z + \sin\alpha\, a_\rho$.

Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero.

$$E_z = \bar{E} \cdot a_z \qquad [\because \bar{A} \cdot \bar{B} = AB\cos\theta]$$

$$E_z = \int dE \cos\alpha \qquad [\because \text{magnitude of } a_z = 1]$$

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0} \int \frac{dv \cos\alpha}{R^2} \quad \text{--- (1)}$$

$$dv = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

$\sin\theta' = \rho / r'$

$\rho = r' \sin\theta'$

$\cos\theta' = \frac{z/2}{r'}$

$z/2 = r'\cos\theta'$

$R^2 = (r'\sin\theta')^2 + (z - r'\cos\theta')^2$

Apply cosine rule for the figure,

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

Differentiating $\cos\theta'$ eq. w.r.t θ' keeping z & r' fixed [constant],

$$\sin\theta'\, d\theta' = \frac{R\,dR}{zr'}$$

Sub. all eq's in eq. (1),

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{R\,dR}{zr'}\, dr'\, \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2}$$

$$E_z = \frac{\rho_v 2\pi}{8\pi\varepsilon_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 - r'^2}{R^2}\right] dR\, dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a r' \left[R - \frac{(z^2 - r'^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a 4r'^2\, dr' = \frac{1}{4\pi\varepsilon_0} \frac{1}{z^2} \left(\frac{4}{3}\pi a^3 \rho_v\right)$$

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 z^2} a_z.$$

Electric Flux density:-

The flux due to the electric field $\bar{E}$ can be calculated using the general definition of flux (ψ). A new vector $\bar{D}$ is independent of the medium is defined by,

$$\bar{D} = \varepsilon_0 \bar{E}.$$

Electric flux (ψ) in terms of $\bar{D}$ as,

$$\psi = \int \bar{D} \cdot ds$$

The electric flux is measured in Coulombs. The vector field $\bar{D}$ is called the electric flux density 4 is measured in Coulombs per square meter. The electric flux density is also called electric displacement.

$$\psi \rightarrow C \text{ (Coulombs)}.$$

$$\bar{D} \rightarrow C/m^2.$$

Line Charge Electric field $\bar{E} = \frac{\rho_L}{2\pi \varepsilon_0 \rho} a_\rho$

Surface Charge Electric field $\bar{E} = \frac{\rho_S}{2 \varepsilon_0} a_n$

Volume Charge Electric field $\bar{E} = \frac{Q}{4\pi \varepsilon_0 z^2} a_z = \left(\frac{4\pi a^3 \rho_v}{3}\right) \frac{1}{4\pi \varepsilon_0 z^2}$

Electric flux density for point charge,

$$\bar{D} = \frac{Q}{4\pi R^2} a_R.$$

$\bar{D}$ for line charge,

$$\bar{D} = \frac{\rho_L}{2\pi \rho} a_\rho$$

$\bar{D}$ for surface charge,

$$\bar{D} = \frac{\rho_S}{2} a_n$$

$\bar{D}$ for volume charge,

$$\bar{D} = \int \frac{\rho_v}{4\pi R^2} a_R \cdot dv$$

Gauss law:-

Gauss law states that the total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface.

$$\Psi = Q_{enc}.$$

$$\Psi = \oint d\Psi = \int_S \bar{D} \cdot ds$$

$$= \text{Total charge enclosed } Q = \int \rho_v dv. \quad -(1)$$

$$\therefore Q = \oint_S \bar{D} \cdot ds = \int_v \rho_v \, dv. \quad -(2)$$

Gauss law in integral form (or) point form.

Apply divergence theorem to the eq. (2),

$$\oint_S \bar{D} \cdot ds = \int_v \nabla \cdot D \, dv \quad -(3)$$

Comparing eq.'s (2) & (3), $\rho_v = \nabla \cdot D$ — (4)

Eq-(4) states that the volume charge density is the same as the divergence of electric flux density.

1. Eq.'s (2) & (4) are basically stating Gauss law in different ways, eq. (2) is the integral form, & eq. (4) is the differential or point form of Gauss law.
2. Gauss law is an alternative statement of Coulomb's law, proper application of the divergence theorem to Coulomb's law results in Gauss law.

3. Gauss law provides an easy means of finding $\bar{E}$ (or) $\bar{D}$ for Symmetrical Charge distributions such as point charge, an infinite line charge, an infinite cylindrical surface charge & a spherical distribution of charge.

Applications of Gauss law:-

1. The main application of Gauss law is to find Electric field Intensity.
2. If symmetric Charge distribution exists, we construct a mathematical closed surface, known as a Gaussian Surface.

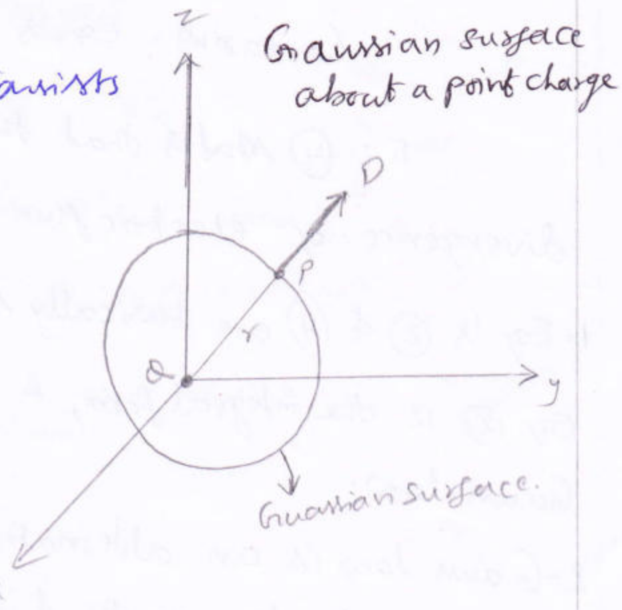
The surface is chosen, such that $\bar{D}$ is normal (or) Tangential to the Gaussian surface.

$\bar{D}$ is Normal to the surface, then, $\bar{D} \cdot ds = D\, ds$. (D → Constant.) $\{\theta = 0^\circ\}$

$\bar{D}$ is Tangential to the surface, then, $\bar{D} \cdot ds = 0$. $\{\theta = 90^\circ\}$

(i) Point Charge:-

Let us consider a point charge consists at the origin of coordinates.



$\because \bar{D}$ is everywhere normal to the gaussian surface.

$$\bar{D} = D_r a_r$$

Apply gauss law,

$\psi = Q_{enc}$ gives,

$$Q = \oint \bar{D} \cdot ds = D_r \oint ds = D_r \cdot 4\pi r^2. \qquad \because \int ds = \text{area}.$$

Where, $\oint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2.$

$$\therefore \bar{D} = \frac{Q}{4\pi r^2} a_r.$$

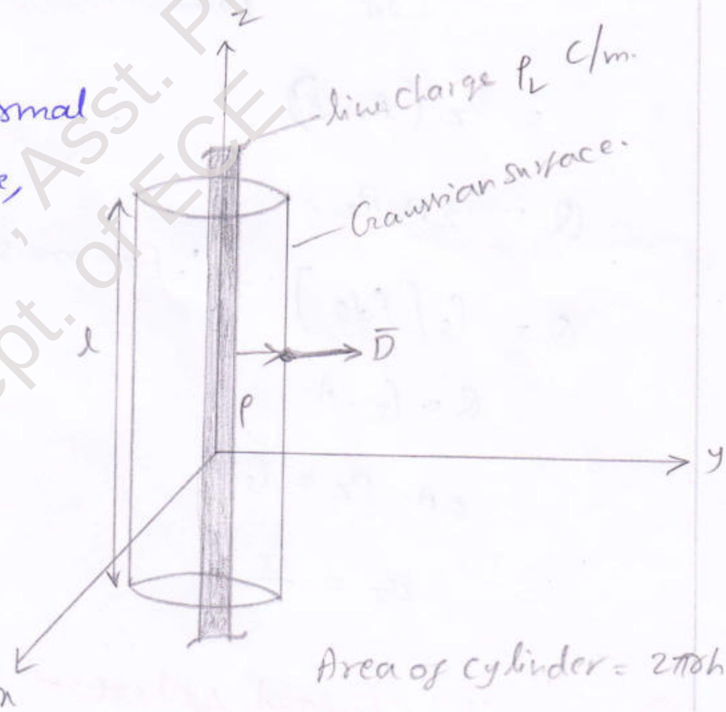
$$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 r^2} a_r.$$

② Infinite Line charge:-

$\bar{D}$ is constant on and normal to the cylindrical Gaussian surface,

i.e.,,

$$\bar{D} = D_\rho a_\rho.$$



$$\rho_L l = Q = \oint \bar{D} \cdot ds$$
$$= D_\rho \oint ds$$
$$= D_\rho \, 2\pi \rho l.$$

Area of cylinder $= 2\pi r h$

$\therefore r = \rho, \; h = l,$

$\Rightarrow 2\pi \rho l.$

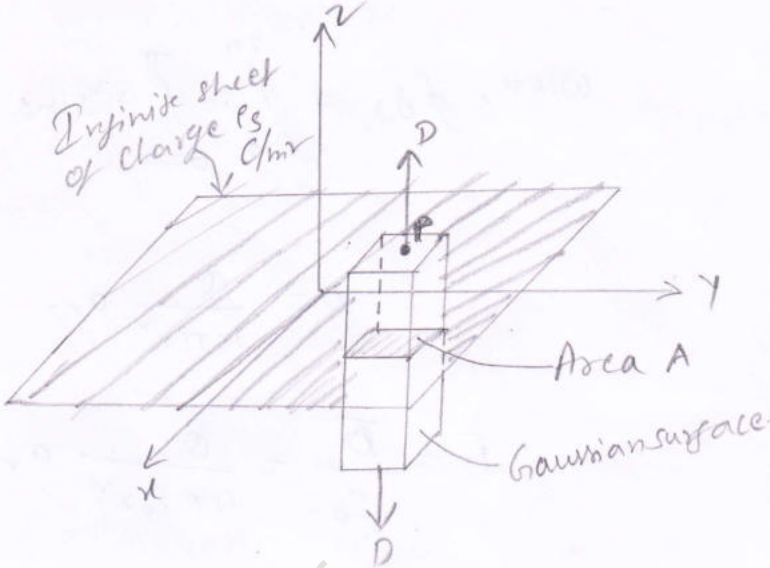
where, $\oint ds = 2\pi\rho l$ is the surface area of gaussian surface.

$$\rho_L l = D_\rho 2\pi \rho l$$

$$\bar{D} = \frac{\rho_L}{2\pi\rho} a_\rho$$

$$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_L}{2\pi\varepsilon_0 \rho} \cdot a_\rho.$$

③ Infinite Surface Charge (or) Infinite Sheet of Charge:-



$\bar{D} = D_z a_z$.

($\bar{D}$ is normal to the sheet).

$Q = \int \bar{D} \cdot ds$.

$= D_z \left[\int ds\right]$

$= D_z \left[\int_{top} ds + \int_{bottom} ds\right]$

$= D_z (A + A)$

$Q = 2A \cdot D_z$

$Q = \rho_s \left[\int ds\right]$ $\left\{\because \text{For one side} \int ds = A\right\}$

$Q = \rho_s \cdot A$.

$2A \cdot D_z = \rho_s \cdot A$.

$D_z = \frac{\rho_s}{2}$.

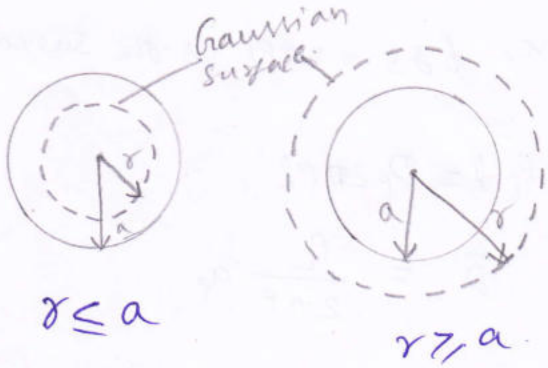
$\bar{D} = \frac{\rho_s}{2} a_z$.

$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0} a_z$.

④ Uniformly Charged sphere:-

Case (i):-

$(r \leq a)$



$Q_{enc} = \int \rho_v \, dv$

$= \rho_v \int dv$

$= \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= \rho_v \left(\frac{r^3}{3}\right) [\phi]_0^{2\pi} \left\{-(\cos\theta)_0^{\pi}\right\}$$

$$= \rho_v \left(\frac{r^3}{3}\right) [2\pi][4]$$

$$= \rho_v \frac{4}{3}\pi r^3$$

$$\psi = \oint \bar{D}\cdot ds = D_r \oint ds$$

$$= D_r \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi r^2$$

$$\therefore \psi = Q_{enc},$$

$$D_r 4\pi r^2 = \rho_v \frac{4}{3}\pi r^3$$

$$D_r = \frac{\rho_v r}{3}$$

$$\bar{D} = D_r \cdot a_r$$

$$\bar{D} = \frac{\rho_v r}{3} a_r \quad 0 < r \leq a.$$

Case(ii):-

$r \geq a;$

$$\bar{D} = D_r \cdot a_r$$

$$Q_{enc} = \int \rho_v \, dv$$

$$= \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$Q_{enc} = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$\psi = \oint \bar{D}\cdot ds \Rightarrow D_r \cdot \int ds$$

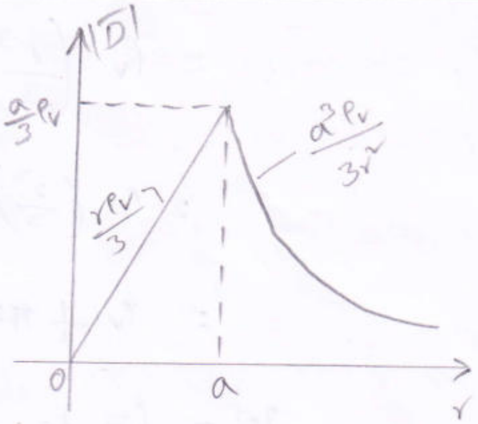
$$\psi = D_r \cdot 4\pi r^2$$

$$\Psi = Q_{enc}.$$

$$D_r \cdot 4\pi r^2 = \rho_v \cdot \frac{4}{3}\pi a^3$$

$$D_r = \frac{\rho_v a^3}{3r^2}$$

$$\bar{D} = \frac{a^3}{3r^2}\rho_v a_r \qquad r \geq a.$$



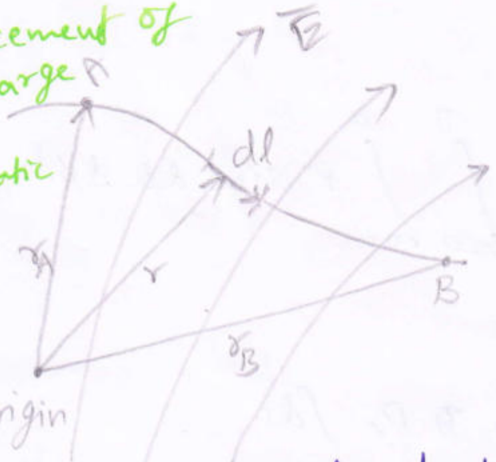
|D| against r for a uniformly charged sphere.

$$\bar{D} = \begin{cases} \frac{r}{3}\rho_v a_r & 0 < r \leq a. \\ \frac{a^3}{3r^2}\rho_v a_r & r \geq a. \end{cases}$$

Electric Potential:-

The electric field intensity $\bar{E}$ due to charge distribution can be obtained from Coulomb's law in general (or) from Gauss's law when charge distribution is symmetric. Another way of obtaining $\bar{E}$ is from the electric scalar potential V.

Let, a point charge Q from point A to point B in an electric field $\bar{E}$ as shown in figure.



Displacement of Point Charge Q in an electrostatic field $\bar{E}$.

From Coulomb's law,

The force on Q is,

$$\bar{F} = Q\bar{E}$$

$\therefore$ work done in displacing the charge by $d\bar{l}$ is,

$$dw = -\bar{F}\cdot d\bar{l}$$

$$dw = -Q\bar{E}\cdot d\bar{l}$$

The negative sign indicates that the work is being done by an external agent.

The total work done, or the potential energy required, in moving Q from A to B is,

$$W = -Q \int_A^B \bar{E} \cdot d\bar{l}$$

Dividing W by Q in the above eq. gives the potential energy per unit charge.

$$\therefore V_{AB} = \frac{W}{Q} = -\int_A^B \bar{E} \cdot d\bar{l}$$

where V_{AB} is known as the potential difference between A & B.

Note:-

1. In determining V_{AB}, A is the initial point while B is the final point.
2. If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B, this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement, an external agent performs the work.
3. V_{AB} is independent of the path taken.
4. V_{AB} is measured in Joules per coulomb, commonly referred to as volts (V).

If the $\bar{E}$ field is due to a point charge Q located at the origin,

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

$$\therefore V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r$$

$$V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{1}{r^2}\,dr \qquad dL = dr\,a_r.$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_A}^{r_B}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{r_A}^{r_B} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B}-\frac{1}{r_A}\right]$$

$$\therefore \quad V_{AB} = V_B - V_A.$$

Where V_B & V_A are the potentials (or absolute potentials) at B and A, respectively.

$V_{AB} \rightarrow$ difference between potentials.

When point A is at infinite distance,

then $V_A = 0$ as $r_A \rightarrow \infty$

& $r_B \rightarrow r$ due to a point charge Q located at origin.

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

$\bar{E}$ points in the radial direction, any contribution from a displacement in the θ or ϕ direction is wiped out by the dot product, $\bar{E}\cdot d\bar{l} = E\cos\theta\, dl = E\,dr$.

Hence, the potential difference V_{AB} is independent of the path.

The potential at any point is the potential difference between that point and a chosen point at which the potential is zero

By assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point.

$$V = -\int_{\infty}^{r} \bar{E} \cdot d\bar{l}$$

If the point Q is not located at the origin but at a point whose position vector $\bar{r}'$, the potential $V(x,y,z)$ (or) simply $V(\bar{r})$ at $\bar{r}$ becomes,

$$V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 |\bar{r} - \bar{r}_0|}$$

If the point charges, $Q_1, Q_2 \ldots Q_n$ located at points with position vectors $\bar{r}_1, \bar{r}_2, \ldots \bar{r}_n$, then the potential $\bar{r}$ is,

$$V(\bar{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\bar{r} - \bar{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0 |\bar{r} - \bar{r}_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 |\bar{r} - \bar{r}_n|}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\bar{r} - \bar{r}_k|} \quad \text{(Point Charges)}$$

Replace Q_k with charge element, $\rho_L dL$, $\rho_s ds$, $\rho_v dv$. $[\Sigma \rightarrow \int]$

For line charge,

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|} \quad \text{(Line charge)}$$

For surface charge,

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s(\bar{r}')\,ds'}{|\bar{r}-\bar{r}'|}$$

For volume charge,

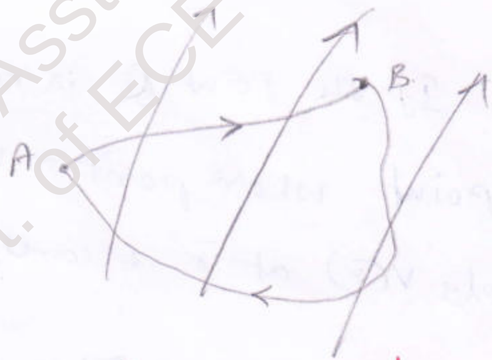
$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_v(\bar{r}')\,dv'}{|\bar{r}-\bar{r}'|}$$

Relation between E & V — Maxwells Equation:-

$$V_{BA} = -V_{AB}$$

$$V_{BA} + V_{AB} = 0$$

$$\oint_L \bar{E}\cdot d\bar{l} = 0 \quad \text{--- (1)}$$



The line integral of $\bar{E}$ along a closed path must be zero. This implies that no network is done in moving a charge along a closed path in an electrostatic field.

Conservative nature of an electrostatic field

Apply stoke's theorem, to the above eq.',

$$\oint \bar{E}\cdot d\bar{l} = \int (\nabla\times\bar{E})\cdot d\bar{s} = 0$$

$$\nabla\times\bar{E} = 0 \quad \text{--- (2)}$$

Eq.'s (1) & (2) is said to be conservative, irrotational,

Thus an electrostatic field is a Conservative field.

Eq.'s ① & ② is referred to as Maxwell's Equation (The second Maxwell's equation) for electrostatic fields.

Eq. ① is in integral form.

Eq. ② is in differential form.

∴ Potential, $V = -\int \bar{E} \cdot d\bar{l}$

$$dV = -\bar{E} \cdot d\bar{l}$$

$$= -E_x dx - E_y dy - E_z dz. \quad —③$$

$$\because dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \quad —④$$

Comparing eq.'s ③ & ④,

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$\therefore \bar{E} = -\nabla V.$$

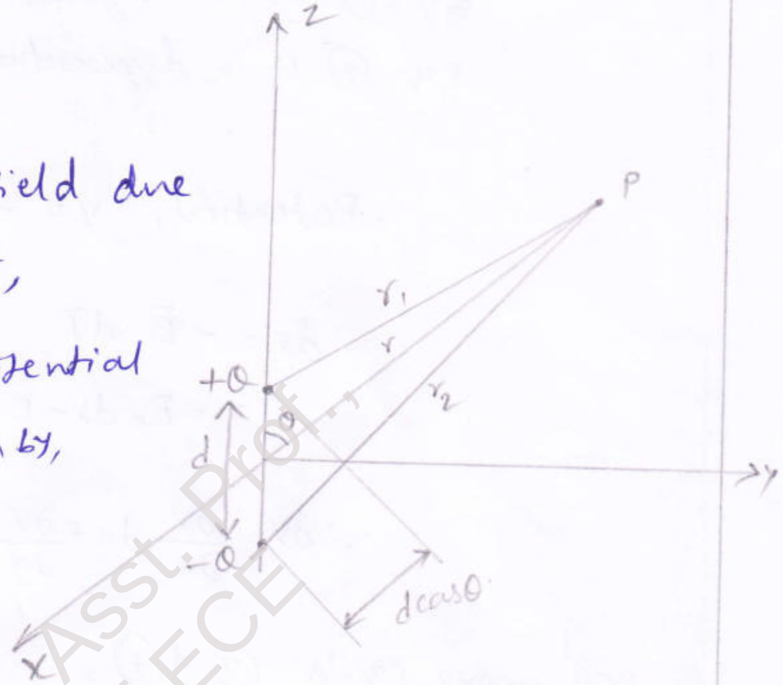
∴ The electric field intensity is the gradient of V.

Electric Dipole and Flux Lines:-

"An electric dipole is formed when two point charges of equal magnitude but opposite sign are seperated by a small distance."

The importance of the field due to a dipole will be evident,

Consider the dipole, the potential at point $P(r, \theta, \phi)$ is given by,



$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad -(1)$$

$\Rightarrow r_1$ & r_2 are the distance between P & +Q, P & -Q,

If $r \gg d$,

$r_2 - r_1 \simeq d\cos\theta$,

$r_2 r_1 \simeq r^2$.

$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \quad -(2)$$

$\because d\cos\theta = \bar{d} \cdot a_r$

$\bar{d} = \bar{d}\, a_z$

$\bar{P} = Q\bar{d} \quad -(3)$

as the dipole moments,

$$V = \frac{\bar{P} \cdot a_r}{4\pi\varepsilon_0 r^2} \quad -④$$

The dipole moment $\bar{P}$ is directed from $-Q$ to $+Q$. If the dipole center is not at the origin but at $\bar{r}'$ becomes,

$$V(\bar{r}) = \frac{\bar{P} \cdot (\bar{r} - \bar{r}')}{4\pi\varepsilon_0 |\bar{r} - \bar{r}'|^3} \quad -⑤$$

The electric field due to the dipole with center of the origin, shown in figure, can be obtained from eq.'s ②,

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3} a_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3} a_\theta$$

$$\therefore \bar{E} = \frac{P}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\, a_r + \sin\theta\, \bar{a}_\theta\right)$$

where, $P = |\bar{P}| = Qd$.

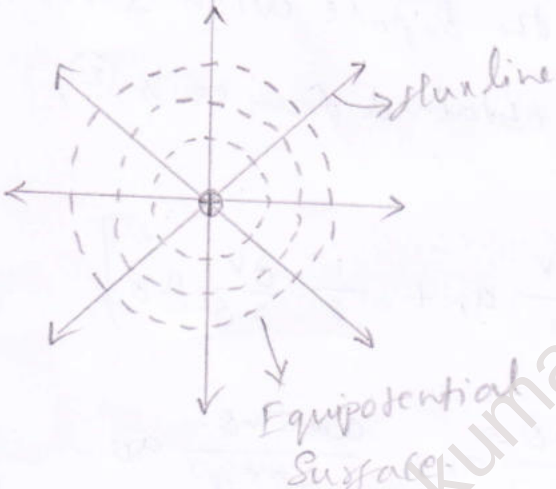
A point charge is a monopole & its electric field varies inversely as r^2 while its potential field varies inversely as r.

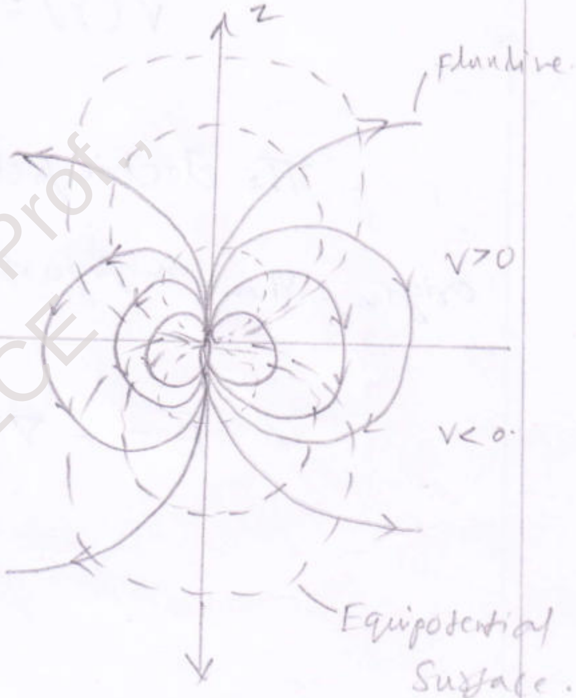
Electric flux:-

[Electric flux lines]

The electric flux line (or) electric lines of force.

The electric flux line is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.





Energy density:-

Let three position charges Q_1, Q_2 and Q_3 in an initially empty space shown in figure. No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.

$$\therefore W = 0$$

The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 & the potential V_{21} at P_2 due to Q_1.

$$\therefore W_2 = Q_2 V_{21}$$

where, V_{21} is potential at P_2 due to Q_1

Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$,

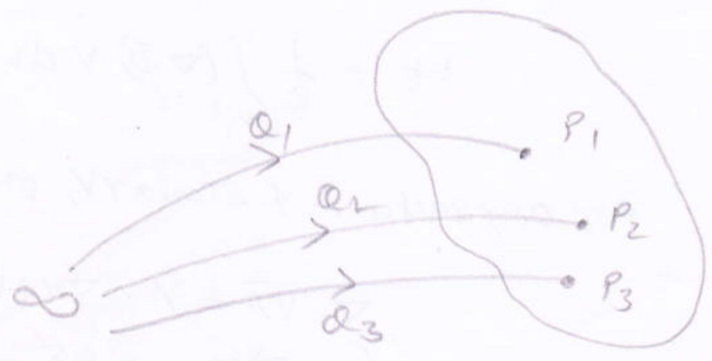
$$\therefore W_3 = Q_3(V_{32} + V_{31})$$

V_{32} & V_{31} are potentials at P_3 due to Q_2 & Q_1.

$\therefore$ Total work done is,

$$W_E = W_1 + W_2 + W_3$$

$$= Q_2 V_{21} + Q_3(V_{31} + V_{32}) \quad \text{--- (1)}$$



If the charges were positioned in reverse order,

$$W_E = W_3 + W_2 + W_1$$

$$= 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \quad -② $$

→ V_{23} is the potential at P_2 due to Q_3

Adding eq ① & ②,

$$2 W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

$$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W_E = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

If there is 'n' point charges,

$$W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k \text{ Joules}$$

Instead of point charges, the region has a continuous charge distribution,

$$W_E = \frac{1}{2} \int \rho_L V dl \quad \text{(line charge)}$$

$$W_E = \frac{1}{2} \int \rho_s V ds \quad \text{(surface charge)}$$

$$W_E = \frac{1}{2} \int \rho_v V dv \quad \text{(volume charge)}$$

$$\because \rho_v = \nabla \cdot \bar{D},$$

$$W_E = \frac{1}{2} \int_v (\nabla \cdot \bar{D}) V dv$$

For any vector $\bar{A}$ & scalar V, the identity,

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

$$(\nabla \cdot \bar{A}) V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

From the Identities, $W_E = \frac{1}{2}\int_v (\nabla \cdot V\bar{D})\, dv - \frac{1}{2}\int_v (\bar{D} \cdot \nabla V)\, dv$

Apply divergence theorem to the first term on the right-hand side of the eq,

$$W_E = \frac{1}{2}\oint_s (V\bar{D}) \cdot d\bar{s} - \frac{1}{2}\int_v (\bar{D} \cdot \nabla V)\, dv \quad \text{--- (3)}$$

V varies as $1/r$ & $\bar{D}$ as $1/r^2$ for point Charges,

V varies as $1/r^2$ & $\bar{D}$ as $1/r^3$ for dipoles,

$\therefore$ $V\bar{D}$ must vary atleast as $1/r^3$ while ds varies as r^2,

Consequently, the first integral in eq. (3) must tend to zero as the surface 's' becomes large.

$$W_E = -\frac{1}{2}\int_v (\bar{D} \cdot \nabla V)\, dv$$

$$= \frac{1}{2}\int_v (\bar{D} \cdot \bar{E})\, dv$$

$\therefore$ $\bar{E} = -\nabla V$ & $\bar{D} = \varepsilon_0 E$.

$$\therefore W_E = \frac{1}{2}\int \bar{D} \cdot \bar{E}\, dv$$

$$= \frac{1}{2}\int \varepsilon_0 E^2\, dv$$

Electrostatic energy density w_E (in J/m^3) as,

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\bar{D} \cdot \bar{E}$$

$$= \frac{1}{2}\varepsilon_0 E^2$$

$$= \frac{D^2}{2\varepsilon_0}.$$

Dielectric Constant & strength:-

$$\bar{D} = \varepsilon_0 (1+\chi_e)\bar{E} = \varepsilon_0 \varepsilon_r \bar{E}$$

$\chi_e \rightarrow$ Electric susceptibility of the material.

$$\bar{D} = \varepsilon \bar{E}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$.

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$

ε is called the permittivity of the dielectric,

ε_0 " " " of free space.

$$\varepsilon_0 \simeq \frac{10^{-9}}{36\pi} \text{ F/m.}$$

ε_r is called the relative permittivity (or) dielectric constant.

The "dielectric constant (or relative permittivity) ε_r is the ratio of the permittivity of the dielectric to that of free space.

Note:
→ ε_r is always greater or equal to unity.
→ Freespace & non dielectric materials $\varepsilon_r = 1$.

Practically speaking, no dielectric is ideal. When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, & the dielectric becomes conducting. Dielectric breakdown is said to have occured when a dielectric becomes conducting. Dielectric breakdown occurs in all kinds of dielectric materials [gases, liquids, solids) &

depends on the nature of the material, temperature, humidity, & the amount of time that the field is applied. The minimum value of the electric field at which dielectric breakdown occurs is called the dielectric strength of the dielectric material.

A material which acts as both insulator & Conductor is called Dielectric material. First a material acts as insulator when the force is applied to the material then it acts as Conductor is called dielectric.

$$\bar{D} \propto \bar{E}$$

$$\bar{D} = \varepsilon_0 \bar{E} \rightarrow \text{for freespace.}$$

If there is no free space,

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$$

Polarization $\bar{P} = \varepsilon_0 \chi_e \bar{E}$

$$\bar{D} = \varepsilon_0 \bar{E} + \varepsilon_0 \chi_e \bar{E}$$

$$= \varepsilon_0 \bar{E} [1 + \chi_e]$$

$$\bar{D} = \varepsilon_0 \varepsilon_r \bar{E}$$

$$\bar{D} = \varepsilon \bar{E}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r .$$

* The minimum value of charge used to convert insulating to the conducting is called Dielectric strength.

Linear, Isotropic, Homogeneous Dielectrics:-

For a material at any point 'ε' is constant it doesn't depends on (x, y, z) Coordinate systems is called Homogeneous. When the Characteristics of $\bar{D}$ & $\bar{E}$ are same is called isotropic.

For a dielectric material satisfies $\bar{D} = \varepsilon_0 \bar{E}$ if the material is Linear when ε_0 not changes to the $\bar{E}$ & doesnot depends on coordinate system (x, y, z) is called Linear, isotropic & homogeneous.

a) Linear:-

A material is said to be linear, if the $\bar{D}$ varies linearly with $\bar{E}$ & non linear otherwise.

b) Homogeneous:-

Materials for which 'ε' does not vary in the region being consider & it is same at all the points, (independent of (x, y, z)) are said to be homogeneous.

When ε depends on Coordinates, then it is said to be non-Homogeneous.

c) Isotropic:

Materials for which $\bar{D}$ & $\bar{E}$ are in same direction are said to be isotropic.

For non-isotropic $\bar{D}$ & $\bar{E}$ & $\bar{P}$ are not in parallel (or) not in the same direction.

For a Conductor which satisfies $\bar{J} = \sigma \mathbf{E}$, '$\sigma$' doesnot vary with $\bar{E}$, homogeneous if 'σ' doesnot vary from point to point, isotropic if 'σ' does not vary with direction.

Continuity Equation:-

Due to the principle of Charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

$$\therefore I_{out} = \oint \bar{J} \cdot d\bar{s}$$

$$= -\frac{dQ_{in}}{dt} \quad \text{--- (1)}$$

$Q_{in} \rightarrow$ Total Charge enclosed by the closed surface.

Using divergence theorem to eq. (1),

$$\oint_S \bar{J} \cdot d\bar{s} = \int_V \nabla \cdot \bar{J}\, dv \quad \text{--- (2)}$$

$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_V \rho_v\, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv \quad \text{--- (3)}$$

Comparing eq. (2) & (3),

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{--- (4)}$$

Eq. (4) is called the continuity of current equation.

The continuity equation is derived from the principle of conservation of charge & essentially states that there can be no accumulation of charge at any point.

For steady currents $\frac{\partial \rho_v}{\partial t} = 0$

$$\therefore \nabla \cdot J = 0.$$

Shows that the total charge leaving a volume is the same as the total charge entering it.

~~Relaxation~~
Relaxation time:-

The point form of Ohm's law is,

$$\bar{J} = \sigma \bar{E},$$

$$\nabla \cdot \bar{D} = \rho_v,$$

$$\nabla \cdot \varepsilon \bar{E} = \rho_v.$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon}.$$

$$\nabla \cdot \sigma \bar{E} = \frac{\sigma \rho_v}{\varepsilon}.$$

$$\nabla \cdot \bar{J} = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}.$$

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0.$$

$$\frac{\partial \rho_v}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_v.$$

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon} \partial t$$

Apply integration on both sides,

$$\int \frac{1}{\rho_v}\, d\rho_v = -\int \frac{\sigma}{\varepsilon}\, dt.$$

$$\ln(\rho_v) = -\frac{\sigma}{\varepsilon} t + \ln \rho_{v0}.$$

where $\ln\rho_{v0}$ is a Constant of integration.

$$\rho_v = \rho_{v0} \cdot e^{-t/T_r}.$$

where, $T_r = \frac{\varepsilon}{\sigma}$

$T_r \rightarrow$ Relaxation time (or) Rearrangement time.

$\rho_{v0} \rightarrow$ Initial charge density. i.e, ρ_v at $t=0$.

Relaxation time is the time it takes a charge placed in the interior of a material to drop to $e^{-1} = 36.8$ percent of its initial value.

Case(i):- For good Conductors,

Eg:- For Copper, $\sigma = 5.8\times10^{7}$ mhos/m,

$\varepsilon_r = 1$.

$$T_r = \frac{\varepsilon_r \varepsilon_0}{\sigma} = 1 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{5.8\times10^{7}}$$

$$= 1.53\times10^{-19}\,s.$$

shows rapid decay of Charge placed inside copper. This implies that for good conductors, the relation time is so short i.e., most of the charge will vanish from any interior point & appear at the surface.

For Fused ~~quartz~~ quartz for instance,

$$\sigma = 10^{-17} \text{ mhos/m}$$

$$\varepsilon_r = 5.0$$

$$T_r = 5 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{10^{-7}}$$

$$= 51.2 \text{ days}$$

It shows a very large relaxation time. Thus for good dielectrics, one may consider the introduced charge to remain wherever placed.

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Asst. Prof.

Magneticstatic Fields

→ ElectroStatic fields are characterized by $\bar{E}$ (or) $\bar{D}$

$\bar{E}$ → Electric field Intensity

$\bar{D}$ → Electric flux density.

→ MagnetoStatic fields are Characterized by $\bar{H}$ (or) $\bar{B}$

$\bar{H}$ → Magnetic field Intensity

$\bar{B}$ → Magnetic flux density

Relation between $\bar{D}$ & $\bar{E}$ as, $\bar{D} = \varepsilon \bar{E}$

Relation between $\bar{H}$ & $\bar{B}$ as, $\bar{B} = \mu \bar{H}$

→ ElectroStatic field is produced by static (or) stationary Charges.

→ If Charges are moving with Constant velocity, a static magnetic [or] Magnetostatic field is produced.

↳ [Constant current flow] [or] [Direct current]

* The Current flow may be due to,

Magnetization Currents as in permanent magnets,

Electron-Beam Currents as in vaccume tubes,

Conduction Currents as in Current-carrying wires.

Magnetic fields in, Motors,

Transformers,

Microphones,

Compasses,

Telephone bell ringers,

Television focusing Controls,

Advertising displays,

Magnetically Leviated high speed vehicles,

Memory stores,

Magnetic separators, - - -

Analogy Between Electric & Magnetic Field

Term	Electric	Magnetic
Basic laws	$\bar{F} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} a_r$ $\oint \bar{D} \cdot d\bar{s} = Q_{enc}$	$d\bar{B} = \frac{\mu_0 I d\bar{l} \times a_R}{4\pi R^2}$ $\oint \bar{H} \cdot d\bar{l} = I_{enc}$
Force law	$\bar{F} = Q\bar{E}$	$\bar{F} = Q\bar{u} \times \bar{B}$
Source Element	dQ	$Q\bar{u} = \bar{I} \cdot d\bar{l}$
Field Intensity	$E = \frac{V}{l}$ (V/m)	$H = \frac{I}{l}$ (A/m)
Flux density	$\bar{D} = \frac{\psi}{S}$ (C/m²)	$\bar{B} = \frac{\psi}{S}$ (Wb/m²)
Relationship between field potentials	$\bar{D} = \varepsilon \bar{E}$ $\bar{E} = -\nabla V$ $V = \int \frac{\rho_L dl}{4\pi \varepsilon r}$	$\bar{B} = \mu \bar{H}$ $\bar{H} = -\nabla V_m$ $(\bar{J} = 0)$ $\bar{A} = \int \frac{\mu I dl}{4\pi R}$
Flux	$\psi = \int \bar{D} \cdot d\bar{s}$ $\psi = Q = CV$ $I = C\frac{dV}{dt}$	$\psi = \int \bar{B} \cdot d\bar{s}$ $\psi = LI$ $V = L \cdot \frac{dI}{dt}$
Energy density	$w_E = \frac{1}{2} \bar{D} \cdot \bar{E}$	$w_m = \frac{1}{2} \bar{B} \cdot \bar{H}$
Poisson's Equation	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$	$\nabla^2 A = -\mu \bar{J}$

Biot – Savart's law:-

Statement:-

The magnetic field Intensity dH produced at a point P, by the differential Current element Idl is proportional to the product Idl & the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element

$$dH \propto \frac{Idl\sin\alpha}{R^2}$$

$$dH = \frac{KIdl\sin\alpha}{R^2}$$

K is constant of proportionality,

$$K = \frac{1}{4\pi}$$

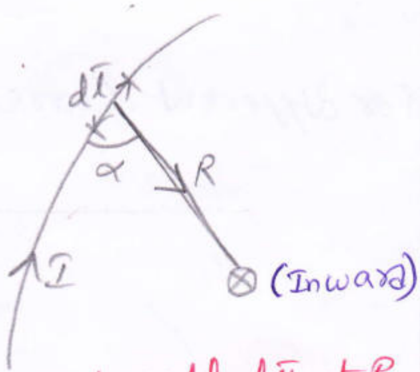
$$dH = \frac{Idl\sin\alpha}{4\pi R^2} \quad \text{— ①}$$

From the definition of cross product, eq. ① is put in vector form as,

$$d\bar{H} = \frac{Id\bar{l} \times a_R}{4\pi R^2}$$

$$= \frac{Id\bar{l} \times \bar{R}}{4\pi R^3}$$

Where, $R = |\bar{R}|$, $a_R = \frac{\bar{R}}{R}$.

