UNIT - 1

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Basics of Control Systems:-

System:- A system is an arrangement or combination of different physical components such that it gives the proper output to given input.

ex: A kite is an example of a physical system, because it is made up of paper and stics.

Control:-

The meaning of control is to regulate, direct or commar a system so that a desired objective is obtained.

plant:-

It is defined as the portion of a system which is to be controlled or regulated. It is also called as a process.

Controller:-

It is the element of the system itself, or may be external to the system, It controls the plant or the process.

Input:-

The applied signal or excitation signal that is applied to a control system to get a specified output is called input.

Output:-

The actual response that is obtained from a control sys due to the application of the i/p.

Disturbance:-

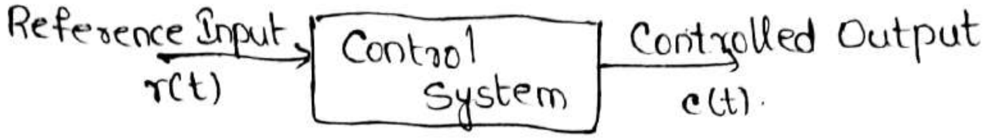
The s/l that has some adverse effect on the value of the o/p of a system is called disturbance.

* If disturbance is produced within the system, it is termed as an internal disturbance.
* Otherwise, it is known as an external disturbance.

Control systems:-

It is an arrangement of different physical components such that it gives the desired o/p for the given i/p by means of regulate or control either direct or indirect method.

A control system must have (i) Input ② Output, ③ Ways to achieve input & o/p objectives & (iv) Control action.



⇒ Any system can be characterized mathematically by

(i) Transfer function.

(ii) State model.

$$\text{Transfer function} = \left.\frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}\right|_{\text{Initial Conditions}=0}.$$

$$= \left.\frac{L[c(t)]}{L[r(t)]} = \frac{C(S)}{R(S)}\right|_{\text{Initial Condition}=0}.$$

* Transfer function is also called as the impulse response of the system.

$$\text{Impulse response (IR)} = L^{-1}[TF]$$

$$\boxed{TF = L[IR]}$$

ex① The IR of a system is $(e^{-t} - e^{-2t})\,u(t)$, then find the TF?

sol

$$TF = L[IR] = \frac{1}{S+1} - \frac{1}{S+2}$$

$$= \frac{1}{(S^2+3S+2)}$$

UNIT-1 : INTRODUCTION

System :-

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Control System :-

Whenever the output quantity is controlled by varying input quantity is called control system.

- The output quantity is called controlled variable or response.
- Input quantity is called command signal or excitation.

Types of control system :-

1. Based on feedback. — 1. open loop control system.
   2. Closed loop control system.

2. Based on analysis & design of circuit

(i) Linear Control system.

(ii) Non-linear Control system.

3. Based on system parameters

(i) Time variant control system

(ii) Time invarient control system.

4. Based on signal variation

(i) Continuous signals

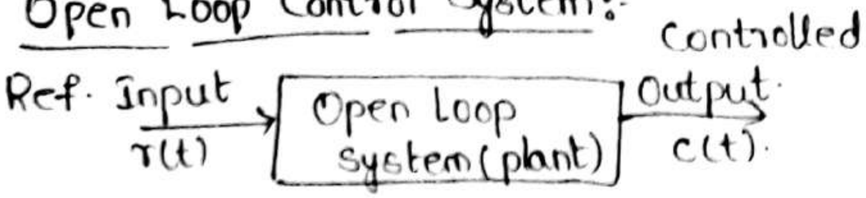
(ii) Discontinuous signals.

5. Based on Number of input's

(i) single i/p single o/p (SISO)

(ii) Multi i/p Multi o/p (MIMO).

Open Loop Control System:-



Definition:- The system in which the o/p quantity has no eff
on i/p quantity then the system is called open loop Control system.

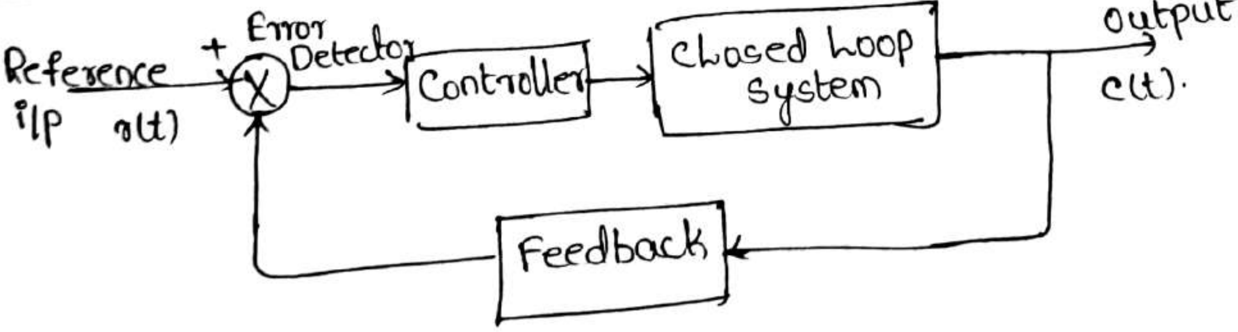
Advantages of open loop Control System:-