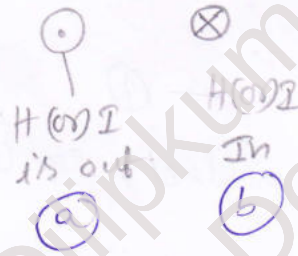


Magnetic field $d\bar{H}$ at P due to current element $Id\bar{l}$.

Direction of $d\bar{H}$ using the right-hand rule

Direction of $d\bar{H}$ using the right handed screw rule

It is customary to represent the direction of the magnetic field intensity $\bar{H}$ (or current I) by a small circle with a dot or cross sign depending on whether $\bar{H}$ (or I) is out of, or into, the page is shown in figure.

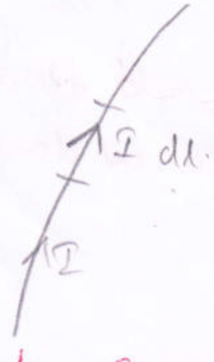


Conventional representation of H or I

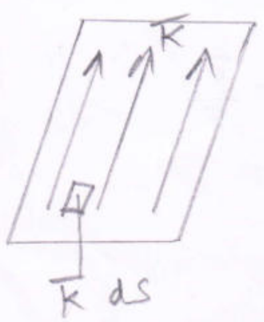
(a) → Out of the page

(b) → Into the page

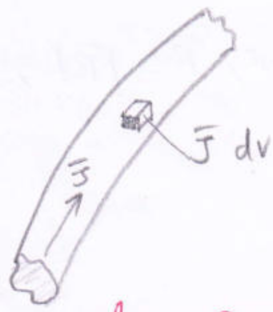
For different current distributions : Line Current
Surface Current
Volume Current.



Line Current



Surface Current



volume Current

$\bar{K} \rightarrow$ Surface Current density (A/m)

$\bar{J} \rightarrow$ Volume current density (A/m²)

$$I\,dl \approx \bar{K}\,ds \approx \bar{J}\,dv$$

∴ The distributed current sources, the Biot-Savart law as,

$$\bar{H} = \int \frac{I\,dl \times a_R}{4\pi R^2} \quad \text{[Line Current]}$$

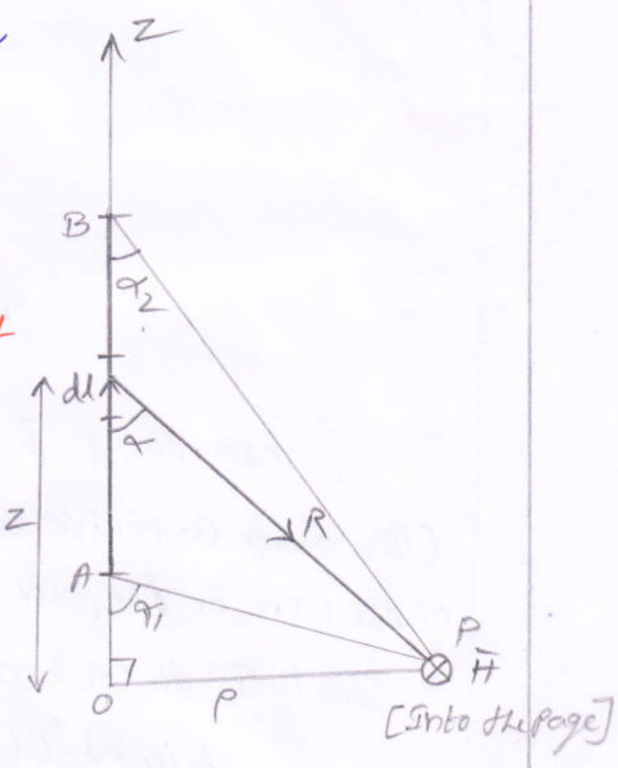
$$\bar{H} = \int \frac{\bar{K}\,ds \times a_R}{4\pi R^2} \quad \text{[Surface Current]}$$

$$\bar{H} = \int \frac{\bar{J}\,dv \times a_R}{4\pi R^2} \quad \text{[Volume Current]}.$$

Magnetic field Intensity in Straight (Current) Filamentary Conductor:-

→ Finite length of Filamentary Conductor of finite length AB.

Field at point P due to a Straight Filamentary Conductor



→ Conduct is along z-axis with its upper & lower ends subtending angles α_2 & α_1 at P.

Consider the Contribution $d\bar{H}$ at P due to an element $d\bar{I}$ at (0, 0, z).

$$d\bar{H} = \frac{I\,dl \times R}{4\pi R^3}$$

But $d\bar{l} = dz\,a_z$;

$$\bar{R} = \rho a_\rho - z a_z$$

$$d\bar{l} \times \bar{R} = \rho\, dz\, a_\phi$$

$$\bar{H} = \int \frac{I \rho\, dz}{4\pi [\rho^2 + z^2]^{3/2}} a_\phi$$

let, $z = \rho \cot\alpha$, $dz = -\rho \operatorname{cosec}^2\alpha$,

$$\therefore \bar{H} = -\frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha}{\rho^3 \operatorname{cosec}^3\alpha}\, d\alpha\; a_\phi$$

$$= -\frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha$$

$$\bar{H} = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1)\, a_\phi \quad \text{— (1)}$$

From the eq-(1), $\bar{H}$ is always along the unit vector a_ϕ (i.e., along Concentric circular paths) irrespective of the length of the wire or the point of interest P.

Case(i): If the conductor is semiinfinite (w.r.t. P)

A is at O(0, 0, 0) $\quad \therefore \alpha_1 = 90°, \alpha_2 = 0°$

B is at $(0, 0, \infty)$

$$\therefore H = \frac{I}{4\pi\rho} a_\phi$$

Case(ii):- If the conductor is of infinite in length,

Point A is at $(0,0,-\infty)$ $\therefore \alpha_1 = 180°, \alpha_2 = 0°$.
B is at $(0,0,\infty)$

$$\therefore \bar{H} = \frac{I}{2\pi\rho} a_\phi$$

$\therefore$ unit vector, $a_\phi = a_l \times a_\rho$.

where, a_l is $\rightarrow$ unit vector along the line current.
$a_\rho \rightarrow$ " " ⊥er line from the line current to field point.

Ampere's Circuit Law — Maxwell's Equations:-

Statement:-

The line Integral of the tangential component of $\bar{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path.

$$\oint \bar{H} \cdot d\bar{l} = I_{enc} \quad — ①$$

* Ampere's law is similar to Gauss's law.
* This expression is useful when the current distribution is symmetrical.

Apply Stoke's Theorem to the eq. ①,

$$I_{enc} = \oint_L \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H}) \cdot d\bar{s} \quad — ②$$

$$I_{enc} = \int_S \bar{J} \cdot ds \quad — ③$$

Comparing eq.'s ② & ③,

$\nabla \times \bar{H} = \bar{J}$ $\longrightarrow$ Third Maxwell's equation.
⇓
Differential form (or) Point form.

Note:- Magnetostatic field is not Conservative
$\therefore \nabla \times \bar{H} = \bar{J} \neq 0$.

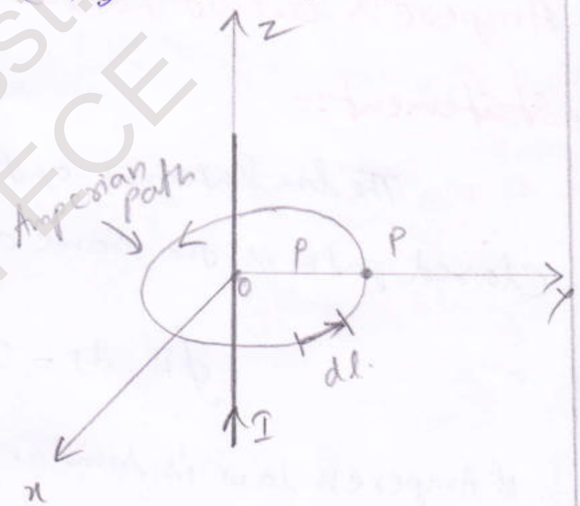
Applications of Ampere's law:-

① Infinite Line Current:-

Consider an infinitely long filamentary current I along the z-axis. $\bar{H}$ at an observation point P, allow a closed path pass through P. This path, on which Ampere's law is to be applied is known as an Amperian path.

If $\bar{H}$ is constant provided ρ is constant. Since this path encloses the whole current I, according to Ampere's law.

~~$I = \int H_\phi a_\phi \cdot \rho d\phi a_\phi = H_\phi \int \rho d\phi = H_\phi 2\pi\rho$~~



$$I = \int H_\phi a_\phi \cdot \rho \, d\phi \, a_\phi$$

$$= H_\phi \rho \int_0^{2\pi} d\phi$$

$$I = H_\phi \cdot 2\pi\rho$$

$$H_\phi = \frac{I}{2\pi\rho} a_\phi$$

② Infinite Sheet of Current:-

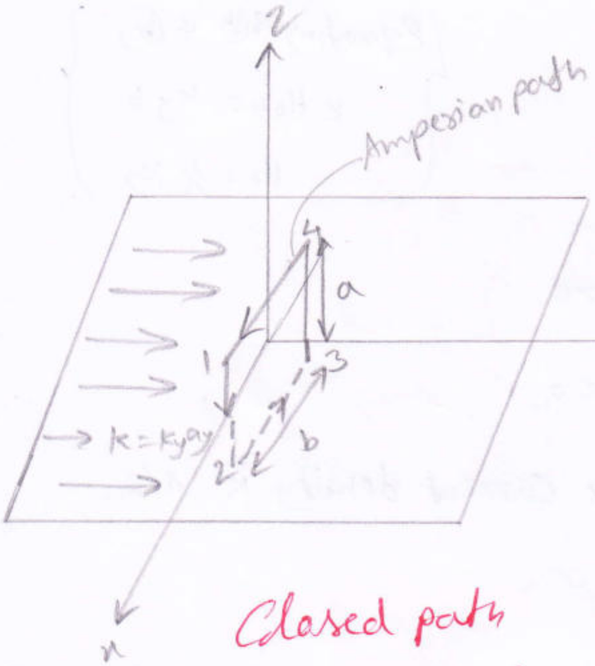
Let z=0 plane.

Uniform current density $\bar{K} = K_y a_y$ A/m

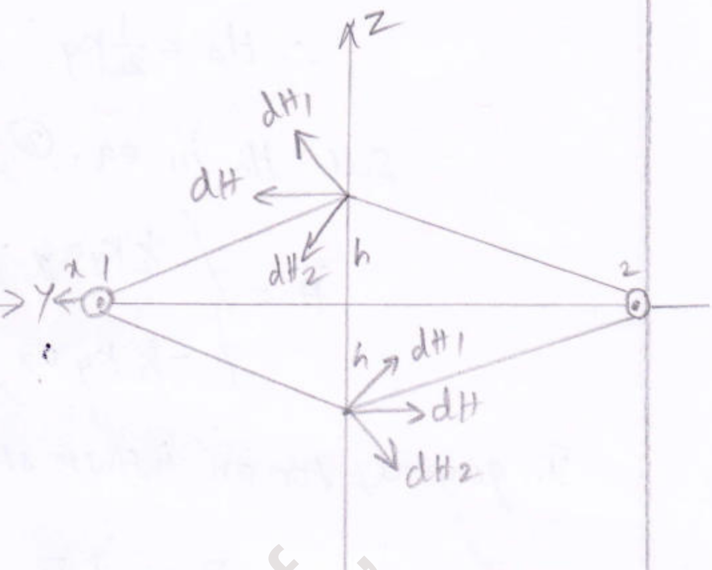
Apply Ampere's law to the rectangular closed path [Amperian path],

gives,

$$\oint \bar{H} \cdot d\bar{l} = I_{enc} = K_y \cdot b \quad —① \;, \rightarrow ⓐ$$



Closed path
1-2-3-4-1



Symmetrical pair of current filaments with current along ay.

To integrate $\bar{H}$, the infinite sheet as comprising of filaments, $d\bar{H}$ above or below the sheet due to a pair of filamentary currents can be found using,

$$\bar{H} = \frac{1}{2\pi\rho} a_\phi \quad \& \quad a_\phi = a_l \times a_\rho$$

The resultant $d\bar{H}$ has only an x-component. $\bar{H}$ on one side of the sheet is the negative of that on the other side. Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of $\bar{H}$ for a pair are the same for the infinite current sheets,

$$\bar{H} = \begin{cases} H_0 a_x & z > 0 \\ -H_0 a_x & z < 0 \end{cases} \quad -(2)$$

$$\oint \bar{H} \cdot d\bar{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \bar{H} \cdot d\bar{l}$$

$$= 0(-a) + (-H_0)(-b) + 0(a) + (H_0)(b) = 2H_0 b \quad -(3)$$

$$\therefore H_0 = \frac{1}{2} K_y. \quad -⑥$$

{ Equating eq ⑤ & ⑥,
$2H_0 b = K_y b$
$H_0 = \frac{1}{2} K_y$ }

Sub. H_0 in eq. ②,

$$\bar{H} = \begin{cases} \frac{1}{2} K_y a_x, & z > 0 \\ -\frac{1}{2} K_y a_x, & z < 0 \end{cases}$$

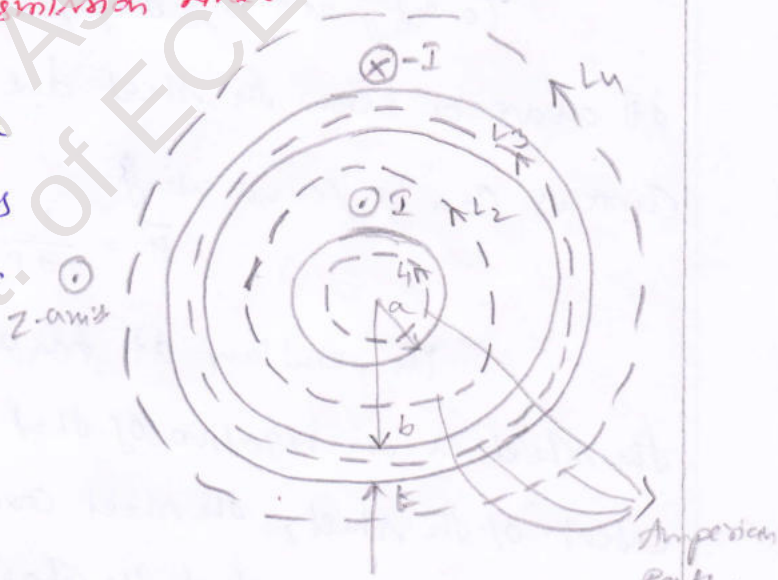
In general, for an infinite sheet of current density $\bar{K}$ A/m.

$$\bar{H} = \frac{1}{2} \bar{K} \times a_n.$$

where, a_n is an unit normal vector directed from the current sheet to the point of interest.

③ Infinitely Long Coaxial Transmission Line:-

* Let An infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis.
* z-axis is out of the page.
* Inner Conductor radius a,
  " Current I,
* outer " Inner radius b, & thickness t
  " Current -I,



Cross section of the transmission line, the positive z-direction is out of the page.

* $\bar{H}$ everywhere assuming that current is uniformly distributed in both conductors.

Ampere's law along the Amperian path for each of the four possible regions: * $0 \le \rho \le a$,

* $a \le \rho \le b$,
* $b \le \rho \le b+t$,
* $\rho \ge b+t$.

* For region $0 \le \rho \le a$, apply Ampere's law to path L_1,

$$\oint_{L_1} \bar{H} \cdot d\bar{l} = \oint I_{enc} = \int \bar{J} \cdot d\bar{S} \quad -(1)$$

Since the current is uniformly distributed over the cross section,

$$\bar{J} = \frac{I}{\pi a^2} a_z, \quad d\bar{S} = \rho \, d\phi \, d\rho \, a_z.$$

$$I_{enc} = \int \bar{J} \cdot d\bar{S}$$

$$= \frac{I}{\pi a^2} \iint \rho \, d\phi \, d\rho \qquad 0 < \phi < 2\pi$$

$$= \frac{I}{\pi a^2} \pi \rho^2$$

$$= \frac{I \rho^2}{a^2}$$

from eq. (1), $H_\phi \int dl = H_\phi 2\pi \rho = \frac{I \rho^2}{a^2}$.

$$H_\phi = \frac{I \rho}{2\pi a^2} \quad -(2)$$

* For region $a \le \rho \le b$, the path L_2 as the Amperian path,

$$\oint_{L_2} \bar{H} \cdot d\bar{l} = I_{enc} = I.$$

$$H_\phi 2\pi \rho = I. \quad ; \quad H_\phi = \frac{I}{2\pi \rho} \quad -(3)$$

* For the region $b \leq \rho \leq b+t$, the path L_3,

$$\oint \bar{H} \cdot d\bar{l} = H_\phi \cdot 2\pi \rho$$
$$= I_{enc} \quad -④$$

$$\because \quad I_{enc} = I + \int \bar{J} \cdot d\bar{s}$$

where, $\bar{J}$ is the current density (current per unit area) of the outer conductor and is along $-a_z$, i.e.,

$$\bar{J} = \frac{-I}{\pi[(b+t)^2 - b^2]} a_z$$

$$I_{enc} = I - \frac{I}{\pi[(b+t)^2 - t^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] \quad -⑤$$

sub. ⑤ in ④,

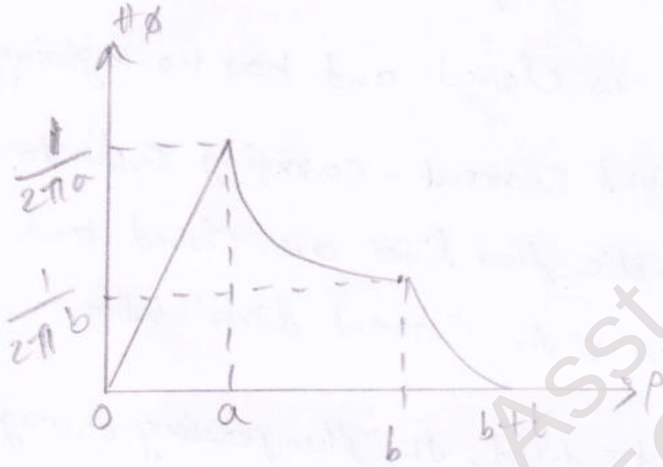
$$H_\phi = \frac{I}{2\pi \rho} \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] \quad -⑥$$

* For region $\rho \geq b+t$, use path L_4,

$$\oint_{L_4} \bar{H} \, d\bar{l} = I - I = 0.$$

$$\therefore \quad H_\phi = 0.$$

$$\bar{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\, a_\phi & 0 \le \rho \le a \\ \dfrac{I}{2\pi\rho}\, a_\phi, & a \le \rho \le b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right] a_\phi & b \le \rho \le b+t \\ 0 & \rho \ge b+t \end{cases}$$



Magnetic flux density:- [Maxwell's Equation]:-

$\bar{B} \rightarrow$ Magnetic flux density.

$\bar{B}$ is related to the Magnetic field Intensity $(\bar{H})$,

$$\bar{B} = \mu_0 \bar{H}.$$

where, $\mu_0 \rightarrow$ permeability of freespace.

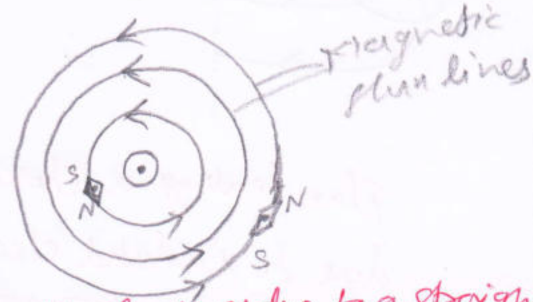
$$\mu_0 = 4\pi \times 10^{-7}\ H/m.\ [\text{Henrys/meter}]$$

The Magnetic flux through a surface S is,

$$\psi = \int_S \bar{B} \cdot d\bar{S}$$

where, $\psi \longrightarrow$ webers (Wb)

$\bar{B} \longrightarrow$ webers/squaremeter (Wb/m^2) (or) Teslas



Magnetic flux lines due to a straight wire with current Coming out of the page.

The magnetic flux line is the path to which $\bar{B}$ is tangential at every point in a magnetic field. It is the line along which the needle of a magnetic compass will orient itself if placed in the magnetic field.

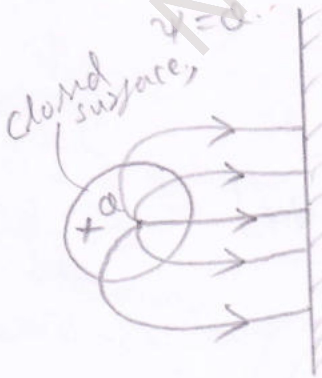
* The direction of $\bar{B}$ is taken as that indicated as "north" by the needle of the magnetic campass.

* Each flux line is closed and has no begining or end. The figure is for a straight current - carrying conductor, it is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

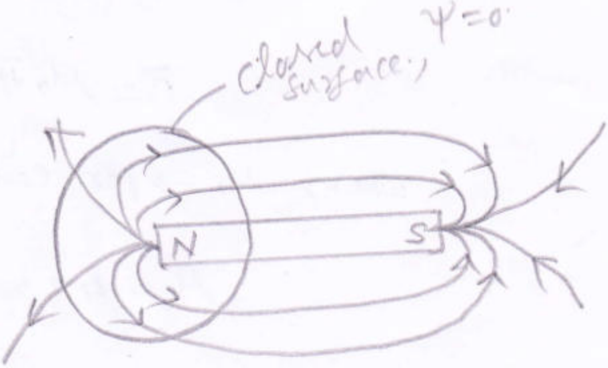
In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed;

$$\Psi = \oint \bar{D} \cdot d\bar{S} = Q_{enc}.$$

* It is possible to have isolated electric charge as shown in figure, which reveals, electric flux lines are not necessarily closed.



Flux leaving a closed surface due to isolated electric charge,
$\Psi = \oint_S \bar{D} \cdot d\bar{S} = Q_{enc}$



Flux leaving a closed surface due to magnetic charge,
$\Psi = \oint_S \bar{B} \cdot d\bar{S} = 0.$

* Unlike electric flux lines, magnetic flux lines always close upon themselves as shown in figure. This is due to the fact it is not possible to have isolated magnetic poles (or magnetic charges).

Eg:- An isolated magnetic pole by dividing a magnetic bar successively in to two, we end up with pieces each having north & south poles. It is impossible to seperate the north pole from the south pole.

* An "isolated magnetic" charge does not exist

The total flux through a closed surface in a magnetic field must be zero,

$$\oint \bar{B} \cdot d\bar{s} = 0. \quad — (a)$$

This eq. is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields., just as,

$\oint \bar{D} \cdot d\bar{s} = \bar{Q}$ in gauss's law for electrostatic fields.

The magnetostatic field is not conservative, magnetic flux is conservative.

Apply divergence theorem to eq. (a),

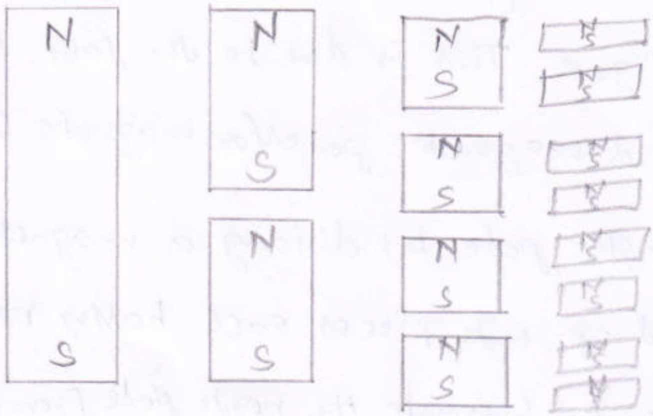
$$\oint_S \bar{B} \cdot d\bar{s} = \int_V \nabla \cdot \bar{B} \, dv = 0. \quad — (b)$$

$$\nabla \cdot \bar{B} = 0. \quad — (c)$$

eq. (c) is the fourth maxwell's eq.

Eq. (a) & (c) shows that magnetostatic fields have no sources (or) sinks.

* Magnetic field lines are always continuous.



Successive division of a bar magnet results in pieces with north & south poles, showing that magnetic poles cannot be isolated.

Maxwell's Equations For Static EM Fields:-

Differential form (or) Point form	Integral form	
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv$	Gauss's law
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$	Non-existance of Magnetic monopole
$\nabla \times \bar{E} = 0$	$\oint_L \bar{E} \cdot d\bar{l} = 0$	Conservativeness of Electrostatic field
$\nabla \times \bar{H} = \bar{J}$	$\oint_L \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{s}$	Ampere's law

Magnetic Scalar and vector Potentials:-

* The potential in electrostatics related with electric field intensity is given by,

$$\bar{E} = -\nabla V.$$

Similary a potential associated with magnetostatic field $\bar{B}$. The magnetic potential Could be Scalar V_m (or) vector $\bar{A}$. To define V_m & $\bar{A}$ involves two identities,

$$\nabla \times (\nabla V) = 0 \quad -(1)$$

$$\nabla \cdot (\nabla \times \bar{A}) = 0. \quad -(2)$$

As, $\bar{E} = -\nabla V$, the magnetic scalar potential V_m (in amperes) as related to $\bar{H}$, according to,

$$\bar{H} = -\nabla V_m \quad \text{if } \bar{J} = 0 \quad -(3)$$

$$\because \nabla \times \bar{H} = \bar{J}$$

$$\Rightarrow \bar{J} = \nabla \times \bar{H}$$

$$= \nabla \times (-\nabla V_m) = 0. \quad -(4)$$

The magnetic scalar potential V_m is only defined in a region where $\bar{J} = 0$. V_m satisfies Laplace's equation just as V does for electrostatic fields.

$$\text{Hence, } \nabla^2 V_m = 0., \quad (\bar{J} = 0) \quad -(5)$$

For magnetostatic field, $\nabla \cdot \bar{B} = 0$,

The vector magnetic potential $\bar{A}$ (in Wb/m), such that,

$$\bar{B} = \nabla \times \bar{A} \quad \text{— (6)}$$

$$V = \int \frac{dQ}{4\pi \varepsilon_0 r} \quad \text{— (7)}$$

$$\bar{A} = \int_L \frac{\mu_0 I \, d\bar{l}}{4\pi R} \text{ for line current} \quad \text{— (8)}$$

$$\bar{A} = \int_S \frac{\mu_0 \bar{K} \, ds}{4\pi R} \text{ for surface current} \quad \text{— (9)}$$

$$\bar{A} = \int_V \frac{\mu_0 \bar{J} \, dv}{4\pi R} \text{ for volume current.} \quad \text{— (10)}$$

$$\bar{H} = \int_L \frac{I \, d\bar{l} \times a_R}{4\pi R^2}$$

$$\bar{B} = \frac{\mu_0}{4\pi} \int_L \frac{I \, d\bar{l}' \times \bar{R}}{R^3} \quad \text{— (11)}$$

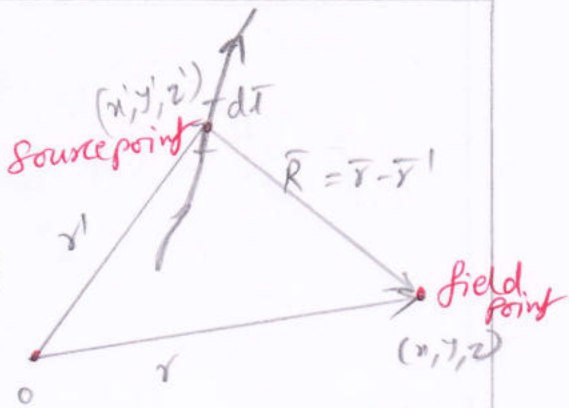
where $\bar{R}$ is the distance vector from the line element $d\bar{l}'$ at the source point (x', y', z') to the field point (x, y, z) as shown in figure, & $R = |\bar{R}|$ i.e.,

$$R = |\bar{r} - \bar{r}'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \quad \text{— (12)}$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x-x')a_x + (y-y')a_y + (z-z')a_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} = -\frac{\bar{R}}{R^3}$$

$$\frac{\bar{R}}{R^3} = -\nabla\left(\frac{1}{R}\right), \quad \left(= \frac{a_R}{R^2}\right) \quad \text{— (13)}$$

Sub., in eq. (13) in eq. (11)



$$\bar{B} = -\frac{\mu_0}{4\pi}\int_L I\, d\bar{l}' \times \nabla\left(\frac{1}{R}\right) \quad -(14)$$

*Basic vector Identity,

$$\nabla \times (f\bar{F}) = f\,\nabla \times \bar{F} + (\nabla f) \times \bar{F} \quad -(15)$$

$f \rightarrow$ scalar field

$\bar{F} \rightarrow$ vector field.

Let us assume, $f = \frac{1}{R}$ & $\bar{F} = d\bar{l}$,

$$d\bar{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}(\nabla \times d\bar{l}') - \nabla \times \left(\frac{d\bar{l}'}{R}\right)$$

$\because \nabla$ operates w.r.t (x, y, z) while $d\bar{l}'$ is a function of (x', y', z'),

$$\nabla \times d\bar{l}' = 0$$

$$d\bar{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\bar{l}'}{R} \quad -(16)$$

$\therefore$ eq. (14) reduces to,

$$\bar{B} = \nabla \times \int_L \frac{\mu_0 I\, d\bar{l}'}{4\pi R} \rightarrow (17)$$

Comparing eq.'s (17), & eq. (6),

$$\bar{A} = \int_L \frac{\mu_0 I\, d\bar{l}'}{4\pi R}$$

$$\Psi = \int_S \bar{B} \cdot d\bar{S}$$

$$\because \bar{B} = \nabla \times \bar{A},$$

$$\Psi = \int_S (\nabla \times \bar{A}) \cdot d\bar{S}$$

Stoke's Theorem,

$$\Psi = \oint_L \bar{A} \cdot d\bar{l}$$

$\Psi \rightarrow$ Magnetic flux through a given area.

Forces due to Magnetic Fields:-

There are at least three ways in which force due to magnetic fields can be experienced.

The force can be (a) Due to a moving charged particle in a $\bar{B}$ field,

(b) On a current element in an external $\bar{B}$ field,

(c) Between two current elements.

A. Force On a Charged Particle:-

$F_e \rightarrow$ Force on a stationary (or) moving electric charge Q in an electric field.

- Relation b/n F_e & E as, $\left.\begin{matrix}\bar{F}\\ \bar{E}\end{matrix}\right\}$ same direction

$$\bar{F}_e = Q\bar{E} \quad -① $$

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\bar{F}_m$ experienced by a charge Q moving with a velocity $\bar{u}$ in a magnetic field $\bar{B}$ is,

$$\bar{F}_m = Q\bar{u} \times \bar{B} \quad -② $$

$\bar{F}_m$ is per to $\bar{u}$ & $\bar{B}$.

* F_e is independent of the velocity of the charge & can perform work on the charge & change its kinetic energy.

$\bar{F}_m \rightarrow$ depends on charge velocity & normal to it.

$\bar{F}_m \cdot d\bar{l} = 0$ } work cannot perform because it is at right angles to the direction of motion of the charge.

For a moving Charge Q in the presence of both electric & magnetic fields,

The total force on the Charge is,

$$\bar{F} = \bar{F}_e + \bar{F}_m$$

$$\boxed{\bar{F} = Q\left[\bar{E} + \bar{u} \times \bar{B}\right]} \quad - (3)$$

Eq. (3) is called Lorentz force Equation. It relates mechanical force to Electrical force. If the mass of the charged particle moving in $\bar{E}$ & $\bar{B}$ fields is m, by Newton's second law of motion,

$$\bar{F} = m\frac{d\bar{u}}{dt} = Q(\bar{E} + \bar{u} \times \bar{B}) \quad - (4)$$

State of particle	$\bar{E}$ field	$\bar{B}$ field	Combined $\bar{E}$ & $\bar{B}$ fields.
Stationary	$Q\bar{E}$		$Q\bar{E}$
Moving	$Q\bar{E}$	$Q\bar{u} \times \bar{B}$	$Q(\bar{E} + \bar{u} \times \bar{B})$

B. Force on a Current Element:-

The force on a Current element $I d\bar{l}$ of a Current-carrying Conductor due to the magnetic field $\bar{B}$,

Using the Convection Current,

$$\bar{J} = \rho_v \bar{u} \quad - (1)$$

$$I\, d\bar{l} = \bar{K}\, ds = \bar{J}\, dv \quad - (2)$$

Comparing eq's (1) & (2),

$$I\,dl = \rho_v \bar{u}\, dv$$

$$= (\rho_v dv)\, \bar{u}$$

$$I\,dl = dQ\, \bar{u}$$

Alternatively, $I\,d\bar{l} = \frac{dQ}{dt} d\bar{l} = dQ \frac{d\bar{l}}{dt} = dQ\,\bar{u}$

$$I\,d\bar{l} = dQ\,\bar{u} \quad -③$$

An element charge dQ moving with velocity $\bar{u}$ [there by producing Convection current element $dQ\bar{u}$) is equal to a conduction current ~~Curr~~ element $I\,d\bar{l}$.

∴ The force on a current element $I\,d\bar{l}$ in a magnetic field $\bar{B}$ is,

$$d\bar{F} = I\,d\bar{l} \times \bar{B}. \quad \left[\begin{matrix} \text{w.r.t. eq ②} \\ \text{from part ⓐ} \end{matrix}\right] \left\{\begin{matrix} \text{Replacing} \\ Q\bar{u} \text{ by } I d\bar{l} \end{matrix}\right\}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by,

$$\bar{F} = \oint_L I\,d\bar{l} \times \bar{B}$$

Instead of $I\,d\bar{l}$, if surface element $\bar{K}\,ds$ then,

$$dF = K\,ds \times \bar{B}$$

$$F = \int_S K\,ds \times \bar{B}$$

if volume current element,

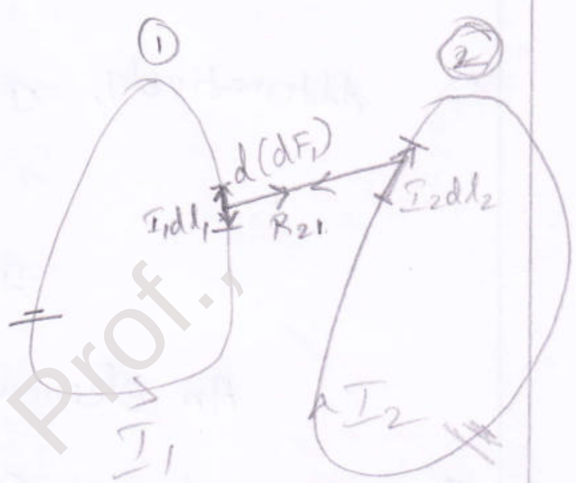
$$d\bar{F} = \bar{J}\,dv \times \bar{B}$$

$$\bar{F} = \int_V \bar{J}\,dv \times \bar{B}$$

* The magnetic field $\bar{B}$ is defined as the force per unit current element.

(c) Force between Two Current Elements:-

Let us consider the force between two elements $I_1 d\bar{l}_1$ & $I_2 d\bar{l}_2$.



Force between two Current loops

A/c to Biot-Savart's law, both current elements produce magnetic fields,

The force $d(dF_1)$ on element $I_1 d\bar{l}_1$ due to the field $d\bar{B}_2$ produced by element $I_2 d\bar{l}_2$

$$d(d\bar{F}_1) = I_1 dl_1 \times d\bar{B}_2 \quad -(1)$$

$$d\bar{B}_2 = \frac{\mu_0 I_2 dl_2 \times a_{R_{21}}}{4\pi R_{21}^2} \quad -(2)$$

$$d(d\bar{F}_1) = \frac{\mu_0 I_1 d\bar{l}_1 \times (I_2 d\bar{l}_2 \times a_{R_{21}})}{4\pi R_{21}^2} \quad -(3)$$

$\bar{F}_1$ is the total force, on current loop 1 due to current loop 2 as,

$$\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{l}_1 \times (d\bar{l}_2 \times \bar{a}_{R_{21}})}{R_{21}^2} \quad -(4)$$

The force $\bar{F}_2$ on loop 2 due to magnetic field $\bar{B}_1$ from loop 1 is from eq-(6), by interchanging subscripts 1 & 2.

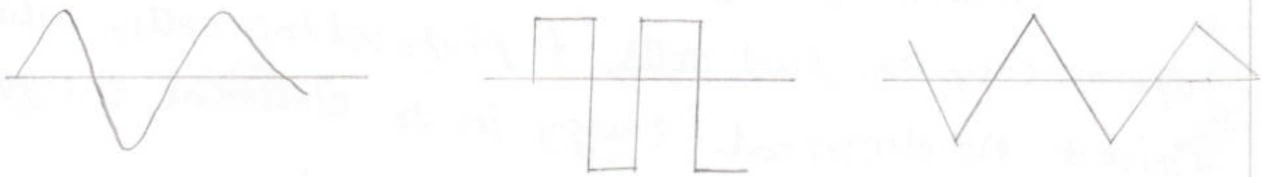
$$\bar{F}_2 = -\bar{F}_1$$

$\bar{F}_1$ & $\bar{F}_2$ obeys Newton's Third law.

Maxwell's Equations
[Time varying Fields]

Time-varying fields (or) waves are usually due to accelerated charges or time varying currents. Any pulsating current will produce radiation [Time varying fields].

* Stationary charges → Electrostatic fields
* Steady currents → Magnetostatic fields.
* Time-varying currents → Electromagnetic fields (or) waves.



Various types of Time-Varying Current

Faraday's Law:-

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying fields produces an induced voltage [called electromotive force or EMF] in a closed circuit, which causes a flow of current.

"Faraday discovered that the induced Emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit."

This is called Faraday's law, it can be expressed as,

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\psi}{dt} \quad -(1)$$

Where N is the number of turns in the circuit and

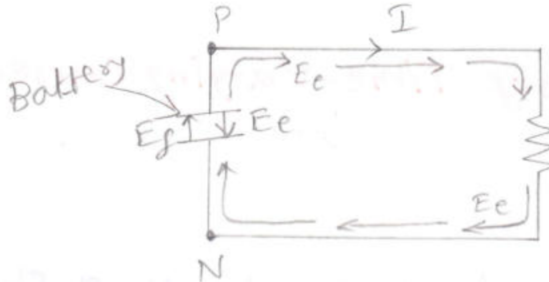
* Ψ is the flux through each turn.

* The -ve sign shows that the induced voltage acts in such a way as to oppose the flux producing it.

This is Eq. ① known as Lenz's law.

Other than Electric fields produced by electric charges, there are other kinds of electric fields not directly caused by electric charges. These are Emf - produced fields.

Sources of Emf include electric generators, batteries, thermo couples, fuel cells, & photovoltaic cells, which all convert nonelectrical energy in to electrical energy.



A circuit showing Emf-producing field E_f & electrostatic field E_e

Consider the electric circuit, where the battery is a source of Emf. The electro chemical action of the battery results in an Emf - produced $\bar{E}_f$. Due to accumulation of charge at the battery terminals, an Electrostatic field $\bar{E}_e (= -\nabla V)$ also exists.

Total electric field at any point,

$$\bar{E} = \bar{E}_f + \bar{E}_e$$

$\bar{E}_f$ is zero outside the battery.

$\bar{E}_f$ & $\bar{E}_e$ have opp. direction in the battery

Integrate over the closed circuit,

$$\oint_L \bar{E}\cdot d\bar{l} = \oint_L \bar{E}_f \cdot d\bar{l} + 0 = \int_N^P \bar{E}_f \cdot d\bar{l} \quad \text{[Through battery]}$$

where $\oint \bar{E}_e \cdot d\bar{l} = 0$ because $\bar{E}_e$ is conservative. The Emf of the battery is the line integral of the Emf-produced field, i.e.,

$$V_{emf} = \int_N^P \bar{E}_f \cdot d\bar{l} = -\int_N^P \bar{E}_e \cdot d\bar{l} = IR$$

$\because$ $\bar{E}_f$ & $\bar{E}_e$ are equal but opposite with in the battery. It may also be regarded as the potential difference $(V_P - V_N)$ between the battery's open-circuit terminals.

Note:-

1. Electrostatic field $\bar{E}_e$ cannot maintain a steady current in a closed circuit since $\oint_L \bar{E}_e \cdot d\bar{l} = 0 = IR$.
2. An Emf-produced field $\bar{E}_f$ is nonconservative.
3. Except in electrostatics, voltage & potential difference are usually not equivalent.

Transformer & Motional EMF's :-

For a circuit with a single turn (N = 1),

$$V_{emf} = -N\frac{d\psi}{dt}$$

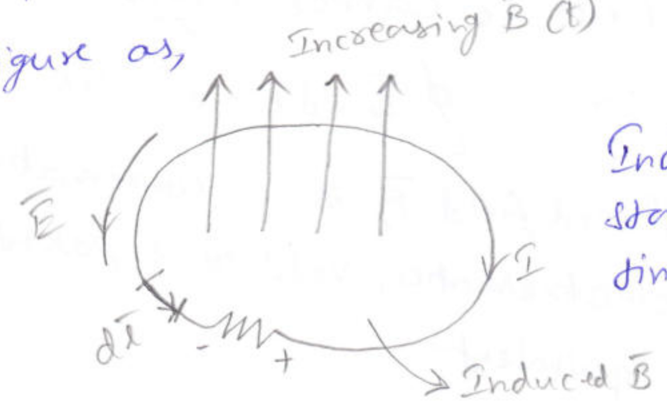
$$= -\frac{d\psi}{dt} \quad — (1)$$

$$V_{emf} = \oint_L \bar{E}\cdot d\bar{l} = -\frac{d}{dt}\int_S \bar{B}\cdot ds \quad - (2).$$

Where ψ has been replaced by $\int_S \bar{B}\cdot d\bar{s}$ and s is the surface area of the circuit bounded by the closed path L.

Eq. (2) in a time varying situation both electric & magnetic fields are present and are interrelated.

Eq. (2) is based on stokes theorem. Eq. (2) is shown in the form of figure as,



Induced Emf due to a stationary loop in a time varying $\bar{B}$ field.

The variation of flux with time may be caused in three ways,

1. By having a stationary loop in a time-varying $\bar{B}$ field.
2. By having a time-varying loop area in a static $\bar{B}$ field.
3. By having a time-varying loop area in a time-varying $\bar{B}$ field.

1. Stationary loop in Time-Varying $\bar{B}$ field [Transformer EMF]

Consider a stationary conducting loop is in a time-varying magnetic field $\bar{B}$ field.

$$V_{emf} = \oint_L \bar{E} \cdot dl \qquad — ①$$

$$= -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}. \qquad — ②$$

This emf induced by the time-varying current (producing the time-varying $\bar{B}$ field) in a stationary loop is often referred to as transformer emf in power analysis since it is due to transformer action.

Apply stoke's theorem for Eq. ①,

$$\oint_L \bar{E} \cdot dl = \int_S (\nabla \times \bar{E}) \cdot d\bar{s} = -\int_S \frac{\partial \bar{B}}{\partial t} d\bar{s}. \quad — ③$$

From eq. ③ equating the integrals,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}.$$

This is one of the Maxwell's Equations for time-varying fields. It shows that the time varying $\bar{E}$ field is not conservative $[\nabla \times \bar{E} \neq 0]$. This does imply that the principles of energy conservation are violated.

The work done in taking a charge about a closed path in a time-varying field, for Example, is due to energy from the time varying magnetic field.

2. Moving loop in static $\bar{B}$ field [Motional Emf]:-

When a Conducting loop is moving in a static $\bar{B}$ field, an Emf is induced in the loop. The force on a Charge moving with uniform velocity $\bar{u}$ in a magnetic field $\bar{B}$ is,

$$\bar{F}_m = Q\bar{u} \times \bar{B} \quad - (1)$$

The Motional electric field $\bar{E}_m$ as,

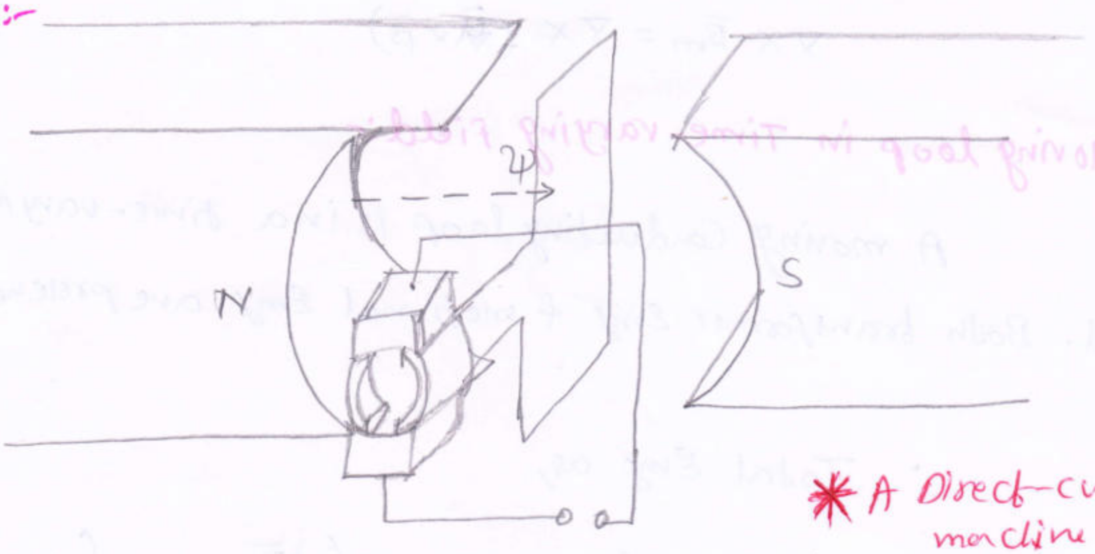
$$\bar{E}_m = \frac{\bar{F}_m}{Q} = \bar{u} \times \bar{B}. \quad - (2)$$

Consider a Conducting loop, moving with uniform velocity $\bar{u}$ as Consisting of a large number of free electrons, the Emf induced in the loop is,

$$V_{emf} = \oint_L \bar{E}_m \cdot d\bar{l} = \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l}. \quad - (3)$$

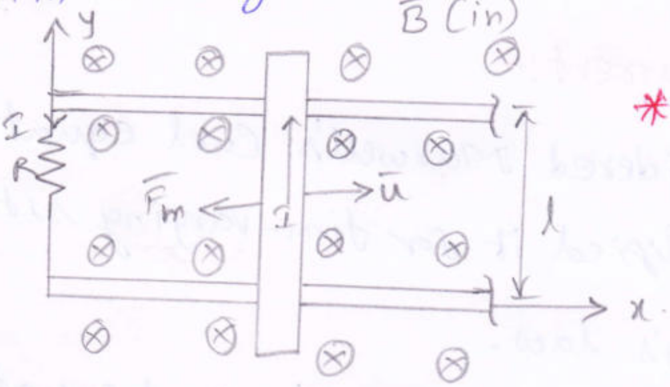
This type of Emf is called motional Emf or flux-cutting Emf because it is due to motional action. It is the kind of Emf found in electrical machines such as motors, generators, and alternators.

Examples:-



* A Direct-current machine

Consider a two-pole dc machine with an armature coil and a two bar commutator. The voltage is generated as the coil rotates with in the magnetic field.



* Induced Emf due to a moving loop in a static B field.

~~Consider the voltage is generated as the coil rotates within the magnetic field.~~ Consider a rod is moving between a pair of rails. In this $\overline{B}$ & $\overline{u}$ are perpendicular, then the force is,

$$\overline{F}_m = I l \times \overline{B}$$

$$F_m = I l B.$$

$$\therefore V_{emf} = uBl.$$

using stokes theorem,

$$\int_s (\nabla \times \overline{E}_m) \cdot d\overline{s} = \int_s \nabla \times (u \times \overline{B}) \cdot d\overline{s}$$

$$\nabla \times \bar{E}_m = \nabla \times (\bar{u} \times \bar{B})$$

3. Moving loop in Time-varying field:-

A moving Conducting loop is in a time-varying magnetic field. Both transformer Emf & motional Emf are present.

∴ Total Emf as,

$$V_{emf} = \oint_L \bar{E} \cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l}$$

$$\therefore \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} + \nabla \times (u \times \bar{B})$$

Displacement Current:-

Reconsidered Maxwell's Curl equation for electrostatic fields and modified it for time-varying situations to satisfy Faraday's law.

Now Consider Maxwell's Curl equation for magnetic fields [Ampere's Circuit law] for time-varying Conditions.

For static EM fields,

$$\nabla \times \bar{H} = \bar{J} \quad \text{—} (1)$$

The divergence of the curl of any vector field is identically zero.

Hence, $\nabla \cdot (\nabla \times \bar{H}) = 0$ [Proof in the previous unit].

$$\therefore \nabla \cdot \bar{J} = 0. \quad \text{—} (2)$$

The Continuity of Current is,

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \neq 0 \quad -③$$

Thus eq.'s ② & ③ are obviously incompatible for time-varying conditions.

Add a term to eq.①, then,

$$\nabla \times \bar{H} = \bar{J} + \bar{J}_d \quad -④$$

The divergence of the curl of any vector is zero,

Hence,

$$\nabla \cdot (\nabla \times \bar{H}) = 0 = \nabla \cdot \bar{J} + \nabla \cdot \bar{J}_d$$

$$\nabla \cdot \bar{J}_d = -\nabla \cdot \bar{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$= \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

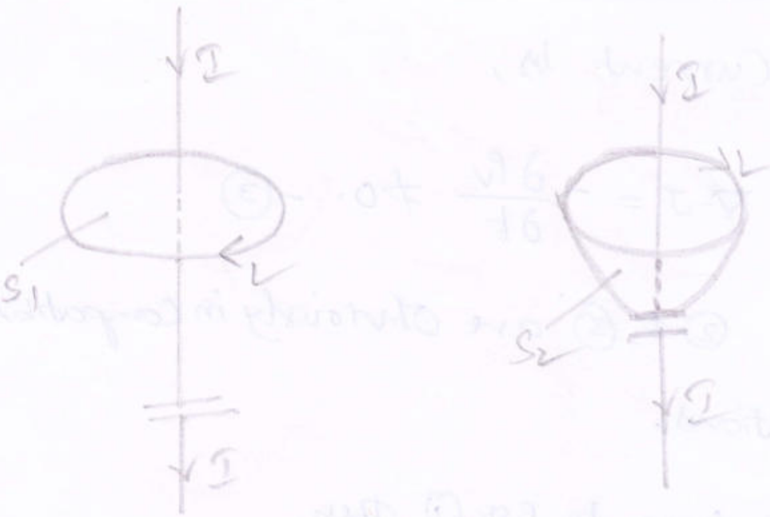
$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t} \quad -⑤$$

sub. eq.⑤ in ④,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad -⑥$$

Eq-⑥ is Maxwell's equation for a time-varying field.

The term $\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$ is known as displacement current density & $\bar{J}$ is the conduction current density $(\bar{J} = \sigma \bar{E})$.



Two surfaces of integration
Showing the need for J_d in Ampere's circuit law.

Based on the displacement Current density, the displacement Current as,

$$I_d = \int \bar{J}_d \cdot d\bar{s} = \int \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$$

The displacement current is a result of time-varying electric field.

Eg:- Current through a capacitor when an alternating voltage source is applied to its plates. Eg. Shown in figure.

Ampere's circuit law to a Closed path L shown in figure, gives,

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_1} \bar{J} \cdot d\bar{s} = I_{enc} = I.$$

where I is the Current through the conductor & S_1 is the flat surface bounded by L. If the balloon-shaped surface S_2 that passes between the capacitor plates, as shown in figure.

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_2} \bar{J} \cdot d\bar{s} = I_{enc} = 0.$$

Because no conduction current ($\bar{J} = 0$) flows through s_2.

The total current density is, $\bar{J} + \bar{J}_d$,)

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{s_2} \bar{J}_d \cdot d\bar{s} = \frac{d}{dt} \int_{s_2} \bar{D} \cdot d\bar{s} = \frac{dQ}{dt} = I.$$

Same current for either surface through it is conduction current in s_1 & displacement current in s_2.

Boundary Conditions :-

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called "boundary conditions".

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. The conditions will be dictated by the types of material the media are made of.

* Dielectric (ε_{r_1}) & Dielectric (ε_{r_2})
* Conductor & Dielectric
* Conductor & free space.

To determine the boundary conditions, use Maxwell's equations,

$$\oint \bar{E} \cdot d\bar{l} = 0. \quad — ①$$

$$\oint \bar{D} \cdot d\bar{s} = Q_{enc} \quad — ②$$

The electric field intensity $\bar{E}$ into two orthogonal components,

$$\bar{E} = \bar{E}_t + \bar{E}_n \quad — ③$$

Where, $\bar{E}_t$ is tangential component of $\bar{E}$ &
$\bar{E}_n$ " normal " " .

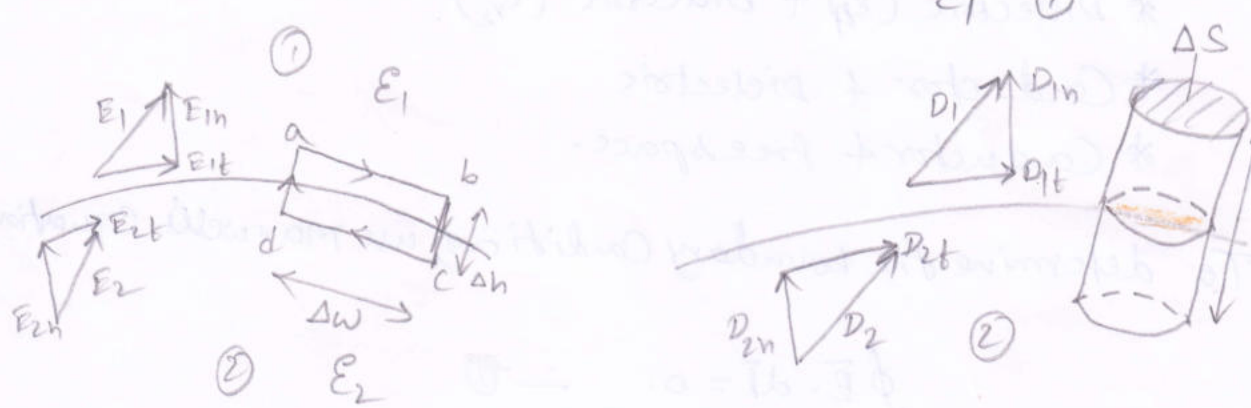
A similar decomposition can be done for the electric flux density $\bar{D}$.

(1) Dielectric - Dielectric Boundary Conditions:-

Consider the $\bar{E}$ field existing in a region Consisting of two different dielectrics characterized by,

$$\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1} \quad \& \quad \bar{E}_1 \text{ in media 1.}$$
$$\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2} \quad \quad \bar{E}_2 \quad " \quad 2.$$

$$\left.\begin{array}{l} \bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1n} \\ \bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2n} \end{array}\right\} - (4)$$



Dielectric - Dielectric Boundary.

Total electric field intensity for the closed path abcda, assume that the path is very small w.r.t the variation of $\bar{E}$.

$$0 = E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} - (5)$$

Where, $E_t = |\bar{E}_t|$ & $E_n = |\bar{E}_n|$ as $\Delta h \to 0$, eq. (5) becomes,

$$E_{1t} = E_{2t} \quad -(6)$$

Thus the tangential components of $\bar{E}$ are the same on the two sides of the boundary. $\bar{E}_t$ undergoes no change on the boundary & it is said to be continuous across the boundary.

$$\therefore \bar{D} = \varepsilon \bar{E} = \bar{D}_t + \bar{D}_n,$$

eq. ③ can be written as,

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \quad — ⑦$$

i.e., D_t undergoes some change across the interface. Hence, D_t is said to be discontinuous across the interface.

Similarly eq. ② to the pillbox [Gaussian surface]. Allowing $\Delta h \to 0$, gives,

$$\Delta Q = \rho_s \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$D_{1n} - D_{2n} = \rho_s. \quad — ⑧$$

ρ_s is the free charge density deliberately at the boundary. If no free charges exist at the interface (i.e., charges are not deliberately placed), $\rho_s = 0$,

$$D_{1n} - D_{2n} = 0$$

$$D_{1n} = D_{2n} \quad — ⑨$$

Thus the normal component of $\bar{D}$ is continuous across the interface, i.e., D_n undergoes no change at the boundary.

$\because \bar{D} = \varepsilon \bar{E}$,

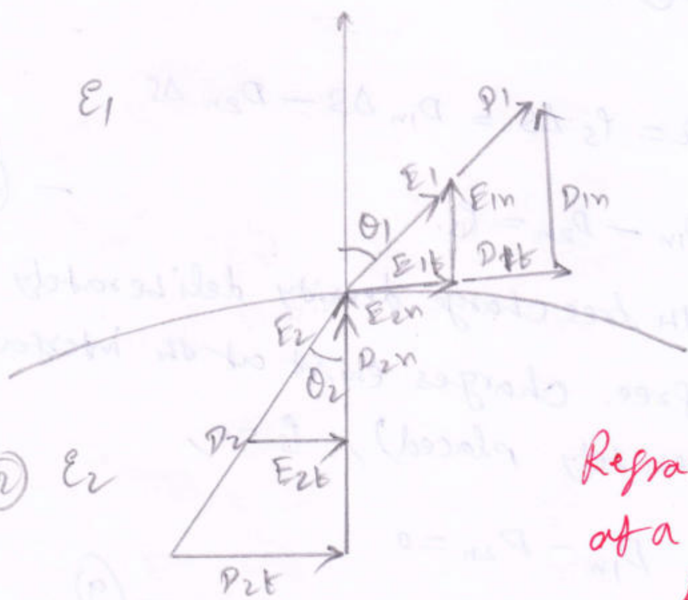
$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \quad \text{--- (10)}$$

The normal component of $\bar{E}$ is discontinuous at the boundary. eq.'s (6), (8), (9) are collectively referred to as boundary conditions, they must be satisfied by an electric field at the boundary seperating two different dielectrics.

The boundary conditions are used to determine the "refraction" of the electricfield across the interface.

Consider $\bar{D}_1$ or $\bar{E}_1$ and $\bar{D}_2$ or $\bar{E}_2$ making angles θ_1 & θ_2 with the normal to the interface as shown in figure,

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$



Refraction of $\bar{D}$ or $\bar{E}$ at a dielectric-dielectric boundary.

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \quad \text{--- (11)}$$

Similarly,

$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n}$$

$$\varepsilon_2 E_2 \cos\theta_2 = D_{2n}$$

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \quad -(12)$$

divide eq's (11) & (12),

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2} \quad -(13)$$

$\because \varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ &

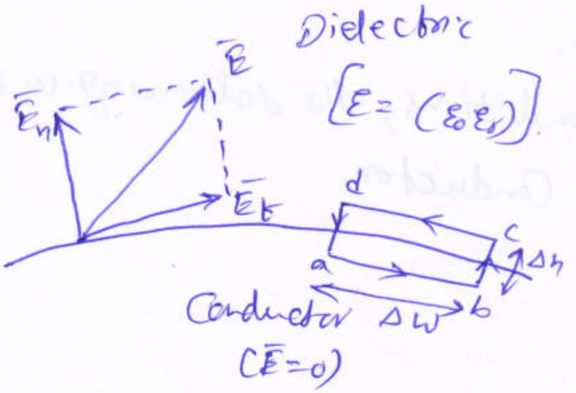
$\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$

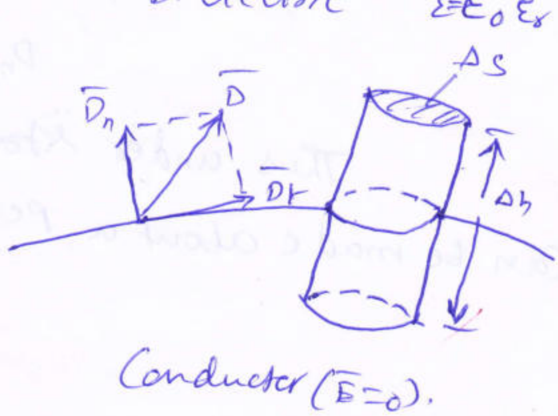
eq-(13) becomes,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \quad -(14)$$

This is the law of refraction of the electric field at a boundary free of charge [$\because \rho_s = 0$]. Interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the sides of the interface.

(2) Conductor - Dielectric Boundary Conditions:-





The Conductor is assumed to be perfect (i.e., $\sigma \to \infty$, (or) $\rho_c = 0$). Such a Conductor is not practically realizable,

Eg:- Copper & silver act as perfect Conductors.

To determine the boundary Conditions for a Conductor-dielectric interface, follow same procedure used for dielectric-dielectric interface except that, $\bar{E} = 0$ inside the conductor.

Apply $\oint_L \bar{E} \cdot d\bar{l} = 0$ for Closed path abcda as;

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2} \quad (15)$$

As $\Delta h \to 0$,

$$E_t = 0.$$

For a pill box, letting $\Delta h \to 0$,

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S.$$

because, $\bar{D} = \varepsilon \bar{E} = 0$, inside the conductor,

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_s$$

$$D_n = \rho_s.$$

Thus under static Conditions, the following conclusions can be made about a perfect Conductor,

1. No electric field may exist with in a conductor, i.e.,

$$\rho_v = 0, \quad \bar{E} = 0.$$

2. Since $\bar{E} = -\nabla V = 0$, there can be no potential difference between any two points in the conductor, i.e., a conductor is an equipotential body.

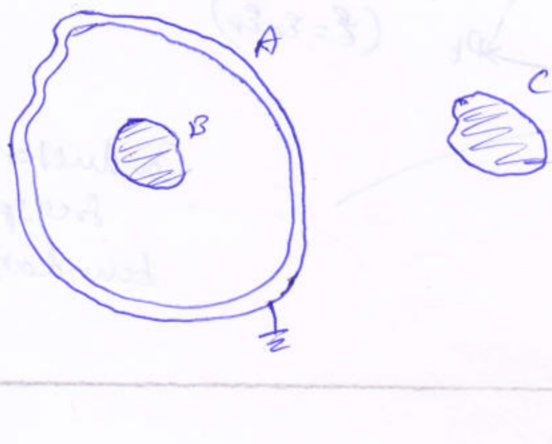
3. The electric field $\bar{E}$ can be external to the conductor & normal to its surface, i.e.,

$$D_t = \varepsilon_0 \varepsilon_r E_t = 0,$$

$$D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s.$$

An important application of the fact that $\bar{E} = 0$ inside a conductor is in electrostatic screeing or shielding. If conductor B as shown in figure, B is said to be electrically screened by A from electric systems, such a conductor C, outside A.

Similarly, conductor C outside A is screened by A from B. Thus conductor A acts like a screen or shield & the electrical conditions inside & outside are completely independent of eachother.



Electrostatic screening

3. Conductor - Free space Boundary Conditions:-

It is a special case of Conductor - dielectric Conditions, as shown in figure. The boundary Conditions at the interface between a Conductor & free space can be obtained as, by replacing ε_r by 1.

∴ Free space may be regarded as the special case dielectric for which $\varepsilon_r = 1$.

The electric field $\bar{E}$ to be external to the conductor and normal to its surface.

Thus the boundary Conditions from the previous Case,

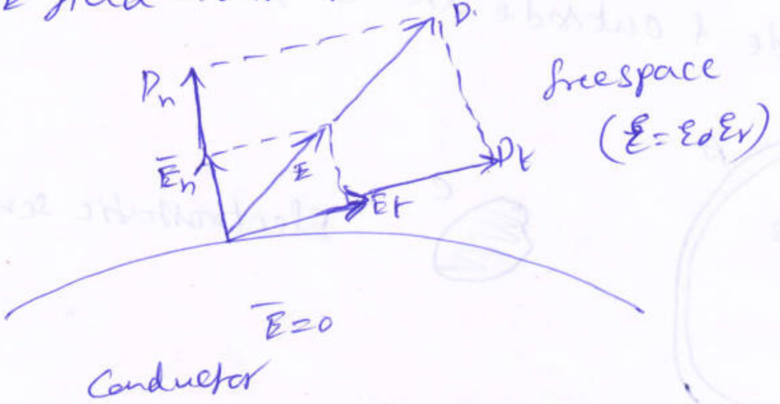
$$D_t = \varepsilon_0 \varepsilon_r E_t = 0$$

$$D_t = \varepsilon_0 E_t = 0. \quad [\because \varepsilon_r = 1].$$

$$D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s$$

$$D_n = \varepsilon_r E_n = \rho_s$$

$\bar{E}$ field must approach a Conducting surface normally.



Conductor - freespace boundary.

UNIT-IV

EM Wave Characteristics

1. Free space $\{\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0\}$ $\{\varepsilon_r = 1\}$
2. Lossless dielectrics $\{\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0, \sigma \ll \omega\varepsilon\}$
3. Lossy dielectrics $\{\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0\}$
4. Good Conductors $\{\sigma \simeq \infty, \varepsilon = \varepsilon_0, \mu = \mu_r\mu_0, \sigma \gg \omega\varepsilon\}$

$\omega \rightarrow$ Angular freq. of the wave.

* A wave is a function of both space & time.

wave motion occurs when a disturbance at point A, at time t_0, is related to what happens at point B, at time $t > t_0$. A wave equation, is a partial differential equation of the second order.

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0 \qquad - (1)$$

$u \rightarrow$ wave velocity.

If the medium is free space then $\rho_v = 0, \bar{J} = 0$.

$$E^- = f(z - ut)$$
$$E^+ = g(z + ut)$$

(or)

$$E = f(z - ut) + g(z + ut) \qquad - (2)$$

f & g functions include $z \pm ut$, $\sin k(z \pm ut)$, $\cos k(z \pm ut)$ and $e^{jk(z \pm ut)}$, K is a constant.

If, assume harmonic (or sinusoidal) time dependance $e^{j\omega t}$,

then,

$$\frac{d^2 E_s}{dz^2} + \beta^2 E_s = 0.$$

where $\beta = \omega/u$

E_s is phasor form of E.

$$E^+ = A e^{j(\omega t - \beta z)}$$

$$E^- = B e^{j(\omega t + \beta z)}$$

$$\& \; E = A e^{j(\omega t - \beta z)} + B e^{j(\omega t + \beta z)} \quad — (3)$$

where, A & B are real Constants.

Taking the imaginary part of the equation,

$$E = A \sin(\omega t - \beta z). \quad — (4)$$

Following Characteristics of the wave in the equation (4)

1. It is time harmonic because, it is assumed time dependence $e^{j\omega t}$ to arrive at the eq. (4).

2. A is called the amplitude of the wave and has the same units as E.

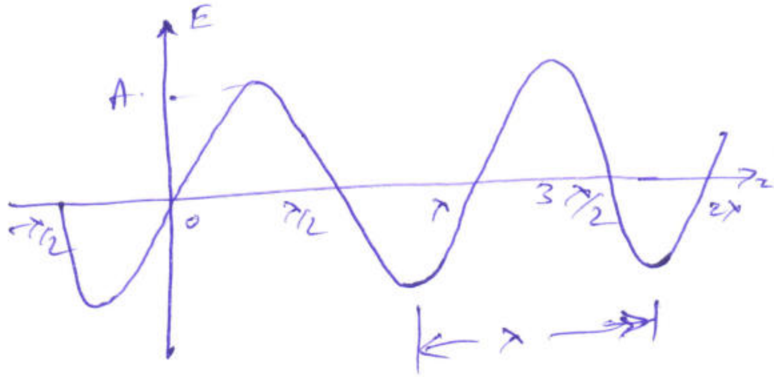
~~3. ($\omega t - \beta z$) is the amplitude of the wave~~

3. ($\omega t - \beta z$) is the phase (in radians) of the wave, it depends on time t & space variable z.

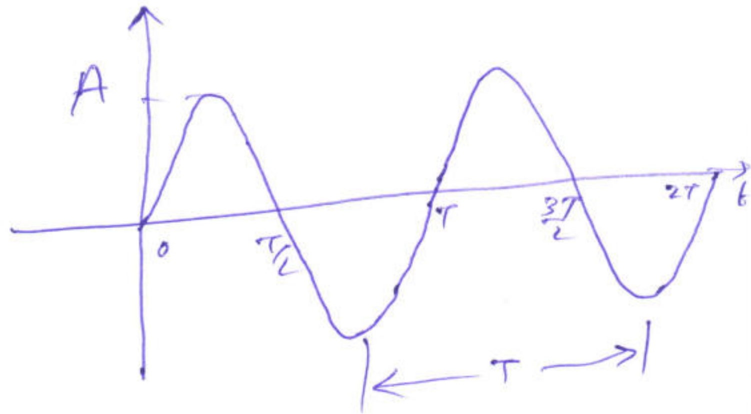
4. ω is the angular frequency (in radians/second); β is the phase constant or wave number (in radians/meter).

Due to the variation of E with both time t and space variable z, we may plot E as a function of t by keeping z constant and vice versa.

The plots of E (z, t = constant) & E (t, z = constant).



with constant t.



with constant z.

Power and the Poynting vector:-

Energy can be transported from one point to another point by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations,

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \quad - (1)$$

$$\nabla \times \bar{H} = \sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t} \quad - (2)$$

Take dot product w.r.t $\bar{E}$ gives,

$$\bar{E} \cdot (\nabla \times \bar{H}) = \sigma E^2 + \bar{E} \cdot \varepsilon \frac{\partial \bar{E}}{\partial t} \quad - (3)$$

Identity:-

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B}) \quad \left\{ \begin{array}{l} A = \bar{H} \\ B = \bar{E} \end{array} \right\}$$

$$\bar{H} \cdot (\nabla \times \bar{E}) + \nabla \cdot (\bar{H} \times \bar{E}) = \sigma E^2 + \bar{E} \cdot \varepsilon \frac{\partial \bar{E}}{\partial t} \quad - (4)$$

$$\bar{H} \cdot (\nabla \times \bar{E}) = \bar{H} \cdot \left(-\mu \frac{\partial \bar{H}}{\partial t}\right)$$

$$= -\frac{\mu}{2} \frac{\partial}{\partial t} (\bar{H} \cdot \bar{H}) \quad - (5)$$

Sub. Eq. (5) in (4),

$$-\frac{\mu}{2} \frac{\partial}{\partial t} (\bar{H} \cdot \bar{H}) - \nabla \cdot (\bar{E} \times \bar{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t} \quad - (6)$$

Rearranging terms and taking the volume integral of both sides, to (6),

$$\int_v \nabla\cdot(\bar{E}\times\bar{H})\,dv = -\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right]dv - \int_v \sigma E^2 dv \quad -⑦$$

Applying the divergence theorem to the LHS gives,

$$\oint_s(\bar{E}\times\bar{H})\,d\bar{s} = -\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right]dv - \int_v \sigma E^2 dv \quad -⑧$$

Total power leaving the volume. = Rate of decrease in energy stored in electric and magnetic fields. − Ohmic power dissipated.

Equation ⑧ is referred to as poynting's theorem. The quantity $\bar{E}\times\bar{H}$ on the LHS of eq. ⑧ is known as the poynting vector 'P' in watt per square meter (W/m²) i.e.,

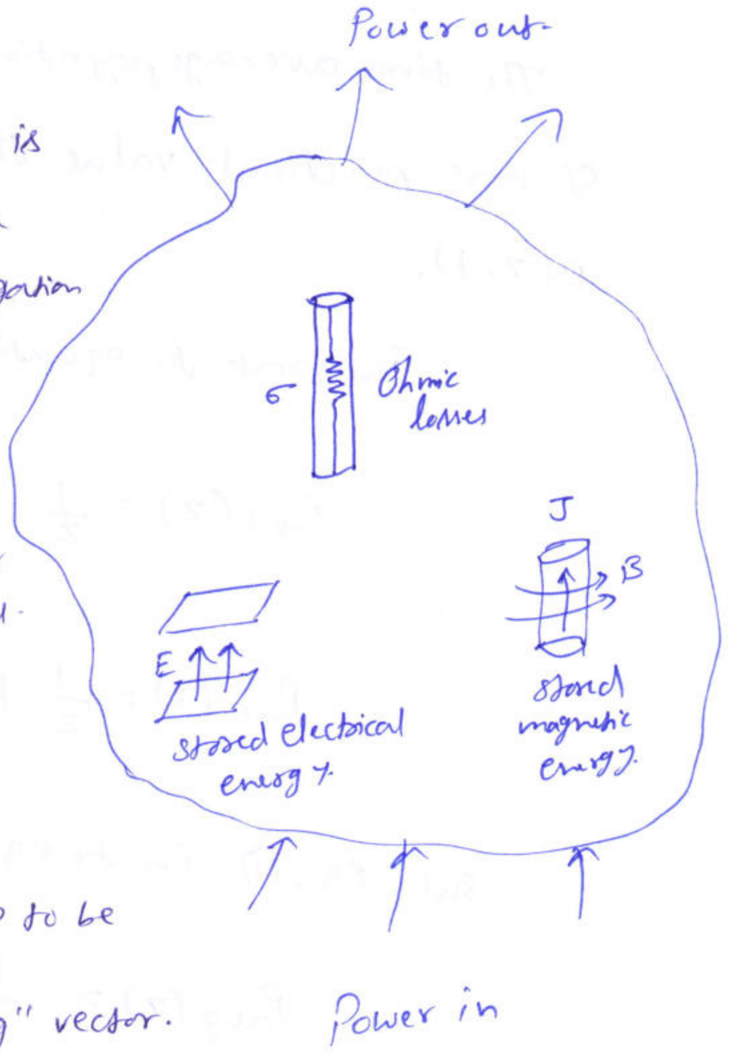
$$\vec{P} = \bar{E}\times\bar{H}.$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the poynting vector over any closed surface gives the net power flowing out of that surface.

Word statement of poynting theorem:-

Poynting's theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease in the energy stored within V minus the conduction losses.

It should be noted that P is normal to both $\bar{E}$ & $\bar{H}$ & is therefore along the direction of wave propagation $\bar{a}_k$ for uniform plane waves.



Power Balance for EM fields.

$$\bar{a}_K = \bar{a}_E \times \bar{a}_H$$

The fact that $\bar{a}_K$ causes P to be regarded decisively as a "pointing" vector.

Assume,

$$\bar{E}(z,t) = E_0 \, e^{-\alpha z} \cos(\omega t - \beta z) \, \bar{a}_x$$

then

$$\bar{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \, \bar{a}_y$$

$$P(z,t) = \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \bar{a}_z$$

$$= \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \left[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)\right] \bar{a}_z \quad -(1)$$

$$\because \cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right].$$

The time-average poynting vector $P_{ave}(z)$ (in W/m^2), which is of more practical value than the instantaneous poynting vector $P(z,t)$,

Integrate the equation over the period $T = \frac{2\pi}{\omega}$ i.e.,

$$P_{ave}(z) = \frac{1}{T}\int_0^T P(z,t)\,dt. \quad -②$$

$$P_{ave}(z) = \frac{1}{2} Re(\bar{E}_s \times \bar{H}_s^*). \quad -③$$

Sub. eq. ① in to eq. ②,

$$P_{avg}(z) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \, \bar{a}_z$$

The total time-average power crossing a given surface 's' is given by,

$$P_{ave} = \int_S P_{ave} \cdot d\bar{S}.$$

Cases	propagation const(γ) $\gamma = \alpha + j\beta$	Characteristic impedance (Z_0)
lossy	$\sqrt{(R+j\omega L)(G+j\omega C)}$	$\frac{R+j\omega L}{\sqrt{(R+j\omega L)(R+j\omega C)}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$
loss less	$j\omega\sqrt{LC}$	$\sqrt{\frac{L}{C}}$
distortion less	$\sqrt{RG}\left(1+j\omega\frac{L}{R}\right)$	$\sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}$

2

1. A telephone line has the following parameters $R = 40\,\Omega/m$ $G = 100\,ms/m$ $L = 0.2\,\mu H/m$ $C = 0.5\,nF/m$ if the line is operated at 10 MHz Calculate the the characteristic impedance and velocity of the signal.

A) Given that

$R = 40\,\Omega/m$ $G = 100\,ms/m$ $f = 10\,MHz$

$L = 0.2\,\mu H/m$ $C = 0.5\,nF/m$

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{40 + j2\pi\times10^{6}\times 0.2\,\mu H/m}{100\,ms/m + j2\pi\times10^{6}\times0.5\times10^{-9}}}$$

$$= \sqrt{\frac{40+j4\pi}{0.1+j0.01\pi}} = \sqrt{\frac{41.92\angle17.4}{0.104\angle17.4}}$$

$$= (403.07)^{1/2}\angle 0/2 \qquad \sqrt{403.07\angle 0^\circ}$$

$= 20.07 + 0i\ \Omega/m$

$\simeq 20\Omega$

$\gamma = \alpha + j\beta$

$= \sqrt{(R+j\omega L)(G+j\omega C)}$

$= \sqrt{(40+j4\pi)(0.1+j0.01\pi)} = \sqrt{(40.92\angle 17.4)(0.104\angle 17.4)}$

$= \sqrt{4.25\angle 34.8}$

$= 2.06 \angle 17.4$

$= 1.96 + 0.616 j\ /m$

$\alpha = 1.96$ N/m

$\beta = 0.616$ radians/m

$U = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{0.616} = 101.94 \times 10^6$ m/s.

2. A distortionless line has $Z_0 = 60\Omega$ attenuation Const $\alpha = 20$ mneper/m signal velocity $u = 1.8 \times 10^8$ m/s find the primary parameters of the transmission line at 100MHz.

A) $Z_0 = 60\Omega$ $\alpha = 20$mN/m $u = 1.8 \times 10^8$ m/s

$$\frac{R}{L} = \frac{G}{C}$$

$$Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

$$\alpha = \sqrt{RG}$$

$$Z_0 \alpha = R$$

$$R = 60 \times 20 \times 10^{-3} = 1.2\ \Omega/m$$

$$Z_0^2 = \frac{R}{G} \qquad G = \frac{R}{Z_0^2} = \frac{1.2}{(60)^2} = 0.3\text{ m}\mho/m$$

$= 333$ µs/m

$$u = \frac{1}{\sqrt{LC}}$$

$$Z_0 \cdot u = \sqrt{\frac{L}{C}} \times \frac{1}{\sqrt{LC}}$$

$$Z_0 \cdot u = \frac{1}{C}$$

$$C = \frac{1}{Z_0 \cdot u}$$

$$= \frac{1}{60 \times 1.8 \times 10^{8} \text{ m/s}}$$

$$= 9.259 \times 10^{-11}$$

$$C = 92.59 \text{ pF/m}$$

$$\frac{Z_0}{u} = \frac{\sqrt{\frac{L}{C}}}{\frac{1}{\sqrt{LC}}} = \sqrt{\frac{L}{C}}\sqrt{LC}$$

$$= L$$

$$\frac{Z_0}{u} = L$$

$$L = \frac{60}{1.8 \times 10^{8} \text{ m/s}}$$

$$= 33.33 \times 10^{-8}$$

$$= 333.3 \times 10^{-9} \text{ H/m}$$

$$= 333.3 \text{ nH/m}$$

$$\lambda = \frac{u}{f} = \frac{1.8 \times 10^{8} \text{ m/s}}{100 \text{ MHz}} = 1.8 \text{ m}$$

3. An air line has characteristic impedence of 70Ω and has phase Const of 3 radians/m at 100MHz Calculate the inductance per meter and capacitance per meter of the line.

[Note :- air line is a loss less line]

4. A transmission line operating at 500 MHz has $Z_0 = 80\Omega$ $\alpha = 0.04$ N/m $\beta = 1.5$ radians/meter find the line parameters R L G & C.