1) The open loop systems are easier to Construct.
2) Open loop System is simple and economical.
3) Open loop system is stable.

Disadvantages of open loop Control system:-

i) The o/p of open loop C.S are inaccurate and Unreliable.
ii) The changes in the o/p due to external disturbances are not corrected automatically.

Closed Loop Control system (CLCS):-



Definition:- The system in which the o/p quantity has an effect upon the i/p quantity then that system is called closed loop Control system.

* The closed loop Control system is also called as automatic control system.

Advantages of closed loop Control system:-

(i) The o/p is accurate.

(ii) The o/p is accurate even in the presence of non-linearities.

(iii) The closed loop systems are less affected by noise.

Disadvantages of closed loop Control system:-

(i) The closed loop systems are complex & costly.

(ii) The feedback in closed loop system may lead to oscillatory response.

(iii) The feedback reduces the overall gain of the system.

Comparision of open-loop and closed loop Control system:-

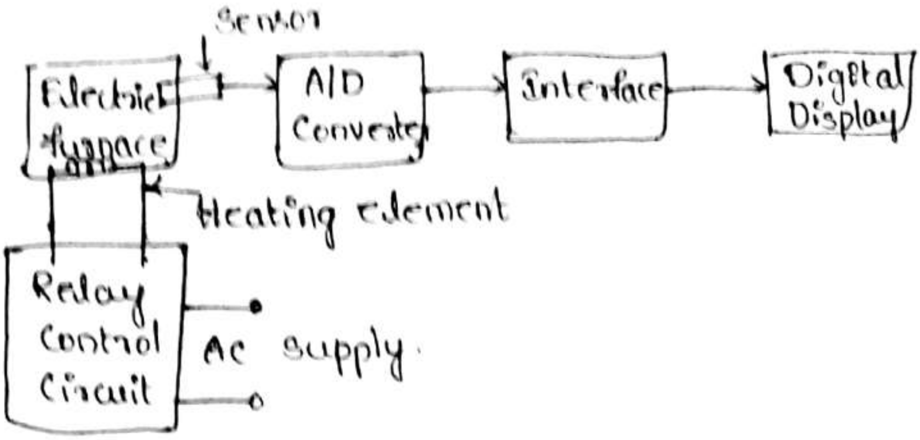
Open-Loop C.S	Closed-Loop C.S.
1. The accuracy of an open-loop sys depends on the calibration of the input. Any departure from pre-determined calibration affects the output. (Inaccurate & unreliable)	1. As the error between the reference input and the output is continuously measured through feedback, the closed-loop system works more accurately.
2. The open-loop system is simple to construct and cheap.	2. The closed-loop system is complicated to construct & costly.
3. The open-loop systems are generally stable.	3. The closed-loop systems can become unstable under certain conditions.
4. The operation of open-loop system is affected due to presence of non-linearity's in its elements. (automatic correction is not occur)	4. In terms of the performance, the closed loop systems adjusts to the effects of non-linearitys present in its elements. (automatic correction is occur).

Examples of Control systems:

Openloop Control system: Traffic lights, fans, any system which is not having the sensor.

* Temperature control system, Numerical control system

Temperature Control system:

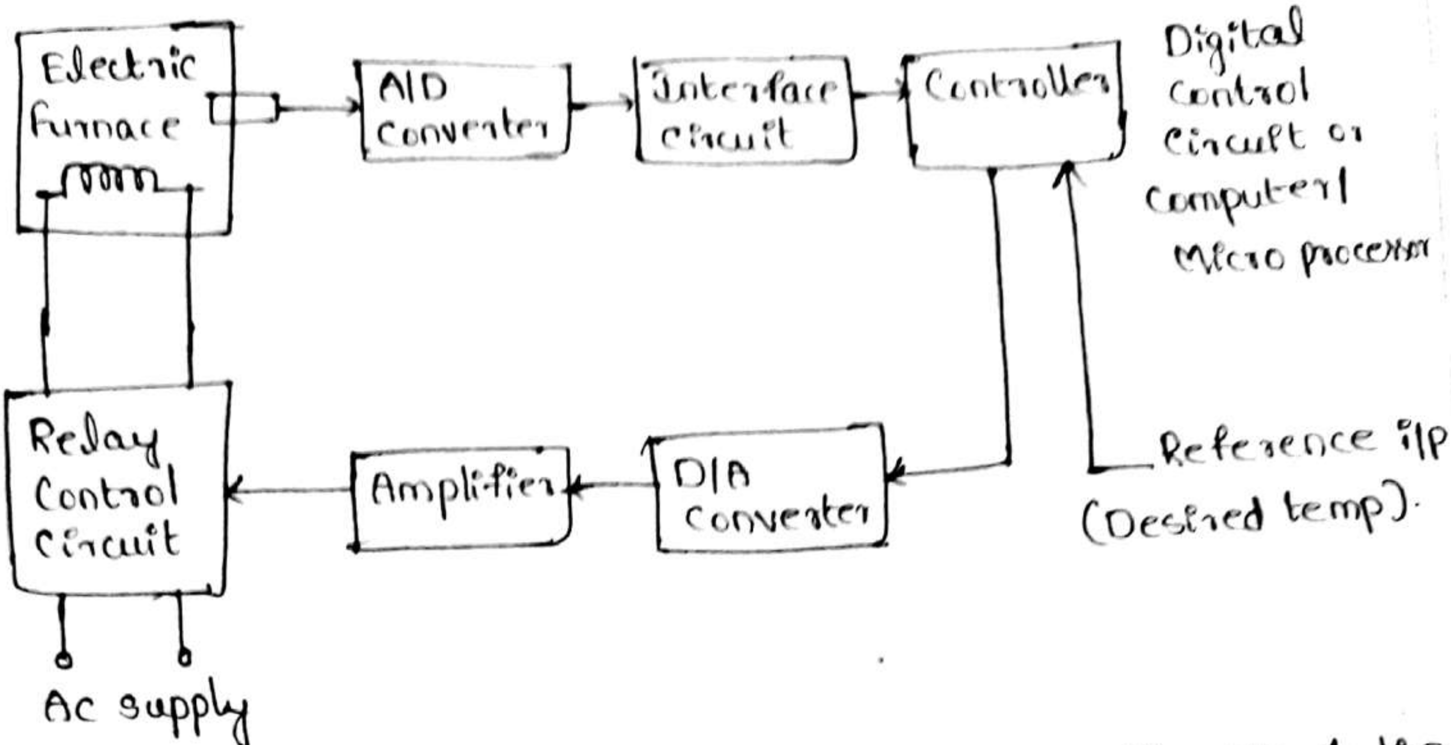


The electric furnace is an open loop system. The o/p in the sys is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heat remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog-to-Digital (AD Converter). The digital signal is given to the digital display device to display ~~to~~ the temperature. In this system if there is any change in o/p temperature then the time setting of the relay is not altered automatically.

Closed Loop Control system:

* Temperature Control system, traffic control signals, Numerical control system.



The electric furnace is a closed loop system. The o/p of the sys is the desired temperature & it depends on the time during which the supply to heat remains ON.

The switching ON and OFF of the relay is controlled by a Controller which is a digital system or computer. The desired temperature is i/p to the sys through keyboard or as a signal corresponding to desired temp via ports. The actual temp is sensed by sensor & converted to digital s/l by the A/D converte The computer reads the actual temp & compares with desired temp. If it finds any difference then it sends s/l to switc ON or OFF the relay through D/A converter & amplifier. Thus the sys automatically corrects any changes in o/p. So it is called as closed loop system.

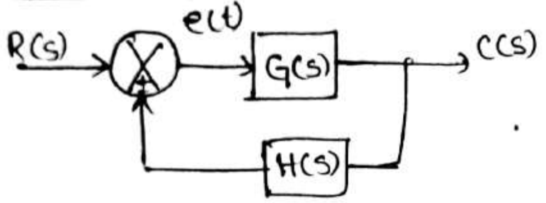
Classification of control system:-

- Control system
  - Open loop Control system
  - Closed-loop Control system
    - Positive feedback Control system
    - Negative feedback. Control system

Open loop C S:-

$$\boxed{\frac{C(S)}{R(S)} = G(S) \quad \text{or} \quad C(S) = G(S)R(S)}$$

Closed-loop C.S:-



* If error s/l e(t) is zero, o/p is Controlled.

* If e(t) is not zero, o/p is not Controlled.

* For positive feedback, error s/l = $r(t) + c(t)$

* For Negative feedback, error s/l = $r(t) - c(t)$.