[Note :- transmission line is distortion less)

5. A telephone line has $R = 30\Omega/Km$

3.A) given ∴ $Z_0 = 70\Omega$

$\beta = 3$ radians/m

$f = 100$ MHz

$$Z_0 = \sqrt{\frac{L}{C}} = R_0$$

$$\frac{L}{C} = Z_0^2$$

$$L = Z_0^2 C$$

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = u$$

$$\beta = \omega \cdot \sqrt{LC}$$

$$\frac{R_0}{\beta} = ? \Rightarrow \frac{Z_0}{\beta} = \frac{\sqrt{\frac{L}{C}}}{\omega\sqrt{LC}} = \sqrt{\frac{1}{C}} \times \frac{1}{\omega\sqrt{C}}$$

$$\frac{Z_0}{\beta} = \frac{1}{\omega C}$$

$$\frac{1}{C} = \frac{Z_0 \omega}{\beta}$$

$$C = \frac{\beta}{Z_0\,\omega}$$

$$= \frac{3}{70 \times 2\pi \times 100 \times 10^{6}}$$

$$= \frac{3}{7 \times 2\pi \times 10^{9}} = \frac{3}{0.4396} \times 10^{-11}$$

$$= 6.824 \times 10^{-11}$$

$$= 68.2 \times 10^{-12}$$

$$C = 68.2\ PF/m$$

$$L = Z_0^2 C$$

$$L = (70)^2 \times 68.2 \times 10^{-12}$$

$$= 4900 \times 68.2 \times 10^{-12}$$

$$= 4.9 \times 68.2 \times 10^{-9}$$

$$= 334.18\ nH/m$$

4.A) given that

$f = 500\,MHz$ $\alpha = 0.04\,N/m$

$\beta = 1.5$ radians/meter

$Z_0 = 80\Omega$

$$\frac{R}{L} = \frac{G}{C}$$

$$Z_0 = \sqrt{\frac{L}{C}} \qquad \alpha = \sqrt{RG}$$

$$\beta = \omega\sqrt{LC}$$

$$\frac{Z_0}{\beta} = \frac{\sqrt{L/C}}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}} \times \frac{1}{\omega\sqrt{LC}}$$

$$\frac{Z_0}{\beta} = \frac{1}{\omega C}$$

$$C = \frac{\beta}{\omega Z_0}$$

$$C = \frac{1.5}{2\pi \times 500 \times 10^6 \times 80}$$

$$= \frac{1.5}{2\pi \times 5 \times 8 \times 10^9}$$

$$= 5.97\ pF/m$$

$$Z_0 \alpha = \sqrt{\frac{R}{G}} \sqrt{RG}$$

$$Z_0 \alpha = R$$

$$R = 80 \times 0.04 = 3.2 \Omega/m$$

$$Z_0 . \beta = \sqrt{\frac{L}{C}} \times \omega \sqrt{LC}$$

$$L = \frac{Z_0 . \beta}{\omega}$$

$$= \frac{80 \times 1.5}{2\pi \times 500\ MHz} = \frac{8 \cdot 120}{3.14 \times 10^9}$$

$$= 38.2\ nH/m$$

$$\frac{Z_0}{\alpha} = \frac{\sqrt{\frac{R}{G}}}{\sqrt{RG}} = \sqrt{\frac{R}{G}} \times \frac{1}{\sqrt{RG}} = \frac{1}{G}$$

$$G = \frac{\alpha}{Z_0} = \frac{0.04}{80} = 5 \times 10^{-4}\ S/m$$

℧/m

Note:

★ $\frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$

★ $(r_1 \angle \theta_1)(r_2 \angle \theta_2) = r_1 r_2 \angle \theta_1 + \theta_2$

★ $(r \angle \theta)^{1/2} = r^{1/2} \angle \theta/2$

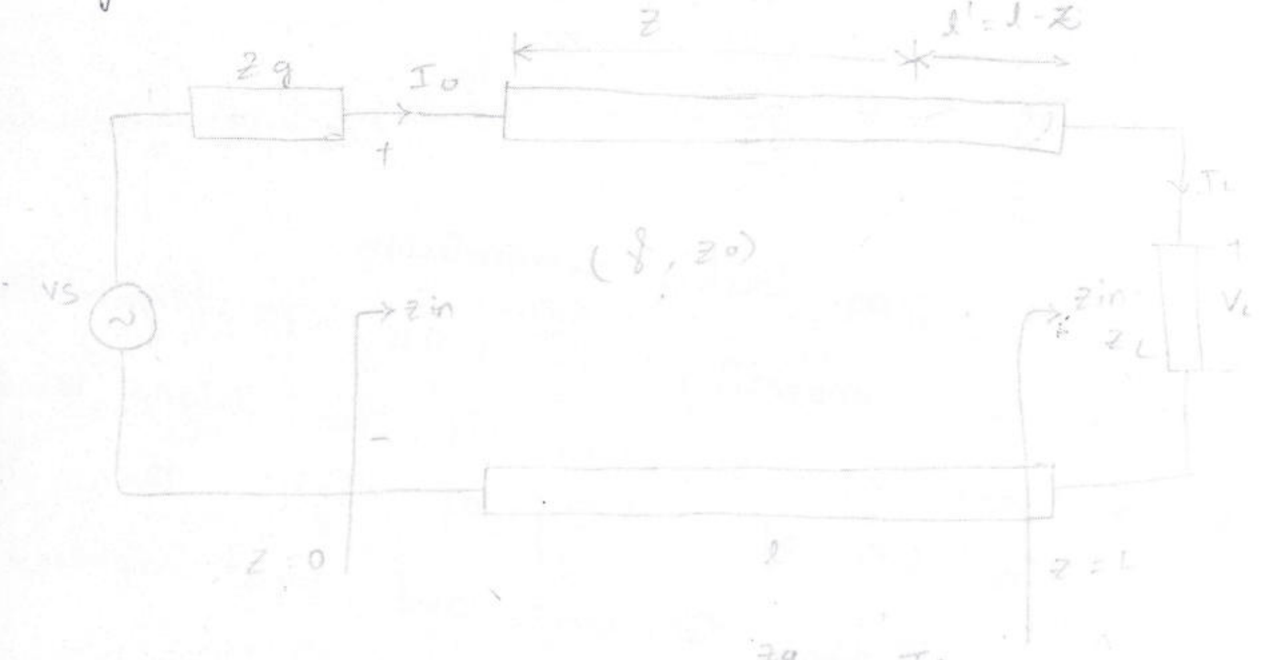
★ for addition $r_1 + r_2 \angle \theta_1 + \theta_2$

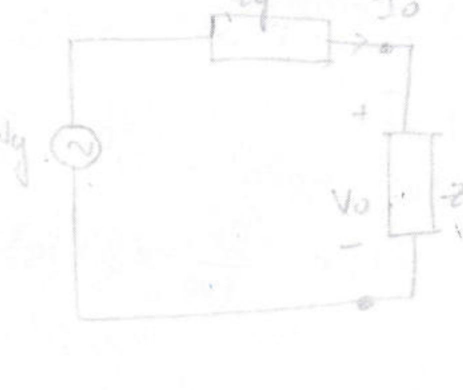
Input impedence SWR & power of a transmission line:-

(∵ SWR - Standing wave ratio)

Let us consider a transmission line applying

V/g source V_s





Case (ii) :- when the transmission line is connected to the load open circuited line $Z_{oc} = Z_{in}/_{Z_L = \infty}$

$$Z_{oc} = Z_{in}/_{Z_L=\infty} = \frac{Z_0}{\tanh \gamma l} = Z_0 \coth \gamma l.$$

$$\Gamma_L = 1$$

$$S = \infty$$

$$\boxed{Z_{oc} Z_{sc} = Z_0^2}$$

Case (iii) :- when the transmission line is connected to matched line.

$Z_L = Z_0$ which is called matched line.

$$Z_{in} = Z_0 = Z_L$$

$$\Gamma_L = 0$$

$$S = 1$$

Prb - 1 :- A 30 m lossless transmission line has $Z_0 =$ ~~50~~ $50\,\Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\,\Omega$ if the signal velocity on the line is 60% of velocity of light then find the reflection coefficient and input impedence.

A) given that $f = 2\,MHz$ $Z_L = 60 + j40$

$Z_0 = 50\,\Omega$

$$u = \frac{60}{100} \times 3 \times 10^8\, m/s = 1.8 \times 10^8\, m/s$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{60 + 40j - 50}{60 + 40j + 50} = \frac{10 + 40j}{110 + 40j}$$

$= 0.197 + 0.292j$

$= 0.352 \angle 55.98$

$= 0.352 \angle 56°$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right]$$

∴ assume $l = 1m$

$$u = \frac{\omega}{\beta}$$

$$\beta = \frac{\omega}{u}$$

$$= \frac{2\pi \times 2 \times 10^6}{1.8 \times 10^8}$$

$\tan(\beta) = 0.0012$

$= 0.0697$ radians/metre

$$Z_{in} = 50 \left[\frac{60 + 40j + j(50)(0.0012)}{50 + j(60 + 40j)(0.0012)} \right]$$

$$= 50 \left[\frac{60 + 40.06j}{50 + 0.072j - 0.048} \right]$$

$$= 50 \left[\frac{60 + 40.06j}{49.950 + 0.072j} \right]$$

$$= 50 (1.202 + 0.8003j)$$

$= 60.12 + 40.015j$

$= 72.21 \angle 33.64$

$\beta l = 2009$

assume $l = 30m$

$\tan \beta l = 0.036$

$\beta = 0.0697$ radians / meter

$$Z_{in} = 50 \left[\frac{60 + 40j + j(50)(0.036)}{50 + j(60 + 40j)(0.036)} \right]$$

$$= 50 \left(\frac{60 + 41.8j}{50 + 2.16j - 1.44} \right)$$

$$= 50 \left(\frac{60 + 41.8j}{48.56 + 2.16j} \right)$$

$$= 50 (1.271 + 0.804j)$$

$$= 63.57 + 40.21j$$

$$= 75.21 \angle 32.314$$

$\beta = \frac{\omega}{\mu}$

$\beta = 4$

$Z_{in} = 24.01 (3.22$

$Z_{in} = 23.97 + j1.35 \Omega$

1. Calculate the reflection coefficient, VSWR for a 50Ω line terminated with (a) matched load (b) short circuited (c) $+j50\Omega$ (d) $-j50\Omega$

A) $Z_0 = 50\Omega$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

(a) $Z_L = Z_0$ for matched load

$\therefore Z_L = 50\Omega$

$\Gamma = 0$

$$S = \frac{1+0}{1-0} = 1$$

(b) $Z_L = 0$ for short circuited

$\Gamma = -1$

$S = \infty$

(c) $Z_L = +j50$

$$\Gamma = \frac{50j-50}{50j+50} = \frac{j-1}{j+1} = \frac{(j-1)^2}{j^2-1}$$

$$= \frac{-1+1-2j}{-2}$$

$$= j$$

$$S = \frac{1+|j|}{1-|j|}$$

$$= \frac{1+1}{1-1}.$$

$S = \infty$

(d) $Z_L = -j50\Omega$

$$\Gamma = \frac{-50j-50}{-50j+50} = \frac{j+1}{j-1}$$

$$= \frac{(j+1)^2}{j^2-1^2} = \frac{-1+1+2j}{-1-1}$$

$$= -j$$

$$\therefore S = \frac{1+|-j|}{1-|-j|} = \frac{1+1}{1-1} = \infty$$

2) A lossless transmission line has $Z_0 = 100\Omega$ is Connected to a load of 300Ω Calculate the reflection Coefficient and VSWR

A) $Z_0 = 100\Omega$ $Z_L = 300\Omega$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{300-100}{300+100} = \frac{200}{400} = 0.5$$

$$S = \frac{1+|0.5|}{1-|0.5|} = \frac{1.5}{0.5} = 3$$

3) The open and short circuited impedence of transmission line at 36kHz are $1800\angle-60^\circ$ and $800\angle-20^\circ\ \Omega$ Calculate the characteristic impedence

A) $f = 36\,KHz$

$Z_{oc} = 1800 \angle -60° \Omega$

$Z_{sc} = 800 \angle -20° \Omega$

$Z_0^2 = Z_{oc} Z_{sc}$

$Z_0 = \sqrt{Z_{oc} Z_{sc}}$

$= \sqrt{1800 \angle -60° . 800 \angle -20°}$

$= \sqrt{1440000 \angle -80}$

$= 1200 \angle -40° \Omega$

3. A Certain transmission line operating at $\omega = 10^6$ radians/sec has $\alpha =$ 8 decibles/meter $\beta = 1$ radian/meter and $Z_0 = 60 + j40 \Omega$ and is 2 meters long if the line is connected to a source of $10 \angle 0°$ Volts $Z_g = 40 \Omega$ and terminated by a load of $20 + j50 \Omega$ determine

(a) input impedence

(b) The secondary end current

(c) the current at the middle of the line

A) given that

$l = 2m$ $\omega = 10^6$ radians/sec

$\beta = 1$ radian/meter $\alpha = 8$ decibles/meter

$Z_0 = 60 + j40 \Omega$ $V_g = 10 \angle 0°\,V$ $Z_g = 40 \Omega$

$Z_L = 20 + j50 \Omega$

(i) Since $1\,NP = 8.686\,dB$

$$\alpha = \frac{8}{8.868} = 0.921\,Np/m$$

$\gamma = \alpha + j\beta = 0.921 + j1/m$

$\gamma l = 2(0.921 + j\cdot 1) = 1.84 + 2j$

using the formula for $\tanh(x+jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y}$

$x = 1.84,\ y = 2$

$\tanh(1.84+2j)$

$\sinh x = \frac{e^{x} - e^{-x}}{2}$ $\quad \cosh x = \frac{e^{x} + e^{-x}}{2}$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\tanh(x \pm jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y}$

$\tanh \gamma l = 1.033 - j0.03929$

$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right]$

$= (60+j40) \left[\frac{20 + j50 + (60+j40)(1.033 - j0.03929)}{60 + j40 + (20+j50)(1.033 - j0.03929)} \right]$

$Z_{in} = 60.25 + j38.79\,\Omega$

b) Sending end current is $I(z=0) = I_0$

$I(z=0) = \frac{V_g}{Z_{in} + Z_g} = \frac{10}{60.25 + j38.79 + 40}$

$= 93.03\angle -21.15^\circ\,mA$

(c) To find the current at any point we need V_0^+ & V_0^-

but $I_0 = I(z=0) = 93.03\angle -21.15^\circ\,mA$

$V_0 = Z_{in} I_0 = (71.66\angle 32.77^\circ)(0.09303\angle -21.15^\circ))$

$= 6.667\angle 11.62^\circ\,V$

$\therefore V_0^+ = \frac{1}{2}(V_0 + Z_0 I_0)$

$= \frac{1}{2}[6.667\angle 11.62^\circ + (60+j40)(0.09303\angle -21.15^\circ)]$

$= 6.687\angle 12.08^\circ$

$V_0^- = \frac{1}{2}(V_0 - Z_0 I_0)$

$= \frac{1}{2}[6.667\angle 11.62^\circ - (60+j40)(0.09303\angle -21.15^\circ)] = 0.0518\angle 260^\circ$

at the middle of the line $z = l/2$, $\gamma z = 0.921 + j1$

Hence current is $I_s(z = l/2) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$

$= \frac{(6.687 e^{j12.08^\circ}) e^{-0.921 - j1} - (0.0518 e^{j260^\circ}) e^{0.921 + j1}}{60 + j40}$ $\quad [\because j1 = j57.3^\circ]$

$I_s(z = l/2) = \frac{6.687 e^{j12.08^\circ} e^{-0.921} e^{-j57.3^\circ}}{72.1 e^{j33.69^\circ}} - \frac{0.0518 e^{j260^\circ} e^{0.921} e^{j57.3^\circ}}{72.1 e^{j33.69^\circ}}$

$= 0.0369 e^{-j78.91^\circ} - 0.001805 e^{j283.61^\circ}$

$= 6.673 - j34.456\,mA$

$= 35.10\angle 281^\circ\,mA$

2/8/18

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ECE-II

EMTL ASSIGNMENT-I

1) Ans **Convenction and conduction currents:—**

* In a broad scense, materials may be classified in terms of their conductivity σ, in mhos per meter (℧/m) or, more usually siemens per meter (s/m), as conductors and non conductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity ($\sigma >> 1$) is reffered to as a metal, whereas one with low conductivity ($\sigma << 1$) is reffered to as an insulator. A material whose conductivity lies somewhere between those of metals and insulators is called a semi conductor.

* Electric voltage (or potential difference) and current are two fundamental quantities in electrical engineering. we considered potential in the last chapter. Before examining how the electric field behaves in a conductor or dielectric. It is appropriate to consider electric current. Electric current is generally caused by the motion of electric charges.

* The current (in amperes) through a given area is the electric charge passing through the area per unit time.

That is,

$$I = \frac{dQ}{dt} \longrightarrow (1)$$

Thus in a current of one ampere, charge is being transfer[red] rate of one coloumb per second.

we now introduce the concept of current density J. If current ΔI flows through a planar surface Δs, the current density is

$$J = \frac{\Delta I}{\Delta S}$$

or

$$\Delta I = J \Delta S \longrightarrow (2)$$

Assuming that the current density is perpendicular to the surface. If the current density is at normal to the surface.

$$\Delta I = \bar{J} \cdot \Delta \bar{S} \longrightarrow (3)$$

Thus, the total current flowing through a surface S is

$$I = \int_S \bar{J} \cdot d\bar{s} \longrightarrow (4)$$

Depending on how I is produced, there are different kinds of current density convenction current density, conduction current density and displacement current density. we will consider convenction and conduction current densities here; displacement current density will be considered as in chapter 9. what we need to keep in mind is that eqn (4) applies to any kind of carrier density. compared with the general definition of flux in eqn (3.13) & eqn (4) shows that the current I through S is merely the flux of the current density J.

Case A :- convection current :—

convection current, as distinct from conduction current, does not involving conduction and consequently does not satisfy ohm's law.

It occurs when current flows through an insulating medium such as liquid, rarefield gas, or a Vaccum. A beam of electrons in a Vacuum tube for example: A convection current.

Consider a filament of figure ①. If there is a flow of charge, of density ρ, at velocity

$$\Delta I = \frac{\Delta Q}{\Delta T} = \rho_v \, \Delta s \frac{\Delta y}{\Delta t} = \rho_v \Delta s u_y \longrightarrow (5)$$

The current density at a given point is the current through a unit normal area at that point.

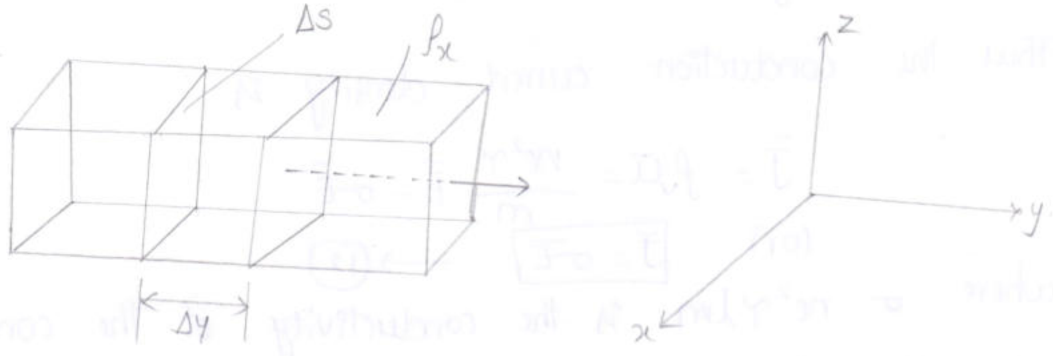
The Y-directed current density J, is given by

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y \longrightarrow ⑥$$

Hence, in general

$$\boxed{\bar{J} = \rho_v \bar{u}} \longrightarrow ⑦$$

The current I is the convection current and J is the convection current density in amperes per square meter (A/m^2)



Case B :- conduction current :-

conduction current requires a conductor. A conductor is characterized by a large number of the electrons that provide conduction current due to an ~~improved~~ impressed electric field. when an electric field E is applied, the force on an electron charge $-e$ is

$$F = -e\bar{E} \longrightarrow ⑧$$

Since the electrons is not in free space, it will not expe[rience] average acceleration under the influence of the electric field. it suffers constant conditions with the atomic lattice and drifts from one atom to another. If an electron with mass m is moving in an electric field E with an average drift velocity u, according to Newton's law, the average change in momentum of the free electrons must match the applied force, thus,

$$\frac{m\bar{u}}{\tau} = -e\bar{E} \longrightarrow (9)$$

(or)

$$\bar{u} = -\frac{e\tau}{m}\bar{E} \longrightarrow (10)$$

where τ is the average time interval between collisions. This indicates that the drift velocity of the electron is directly propotional to the applied field. If there are n electrons per unit volume. The electronic charge density is given by

$$\rho_s = -ne \longrightarrow (11)$$

Thus the conduction current density is

$$\bar{J} = \rho_v\bar{u} = \frac{ne^2\tau}{m}\bar{E} = \sigma\bar{E}$$

(or) $$\boxed{\bar{J} = \sigma\bar{E}} \longrightarrow (12)$$

where $\sigma = ne^2\tau/m$ is the conductivity of the conductor. As mentioned earlier, the values of σ is common materials are provided. The relationship in eqn (11) known as the Point form of ohm's law

Conductors:-

* A perfect conductor ($\sigma = \infty$) cannot contain an electrostatic field within it. ($E = 0$)

$$J = \frac{I}{S} \longrightarrow (13)$$

sub eqn (12) in (14)

$$\frac{I}{S} = \sigma E = \frac{\sigma V}{l} \longrightarrow (14)$$

$$\boxed{R = \frac{V}{I} = \frac{l}{\sigma S}} \longrightarrow (15)$$

$$R = \frac{\rho l}{S}$$

Wave propagation in Lossy dielectrics:-

"A lossy dielectric is a medium in which EM wave, as it propagates, loses power owing to imperfect dielectric."

lossy dielectric } $\sigma \neq 0$ — Imperfect { → dielectric, → conduct }

perfect (or) good } dielectric → $\sigma = 0$.

Linear
Isotropic
Homogeneous
Lossy dielectric.
Time factor $e^{j\omega t}$

} Consider.

$\nabla \cdot E_s = 0$.

$\nabla \cdot H_s = 0$.

$\nabla \times E_s = -j\omega\mu H_s$

$\nabla \times \overline{H}_s = (\sigma + j\omega\varepsilon)E_s$

$\nabla^2 \overline{E}_s - \gamma^2 \overline{E}_s = 0$. $||^{ly}$, $\nabla^2 \overline{H}_s - \gamma^2 \overline{H}_s = 0$. — ⓐ

$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

γ is called propagation Constant

Eq. ⓐ are called Helmholtz equ's. (or) wave eq.'s.

$$\gamma = \alpha + j\beta.$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

For free space,

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$H_0 = \frac{E_0}{\eta}$$

$$\eta = \frac{E_0}{H_0}.$$

$\eta \rightarrow$ Complex quantity. Known as Intrinsic Impedance. (Ω)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle\theta_\eta = |\eta| e^{j\theta_\eta}$$

"As the wave propagates along a direction, it decreases (or) attenuates in amplitude by a factor $e^{-\alpha z}$,

$\alpha \rightarrow$ Attenuation Constant. of the medium.

It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) & can be expressed in decibels per meter (dB/m)."

$$1NP = 20 \log_{10}^{e} = 8.686\,dB.$$

For a free space, $\sigma = 0$, $\alpha = 0$, then no attenuation as the wave propagates.

$\beta \rightarrow$ Phase constant (or) wave Number.

$$u = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta}, \quad \beta = \frac{2\pi}{\lambda}.$$

$$\frac{|J_{cs}|}{|J_{ds}|} = \frac{|\sigma E_s|}{|j\omega\varepsilon E_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

$$\therefore \tan\theta = \frac{\sigma}{\omega\varepsilon}.$$

Plane waves in Lossless dielectrics:-

$$\sigma = 0, \quad \varepsilon = \varepsilon_0 \varepsilon_r, \quad \mu = \mu_0 \mu_r$$

Sub. this values in α & β, then,

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta}.$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0^\circ.$$

* $\bar{E}$ & $\bar{H}$ are in time phase with each other.

Plane waves in Free Space:-

$$\sigma = 0, \quad \varepsilon_r = 1, \quad \varepsilon = \varepsilon_0, \quad \mu = \mu_0.$$

Sub. this values in α & β,

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c}$$

$$u = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \quad c = u, \quad \lambda = \frac{2\pi}{\beta}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega.$$

$$\left\{ \begin{aligned} \bar{E} &= E_0 \cos(\omega t - \beta z) a_x. \\ \bar{H} &= H_0 \cos(\omega t - \beta z) a_y. \end{aligned} \right\}$$

$$\bar{H} = \frac{E_0}{\eta_0} \cos(\omega t - \beta z) a_y.$$

$$a_K \times a_E = a_H$$

$$a_K \times a_H = -a_E$$

$$a_E \times a_H = a_K.$$

$a_K \rightarrow$ direction of wave propagation.

* The field lines lie in a plane i.e., transverse or orthogonal to the direction of wave propagation. {No electric & magnetic field along the direction of propagation, such a wave is called a transverse electromagnetic wave TEM}.

Plane waves in Good Conductors :-

A perfect or Good Conductor, $\sigma \gg \omega\varepsilon$, $\frac{\sigma}{\omega\varepsilon} \gg 1$.

$$\sigma \approx \infty, \quad \mu = \mu_0 \mu_r$$

Sub. this values in α & β,

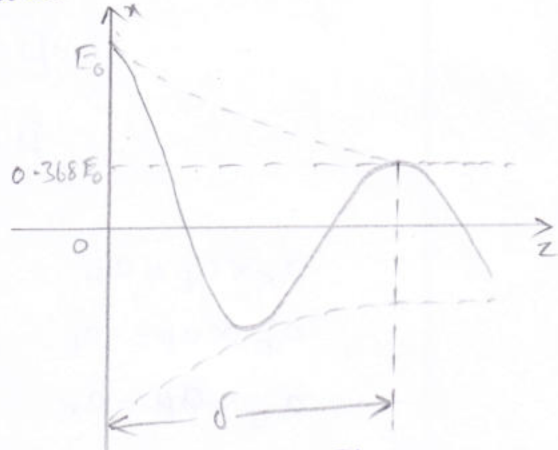
$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \quad \lambda = \frac{2\pi}{\beta}.$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ.$$

As the $\bar{E}$ & $\bar{H}$ wave travels in Conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$,

The distance δ, in which the wave amplitude decreases to a factor e^{-1} (about 37% of the original vale) is called Skin depth (or) penetration depth of the medium.



$$\therefore E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\alpha\delta = 1 ; \quad \delta = \frac{1}{\alpha}.$$

"The Skin depth is a measure of the depth to which EM wave Can penetrate the medium".

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$\alpha = \beta = \frac{1}{\delta}.$$

* E & H can hardly propagate through good conductors.

* The skin depth is useful in calculating the ac resistance due to skin effect. The resistance is called the dc resistance,

$$R_{dc} = \frac{l}{\sigma S}.$$

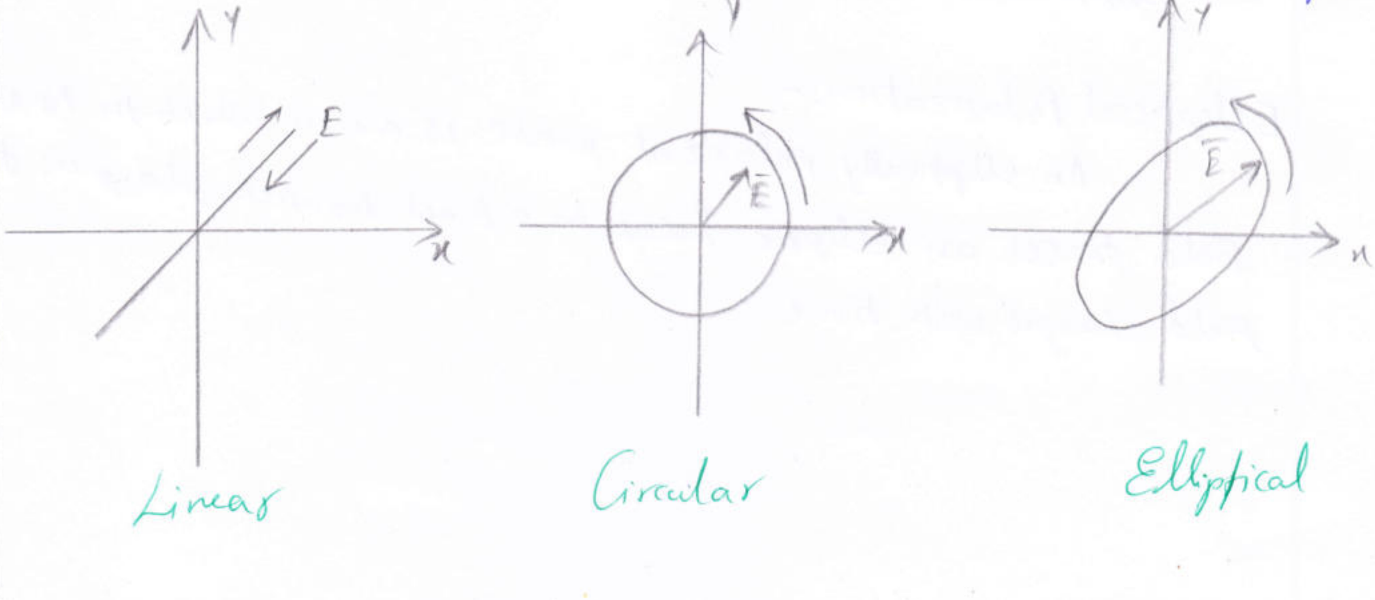
$$\text{(skin resistance) } R_s = \frac{1}{\sigma \delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

Wave Polarization:-

"Polarization may be regarded as the locus of the tip of the electric field (in a plane perpendicular to the direction of propagation) at a given point as a function of time."

Types → Linear
→ Circular
→ Elliptical.

Applications: TV & Radio.
→ AM (vertical polarized)
→ FM (Circularly polarized)



Linear Polarization:-

Transverse Components ($\bar{E}$ & $\bar{H}$) are in phase.

$$E_x = E_{0x} \cos(\omega t - \beta z + \phi_x)$$

$$E_y = E_{0y} \cos(\omega t - \beta z + \phi_y)$$

$$\bar{E} = E_x + E_y$$

$\{E_{0x} = E_{0y}\}\{45^\circ\}$

* Magnitudes are unequal the angle is $\tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)$

It is linearly polarized when,

$$\Delta\phi = \phi_y - \phi_x = n\pi, \text{ for } n = 0, 1, 2 ---$$

The tip of Electric field follows a line, Called Linar polarization

Eg:- Dipole antenna & Laser generates Linar polarized waves.

Circular polarization:-

* x & Y-components are Same in magnitude $\{E_{0x} = E_{0y} = E_0\}$
* Phase difference between them is an odd multiple of $\pi/2$.

$$\Delta\phi = \phi_y - \phi_x = \pm(2n+1)\pi/2, \quad n = 0, 1, 2 ---$$

Eg:- $E_x = E_0 \cos(\omega t - \beta z)$
$E_y = E_0 \cos(\omega t - \beta z + \pi/2)$ } The locus of total field traces a Circle can be seen,

let, $z = 0$,

$E_x = E_0 \cos(\omega t)$

$E_y = -E_0 \sin(\omega t)$

$|E^2| = |E_x^2| + |E_y^2| = E_0^2$ } which is a equation of Circle.

Applications:-

1) Helical Antenna
2) Two linear Sources with out of phase (90°).

Elliptical Polarization:-

An elliptically polarized wave is one in which the tip of the field traces an elliptic locus in a fixed transverse plane as the field changes with time.

(4)

Elliptical polarization is achieved, when the x + y components are not equal in magnitude $E_{ox} \neq E_{oy}$ & phase difference between them is an odd multiple of $\pi/2$, i.e.,

$$\Delta\phi = \phi_y - \phi_x = \pm(2n+1)\pi/2, \quad n = 0, 1, 2 \ldots$$

eg:- let $\Delta\phi = \phi_y - \phi_x = \pi/2$, & $z = 0$,

$$E_x = E_{ox}\cos(\omega t - \beta z + \phi_x) \Rightarrow \frac{E_x}{E_{ox}} = \cos\omega t$$

$$E_y = E_{oy}\cos(\omega t - \beta z + \phi_y) \Rightarrow \frac{E_y}{E_{oy}} = -\sin(\omega t)$$

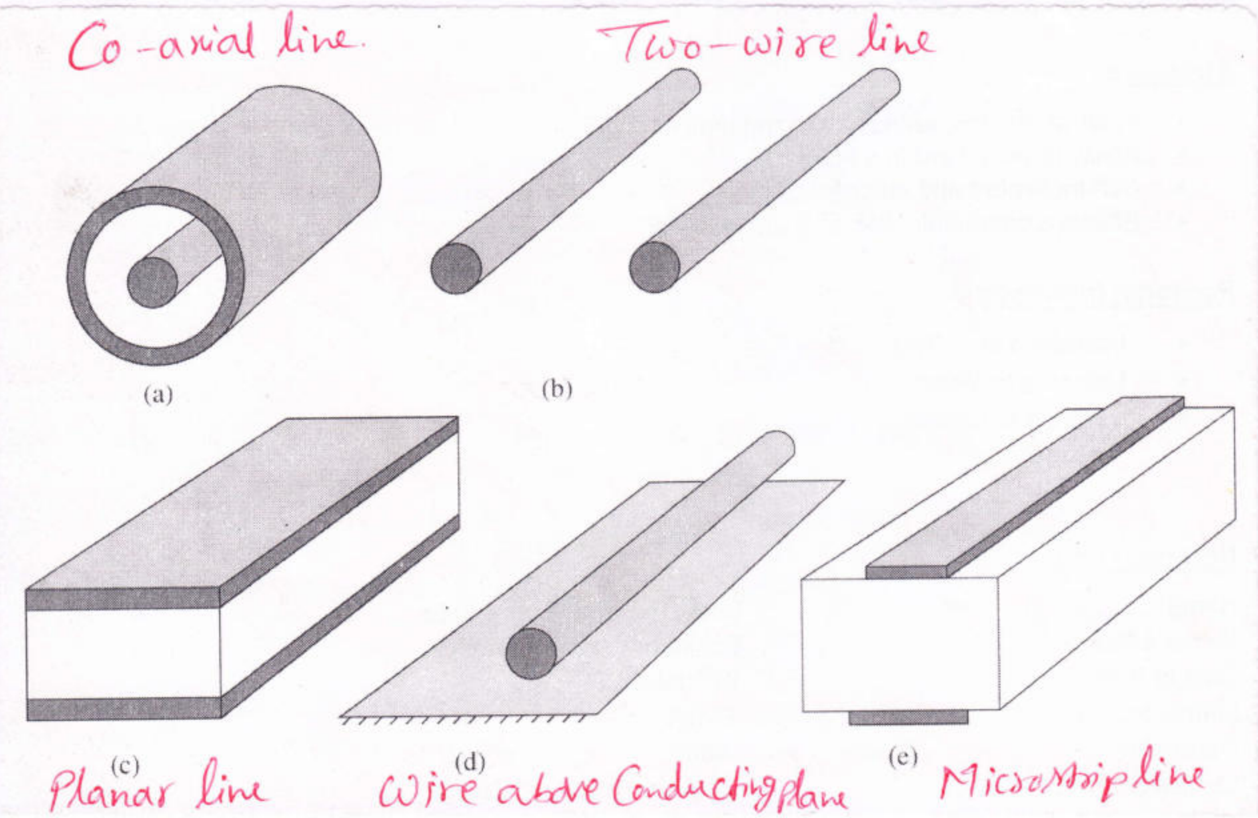
sq. & add on both sides,

$$\cos^2(\omega t) + \sin^2(\omega t) = 1, \quad \frac{E_x^2}{E_{ox}^2} + \frac{E_y^2}{E_{oy}^2} = 1.$$

UNIT-V
TRANSMISSION LINES

Transmission lines are commonly used in power distribution (at low frequencies) and in communications (at high frequencies). Various kinds of transmission lines such as the twisted-pair & coaxial cables (thinnest & thicknet) are used in computer networks such as the Ethernet & internet.

A transmission line basically consists of two or more parallel conductors used to connect a source to a load.



Source	Load
Hydroelectric generator	A Factory
A Transmitter	An Antenna
An oscillator	an oscilloscope.

Transmission line parameters:-

Parameters	Coaxial line	Two-wire Line	Planar Line
$R(\Omega/m)$	$\frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a}+\frac{1}{b}\right]$ ($\delta \ll a, c-b$)	$\frac{1}{\pi a\delta\sigma_c}$ ($\delta \ll a$)	$\frac{2}{w\delta\sigma_c}$ ($\delta \ll t$)
$L(H/m)$	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a}$	$\frac{\mu d}{w}$
$G(S/m)$	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\cosh^{-1}\left(\frac{d}{2a}\right)}$	$\frac{\sigma w}{d}$
$C(F/m)$	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}}$	$\frac{\varepsilon w}{d}$ ($w \gg d$)

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} = \text{skin depth of the Conductor}$$

$$\cosh^{-1}\frac{d}{2a} \approx \ln\left(\frac{d}{a}\right) \text{ if } \left(\frac{d}{2a}\right)^2 \gg 1.$$

Transmission line in terms of its line parameters, which are its resistance per unit length R, inductance per unit length L, Conductance per unit length G, & Capacitance per unit length C.

Note:-

1. The line parameters R, L, G & C are not discrete or lumped but distributed as shown in figure. The parameters are uniformly distributed along the entire length of the line.
2. For each line, the Conductors are characterized by $\sigma_c, \mu_c, \varepsilon_c = \varepsilon_0$, and the homogeneous dielectric separating the Conductors is characterized by σ, μ, ε.

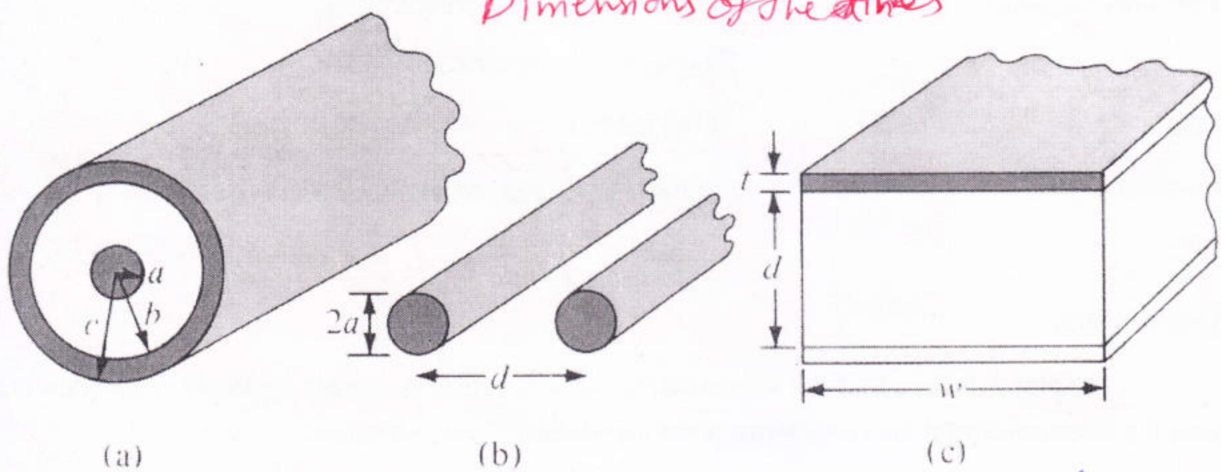
3. $G \neq \frac{1}{R}$; R is the ac resistance per unit length of the conductors comprising the line & G is the conductance per unit length due to the dielectric medium separating the conductors.

4. The value of L is the external inductance per unit length, i.e., $L = L_{ext}$. The effects of internal inductance $L_{in} (= R/\omega)$ are negligible as high frequencies at which most communication system operate.

5. For each line,

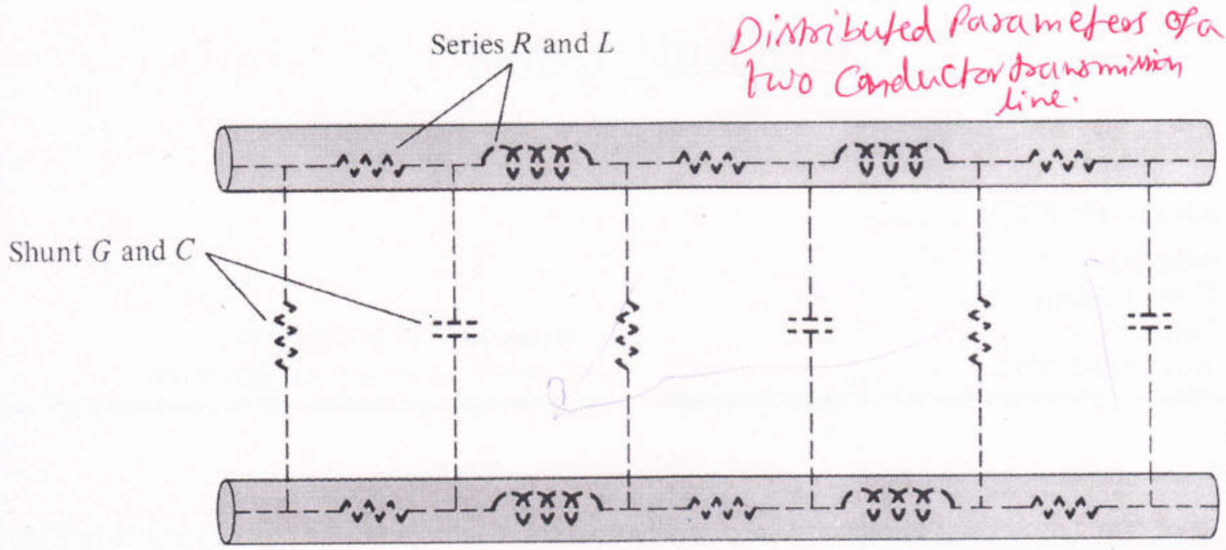
$$LC = \mu\varepsilon \ \& \ \frac{G}{C} = \frac{\sigma}{\varepsilon}.$$

Dimensions of the lines



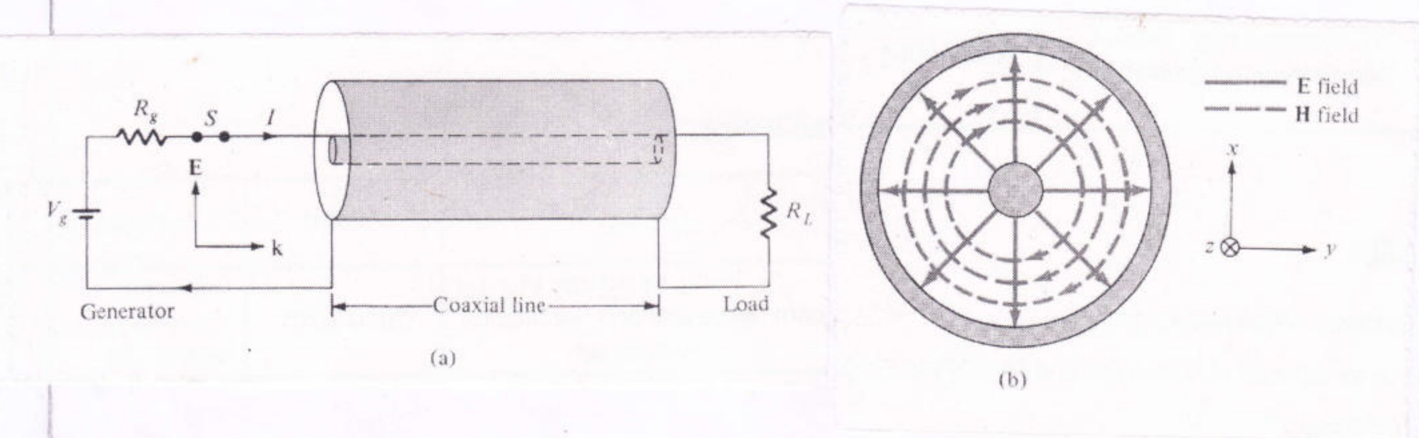
(a) Coaxial line (b) Two-wire line (c) Planar line.

Distributed Parameters of a two conductor transmission line.



Example:-

Consider a Coaxial line Connecting the generator or source to the load as shown in figure.



(a) (b)

When switch S is closed, the inner Conductor is made positive w.r.t the outer one so that the $\bar{E}$ field is radially outward as shown in figure.

According to Ampere's law, the $\bar{H}$ field encircles the current carrying conductor as shown in figure. The poynting vector $(\bar{E}\times\bar{H})$ points along the transmission line. Thus, closing the switch simply establishes a disturbance, which appears as a transverse electromagnetic (TEM) wave propagating along the line. This wave is a non-uniform plane wave & by means of it power is transmitted through the line.

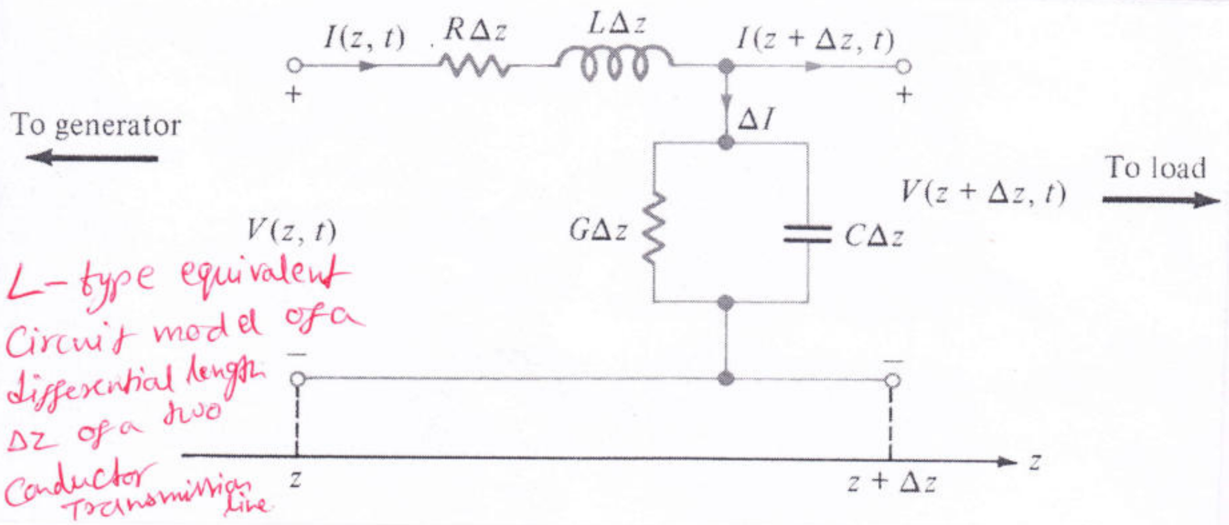
Transmission Line Equations:-

A two-Conductor transmission line supports a TEM wave, i.e., the electric & magnetic fields on the line are transverse to the direction of wave propagation.

An important property of TEM waves is that the fields $\bar{E}$ & $\bar{H}$ are uniquely related to voltage V & current I, respectively,

$$V = -\int \bar{E} \cdot d\bar{l}$$

$$I = \oint \bar{H} \cdot d\bar{l}.$$



L-type equivalent Circuit model of a differential length Δz of a two Conductor Transmission Line

Let us Consider an increamental portion of length Δz of a two-Conductor transmission line. The model of the equivalent Circuit shown in figure, the line parameters are R, L, G, C. The model is called the L-type equivalent Circuit.

In this model, assume, that the wave propagates along the +z direction, from the generator to the load.

By applying Kirchoff's voltage law to the outer loop of the circuit.

$$v(z,t) = R\,\Delta z\, I(z,t) + L\,\Delta z\,\frac{\partial I(z,t)}{\partial t} + v(z+\Delta z,t)$$

(or)

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R\,I(z,t) + L\,\frac{\partial I(z,t)}{\partial t} \quad -(1)$$

Taking limit $\Delta z \to 0$, leads to,
When $\Delta z \to 0$ then,

$$-\frac{\partial v(z,t)}{\partial z} = R\,I(z,t) + L\,\frac{\partial I(z,t)}{\partial t} \quad -(2)$$

Similarly, applying Kirchoff's Current law to the main node of the circuit, gives,

$$I(z,t) = I(z+\Delta z,t) + \Delta I.$$

$$I(z,t) = I(z+\Delta z,t) + G\,\Delta z\, v(z+\Delta z,t) + C\,\Delta z\,\frac{\partial v(z+\Delta z,t)}{\partial t} \quad -(3)$$

(or)

$$\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = G\,v(z+\Delta z,t) + C\,\frac{\partial v(z+\Delta z,t)}{\partial t}$$

as, $\Delta z \to 0$, eq. (3) becomes,

$$-\frac{\partial I(z,t)}{\partial z} = G\,V(z,t) + C\,\frac{\partial v(z,t)}{\partial t} \quad -(4)$$

If, assume harmonic time dependence,

$$\left.\begin{aligned} V(z,t) &= Re\left[v_s(z)\,e^{j\omega t}\right] \\ I(z,t) &= Re\left[I_s(z)\,e^{j\omega t}\right] \end{aligned}\right\} \quad -(5)$$

Where, $V_s(z)$ & $I_s(z)$ are the phasor forms of $V(z,t)$ & $I(z,t)$, then eq. (2) & (4) becomes,

$$-\frac{\partial V_s}{\partial z} = R I_s + j\omega L I_s$$

$$-\frac{\partial V_s}{\partial z} = (R+j\omega L) I_s \quad -(6)$$

Similarly, $-\frac{\partial I_s}{\partial z} = (G+j\omega C) V_s \quad -(7)$

In eq.'s (6) & (7), V_s & I_s are coupled. To separate them, take the second derivative of V_s in eq. (6) & employ in (7), then,

$$\frac{d^2 V_s}{dz^2} = (R+j\omega L)(G+j\omega C) V_s.$$

Let, $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$

$$\therefore \frac{d^2 V_s}{dz^2} - \gamma^2 V_s = 0 \quad -(8)$$

By ~~taking~~ the second derivative of I_s in eq. (7),

$$\frac{d^2 I_s}{dz^2} - \gamma^2 I_s = 0 \quad -(9)$$

Eq.'s (8) & (9) are the wave equations for voltage & current. γ is the propagation constant (in per meter), α is the attenuation constant (in nepers per meter/decibels per meter). β is the phase constant (in radians per meter).

The wavelength λ & wave velocity u are,

$$\lambda = \frac{2\pi}{\beta}$$

$$u = \frac{\omega}{\beta} = f\lambda. \qquad \text{— (10)}$$

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{— (11)}$$

$\longrightarrow +z \qquad \longleftarrow -z$

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \text{— (12)}$$

$\longrightarrow +z \qquad \longleftarrow -z$

where, V_0^+, V_0^-, I_0^+ & I_0^- are wave amplitudes,
+ & − signs denote wave travelling +z & −z directions.

The instantaneous expression for voltage as,

$$V(z,t) = Re\left[V_s(z)\, e^{j\omega t}\right]$$

~~$= V_0^+ e$ cos~~

$$= V_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + V_0^- e^{\alpha z}\cos(\omega t + \beta z) \qquad \text{— (13)}$$

Characteristic Impedance:-

* The characteristic impedance Z_0 of the line is the ratio of positively traveling voltage wave to current wave at any point on the line.

Z_0 is analogous to η, the intrinsic impedance of the medium of wave propagation.

By substituting eq.'s, (11) & (12) into (6) & (7) & equating Co-efficients of terms $e^{\gamma z}$ & $e^{-\gamma z}$,

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C}$$

(or)

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = R_0 + jX_0. \quad — (14)$$

* R_0 & X_0 are the real & imaginary parts of Z_0.
* The reciprocal of Z_0 is the characteristic admittance Y_0, i.e., $Y_0 = 1/Z_0$.

The transmission line considered here is the lossy type in that the conductors comprising the line are imperfect ($\sigma_c \neq \infty$) & the dielectric in which the conductors are embedded is lossy ($\sigma \neq 0$).

Lossless Line ($R = 0 = G$):-

A transmission line is said to be "lossless" if the Conductors of the line are perfect ($\sigma_c \approx \infty$) and the dielectric medium separating them is lossless ($\sigma \approx 0$).

$$\therefore \sigma_c \approx \infty \ \& \ \sigma \approx 0.$$

$$\therefore R = 0 = G.$$

$$\text{If } \alpha = 0, \quad \gamma = j\beta = j\omega\sqrt{LC}$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} = f\lambda.$$

$X_0 = 0, \quad Z_0 = R_0 = \sqrt{\frac{L}{C}}.$

Distortion less line $(R/L = G/C)$:-

A signal normally consists of a band of frequencies, wave amplitudes of different frequency components will be attenuated in a lossy line as α is frequency dependent. This results in distortion.

"A distortionless line is one in which the attenuation constant α is frequency independent while the phase constant β is linearly dependent on frequency".

A distortionless line results if the line parameters,

$$\frac{R}{L} = \frac{G}{C}$$

Thus, for a distortionless line,

$$\gamma = \sqrt{RG\left(1+\frac{j\omega L}{R}\right)\left(1+\frac{j\omega C}{G}\right)}$$

$$= \sqrt{RG}\left(1+\frac{j\omega C}{G}\right)$$

$$\gamma = \alpha + j\beta$$

$$\therefore \alpha = \sqrt{RG}$$

$$\beta = \omega\sqrt{LC}$$

$\therefore$ α does not depend on frequency where as β is a linear function of frequency.

$$\therefore Z_0 = \sqrt{\frac{R(1+j\omega L/R)}{G(1+j\omega C/G)}}$$

$$= \sqrt{\frac{R}{G}}$$

$$= \sqrt{\frac{L}{C}} = R_0 + jX_0$$

$$\therefore R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}, \quad X_0 = 0.$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda.$$

Note:-

1. The phase velocity is independent of frequency because the phase constant β linearly depends on frequency. The shape distortion of signals exists unless α & u are independent of frequency.

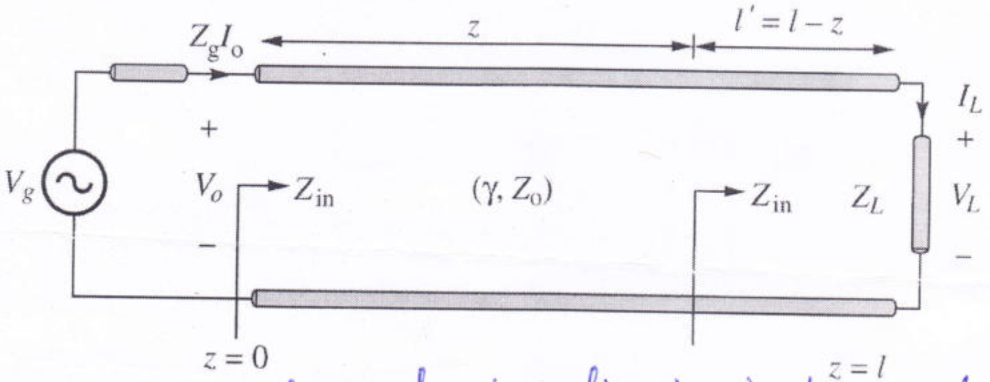
2. u & Z_0 remain the same as for lossless lines.

3. A lossless line is also a distortionless line, but a distortionless line is not necessarily lossless. Although lossless lines are desirable in power transmission, telephone lines are required to be distortionless.

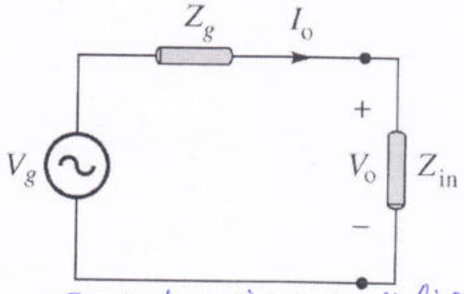
Transmission Line Characteristics

Case	Propagation Constant $\gamma = \alpha + j\beta$	Characteristic Impedance $Z_0 = R_0 + jX_0$
General	$\sqrt{(R+j\omega L)(G+j\omega C)}$	$\sqrt{\frac{R+j\omega L}{G+j\omega C}}$
Lossless	$0 + j\omega\sqrt{LC}$	$\sqrt{\frac{L}{C}} + j0$
Distortionless	$\sqrt{RG} + j\omega\sqrt{LC}$	$\sqrt{\frac{L}{C}} + j0$

Input Impedance, SWR & Power:-



(a) Input impedance due to a line terminated by a load.



(b) Equivalent circuit for finding V_0 & I_0 in terms of Z_{in} at the input.

Consider a transmission line of line length l, characterized by γ & Z_0, connected to a load Z_L as shown in figure. The generator sees the line with the load as in input impedance Z_{in}.

Let the transmission line extend from $z=0$ at the generator to $z=l$ at the load. The voltage & current equations are given by,

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{—} \ (1)$$

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \Rightarrow I_s(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \quad \text{—} \ (2)$$

→ −ve sign indicates direction of current.

Let us consider, $V_0 = V(z=0)$

At generator side, $[z=0]$.

$$I_0 = I(z=0)$$

From eq. ① & ②, $V_0 = V_0^+ + V_0^-$ — ⓐ , $I_0 = I_0^+ + I_0^-$

$$\left\{\begin{aligned} V_0^+ &= \frac{1}{2}\left(V_0 + Z_0 I_0\right) \\ V_0^- &= \frac{1}{2}\left(V_0 - Z_0 I_0\right) \end{aligned}\right\} \quad I_0 = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = \frac{1}{Z_0}\left[V_0^+ - V_0^-\right]$$

$$Z_0 I_0 = V_0^+ - V_0^- \quad \text{— ⓑ}$$

$\therefore$ ⓐ + ⓑ,

$$V_0 + Z_0 I_0 = 2V_0^+$$

$$V_0^+ = \frac{1}{2}\left(V_0 + Z_0 I_0\right)$$

$\therefore$ ⓐ − ⓑ,

$$V_0 - Z_0 I_0 = 2V_0^-$$

$$V_0^- = \frac{1}{2}\left(V_0 - Z_0 I_0\right).$$

If the input impedance at the input terminals is Z_{in}, the input voltage V_0 & the input current I_0 is obtained from figure (b),

$$V_g = I_0 Z_g + I_0 Z_{in}$$

By voltage divider rule,

$$V_0 = \frac{Z_{in}}{Z_g + Z_{in}} V_g .$$

$$I_0 = \frac{V_g}{Z_g + Z_{in}}.$$

At the load is given by

$$z = L$$

$$V_L = V(z), \; z = l. \quad \{V(z=l)\}$$

$$I_L = I(z), \; z = l. \quad \{I(z=l)\}$$

$$V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l}$$

$$Z_0 I_L = V_0^+ e^{-\gamma l} - V_0^- e^{\gamma l}$$

$$V_0^+ = \frac{1}{2}\left(V_L + Z_0 I_L\right) e^{\gamma l}$$

$$V_0^- = \frac{1}{2}\left(V_L - Z_0 I_L\right) e^{-\gamma l}$$

Let us consider the input impedance $Z_{in} = \frac{V_s(z)}{I_s(z)}$

$$Z_{in} = \frac{V_s(z)}{I_s(z)}$$

$$= Z_0 \left[\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right]$$

$$= Z_0 \left[\frac{(V_L + Z_0 I_L) e^{\gamma l} + (V_L - Z_0 I_L) e^{-\gamma l}}{(V_L + Z_0 I_L) e^{\gamma l} - (V_L - Z_0 I_L) e^{-\gamma l}}\right]$$

$$= Z_0 \left[\frac{V_L (e^{\gamma l} + e^{-\gamma l}) + Z_0 I_L (e^{\gamma l} - e^{-\gamma l})}{V_L (e^{\gamma l} - e^{-\gamma l}) + Z_0 I_L (e^{\gamma l} + e^{-\gamma l})}\right]$$

Divide by 2 in each term

$$= Z_0 \left[\frac{V_L \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right) + Z_0 I_L \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right)}{V_L \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right) + Z_0 I_L \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right)}\right]$$

$$Z_{in} = Z_0 \left[\frac{V_L \cosh \gamma l + Z_0 I_L \sinh \gamma l}{V_L \sinh \gamma l + Z_0 I_L \cosh \gamma l}\right]$$

Divide numerator & denominator with $Z_L \cosh \gamma l$,

∴ Input impedance of transmission line from lossy tx'sion line,

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right] \quad \text{(lossy)}$$

For lossless transmission line,

$$\gamma = j\beta \;, \Rightarrow \tanh j\beta l = j \tan \beta l \;, \; Z_0 = R_0 .$$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right] \quad \text{(lossless)}$$

Input impedance varies periodically with distance l, from the load.

The quantity βl is usually referred to as the electrical length of the line & can be expressed in degrees or radians.

Voltage reflection Co-efficient (Γ) :-

Voltage reflection Co-efficient (at the load) (Γ_l) is the ratio of the voltage reflection wave to the incident wave at the load.

$$\Gamma_l = \frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}$$

$$\Gamma_l = \frac{V_0^-}{V_0^+} e^{2\gamma l}$$

$$= \frac{\frac{1}{2}(V_L - Z_0 I_L)\, e^{-\gamma l}\, e^{\gamma l}}{\frac{1}{2}(V_L + Z_0 I_L)\, e^{-\gamma l}\, e^{\gamma l}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

[I_L is taking common]

* The Voltage reflection Coefficient at any point on the line is the ratio of the magnitude of the reflected voltage wave to that of the incident wave.

$$\Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}}$$

$$= \frac{V_0^-}{V_0^+} e^{2\gamma z}$$

$z = l - l'$, substitute,

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{2\gamma l} e^{-2\gamma l'}$$

$$= \Gamma_L e^{-2\gamma l'}$$

Current reflection Co-efficient :-

The Current reflection coefficient at any point on the line is negative of the voltage reflection coefficient at that point.

The Current reflection coefficient at the load is,

$$-\Gamma_L = \frac{I_0^- e^{\gamma l}}{I_0^+ e^{-\gamma l}}$$

Standing wave ratio :-
(SWR)

$$S = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

$$I_{max} = \frac{V_{max}}{Z_0}$$

$$I_{min} = \frac{V_{min}}{Z_0}$$

The input impedance Z_{in} has maxima & minima that occur, respectively, at the maxima & minima of the voltage & current standing wave.

$$|Z_{in}|_{max} = \frac{V_{max}}{I_{min}} = S Z_0.$$

$$|Z_{in}|_{min} = \frac{V_{min}}{I_{max}} = \frac{Z_0}{S}.$$

Example:-

Consider a lossless line with characteristic impedance of $Z_0 = 50\Omega$, the line is terminated in a pure resistive load $Z_L = 100\Omega$ & voltage at the load is 100V (rms).

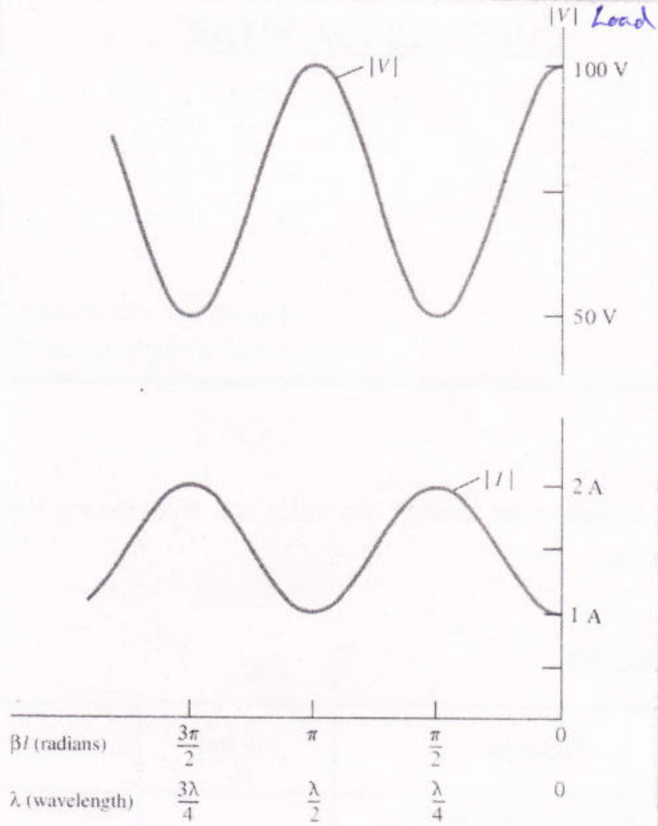


fig: Voltage & Current wave patterns on a lossless line terminated by a resistive load.

Power of a Transmission Line:-

The average input power at a distance 'l' from the load is given by,

$$P_{avg} = \frac{1}{2} Re\left[V_s(z) I_s^*(z)\right]$$

1/2 is a factor, due to peak values instead of the rms values.

$z = l$,

$$P_{avg} = \frac{1}{2} Re\left[V_s(l) I_s^*(l)\right]$$

$$= \frac{1}{2} Re\left[\left(V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l}\right)\left(\frac{V_0^{+*}}{Z_0} e^{\gamma l} - \frac{V_0^{-*}}{Z_0} e^{-\gamma l}\right)\right]$$

$$= \frac{1}{2} Re\left[\left\{V_0^+\left(e^{-\gamma l} + \frac{V_0^-}{V_0^+} e^{\gamma l}\right)\right\}\left\{\frac{V_0^{+*}}{Z_0}\left(e^{\gamma l} - \frac{V_0^{-*}}{V_0^{+*}} e^{-\gamma l}\right)\right\}\right]$$

$$= \frac{1}{2} Re\left[\frac{|V_0^+|^2}{Z_0}\left(e^{-\gamma l} + \Gamma e^{\gamma l}\right)\left(e^{\gamma l} - \Gamma^* e^{-\gamma l}\right)\right]$$

$$= \frac{1}{2} Re\left[\frac{|V_0^+|^2}{Z_0}(1+\Gamma)(1-\Gamma)\right]$$

$$P_{avg} = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma|^2\right)$$

The first term is the incident power P_i, while the second term is the reflected power P_r,

$$\therefore P_t = P_i - P_r.$$

$P_t \rightarrow$ Input or transmitted power & negative sign is due to the negative going wave.

Case(i):-

Shorted line ($Z_L = 0$):-

$$Z_{SC} = Z_{in}\Big|_{Z_L=0} = jZ_0 \tan\beta l.$$

$$\Gamma_L = -1.$$

$$S = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \infty.$$

Case(ii):-

Open-circuited line ($Z_L = \infty$):-

When transmission line is connected to the load open circuited line,

$$Z_{OC} = \lim_{Z_L \to \infty} Z_{in} = \frac{Z_0}{j\tan\beta l} = -jZ_0 \cot\beta l.$$

$Z_{OC} = Z_{in}\Big|_{Z_L=\infty}$

$$\Gamma_L = 1, \quad S = \infty.$$

$$Z_{SC} Z_{OC} = Z_0^2.$$

Case(iii):-

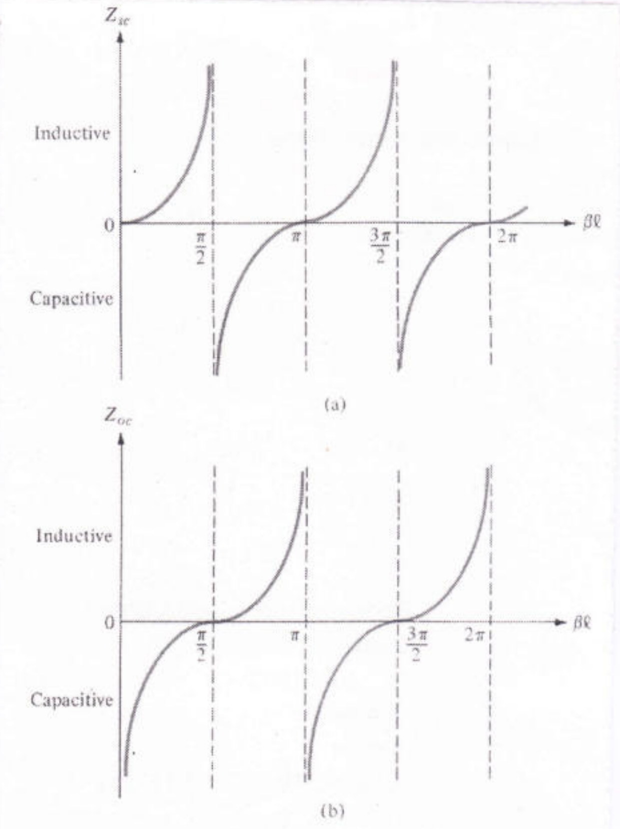
Matched line ($Z_L = Z_0$):-

When the transmission line is connected to $Z_L = Z_0$ which is called matched line.

$$Z_{in} = Z_0 = Z_L$$

$$\Gamma_L = 0.$$

$$S = 1.$$



(a) Input impedance of a lossless line. when shorted.
(b) " " when open.

The incident power is fully absorbed by the load. Thus maximum power transfer is possible when a transmission line is matched to the load.

Smith Chart:-

To reduce the complexity in finding transmission line characteristics [Z_{in}, Γ, SWR].

The smith chart is constructed with in a unit circle with radius $|\Gamma| \leq 1$ then the value of reflection coefficient is ~~transmitted~~ represented as,

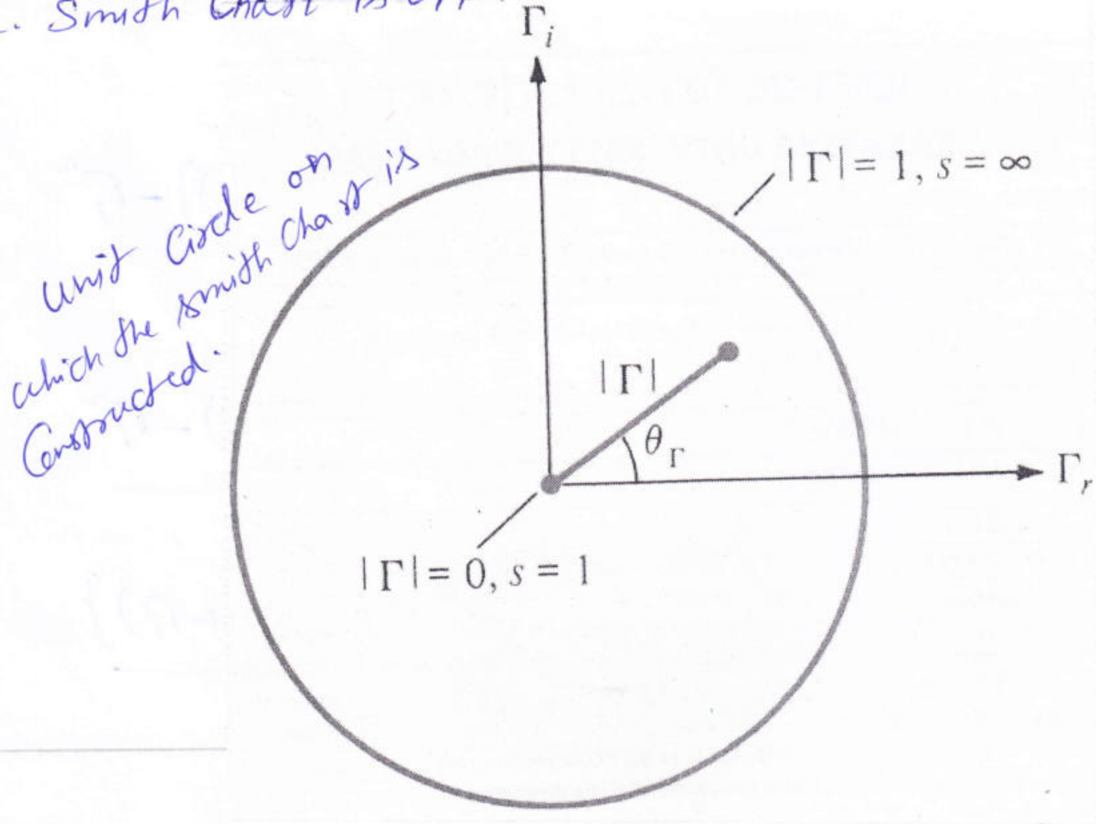
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma = |\Gamma| \angle \theta_r = \Gamma_r + j\Gamma_i$$

Note:-

1. The reflection coefficient lies between 0 & 1, i.e., $0 \leq \Gamma \leq 1$
   &
   S lies between 1 & ∞ i.e., $1 \leq S \leq \infty$

2. Smith Chart is applicable for only lossless transmission line $Z_0 = R_0$.



Normalized Impedance:- (z_L)

$$z_L = \frac{Z_L}{Z_0} = r + jx.$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1} = \Gamma_r + j\Gamma_i$$

$$(\Gamma_r + j\Gamma_i)(z_L + 1) = z_L - 1$$

$$\Gamma_r z_L + \Gamma_r + j[\Gamma_i z_L + \Gamma_i] = z_L - 1.$$

$$\Gamma_r + j\Gamma_i + 1 = z_L[1 - \Gamma_r - j\Gamma_i]$$

$$z_L = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} = r + jx.$$

Mul. & div. with $(1-\Gamma_r) + j\Gamma_i$

$$z_L = \frac{[(1+\Gamma_r) + j\Gamma_i][(1-\Gamma_r) + j\Gamma_i]}{[(1-\Gamma_r) - j\Gamma_i][(1-\Gamma_r) + j\Gamma_i]}$$

$$= \frac{1 - \Gamma_r^2 + j\{\Gamma_i(1+\Gamma_r)\} + j\{\Gamma_i(1-\Gamma_r)\} - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$= \frac{1 - \Gamma_r^2 + j(1+\Gamma_r)(\Gamma_i) + j\Gamma_i(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$= \frac{1 - \Gamma_r^2 - \Gamma_i^2 + j\{(1+\Gamma_r)(\Gamma_i) + \Gamma_i(1-\Gamma_r)\}}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$z_L = \frac{1-\Gamma_r^2-\Gamma_i^2 + j\Gamma_i\{(1+\Gamma_r)+(1-\Gamma_r)\}}{(1-\Gamma_r)^2+\Gamma_i^2}$$

$$= \frac{1-\Gamma_r^2-\Gamma_i^2 + j2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

$$z_L = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} + j\frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2} = r + jx.$$

$$r = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}\ ; \quad x = \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

⋮

$$\left[\Gamma_r - \frac{r}{1+r}\right]^2 + \Gamma_i^2 = \left[\frac{1}{1+r}\right]^2 \quad \text{— (1)}$$

$$\left[\Gamma_r - 1\right]^2 + \left[\Gamma_i - \frac{1}{x}\right]^2 = \left[\frac{1}{x}\right]^2 \quad \text{— (2)}$$

Eq. (1) is called resistance circle [r-circle].

Eq. (1) & (2) is similar to,

$$(x-h)^2 + (y-k)^2 = a^2.$$

which is the general equation of a circle of radius a, centered at (h, k),

Eq. (2) is called reactance circle [x-circle].

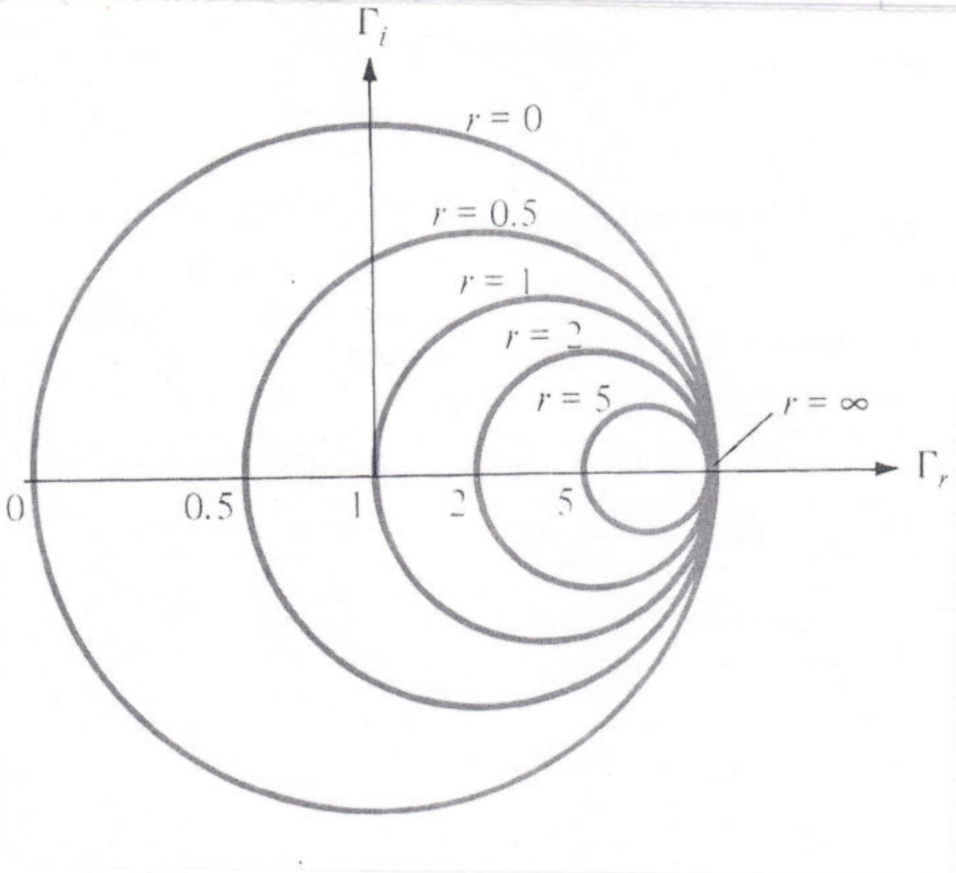
from (1) & (2),

Center at $(\Gamma_r, \Gamma_i) = \left(\frac{r}{r+1}, 0\right)$ & $a = \frac{1}{1+r}$

Center at $(\Gamma_r, \Gamma_i) = \left(1, \frac{1}{x}\right)$ & $a = 1/x$.

Radii & Centers of r-Circles for Typical values of r.

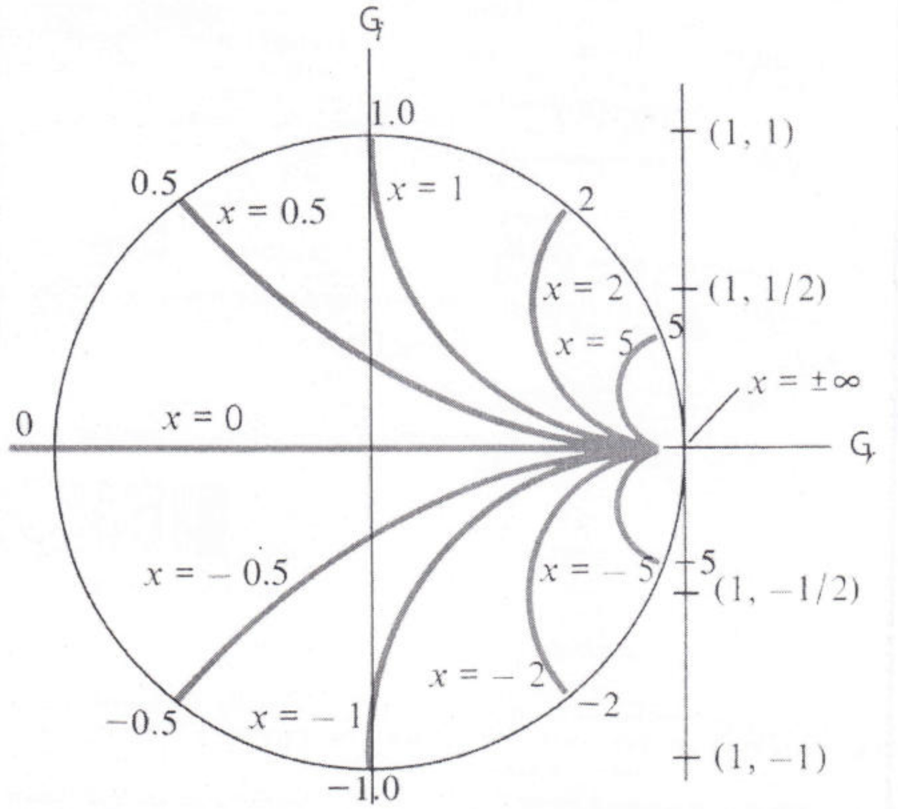
Normalized Resistance (r)	Center $\left(\frac{r}{1+r}, 0\right)$	radius $\left(\frac{1}{1+r}\right)$
0	(0, 0)	1
1/2	(1/3, 0)	2/3
1	(1/2, 0)	1/2
2	(2/3, 0)	1/3
5	(5/6, 0)	1/6
∞	(1, 0)	0



Typical r-circles for r = 0, 0.5, 1, 2, 5, ∞

Radii & centers for x-circles for Typical values of x.

Normalized Reactance(x)	Radius (1/x)	Center (1, 1/x)
0	∞	(1, ∞)
±1/2	2	(1, ±2)
±1	1	(1, ±1)
±2	1/2	(1, ±1/2)
±5	1/5	(1, ±1/5)
±∞	0	(1, 0)



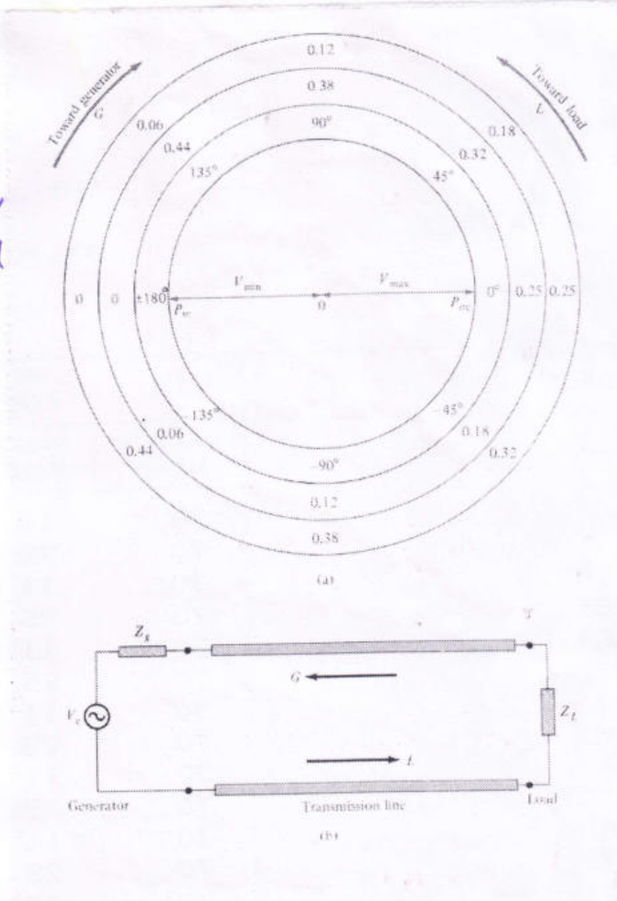
Note:

→ r is always positive

→ x Can be positive [For Inductive impedance]

→ x " Negative [" Capacitive Impedance].

a) Smith Chart illustrating scales around the periphery & movements around the chart.



b) Corresponding movements along the transmission line.

Based on the Important properties, the smith Chart may be used to determine among other things

a) $\Gamma = |\Gamma| \angle \theta_r$ & S

b) Z_{in} or Y_{in}.

c) The locations of V_{max} & V_{min}

$360^\circ \rightarrow \lambda/2$

$720^\circ \rightarrow \lambda$

$r = 0$, $x = 0$, $Z_L = 0 + j0$ } P_{sc}

$r = \infty$, $x = \infty$, $Z_L = \infty + j\infty$ } P_{oc}

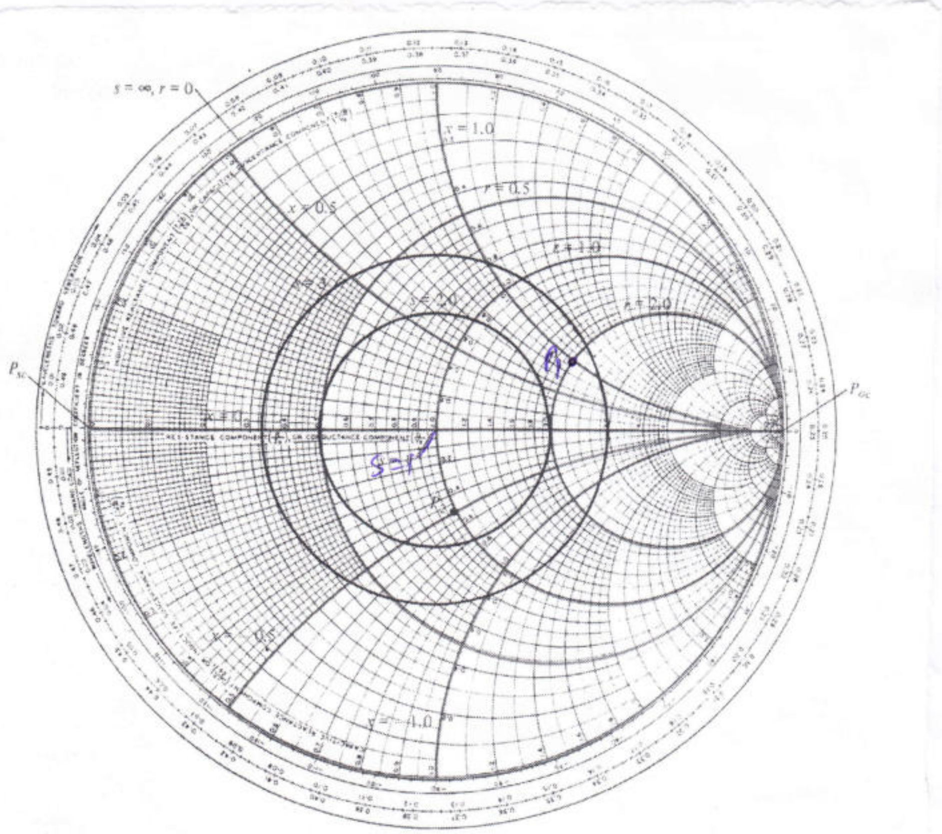


Illustration of the r-, x- & s-circles on the Smith Chart.

Figure shows the Normalized impedance $Z = 2 + j$

i.e., $r = 2$, $x = 1$.

The point of intersection r & x is point P_1.

Similarly, $Z = 1 - j0.5$

$r = 1$, $x = -0.5$ } point P_2.

The SWR is determined by locating where an s-circle crosses the Γ_r axis. Figure shows the s-circles for $s = 1, 2, 3$ & ∞.

* s-circles sometimes referred to as $|\Gamma|$-circles with $|\Gamma|$ varying linearly from 0 to 1 as it move away from the center 0 toward the periphery of the Chart while s varies non-linearly from 1 to ∞.

Problem:

1. A 30m long lossless transmission line with $Z_0 = 50\Omega$ operating at 2MHZ is terminated with a load $Z_L = 60 + j40\Omega$. If $u = 0.6c$ on the line find,

a) The reflection Coefficient Γ

b) The SWR, S.

c) The input Impedance.

Sol:

$Z_0 = 50\Omega$ $u = 0.6c$.

$Z_L = 60 + j40\Omega$.

$f = 2MHZ$.

$l = 30m$.

$$z_L = \frac{Z_L}{Z_0} = \frac{60 + 40j}{50} = 1.2 + 0.8j$$

$r = 1.2$, $x = 0.8$.

From the Smith Chart,

$\theta = 56°$.

Angle θ_r is read directly on the chart as the angle between OS & OP, i.e., θ_r = angle POS = 56°.

From the smith Chart extend OP towards the r=0 Circle at Q.

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.1\,cm}{8.8\,cm} = 0.35$$

$$\Gamma = |\Gamma| \angle\theta_r = 0.35\angle 56° = 0.195 + 0.29j$$

b) Draw a circle with radius OP & center at O.

This is the Constant S or $|\Gamma|$ circle.

Locate point S where the S-circle meets the Γ_r-axis.

$\Gamma_i = 0$ from the Smith Chart.

The value of r at this point is S,

$$\text{i.e.,} \quad S = r \; (\text{for } r \geq 1)$$
$$= 2.1.$$

c) To obtain Z_{in}, first express l in terms of λ or in degrees,

$$\lambda = \frac{u}{f} = \frac{0.6(3\times10^8)}{2\times10^6} = 90\,m.$$

from the question, $\lambda = 3l$.

$$l = 30m = \frac{30}{90}\lambda = \frac{\lambda}{3}$$
$$= \frac{720^\circ}{3}$$
$$= 240^\circ.$$

$\because$ λ corresponds to an angular movement of 720° on the Chart, the length of the line corresponds to an angular movement of 240°. i.e., Move towards the generator (or away from the load, in the clockwise direction) 240° on the S-circle from point P to point G.

At G, the Normalized input impedance,

$$z_{in} = 0.47 + j\,0.035$$

Hence, $Z_{in} = Z_0 z_{in} = 50(0.47 + j\,0.035)$

$$= 23.5 + j1.75\,\Omega.$$

* V_{max} occurs where $Z_{in,max}$ is located on the chart, i.e., on the positive Γ_r axis or on OP_{oc}. V_{min} is located at the same point where $Z_{in,min}$ on the chart. i.e, on the negative Γ_r axis or on OP_{sc}''.

(2) A 70Ω lossless line has $S = 1.6$ & $\theta_r = 300°$ if the line is 0.6λ long obtain,

a) reflection co-efficient, load impedance, input impedance

b) The distance of the first min. voltage from the load.

Given, $S = 1.6$, $\theta_r = 300°$, $l = 0.6\lambda$.

From the smith chart,

the magnitude of reflection co-efficient,

$$|\Gamma| = \frac{OP}{OQ} = \frac{2.1}{8.8} = 0.23$$

$$\Gamma = |\Gamma| \angle \theta$$

$$= 0.23 \angle 300°$$

$$= 0.115 - 0.199j$$

we know that normalized impedance $z_L = \frac{Z_L}{Z_0}$.

$$z_L = 1.15 - j0.5$$

$$Z_L = Z_0 \cdot z_L$$

$$= 70(1.15 - j0.5)$$

$$= 80.5 - 35j\,\Omega.$$

$$l = 0.6\lambda$$

$$= 0.6(720) = 432.$$

From the smith chart,

the normalized input impedance is,

$$z_{in} = 0.68 - 0.26j$$

$$Z_{in} = Z_0 \, z_{in}$$

$$= 70(0.68 - 0.26j)$$

$$= 47.6 - 18.2j$$

From V_{min} to load the distance in degrees is 120°.

$$\lambda = 720^\circ$$

$$\frac{\lambda}{6} = 120^\circ$$

$$\therefore V_{min} = 0.1666\lambda$$

For V_{max}, $\Rightarrow$ $\lambda = 720^\circ$

$$\frac{\lambda}{2.4} = 300$$

$$\therefore V_{max} = 0.416\lambda$$

(3) A load of $100 + j150\,\Omega$ is connected to a $75\,\Omega$ lossless line find,

a) Γ b) S c) The load admittance Y_L

d) Z_{in} at 0.4λ from the load.

e) The locations of V_{max} and V_{min} w.r.t the load if the line is 0.6λ long. , f) Z_{in} at the generator.

$Z_L = 100 + j150\Omega$

$Z_0 = 75\Omega$

a) $z_L = \frac{Z_L}{Z_0} = \frac{100 + j150}{75} = 1.33 + 2j.$

from the smith chart,

$|\Gamma| = \frac{OP}{OQ} = \frac{5.9}{8.9} = 0.66.$

$\theta_r = 40°.$

$\Gamma = |\Gamma| \underline{|\theta_r} = 0.66 \angle 40° = 0.505 + 0.424j$

b) from smith Chart, $S = 4.8$,

(S circle passing through P)

c) For admittance extend OP to POP' and note point P' where the Constant S-Circle meets POP', at P'

normalized, $y_L = 0.22 - 0.34j.$

$Y_L = Y_0 y_L = \frac{1}{75}(0.22 - 0.34j)$

$= 2.93 - 4.53j$ ms.

d) The 0.4λ Corresponds to an angular movement of $0.4 * 720° = 288°$. on the Constant 'S' circle.

from P, we move 288° towards the generator (clockwise) on the S circle to reach point R.

At R, $z_{in} = 0.29 + 0.65j$

$Z_{in} = Z_0 \cdot z_{in} = 75(0.29 + j0.65)$

$= 21.75 + 48.75j\,\Omega$

e) The 0.6λ corresponds to an angular movement of,

$$0.6 * 720^\circ = 432^\circ = 1 \text{ revolution} + 72^\circ.$$

Thus we start from (load end) move along the 'S' circle 432° or one revolution plus 72° and reach the generator at point G. Note that to reach G from P, we have passed through point T (location of V_{min}) once and point S twice thus from the load.

1^{st} V_{max} is located at $\frac{40^\circ}{720^\circ}\lambda = 0.055\lambda$.

2^{nd} V_{max} is located at $0.055\lambda + \frac{\lambda}{2} = 0.555\lambda$.

and the only V_{min} is located at $0.055 + \lambda/4 = 0.305\lambda$.

f) At G (Generator end).

$$z_{in} = 1.8 - 2.2j.$$

$$Z_{in} = Z_0 (z_{in})$$
$$= 75 (1.8 - j2.2)$$
$$= 135 - j165 \Omega.$$

(4) A line characteristic impedance 300Ω is terminated with a load of $175 + j207\Omega$ an electrical signal of 200MHZ is transmitted along the line in the free space determine

a) VSWR b) Load admittance c) Distance between the load and the first voltage min. along the transmission line.

$Z_L = 175 + j207 \Omega$.

$f = 200MHZ \quad Z_0 = 300\Omega$

a) $z_L = \frac{Z_L}{Z_0} = 0.583 + 0.69j$

$\theta_\Gamma = 97°$.

from the smith chart,

$\Gamma = \frac{OP}{OQ} = \frac{4.1}{4.8} = 0.465$

$\Gamma = 0.465 \angle 97° = -0.056 + 0.461j$

$s = 2.7$.

b) $y_L = 0.72 - 0.85j$

$Y_L = Y_0 y_L$

$= \frac{1}{300}(0.72 - 0.85j) = 2.4 - 2.8j \, m\mho$

$V_{min} = 0.36V$

$V_{max} = 2.70$

c) we have to calculate from load Q to V_{min} in degrees and Convert degrees to wavelength.

$277 \times \frac{\lambda}{720} = 0.384\lambda$

$= 0.384 * 1.5m$

$= 0.576m$.

$\lambda = \frac{c}{f} = \frac{3*10^8}{2\times10^8}$

$= 1.5m$.

For V_{max} the degrees are 97°,

$$97 * \frac{\lambda}{720} = 0.134\lambda$$

$\lambda = 1.5m.$

$$= 0.134 * 1.5m$$

$$= 0.202m.$$

(5) A 50Ω line is terminated with a receiving end impedance of 100 + j121 Ω the wave length of the electrical signal along the line is 2.5m determine (a) Reflection coefficient

(b) S

(c) load admittance

(d) Impedance of transmission line at voltage max & min.

(e) Distance between load & 1st voltage max, voltage min for a Tx'sion line.

$Z_L = 100 + j121\,\Omega.$

$Z_0 = 50\,\Omega.$

$\lambda = 2.5m.$

$$z_L = \frac{Z_L}{Z_0} = \frac{100 + 121j}{50} = 2 + 2.42j$$

$\theta_\Gamma = 28.5°.$

$$|\Gamma| = \frac{OP}{OQ} = \frac{6}{8.8} = 0.68$$

$S = 5.$

$\Gamma = 0.68 \angle 28.5$

$y_L = 0.2 - 0.25j$

$Y_L = Y_0 * y_L$

$= 4 - 5j \, m\mho$

$Z_{max} = S \cdot Z_0$ $\qquad Z_{min} = \frac{Z_0}{S} = \frac{50}{5}$

$= 5 * 50$ $\qquad = 10$

$= 250$

(6) A lossless transmission line of length 0.434λ and $Z_0 = 100\,\Omega$ is terminated at an impedance of $260 + j180\,\Omega$ find a) VSWR b) Reflection coefficient c) Z_{in} d) location of V_{max} & V_{min}.

Applications of Transmission lines:-

Transmission lines are mostly used for

→ Load impedance matching

→ Impedance Measurement.

A. Quarter-Wave Transformer (Matching):-

When $Z_0 \neq Z_L$, the load is mismatched and a reflected wave exists on the line. For maximum Power transfer, it is desired that the load be matched to the transmission line ($Z_0 = Z_L$), so that there is no reflection [$|\Gamma| = 0$ or $s = 1$]. The matching is achieved by using shorted sections of transmission lines.

When $l = \lambda/4$ (or) $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \pi/2$.

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right]$$

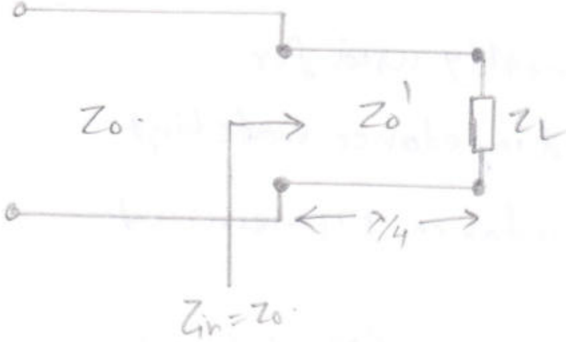
$$= Z_0 \left[\frac{Z_L\left(\frac{1}{\tan\pi/2}\right) + jZ_0}{Z_0\left(\frac{1}{\tan\pi/2}\right) + jZ_L}\right]$$

$$Z_{in} = \frac{Z_0^2}{Z_L}.$$

i.e., $\frac{Z_{in}}{Z_0} = \frac{Z_0}{Z_L}$.

(or) $Z_{in} = \frac{1}{Z_L} \rightarrow Y_{in} = Z_L$.

Thus by adding a $\lambda/4$ line on the smith chart, the input impedance corresponding to a given load impedance.



Load matching using a $\lambda/4$ transformer.

A mismatched load Z_L can be properly matched to a line (with characteristic impedance Z_0) by inserting prior to the load a transmission line $\lambda/4$ ~~also~~ long (with characteristic impedance Z_0') as shown in figure. The $\lambda/4$ section of the transmission line is called a quater-wave transformer because it is used for impedance matching like an ordinary transformer.

Z_0' is selected such that $(Z_{in} = Z_0)$

$$Z_0' = \sqrt{Z_0 Z_L}$$

Where, Z_0', Z_0 & Z_L are all real.

<u>Example:-</u>

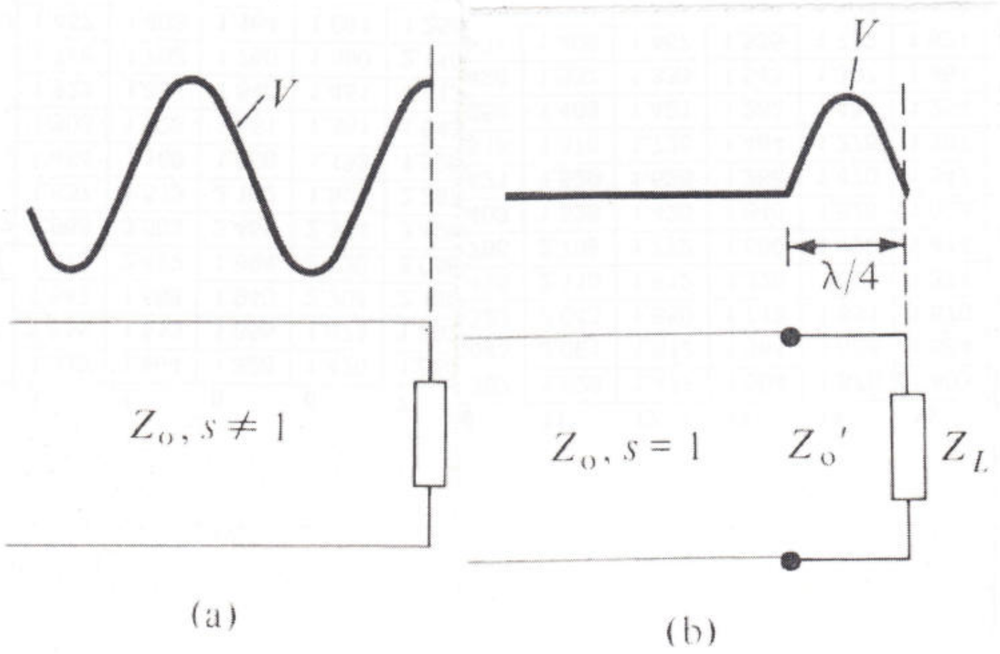
A 120-Ω load is to be matched to a 75 Ω line,

∴ The Quarter-wave transformer must have characteristic impedance

of, $\sqrt{(75)(120)} = 95\,\Omega$.

This 95 Ω quarter-wave transformer will also match a 75-Ω load to a 120 Ω line.

voltage standing wave pattern of mismatched load



without a λ/4 transformer   with a λ/4 transformer

Although a standing wave still exists between the transformer and the load, there is no standing wave to the left of the transformer due to the matching.

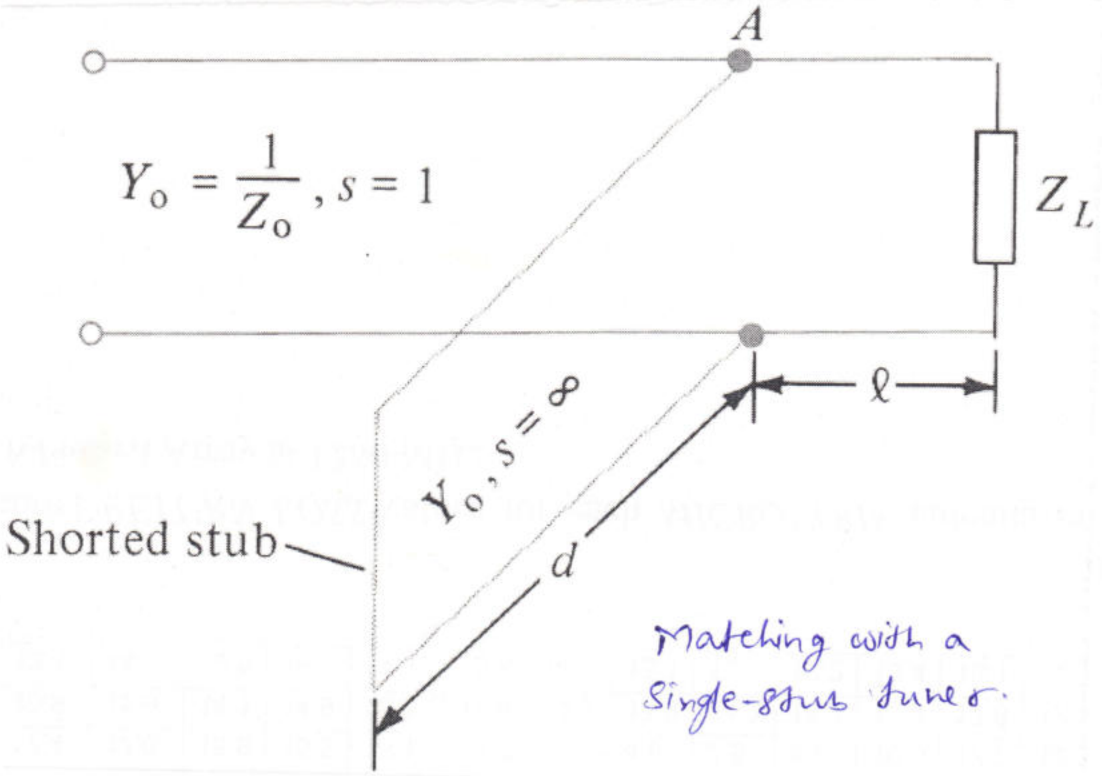
The Reflected wave (or standing wave) is eliminated only at the desired wavelength (or frequency f), there will be reflection at a slightly different wavelength.

The main disadvantage of the quarter-wave transformer is a narrow-band or frequency-sensitive device.

Single-Stub Tuner (Matching):-

The major drawback of using a quarter-wave transformer as a line-matching device is eliminated by using a single-stub tuner.

The tuner consists of an open or shorted section of transmission line of length d connected in parallel with the main line at some distance l from the load as shown in figure.

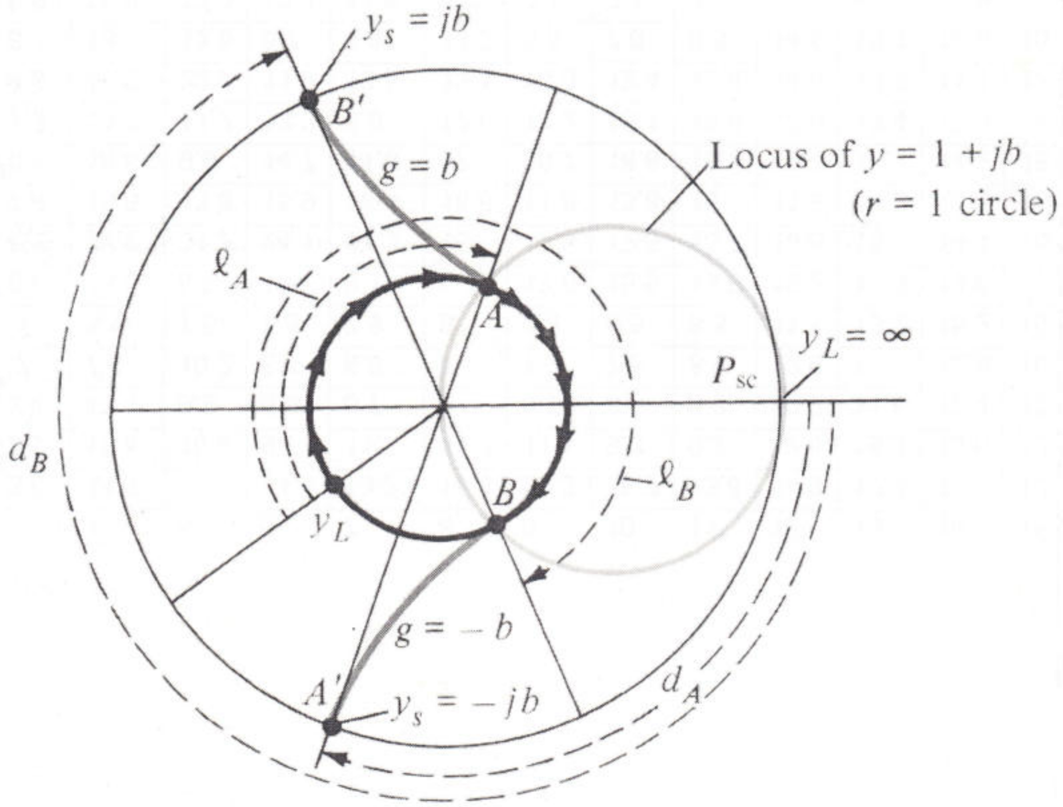


Matching with a single-stub tuner.

The stub has the same characteristic impedance as the main line. It is more difficult to use a series stub although it is theoretically feasible. An open circuited stub radiates some energy at high frequencies. Consequently, shunt short-circuited parallel stubs are preferred.

As that $Z_{in} = Z_0$, i.e., $Z_{in} = 1$ (or) $y_{in} = 1$ at point A on the line, first draw the locus $y = 1 + jb$ ($r = 1$ circle) on the smith chart as shown in figure. If a shunt stub of admittance $y_s = -jb$ is introduced at A, then,

$$y_{in} = 1 + jb + y_s = 1 + jb - jb = 1 + j0.$$



Using the smith chart to determine l & d of a shunt-shorted single-stub tuner.

Since b could be negative, two possible values of l ($< \lambda/2$) can be found on the line.

At A, $y_s = -jb$,

$l = l_A$

at, B, $y_s = jb$ as shown in figure.

$l = l_B$

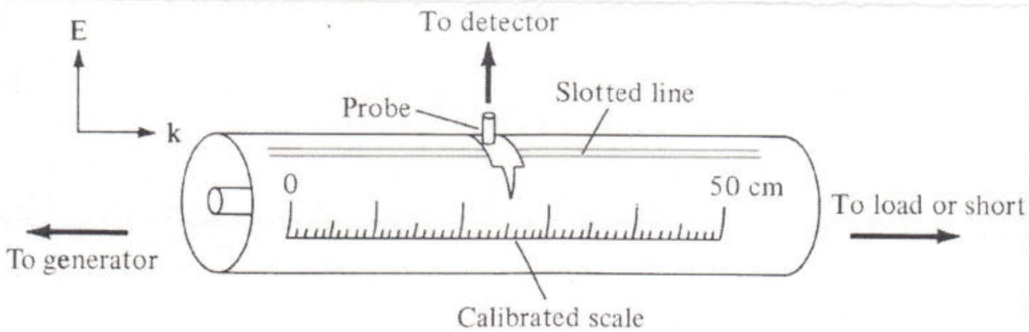
Due to the fact that the stub is shorted, ($y_L' = \infty$), the length d of the stub by finding the distance from P_{sc} (at which $z_L' = 0 + j0$) to the required stub admittance y_s.

For the stub at A, $d = d_A$ as the distance from P to A', where A' corresponds to $y_s = -jb$ located on the periphery of the chart. Similarly $d = d_B$ as the distance from P_{sc} to B' ($y_s = jb$).

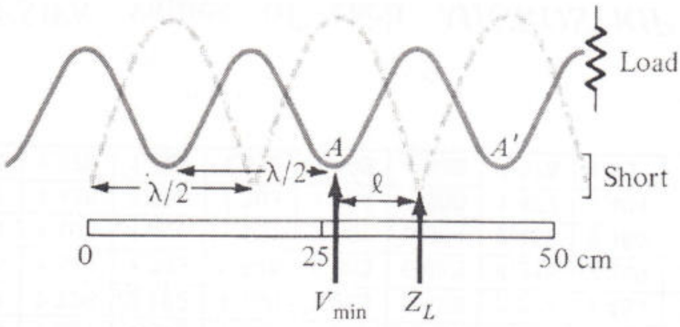
Thus $d = d_A$ & $d = d_B$, corresponding to A & B, respectively, as shown in figure. Note that $d_A + d_B = \lambda/2$ always. Since two possible shunted stubs, normally choose to match the shorter stub or one at a position closer to the load. Instead of having a single stub shunted across the line, we may have two stubs. This is called double-stub matching and allows for the adjustment of the load impedance.

Slotted line (Impedance Measurement):-

At high frequencies, it is very difficult to measure current & voltage because measuring devices become significant in size & every circuit becomes a transmission line. The slotted line is a simple device used in determining the impedance of an unknown load at high frequencies upto into the region of gigahertz (GHz).



(a) Typical slotted line



(b)

Determining the location of the load Z_L & V_{min} on the line

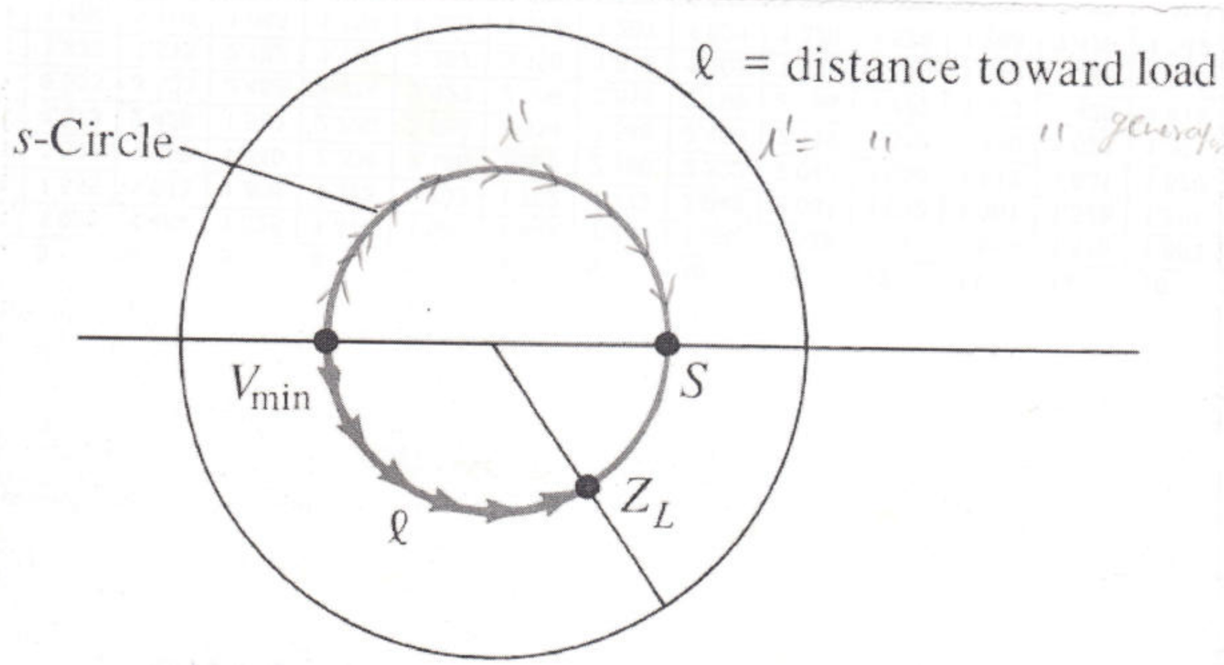
It consists of a section of an air (loss less) line with a slot in the outer conductor as shown in figure. The line has a probe, along the $\bar{E}$ field, which samples the $\bar{E}$ field and consequently measures the potential difference between the probel and its outer shield.

The slotted line is primarily used in conjunction with the smith chart to determine the standing wave ratio s (the ratio of maximum voltage to the minimum voltage) and the load impedance Z_L. The value of s is read directly on the detection meter when the load is connected.

To determin Z_L, first replace the load by a short circuit and not the locations of voltage minima (which are more accurately determined than the maxima because of the sharpness of the burning point) on the scale.

Since impedances repeat every half wavelength, any of the minima may be selected as the load reference point. Now the distance from the selected reference point to the load by replacing the short circuit by the load and noting the locations of the voltage minima.

The distance l (distance of V_{min} toward the load) expressed in terms of λ is used to locate the position of the load of an s-circle on the chart as shown in figure.



And also locate the load by using l', which is the distance of V_{min} toward the generator. Either l or l' may be used to locate Z_L.

The procedure involved in using the Slotted Line can be sumarize as follows:

1. With the load connected, read s on the detection meter. With the value of s, draw the s-circle on the smith chart.
2. With the load replaced by a short circuit, locate a reference position for Z_L at a vol. min. point.
3. With the load on the line, note the position of V_{min} & determine l.
4. On the smith chart, move toward the load a distance l from the location of V_{min}, find Z_L at that point.

1. Antenna with impedance $40 + j30 \Omega$ is to be matched to a 100Ω lossless line with a shorted stub, Determine

a) The required stub admittance

b) The distance between the stub and the antenna

c) The stub length

d) The standing wave ratio on each ratio of the system.

Sol:- a) $z_L = \frac{Z_L}{Z_0} = \frac{40+j30}{100} = 0.4 + j0.3$

Locate z_L on the smith chart as shown in figure, from this draw the s-circle, so that y_L can be located diametrically opposite z_L. Thus, $y_L = 1.6 - j1.2$.

$$y_L = \frac{Z_0}{Z_L} = \frac{100}{40+j30} = 1.6 - j1.2$$

Locate points A & B where the s-circle intersects the $g=1$ circle. At A, $y_s = -j1.04$ & at B, $y_s = +j1.04$.

Thus the required stub admittance is,

$$Y_s = Y_0 y_s = \pm j\, 1.04 \frac{1}{100} = \pm j\, 10.4\, ms.$$

Both $j\, 10.4 ms$ & $-j10.4 ms$ are possible values.

b) The distance between the load (antenna in this case) Y_L & the stub, at A,

$$l_A = 0\frac{\lambda}{2} - \frac{(62^\circ - -39^\circ)\lambda}{720^\circ} = 0.36\lambda \quad (\text{or}) \quad l_A = \frac{259\lambda}{720} = 0.35\lambda.$$

At B:

$$l_B = \frac{(62^\circ - 39^\circ)\lambda}{720^\circ} = 0.032\lambda \quad (\text{or}) \quad l_B = \frac{25\lambda}{720} = 0.634\lambda.$$

c) Locate points A' & B' corresponding to stub admittance $-j1.04$ & $j1.04$, respectively,

Determine the stub length (distance from P_{sc} to A' & B'):

$$d_A = \frac{88^\circ}{720^\circ}\lambda = 0.122\lambda$$

$$d_B = \frac{272^\circ}{720}\lambda = 0.3778\lambda.$$

Notice that $d_A + d_B = 0.5\lambda$ as expected.

d) From the chart, $S = 2.7$.

This is the SWR on the line segment between the stub & the load. $S = 1$ to the left of the stub because the line is matched, & $S = \infty$ along the stub because the stub is shorted.

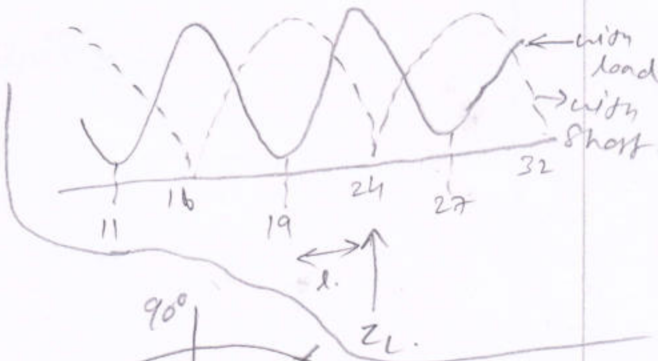
Prob: With an unknown load connected to a slotted air line, $S = 2$ is recorded by a standing wave indicator & minima are found at 11cm, 19cm... on the scale. When the load is replaced by a short circuit, the minima are at 16cm, 24cm.... If $Z_0 = 50\Omega$, Cal. λ, f, Z_L.

Sol:- Consider the standing wave patterns as shown in figure,

$$\frac{\lambda}{2} = 19 - 11 = 8\,cm \quad (or) \quad \lambda = 16\,cm.$$

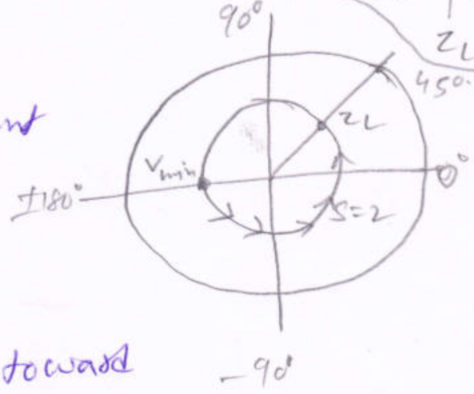
$$f = \frac{u}{\lambda} = \frac{3 \times 10^8}{16 \times 10^{-2}} = 1.875\,GHz$$

Electrically speaking, the load can be located at 16cm or 24cm. If we assume that the load is at 24cm, the load is at a distance l from V_{min},



$$l = 24 - 19 = 5\,cm$$

$$= \frac{5}{16}\lambda = 0.3125\,\lambda.$$



This corresponds to an angular movement of $0.3125 \times 720° = 225°$ on the $S = 2$ circle, By starting at the location of V_{min} and moving 225° toward the load (Counter clockwise), we reach the location of Z_L as shown in figure.

$$z_L = 1.4 + j0.75$$

$$Z_L = Z_0 z_L = 50(1.4 + j0.75) = 70 + j37.5\Omega.$$

1. The following measurements were taken using the slotted line technique. with load, $S = 1.8$, V_{max} occurred 23cm, 33.5cm ---; with short, $S = \infty$, V_{max} occured at 25cm, 37.5cm ---. If $Z_0 = 50\,\Omega$, determine Z_L:

{Sol:- $32.5 - j17.5\,\Omega$}

2. A $75\,\Omega$ lossless line is to be matched to a $100 - j80\,\Omega$ load with a shorted stub. Calculate the stub length, its distance from the load, and the necessary stub admittance.

Sol: $l_A = 0.093\,\lambda$, $d_A = 0.126\lambda$, $\pm j\,12.67\,mS$}

$l_B = 0.272\,\lambda$, $d_B = 0.374\lambda$

determined by locating where an s-circle crosses the Γ_r axis. Typical examples of s-circles for $s = 1, 2, 3$, and ∞ are shown in Figure 11.13. Since $|\Gamma|$ and s are related according to eq. (11.38), the s-circles are sometimes referred to as $|\Gamma|$-circles with $|\Gamma|$ varying linearly from 0 to 1 as we move away from the center O toward the periphery of the chart while s varies nonlinearly from 1 to ∞.

The following points should be noted about the Smith chart:

1. At point P_{sc} on the chart $r = 0, x = 0$; that is, $Z_L = 0 + j0$ showing that P_{sc} represents a short circuit on the transmission line. At point P_{oc}, $r = \infty$ and $x = \infty$, or $Z_L = \infty + j\infty$, which implies that P_{oc} corresponds to an open circuit on the line. Also at P_{oc}, $r = 0$ and $x = 0$, showing that P_{oc} is another location of a short circuit on the line.
2. A complete revolution (360°) around the Smith chart represents a distance of $\lambda/2$ on the line. Clockwise movement on the chart is regarded as moving toward the generator (or away from the load) as shown by the arrow G in Figure 11.14(a) and (b). Similarly, counterclockwise movement on the chart corresponds to moving toward the load (or away from the generator) as indicated by the arrow L in Figure 11.14. Notice from Figure 11.14(b) that at the load, moving toward the load does not make sense (because we are already at the load). The same can be said of the case when we are at the generator end.
3. There are three scales around the periphery of the Smith chart as illustrated in Figure 11.14(a). The three scales are included for the sake of convenience but they are actually meant to serve the same purpose; one scale should be sufficient. The scales are used in determining the distance from the load or generator in degrees or wavelengths. The outermost scale is used to determine the distance on the line from the generator end in terms of wavelengths, and the next scale determines the distance from the load end in terms of wavelengths. The innermost scale is a protractor (in degrees) and is primarily used in determining θ_Γ; it can also be used to determine the distance from the load or generator. Since a $\lambda/2$ distance on the line corresponds to a movement of 360° on the chart, λ *distance on the line corresponds to a 720° movement on the chart.*

$$\boxed{\lambda \rightarrow 720^\circ} \tag{11.55}$$

   Thus we may ignore the other outer scales and use the protractor (the innermost scale) for all our θ_Γ and distance calculations.
4. V_{max} occurs where $Z_{in,max}$ is located on the chart [see eq. (11.39a)], and that is on the positive Γ_r axis or on OP_{oc} in Figure 11.14(a). V_{min} is located at the same point where we have $Z_{in,min}$ on the chart; that is, on the negative Γ_r axis or on OP_{sc} in Figure 11.14(a). Notice that V_{max} and V_{min} (or $Z_{in,max}$ and $Z_{in,min}$) are $\lambda/4$ (or 180°) apart.
5. The Smith chart is used both as impedance chart and admittance chart ($Y = 1/Z$). As admittance chart (normalized impedance $y = Y/Y_o = g + jb$), the g- and b-circles correspond to r- and x-circles, respectively.

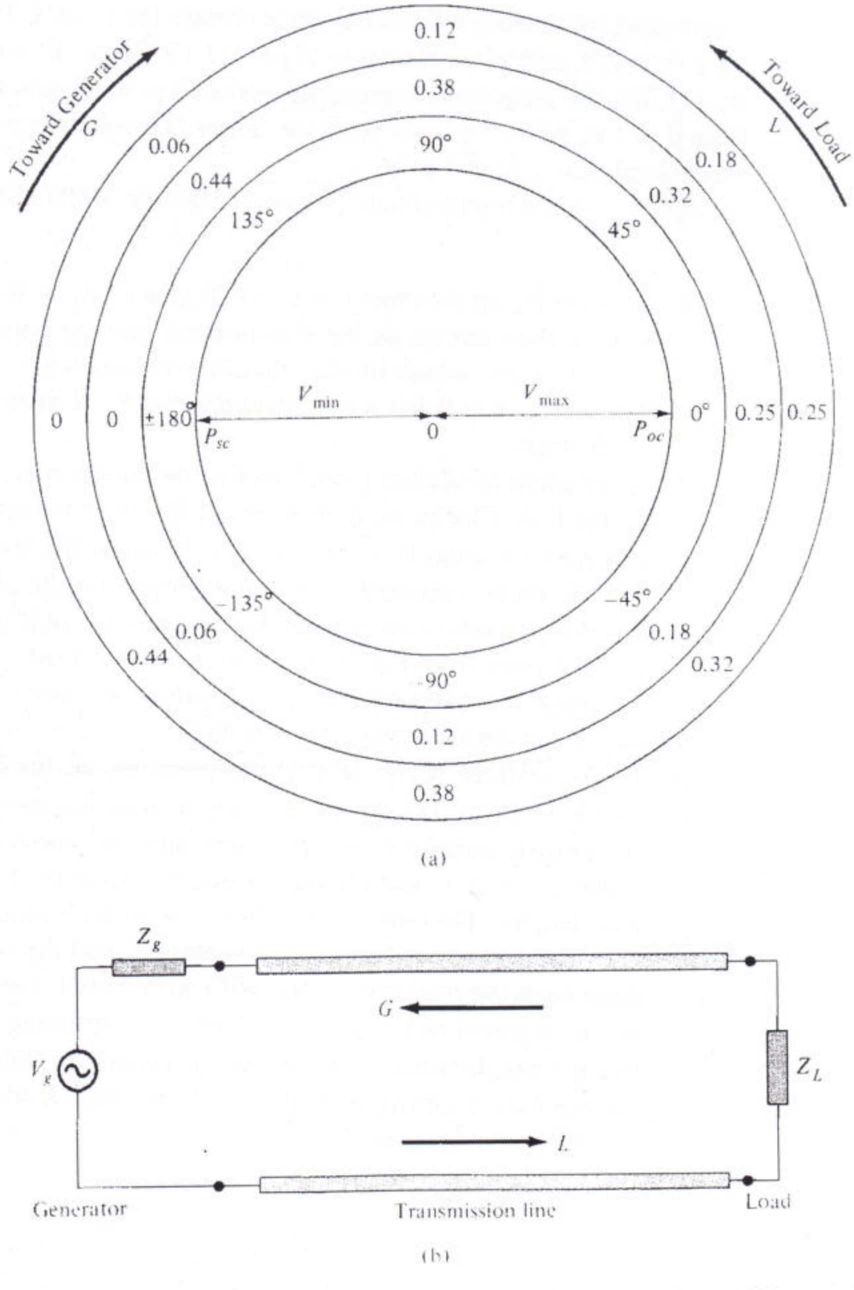


Figure 11.14 (a) Smith chart illustrating scales around the periphery and movements around the chart, **(b)** corresponding movements along the transmission line.

Based on these important properties, the Smith chart may be used to determine, among other things, (a) $\Gamma = |\Gamma|\underline{/\theta_\Gamma}$ and s; (b) Z_{in} or Y_{in}; and (c) the locations of V_{max} and V_{min} provided that we are given Z_o, Z_L, and the length of the line. Some examples will clearly show how we can do all these and much more with the aid of the Smith chart, a compass, and a plain straightedge.

EXAMPLE 11.4

A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

Solution:

This problem will be solved with and without using the Smith chart.

Method 1: (Without the Smith chart)

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40}$$
$$= 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2 \times 10^6)(30)}{0.6\,(3 \times 10^8)} = \frac{2\pi}{3} = 120^\circ$$

Note that $\beta\ell$ is the electrical length of the line.

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell}\right]$$
$$= \frac{50\,(60 + j40 + j50 \tan 120^\circ)}{[50 + j(60 + j40) \tan 120^\circ]}$$
$$= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\underline{/3.22^\circ}$$
$$= 23.97 + j1.35\ \Omega$$

Method 2: (Using the Smith chart).

(a) Calculate the normalized load impedance

$$z_L = \frac{Z_L}{Z_o} = \frac{60 + j40}{50}$$
$$= 1.2 + j0.8$$

Locate z_L on the Smith chart of Figure 11.15 at point P where the $r = 1.2$ circle and the $x = 0.8$ circle meet. To get Γ at z_L, extend OP to meet the $r = 0$ circle at Q and measure OP and OQ. Since OQ corresponds to $|\Gamma| = 1$, then at P,

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.2\text{ cm}}{9.1\text{ cm}} = 0.3516$$

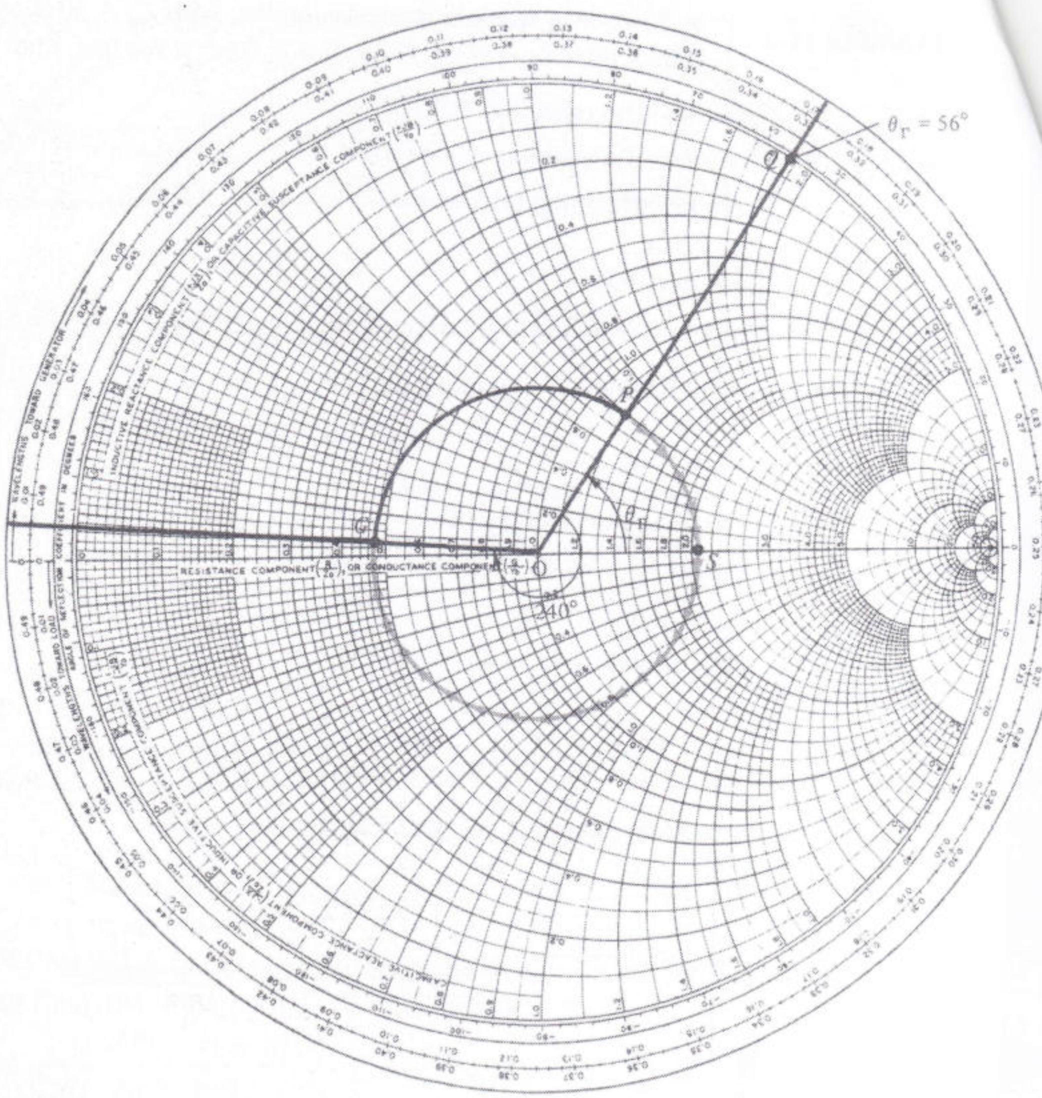


Figure 11.15 For Example 11.4.

Note that $OP = 3.2$ cm and $OQ = 9.1$ cm were taken from the Smith chart used by the author; the Smith chart in Figure 11.15 is reduced but the ratio of OP/OQ remains the same.

Angle θ_Γ is read directly on the chart as the angle between OS and OP; that is

$$\theta_\Gamma = \text{angle } POS = 56°$$

Thus

$$\Gamma = 0.3516 \underline{/56°}$$

(b) To obtain the standing wave ratio s, draw a circle with radius OP and center at O. This is the constant s or $|\Gamma|$ circle. Locate point S where the s-circle meets the Γ_r-axis.

[This is easily shown by setting $\Gamma_i = 0$ in eq. (11.49a).] The value of r at this point is s; that is

$$s = r \text{ (for } r \geq 1) \\ = 2.1$$

(c) To obtain Z_{in}, first express ℓ in terms of λ or in degrees.

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{2 \times 10^6} = 90 \text{ m}$$

$$\ell = 30 \text{ m} = \frac{30}{90}\lambda = \frac{\lambda}{3} \rightarrow \frac{720°}{3} = 240°$$

Since λ corresponds to an angular movement of 720° on the chart, the length of the line corresponds to an angular movement of 240°. That means we move toward the generator (or away from the load, in the clockwise direction) 240° on the s-circle from point P to point G. At G, we obtain

$$z_{in} = 0.47 + j0.035$$

Hence

$$Z_{in} = Z_o z_{in} = 50(0.47 + j0.035) = 23.5 + j1.75\ \Omega$$

Although the results obtained using the Smith chart are only approximate, for engineering purposes they are close enough to the exact ones obtained in Method 1.

PRACTICE EXERCISE 11.4

A 70-Ω lossless line has $s = 1.6$ and $\theta_\Gamma = 300°$. If the line is 0.6λ long, obtain

(a) Γ, Z_L, Z_{in}

(b) The distance of the first minimum voltage from the load

Answer: (a) $0.228\,\underline{/300°}$, $80.5 - j33.6\ \Omega$, $47.6 - j17.5\ \Omega$, (b) $\lambda/6$.

EXAMPLE 11.5

A $100 + j150$–Ω load is connected to a 75-Ω lossless line. Find:

(a) Γ

(b) s

(c) The load admittance Y_L

(d) Z_{in} at 0.4λ from the load

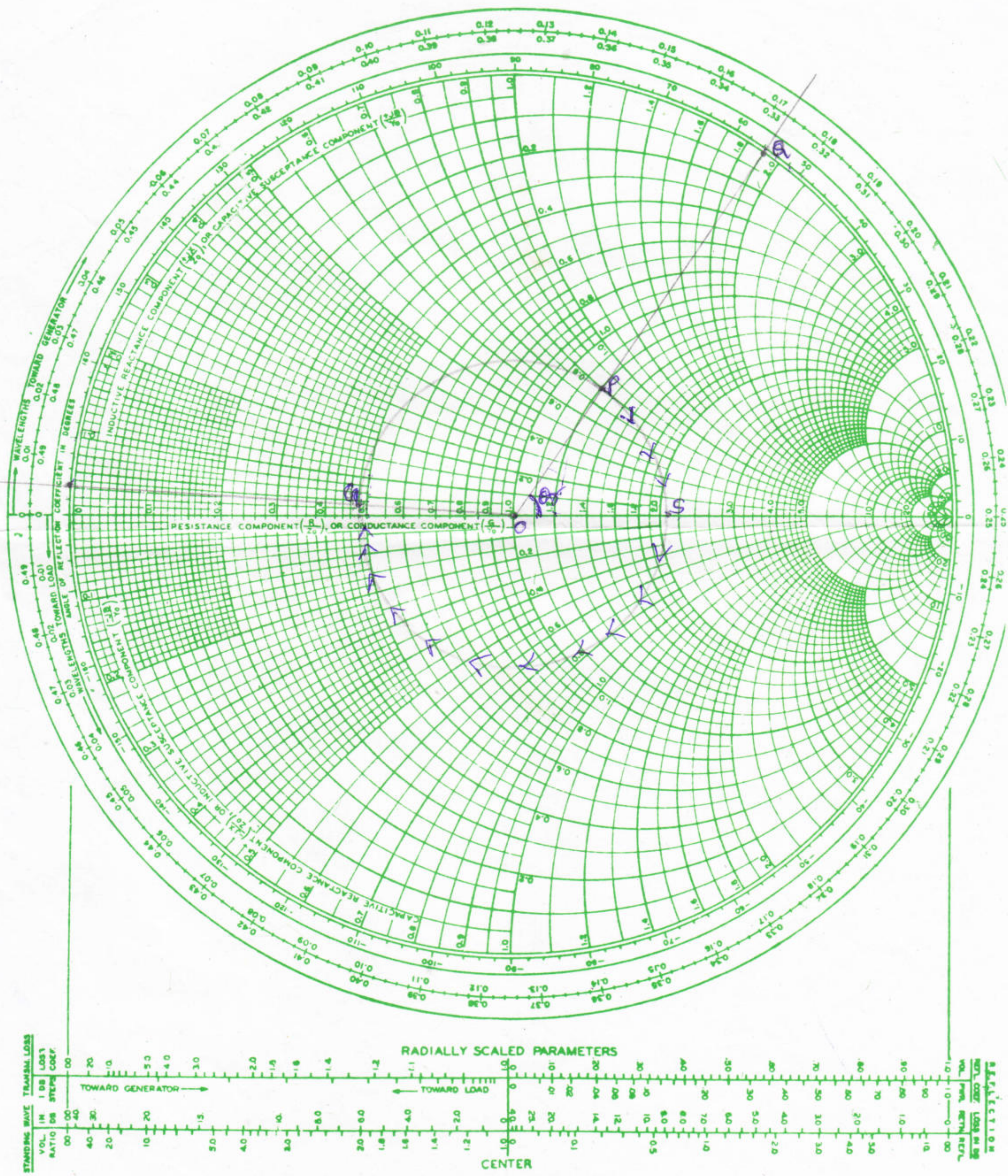
(e) The locations of V_{max} and V_{min} with respect to the load if the line is 0.6λ long

(f) Z_{in} at the generator.

(1)

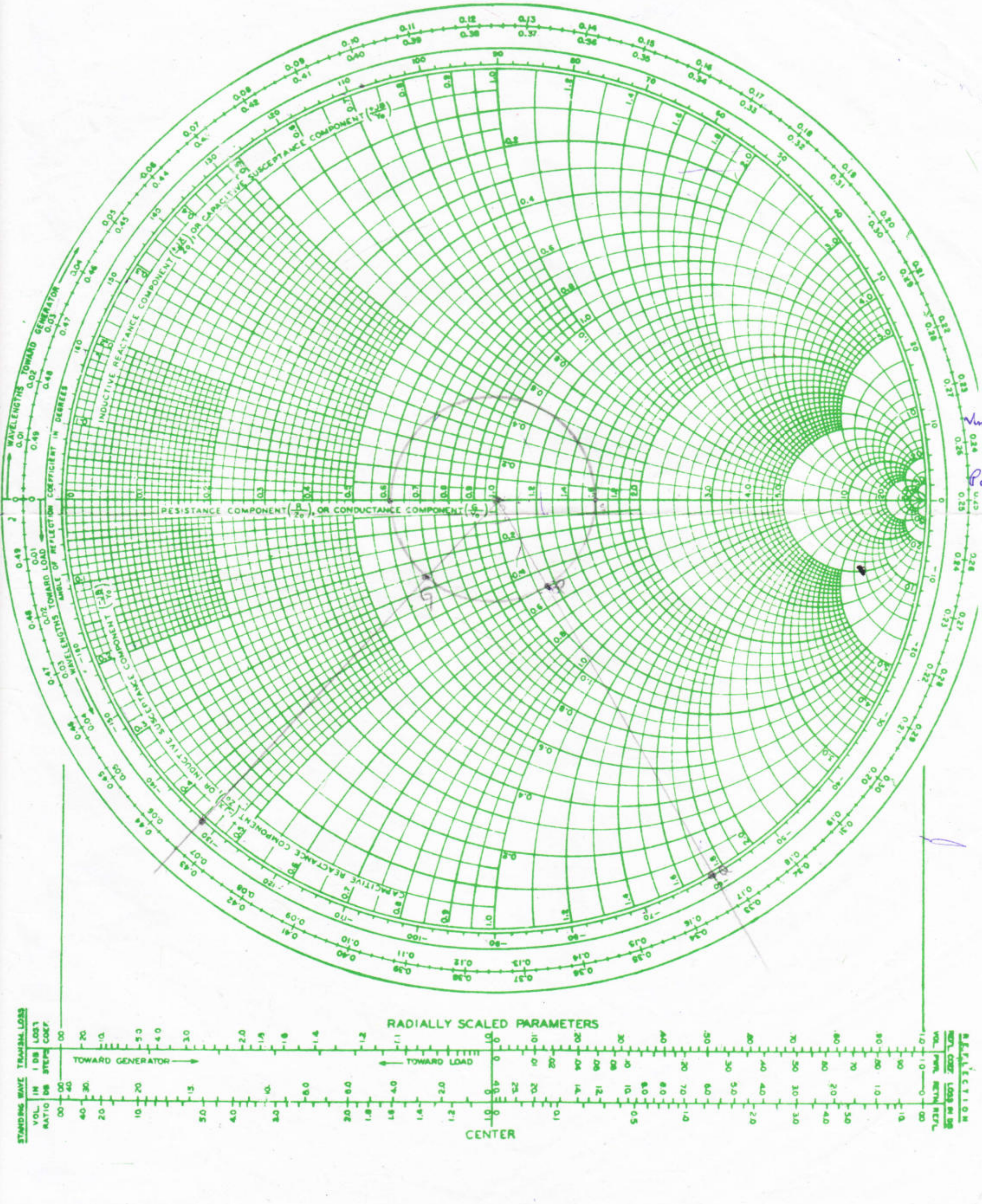
	SMITH-CHART	

IMPEDANCE OR ADMITTANCE COORDINATES



SMITH-CHART

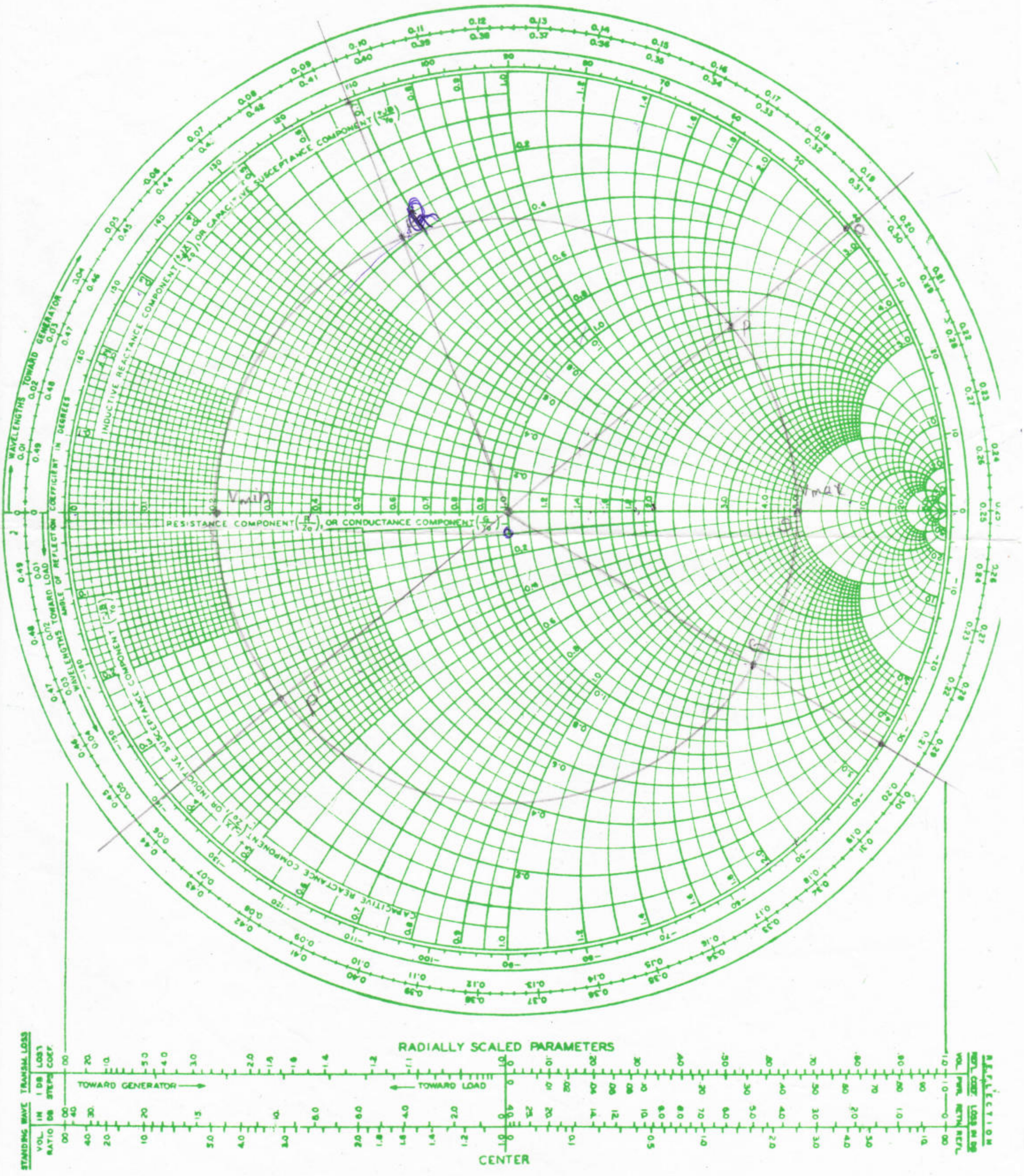
IMPEDANCE OR ADMITTANCE COORDINATES



(3) eg 10.5

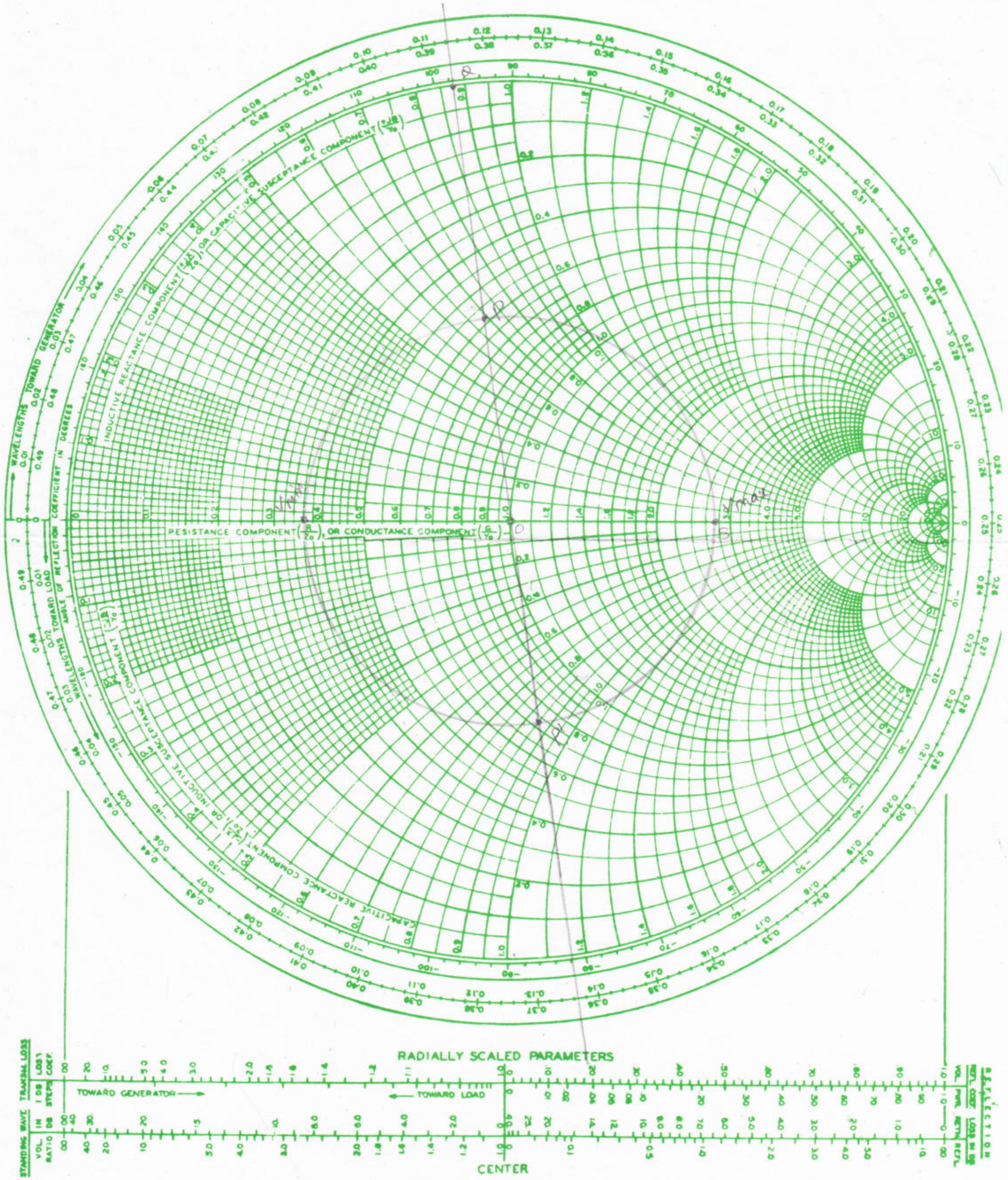
SMITH-CHART

IMPEDANCE OR ADMITTANCE COORDINATES



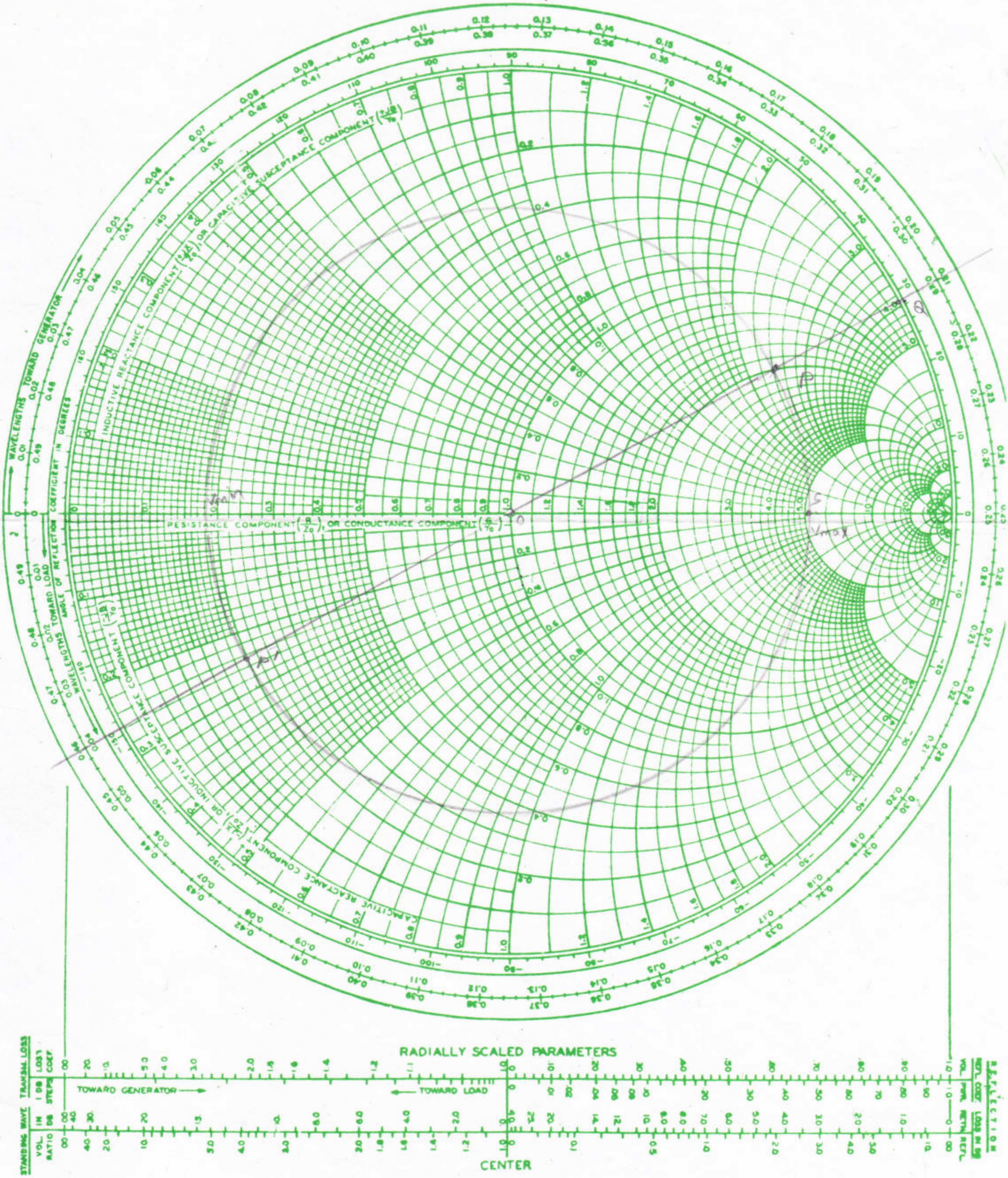
SMITH-CHART

IMPEDANCE OR ADMITTANCE COORDINATES



SMITH-CHART

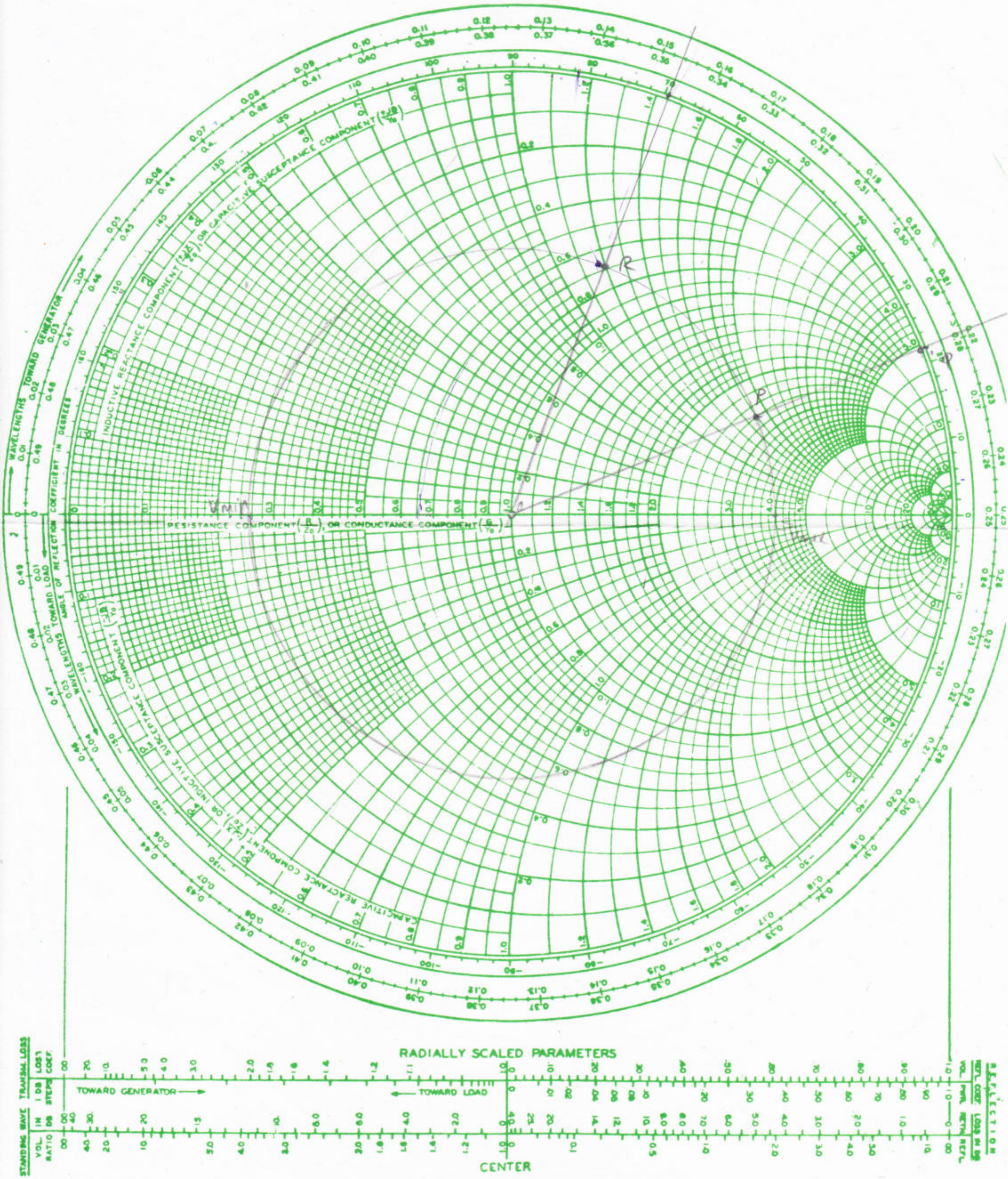
IMPEDANCE OR ADMITTANCE COORDINATES



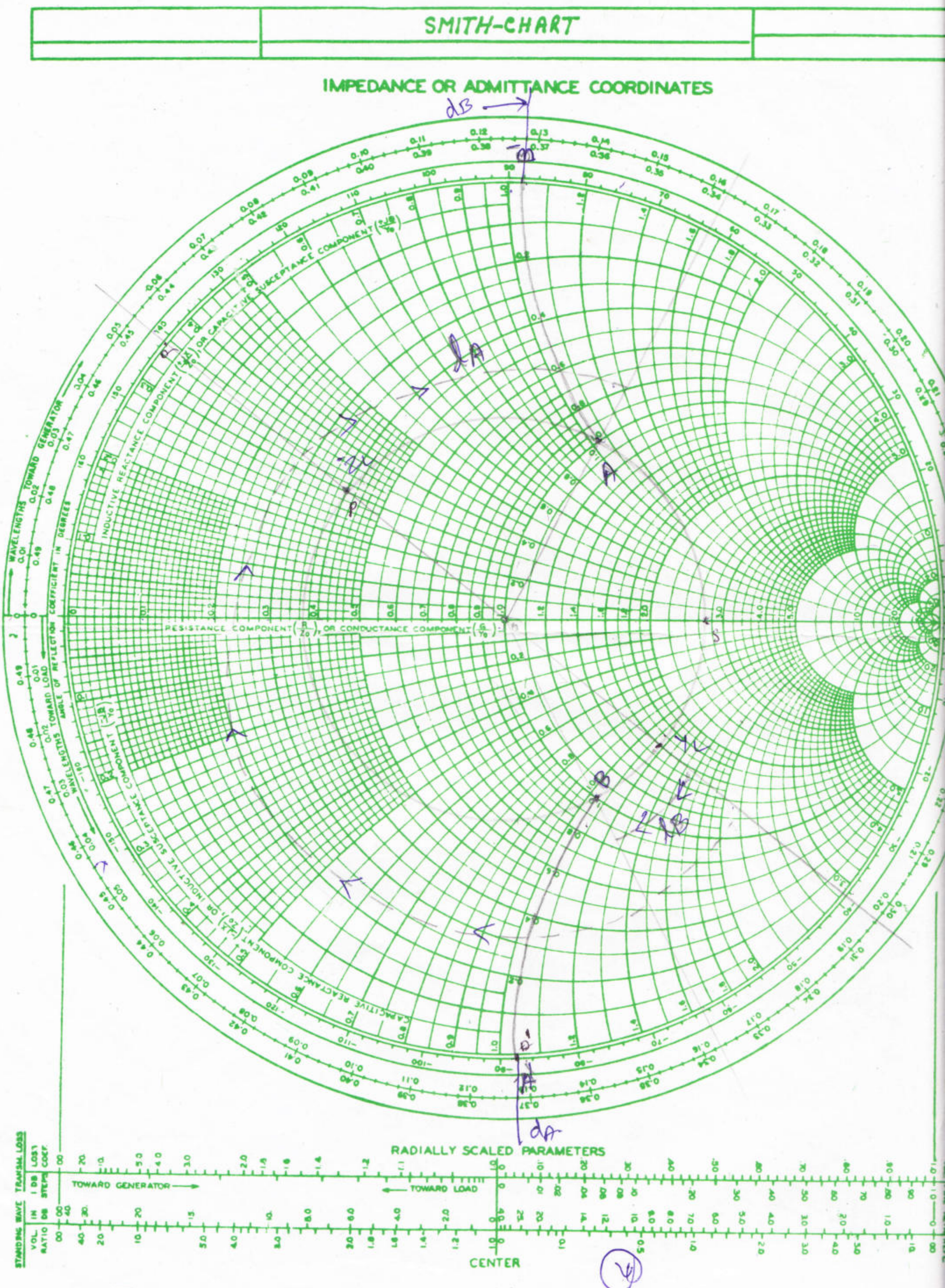
6

SMITH-CHART

IMPEDANCE OR ADMITTANCE COORDINATES



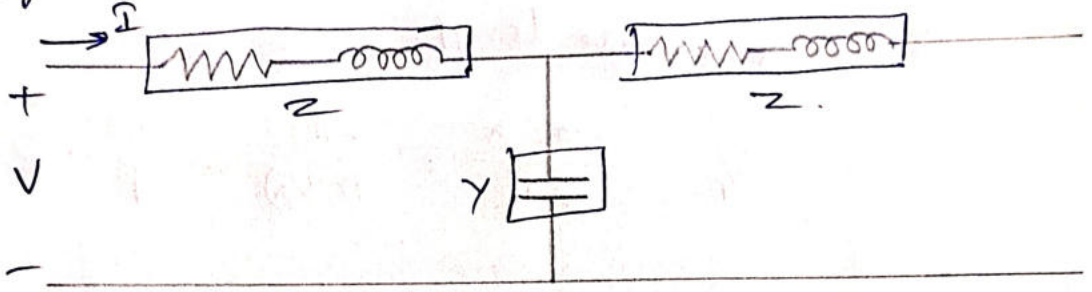
2 –
① stub


SMITH-CHART
IMPEDANCE OR ADMITTANCE COORDINATES
dB
dA
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
CENTER
STANDING WAVE
VOL. RATIO
IN DB
TRANSM. LOSS
1 DB STEPS
LOSS COEF.
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE COMPONENT (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
④

T & π Equivalent Circuits:-

In transmission line theory, the T- Equivalent & π Equivalent Circuits are simplified lumped models used to represent a small section of a transmission line using basic circuit Elements. (R, L, G, C).

*T- Equivalent Circuit:-

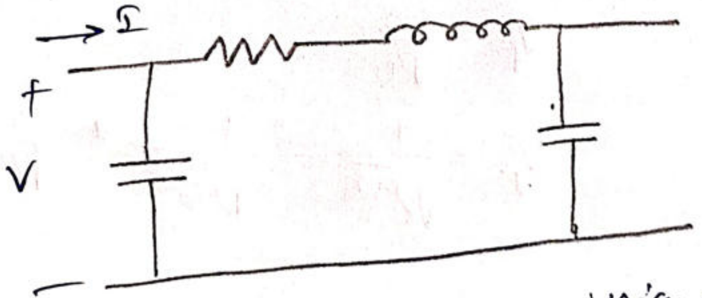


A small section of Transmission line of length Δx is represented as two series arms (R, L) & one shunt arm (G+C).

Series Impedance: $Z = R + j\omega L$

Shunt Admittance: $Y = G + j\omega C$

*(ii) π - Equivalent Circuit:-



A small section of Transmission line is represented as one series & two shunt arms.

Series Impedance: $Z = (R + j\omega L)\Delta x$

Shunt Admittance: $\frac{Y}{2} = (G + j\omega C)\frac{\Delta x}{2}$

Phase velocity (V_p):-

Phase velocity is defined as the rate of change in phase of the propagating wave.

$$V_p = \lambda_g / t$$

$$V_p = \lambda_g f \quad -(1)$$

$\lambda \rightarrow$ wave length.

$$V_p = \frac{2\pi f \lambda}{2\pi} = \frac{2\pi f}{(2\pi/\lambda)} = \frac{\omega}{\beta} \quad -(2) \quad \begin{pmatrix} \omega = 2\pi f \\ \beta = 2\pi/\lambda \end{pmatrix}$$

Group velocity (V_g):-

The rate of change in Energy of Electromagnetic waves along the axis of line is termed as group velocity.

$$V_g = \frac{d\omega}{d\beta} \quad -(3)$$

Relation between Phase velocity & Group velocity:-

$$V_p = \frac{\omega}{\beta}$$

$$V_p = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} * \frac{1}{\sqrt{\mu_r \varepsilon_r}}$$

$$= \frac{c}{\sqrt{\mu_r \varepsilon_r}} \quad \left[\because \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c\right]$$

$$V_p = \frac{c}{\eta}$$

Group velocity is given by,

$$V_g = \frac{d\omega}{d\beta}$$

$$= \frac{d}{d\beta}(\beta V_p) \qquad \left[\because V_p = \frac{\omega}{\beta}\right]$$

$$V_g = V_p + \frac{d}{d\beta}(\beta V_p) \quad -④ \text{ [differentiation]}$$

$$\beta = \frac{2\pi}{\lambda}.$$

differentiate w.r.t λ on both sides,

$$\frac{d\beta}{d\lambda} = -\frac{2\pi}{\lambda^2}$$

$$\frac{d\beta}{d\lambda} = -\frac{\beta}{\lambda}.$$

$$d\beta = -\beta\frac{d\lambda}{\lambda} \Rightarrow -\frac{d\lambda}{\lambda} = \frac{d\beta}{\beta}$$

Substituting the values of $d\beta$ in eq. ④,

$$V_g = V_p + \frac{dV_p}{d\lambda}(-\lambda)$$

$$V_g = V_p - \lambda\frac{dV_p}{d\lambda} \quad -⑤$$

Group velocity Can also be written as,

$$V_g = \frac{d}{d\beta}\left(\beta\frac{c}{\eta}\right)$$

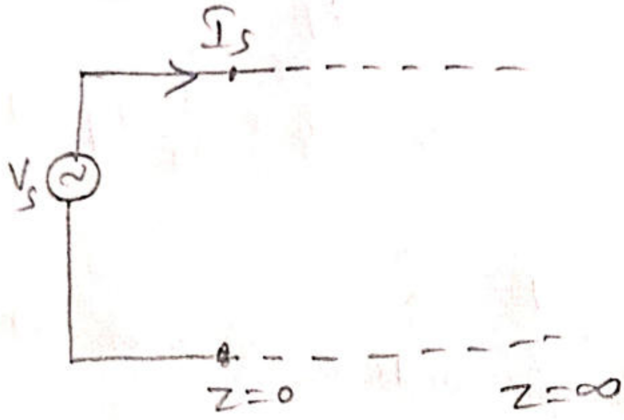
$$= \frac{c}{\eta} - \frac{c\beta}{\eta^2}\left(\frac{d\eta}{d\beta}\right) = \frac{c}{\eta}\left[1 - \frac{\beta}{\eta}\left(\frac{d\eta}{d\beta}\right)\right]$$

$$V_g = V_p\left[1 - \frac{\beta}{\eta}\frac{d\eta}{d\beta}\right] \quad -⑥$$

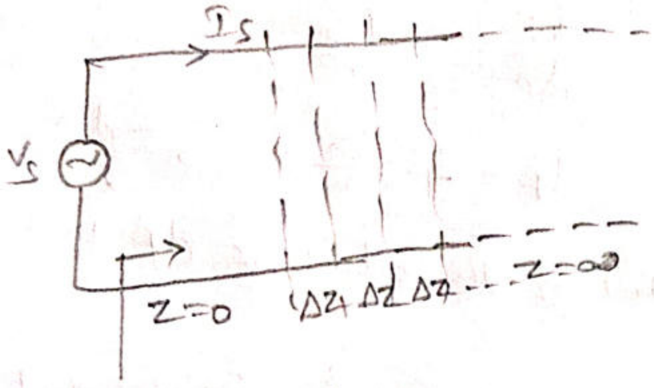
Eq.ns ⑤ & ⑥ are relations between phase & group velocities.

Infinite line :-

An infinite Transmission line is a theoretical line whose length is so large ($\to \infty$) that the wave never reaches the end, so no reflections occur.



Basically, an infinite line is formed by cascading a number of symmetrical sections.



Input impedance (Z_{in}) of an infinite line is Equal to Characteristic impedance. (Z_0). i.e., $Z_{in} = Z_0$. Because any section of an infinite line looks the same.

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right]$$

For, $l = \infty$, $\tanh(\gamma l) \Rightarrow 1$

$$\therefore Z_{in} = Z_0.$$

$\Gamma = 0$, $SWR = 1$, Forward wave, $V(x) = V^+ e^{-\gamma x}$

* Power Continuously absorbed along the line & no power returns back.

7.15 Low-loss Radio Frequency and UHF Transmission Lines. The low-loss transmission line is of special interest to the engineer concerned with the transmission of energy at radio and ultrahigh frequencies. There are two reasons for this. First, most practical lines designed for use at these frequencies will be low-loss lines. Second, at ultrahigh frequencies, sections of low-loss line are used as circuit elements, and a knowledge of the operation of such "distributed-constants circuits" is of considerable importance.

A *low-loss transmission line* is one for which

$$R \ll \omega L \qquad (7\text{-}105)$$

$$G \ll \omega C$$

where R, L, C, and G are the resistance, inductance, capacitance, and conductance per unit length of the line. When the above inequalities hold, the following approximations are valid:

$$Z = R + j\omega L \approx j\omega L$$

$$Y = G + j\omega C \approx j\omega C$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}} \qquad (7\text{-}106)$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \approx j\omega \sqrt{LC} \qquad (7\text{-}107)$$

Since $\gamma = \alpha + j\beta$, this last expression gives

$$\alpha \approx 0 \qquad (7\text{-}108)$$

$$\beta \approx \omega \sqrt{LC} \qquad (7\text{-}109)$$

The approximation for β is very good for low-loss lines, but occasionally the approximation of zero for α may not be good enough, even though α is very small compared with β. A closer approximation for α may be obtained by rearranging the expression for γ and using the binomial expansion. Thus

$$\gamma = j\omega \sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)}$$

$$\approx j\omega \sqrt{LC} \left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right)$$

$$\approx j\omega \sqrt{LC}\left(1 + \frac{R}{2j\omega L} + \frac{G}{2j\omega C}\right)$$

$$\approx \frac{R}{2\sqrt{L/C}} + \frac{G\sqrt{L/C}}{2} + j\omega \sqrt{LC}$$

which gives

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \tag{7-110}$$

$$\beta \approx \omega\sqrt{LC} \tag{7-111}$$

The more correct value for α given by (110) need only be used in place of (108) when the line losses are being considered. As far as voltage and current distributions are concerned, the attenuation of most low-loss ultrahigh frequency lines is so small that the approximation $\alpha = 0$ gives satisfactory results. This may seem strange in view of the fact that R, and therefore α, *increases* with frequency, and α is not usually neglected at low (power and audio) frequencies. The explanation for this apparent paradox is that although α, the attenuation *per unit length*, increases approximately as the square root of frequency, the attenuation *per wavelength* decreases as the square root of the frequency. Transmission lines are ordinarily a few wavelengths long at most, and αl can usually be neglected (compared with βl) at the ultrahigh frequencies. Thus for many purposes, low-loss lines may be treated as though they were lossless; that is, as if $R = G = \alpha = 0$.

Using the approximate values for the secondary constants given by (106), (107), (108), and (109), the general transmission line equations become for this low-loss, high-frequency case

$$V_S = V_R \cos\beta l + jI_R Z_0 \sin\beta l \tag{7-112}$$

$$I_S = I_R \cos\beta l + j\frac{V_R}{Z_0}\sin\beta l \tag{7-113}$$

where now $Z_0 \approx \sqrt{L/C}$ is a pure resistance.

The input impedance of such a line is

$$\begin{aligned} Z_S &= \frac{V_S}{I_S} \\ &= Z_R\left(\frac{\cos\beta l + j(Z_0/Z_R)\sin\beta l}{\cos\beta l + j(Z_R/Z_0)\sin\beta l}\right) \\ &= Z_0\left(\frac{Z_R\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_R\sin\beta l}\right) \end{aligned} \tag{7-114}$$

The voltage and current distributions along the line are obtained from eqs. (112) and (113) by replacing l, the length of line, by x, the distance from the terminating impedance Z_R. Since voltmeters and ammeters read magnitude without regard to phase, the magnitudes of expressions (112) and (113) have been used in Fig. 7-16 to show the standing-wave distributions for various conditions of the terminating impedance Z_R.

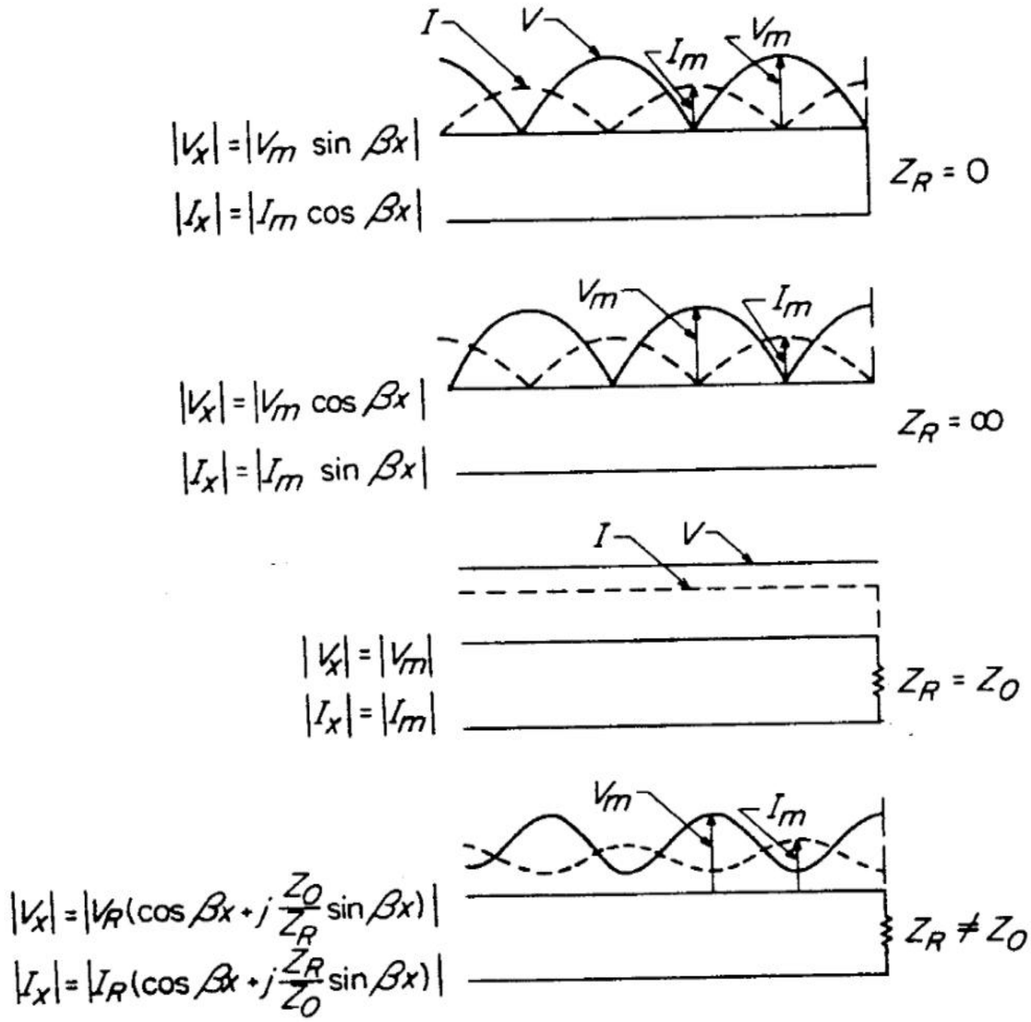


Figure 7-16. Voltage and current distribution along a lossless line.

In general the terminating impedance Z_R will be a complex impedance having both resistance and reactance, but it will be shown later that the results for the general case may be inferred from those obtained for the particular case of a pure-resistance termination. For this latter case where $Z_R = R$, eqs. (112) and (113) may be written as

$$|V_x| = V_R \sqrt{\cos^2 \beta x + (R_0/R)^2 \sin^2 \beta x} \tag{7-115}$$

$$|I_x| = I_R \sqrt{\cos^2 \beta x + (R/R_0)^2 \sin^2 \beta x} \tag{7-116}$$

For the lossless line being considered Z_0 is a pure resistance

$$Z_0 = R_0 = \sqrt{\frac{L}{C}}$$

Examination of eqs. (115) and (116) shows that the voltage and current distributions are given by the square root of the sum of a cosine-squared term and a sine-squared term. It is evident that the maximum value of voltage or current will occur at that value of x that makes the larger of these terms a maximum. In the particular case of a line terminated in R_0, that is, for which $R = R_0$, the sine and cosine terms have equal amplitudes and the square root of the sum of their squares has constant value for all values of x. That is, there are no standing waves on the line. For all other cases, however, the magnitude will vary along the length of the line. When R is less than R_0, the amplitude of the sine term of (115) will be larger than that of the cosine term and the voltage maxima will occur at those values of x that make $\sin \beta x$ a maximum, viz., at $x = \lambda/4$, $3\lambda/4$, and so on. Also the voltage minima will occur at those values of x that make the sine term a minimum, viz., $x = 0$, $\lambda/2$, and so on, also for this case of $R < R_0$, the *current* maxima will occur at $x = 0$, $\lambda/2$, and so on, and the current minima at $x = \lambda/4$, $3\lambda/4$, and so on. When the terminating resistor is larger than R_0, the conditions for both voltage and current are reversed.

One of the important measurable quantities on a transmission line is the standing-wave ratio of voltage or current. When R is less than R_0, eq. (115) shows that the voltage maximum, which occurs when $\sin \beta x = 1$, will have a value

$$V_{\max} = V_R \frac{R_0}{R}$$

Also the voltage minimum, which occurs when $\sin \beta x = 0$, will have a value

$$V_{\min} = V_R$$

The ratio of maximum voltage to minimum voltage is therefore

$$\frac{V_{\max}}{V_{\min}} = \frac{R_0}{R} \qquad (\text{for } R < R_0)$$

Similarly the standing wave of current ratio is given by

$$\frac{I_{\max}}{I_{\min}} = \frac{R_0}{R} \qquad (\text{for } R < R_0)$$

For $R > R_0$ these expressions are just reversed, that is

$$\frac{V_{\max}}{V_{\min}} = \frac{I_{\max}}{I_{\min}} = \frac{R}{R_0} \qquad (\text{for } R > R_0)$$

Using these expressions, the value of a terminating resistance may be determined in terms of R_0 from relative measurements of voltage or current along the line. R_0 is readily calculable from the line dimensions.

Case where Z_R is not a Pure Resistance. When the terminating impedance Z_R is *not* a pure resistance, standing-wave measurements can be still used, and in this case will yield values of both resistance and reactance of the termination. From eqs. (115) and (116) it was seen that with a resistance termination a voltage maximum or minimum always occurred right at the termination ($x = 0$). However, when the terminating impedance has reactance as well as resistance, the maximum or minimum is always displaced from the position $x = 0$, and the direction and amount of this displacement can be used to determine the sign and magnitude of the reactance of the load.

Figure 7-17 shows a transmission line terminated in an impedance

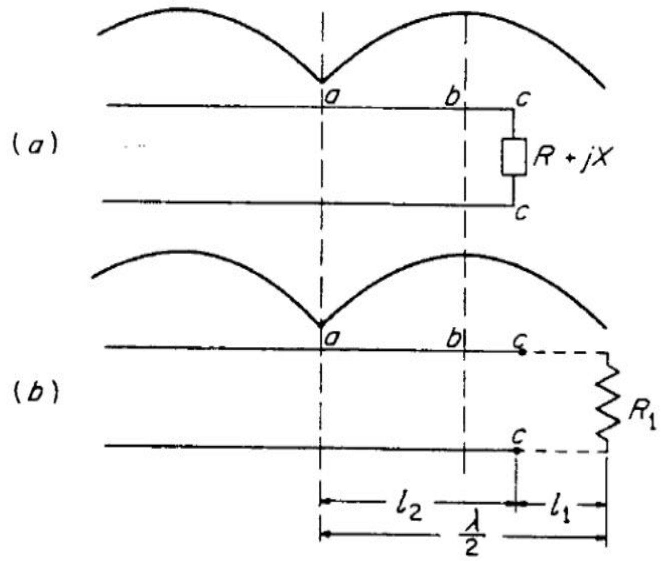


Figure 7-17. A complex terminating impedance in (*a*) is replaced by a pure resistance termination in (*b*).

that has a reactive component. The voltage distribution along the line is shown. Because the impedance is not a pure resistance, the voltage maximum (or minimum) does not occur at the termination. Now any complex impedance can be obtained by placing a pure resistance of proper value at the end of an appropriate length of (lossless) transmission line. In part (*b*) of Fig. 7-17, the complex impedance $R + jX$ has been replaced by the proper value of resistance R_1 at the end of a length l_1 of line, such that the impedance at *c-c* looking towards R_1 is equal to $R + jX$. The standing wave back from *c-c* toward the source will be unchanged and that toward R_1 will be just a continuation of it. Quite evidently the proper position for R_1 is at a distance of one-half wavelength from the minimum point *a* (or the maximum point *b* if R_1 is greater than R_0), and the proper value of R_1 is given by the standing-wave ratio on the line, that is, by

$$\frac{R_1}{R_0} = \frac{V_{min}}{V_{max}} \qquad \text{or} \qquad \frac{R_1}{R_0} = \frac{V_{max}}{V_{min}}$$

Because any resistance greater than R_0 can be obtained by a resistance less than R_0 at the end of a quarter-wave section of line (see below), it is really only necessary to consider for R_1 resistances less than or equal to R_0. It is then possible to state that any impedance whatsoever can be obtained by means of a pure resistance R_1 (not greater than R_0) at the end of a length l_1 of lossless transmission line, less than one-half wavelength long.

The value of the impedance $Z = R + jX$ is given in terms of R_1 and l_1 by eq. (114). Rationalizing and separating into real and imaginary parts, eq. (114) becomes

$$R = \frac{R_0^2 R_1}{R_0^2 \cos^2 \beta l_1 + R_1^2 \sin^2 \beta l_1} \tag{7-117}$$

$$X = \frac{R_0(R_0^2 - R_1^2) \sin \beta l_1 \cos \beta l_1}{R_0^2 \cos^2 \beta l_1 + R_1^2 \sin^2 \beta l_1} \tag{7-118}$$

Equations (117) and (118) make it possible to determine both the resistance and reactance values of a terminating impedance from standing-wave measurements on the transmission line. The sign of the reactance, that is, whether inductive (positive) or capacitive (negative) can be obtained by inspection as shown in Fig. 7-18.

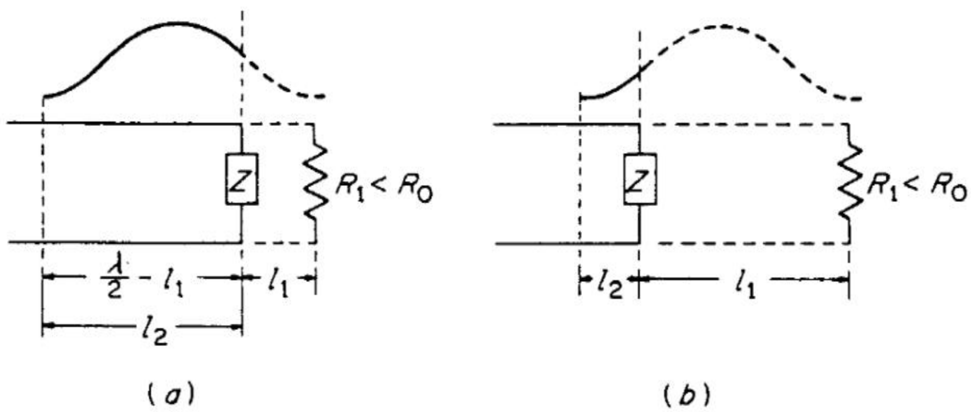


Figure 7-18. A terminating impedance that is inductive (*a*) or capacitive (*b*).

Considering the value of R_1 to be less than R_0, eq. (118) shows that when l_1 is less than one-quarter wavelength, the reactance X is positive (i.e., inductive), whereas if l_1 is between one-quarter and one-half wavelength, X will be negative (capacitive). From this results the conclusion that, if the standing wave of voltage slopes down toward the terminating impedance [Fig. 7-18(*a*)], the impedance is inductive; if the slope is up toward the impedance [Fig. 7-18(*b*)], the impedance is capacitive. Of course, if the slope is zero at the termination, the terminating impedance is a pure resistance.

In practice the measurable quantities are l_2, the distance from the termination to the minimum point a, and the *standing-wave ratio*

$$\rho = \frac{R_0}{R_1} = \frac{V_{max}}{V_{min}}$$

In terms of these measurable quantities, the resistance and reactance of the terminating impedance are given by

$$R = \frac{\rho R_0}{\rho^2 \cos^2 \beta l_2 + \sin^2 \beta l_2} \tag{7-119}$$

$$X = \frac{-R_0(\rho^2 - 1) \sin \beta l_2 \cos \beta l_2}{\rho^2 \cos^2 \beta l_2 + \sin^2 \beta l_2} \tag{7-120}$$

7.16 UHF Lines as Circuit Elements. The transfer of energy from one point to another is only one use of transmission lines. At the ultrahigh frequencies an equally important application is the use of sections of lines as circuit elements. Above 150 MHz the ordinary lumped-circuit elements become difficult to construct and, at the same time, the required physical size of sections of transmission lines has become small enough to warrant their use as circuit elements. They can be used in this manner up to about 3000 MHz where their physical size then becomes too small and wave-guide technique begins to take over.

In Fig. 7-19 are shown some line sections and their low-frequency equivalents. The magnitude of the input reactance of the first four of these sections is given by eq. (114) when the appropriate value of Z_R is inserted; that is, $Z_R = 0$ for the shorted sections and $Z_R = \infty$ for the open sections. The resistive component of the input impedance is negligible for the usual low-loss lines used at UHF. Thus it is seen that for line lengths less than a quarter of a wavelength the shorted section is equivalent to an inductance, and the open section to a capacitance. For lengths of line between a quarter and a half wavelength, the shorted section is capacitive and the open section is inductive. However, it should be noted that unlike their low-frequency equivalents, these "inductances" and "capacitances" change value with frequency.

The Quarter-wave and Half-wave Sections. For the particular case of the shorted quarter-wave line or the open half-wave line, the input reactance, given by (114), goes to infinity, and the resistive component of the input impedance must be taken into account. This corresponds

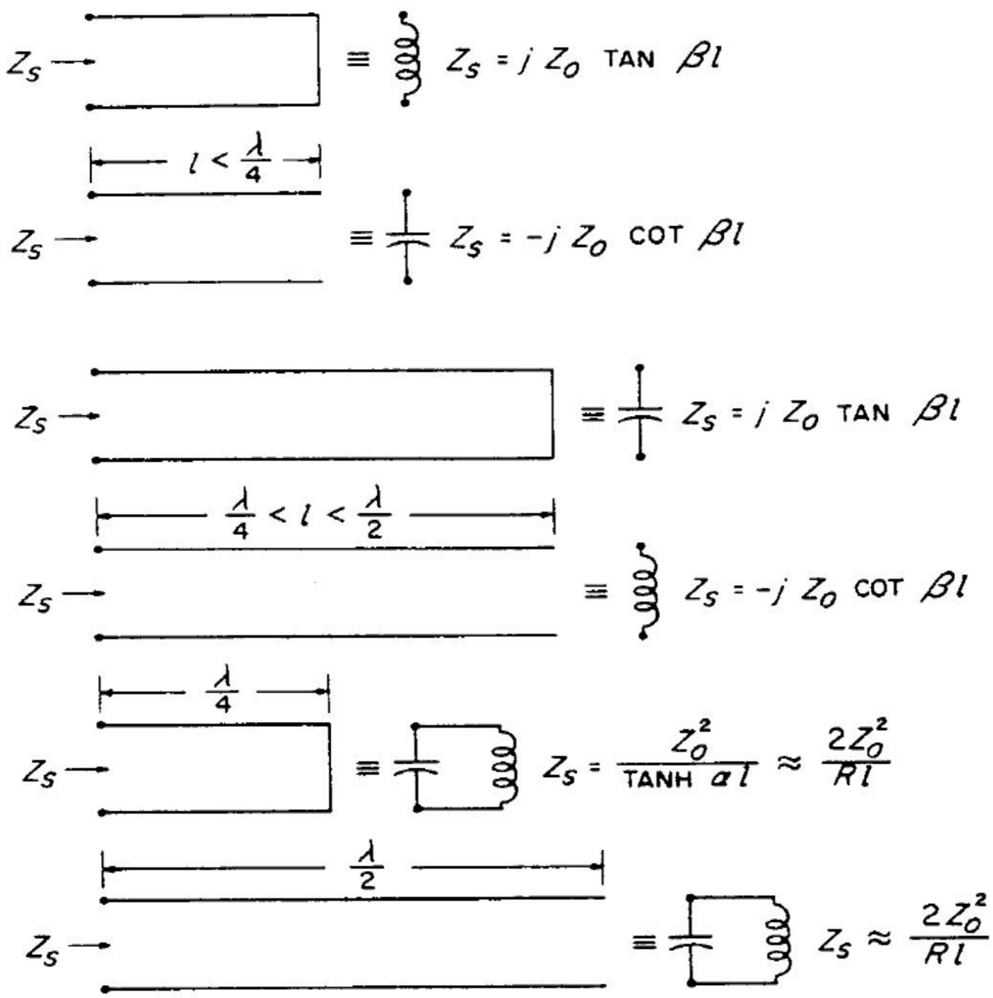


Figure 7-19. Input impedance of various transmission-line sections.

to conditions in the parallel-resonant circuit (the low-frequency analogue), which has an infinite impedance if resistance is neglected. In both cases (the quarter-wave line and the parallel-resonant circuit) the actual input impedance when the series resistance is not neglected is a pure resistance of very high value. In the case of the line its value is given approximately by

$$R_{ar} = \frac{2Z_0^2}{Rl},$$

where R_{ar} is the input resistance of the line at a resonant length and R is the series resistance per unit length of the line. l is the length of the resonant section, which will be an odd multiple of a quarter wavelength for a shorted line or an even multiple of one-quarter wavelength for an open line. This expression is obtained directly from eqs. (100) and (101) in which the actual line loss is not neglected as follows:

For a shorted line for which $V_R = 0$, eqs. (100) and (101) become

$$V_S = I_R Z_0 \sinh \gamma l$$
$$I_S = I_R \cosh \gamma l$$

Dividing the voltage equation by the current equation gives the input impedance of a short-circuited line as

$$\begin{aligned} Z_S &= Z_0 \tanh \gamma l \\ &= Z_0 \frac{\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l}{\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l} \end{aligned}$$

For line lengths that are an odd multiple of a quarter wavelength, $\sin \beta l = \pm 1$ and $\cos \beta l = 0$. Under these conditions the input impedance becomes

$$Z_S = Z_0 \frac{\cosh \alpha l}{\sinh \alpha l}$$

If αl is very small, as is generally true for sections of low-loss line, $\cosh \alpha l \approx 1$ and $\sin \alpha l \approx \alpha l$ so that

$$Z_S \approx \frac{Z_0}{\alpha l}$$

When $\omega L \gg R$ and $\omega C \gg G$, α is given in terms of the line constants by

$$\alpha = \frac{1}{2} \left(R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right) \tag{7-110}$$

For the air-dielectric lines commonly used the losses due to the conductance G are negligible, so that G can be neglected and

$$\alpha = \frac{R}{2} \sqrt{\frac{C}{L}} = \frac{R}{2Z_0} \tag{7-121}$$

Substituting this in the above expression for input impedance of a short-circuited line, whose length is an odd multiple of a quarter wavelength, gives

$$Z_S = \frac{Z_0}{\alpha l} = \frac{2Z_0^2}{Rl} \tag{7-122}$$

An identical expression is obtained for an open-ended section that is a multiple of a half-wave long.

Resonance in Line Sections. The shorted quarter-wave section has other properties of the parallel-resonant circuit. It is a *resonant* circuit and produces the resonant rise of voltage or current which exists in such circuits. The mechanism of resonance is particularly easy to visualize in this case. If it is assumed that a small voltage is induced into the line near the shorted end, there will be a voltage wave sent down the line and reflected without change of phase at the open end. This reflected wave travels back and is reflected again at the shorted end with reversal of phase. Because it required one-half cycle to travel up and back the line, this twice-reflected wave now will be in phase with the original induced voltage and so adds directly to it. Evidently those additions continue to increase the voltage (and current) in the line until the I^2R loss is equal to the power being put into the line. A voltage step-up of several hundred times is possible depending upon the Q of the line.

Input Impedance of the Tuned Line. When the quarter-wave section is tapped at some point x along its length, a further correspondence between this circuit and the simple low-frequency parallel-resonant circuit is observed. The reactance looking toward the shorted end will be inductive and of value $Z_{sc} = jZ_0 \tan \beta x$. The reactance looking toward the open end will be of equal magnitude but opposite sign, i.e., a capacitive reactance. Its value is given by $Z_{oc} = -jZ_0 \cot \beta(\lambda/4 - x) = -jZ_0 \tan \beta x$. The equal but opposite reactances are in parallel just as they are in Fig. 7-20(*b*) and the input impedance will be

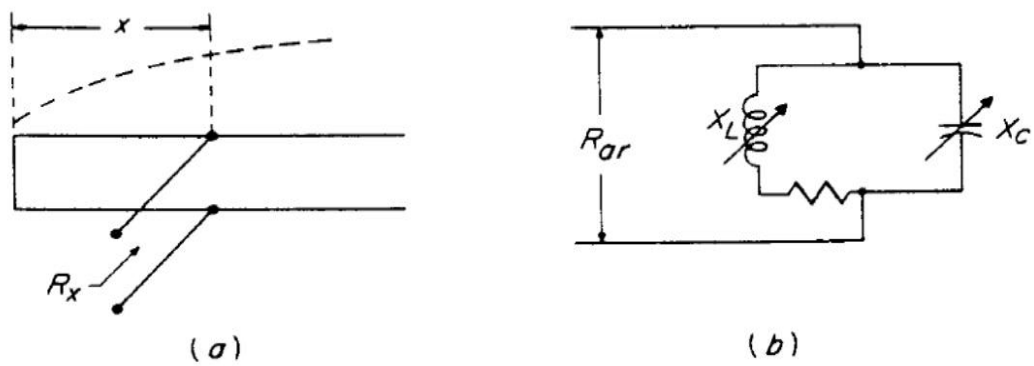


Figure 7-20. (*a*) Tapped quarter-wave line and (*b*) its equivalent circuit.

purely resistive. As the tap point is moved from the shorted end toward the open end of the line, the impedance seen at the tap point is a pure resistance that varies from zero to the quite high value already given ($R_S = 2Z_0^2/Rl$). This corresponds in the circuit of Fig. 7-20(*b*) to varying the reactances X_L and X_C from low to high values, meanwhile always keeping the circuit tuned (i.e., $X_L = X_C$).

It is of interest to know how the input resistance varies as the tap point is moved along the quarter-wave section. For the relatively high Q circuits used in such applications the voltage distribution along the line may be considered sinusoidal and it is a simple matter to determine the input resistance at any point a distance x from the shorted end. For a given magnitude of voltage and current on the quarter-wave section a certain fixed amount of power input will be required to supply the I^2R losses, regardless of where this power is fed in. This power input is equal to

$$\frac{V_S^2}{R_S} = \frac{V_S^2 Rl}{2Z_0^2}$$

where V_S and R_S are, respectively, the voltage and input resistance at the open end of the section. When the tap point of the feed line is at a distance x from the shorted end [Fig. 7-20(a)], the power input is given by V_x^2/R_x, where R_x is the input resistance at the point x. V_x is the voltage at this point and equals $V_S \sin \beta x$. Therefore

$$\frac{V_x^2}{R_x} = \frac{V_S^2 \sin^2 \beta x}{R_x} = \frac{V_S^2 Rl}{2Z_0^2}$$

which gives

$$R_x = \frac{2Z_0^2}{Rl} \sin^2 \beta x$$

Thus the input resistance varies as the square of the sine of the angular distance from the shorted end.

Q of Resonant Transmission-line Sections. One of the important properties of any resonant circuit is its selectivity or its ability to pass freely some frequencies, but to discriminate against others. The selectivity of a resonant circuit may be conveniently stated in terms of the ratio $\Delta f/f_0$, where f_0 is the resonant frequency and $\Delta f = f_2 - f_1$ is the frequency difference between the "half-power" frequencies. In the case of a series-resonant circuit $\Delta f/2$ represents the amount the frequency must be shifted away from the resonant frequency in order to reduce the current to 70.7 per cent of I_0, its value at the resonant frequency. (A constant voltage source is assumed.) Evidently this occurs when the reactance of the circuit becomes equal to the resistance and the phase angle of the circuit is 45 degrees. For the parallel-resonant case $\Delta f/2$ represents the frequency shift away from unity power factor resonance necessary to reduce the voltage across the parallel circuit to 70.7 per cent of its value at resonance. (A constant current source is assumed.) This occurs when the absolute magnitude of the impedance is 70.7 per cent of the impedance at resonance.

The ratio $f_0/\Delta f$ may be used to define the Q of a resonant circuit. The Q of a resonant transmission-line section can be determined as follows:

The input impedance of any shorted line section is given by

$$Z_S = Z_0 \tanh \gamma l$$

$$= Z_0 \frac{\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l}{\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l}$$

When the frequency is a resonant frequency f_0, then $\beta l = n\pi/2$ (where n is an odd integer), $\cos \beta l = 0$ and $\sin \beta l = \pm 1$. The expression for the input impedance becomes

$$Z_S = Z_0 \frac{\cosh \alpha l}{\sinh \alpha l} = \frac{Z_0}{\tanh \alpha l} \approx \frac{Z_0}{\alpha l}$$

When the frequency is shifted off resonance by a small amount δf, that is when $f = f_0 + \delta f$, then

$$\beta l = \frac{2\pi f}{v} l = \frac{2\pi(f_0 + \delta f)}{v} l = \frac{n\pi}{2} + \frac{2\pi\ \delta f l}{v}$$

Under these conditions (with $n = 1$)

$$\cos \beta l = -\sin \frac{(2\pi\ \delta f l)}{v}$$

$$\sin \beta l = \cos \frac{(2\pi\ \delta f l)}{v}$$

and

$$Z_S = Z_0 \frac{-\sinh \alpha l \sin \left(\frac{2\pi\ \delta f l}{v}\right) + j \cosh \alpha l \cos \left(\frac{2\pi\ \delta f l}{v}\right)}{-\cosh \alpha l \sin \left(\frac{2\pi\ \delta f l}{v}\right) + j \sinh \alpha l \cos \left(\frac{2\pi\ \delta f l}{v}\right)}$$

For moderately high Q circuits the first term in the numerator is the product of two small quantities and may be neglected in comparison with other terms. Putting

$$\cosh \alpha l \approx 1, \qquad \sinh \alpha l \approx \alpha l, \qquad \cos \left(\frac{2\pi\ \delta f l}{v}\right) \approx 1,$$

$$\sin \left(\frac{2\pi\ \delta f l}{v}\right) \approx \left(\frac{2\pi\ \delta f l}{v}\right)$$

gives

$$Z_S = \frac{Z_0}{\alpha l + j \left(\frac{2\pi\ \delta f l}{v}\right)}$$

When the imaginary term in the denominator is equal to the real term, the impedance Z_S will be 70.7 per cent of its value for a resonant length, and the frequency shift required to make this true will be $\Delta f/2$. Therefore

$$\frac{2\pi\ \Delta f l}{2v} = \alpha l$$

$$\Delta f = \frac{\alpha v}{\pi} = \frac{2\alpha f_0}{\beta}$$

The Q of the resonant section is

$$Q = \frac{f_0}{\Delta f} = \frac{\beta}{2\alpha} \qquad (7\text{-}123)$$

Alternative forms of this expression are

$$Q = \frac{\pi f_0}{\alpha v} = \frac{2\pi f_0 Z_0}{Rv} = \frac{\omega L}{R} \tag{7-124}$$

The Q is independent of the number of quarter wavelengths in the resonant section as long as αl is a small quantity. It is interesting to observe that the Q of a resonant section of transmission line is equal to the ratio of inductive reactance per unit length to resistance per unit length.

A similar analysis could be carried through for an open-ended resonant section (for which the length would be some multiple of a half wave-length). The expression for Q in this case would be identical with the above.

The Quarter-wave Line as a Transformer. When a section of transmission line is used as a reactance, or as a resonant circuit, it is a two-terminal network. The input terminals of the section are connected across the generator or load and the other terminals are left open or shorted as the case may be. However, a section of line is often used as a four-terminal network, in which case it is inserted in series between generator and load. Because the input impedance is in general different from the load impedance connected across the output terminals, the line section is an impedance-transforming network. This is true for all lengths of line, but the quarter-wave section has certain particular properties that make it very useful in this respect.

For any impedance termination Z_R, the input impedance of a section of lossless line is given by eq. (114) as

$$Z_S = Z_R\left(\frac{\cos \beta l + jZ_0/Z_R \sin \beta l}{\cos \beta l + jZ_R/Z_0 \sin \beta l}\right)$$

For the particular case of a quarter-wave section, $\beta l = \pi/2$, and this reduces to

$$Z_S = \frac{Z_0^2}{Z_R}$$

For the case under consideration, where Z_0 is a pure resistance R_0 this is

$$Z_S = \frac{R_0^2}{Z_R} \tag{7-125}$$

Thus the quarter-wave section is an impedance transformer, or more correctly an impedance inverter. Whatever the terminating impedance may be, the inverse impedance will appear at the input. If the output impedance consisted of a resistance R_2 in series with an inductive reactance X_{L_2}, the input impedance would be given by a resistance R_1 in *parallel* with a capacitive reactance X_{C_1}, where

$$R_1 = \frac{R_0^2}{R_2} \qquad \text{and} \qquad X_{C_1} = \frac{R_0^2}{X_{L_2}}$$

A pure resistance termination R is transformed into a pure resistance of value R_0^2/R.

This property of matching any two impedances Z_1, Z_2 such that $Z_1 Z_2 = Z_0^2$ finds many practical applications. It can be used to join together, without impedance mismatch, lines having different characteristic impedances; it is only necessary to make the characteristic impedance of the quarter-wave matching section the geometric mean of the Z_0's to be matched. By means of the quarter-wave section a pure resistance load can be matched to a generator having a generator impedance that is resistive so long as the geometric mean between the resistances gives a value for the required characteristic impedance that is practicable to obtain.

Voltage Step-up of the Quarter-wave Transformer. As long as the quarter-wave transforming section is considered as being lossless, the ratio between input and output voltages will just be the square root of the ratio of the input and output impedances being matched. From the voltage equation (112), for the quarter-wave section

$$\frac{V_S}{V_R} = \frac{jI_R Z_0}{V_R} = \frac{jZ_0}{Z_R} = j\sqrt{\frac{Z_S}{Z_R}}$$

or calling V_R/V_S the voltage step-up

$$\left|\frac{V_R}{V_S}\right| = \sqrt{\frac{Z_R}{Z_S}}$$

For the infinite impedance termination, that is an open circuit, this simple relation indicates an infinite voltage step-up, and it becomes necessary to resort to the exact eqs. (100) and (101) for the correct answer in this case. For the quarter-wave section the voltage equation of (100) becomes

$$V_S = jV_R \sinh \alpha l + jI_R Z_0 \cosh \alpha l$$

In open circuit I_R is zero and the voltage step-up is

$$\left|\frac{V_R}{V_S}\right| = \frac{1}{\sinh \alpha l} \approx \frac{1}{\alpha l} = \frac{2Z_0}{Rl}$$

For the quarter-wave section this may be written

$$\left|\frac{V_R}{V_S}\right| = \frac{8Z_0}{R\lambda} = \frac{8Z_0 f}{Rv}$$

while for a three-quarter-wave section the voltage step-up would be

$$\left|\frac{V_R}{V_S}\right| = \frac{8Z_0 f}{3Rv}$$