* The purpose of feedback is to reduce the error b/w the ref. i/p & the system o/p.

- +ve feedback
  - Unity feedback $(H(S) = 1)$ $$\boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1-G(S)}}$$
  - Non-Unity F/B. $(H(S) \neq 1)$ $$\boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1-G(S)H(S)}}$$
- -Ve feedback.
  - Unity F/B $$\boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)}}$$
  - Non Unity F/B. $$\boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}}$$

Where G(S) = T.F of the forward path.

H(S) = T.F of the feedback path.

* The characteristic equation of a general feedback system is

$1 + G(S)H(S) = 0$ → for non unity feedback sys.

$1 + G(S) = 0$ → unity feedback sys. $(\because H(S) = 1)$

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Feedback characteristics :-

* The feedback has effects on the system performance characteristics such as stability, bandwidth, Overall gain, impedance and sensitivity.

Effects of feedback :-

* Gain is reduced by a factor. $\frac{1}{1+G(s)H(s)}$
* There is reduction of parameter variation by a factor $1+G(s)H(s)$.
* There is improvement in sensitivity.
* There may be reduction of stability.

The disadvantage of reduction of gain and reduction of stability can be overcome by gain amplification & good design.

* feedback reduces the effect of noise & disturbance on system Performance.
* Bandwidth increases by a factor of $1+G(s)H(s)$.
* The system becomes more accurate.

Effect of feedback on stability :-

* stability is a notion that describes whether the system will be able to follow the input command.
* A system is said to be unstable, if its output is out of cont
* The closed loop system stability depends on loop gain.
* If loop gain $GH=-1$, the o/p of a system becomes infinity for any finite input, and the system is said to be unstable.
* If the loop gain >0, then system stability is improved.
* The feedback can improve stability or be harmful to stabilit if it is not properly applied.

Effect of feedback on Overall gain :-

* Feedback affects the gain G of a non-feedback system by a factor of 1±GH.

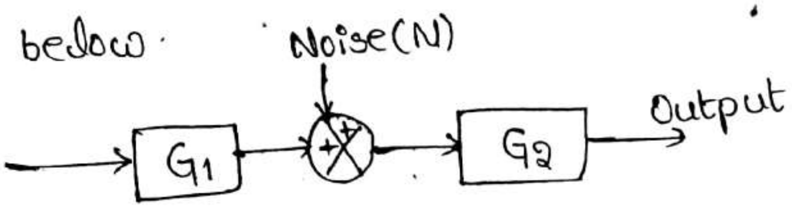
* The general effect of feedback is that it may increase or decrease the gain.

* In a practical control system, G and H are functions of frequency. So the magnitude of 1+GH may be >1 in one frequency range but <1 in another.

* Feedback could increase the gain of the system in one frequency range but decrease it in another.

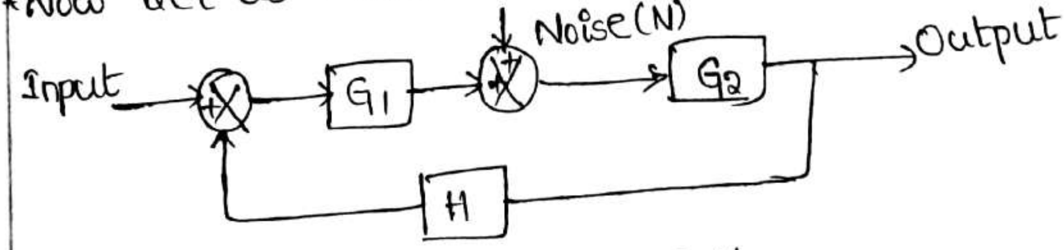
Effect of feedback on Noise :-

Let us consider the non-feedback control system as shown in fig. below.



The output due to Noise = G_2N ---- (1)

*Now let us consider the feedback control system as shown below



The output due to noise = $\frac{G_2N}{1+G_1G_2H}$ ---(2)

* Comparing eq(1) & (2), it is clear that the feedback control system reduces the noise.

* Further the noise can be reduced by increasing the value of G_1.

Effect of feedback on Sensitivity :-

* In general a good control system should insensitive to Parameter variations but sensitive to input command.

* The sensitivity of the gain of the overall system M to the variation in G is defined as.

$$S_G^M = \frac{\text{\% changes in } M}{\text{\% change in } G} = \frac{(\delta M/M)\,100\%}{(\delta G/G)\,100\%}.$$

$$\boxed{S_G^M = \frac{\partial M/M}{\partial G/G}} \quad \text{Where } M = \frac{G}{1+GH}.$$

* Where ∂M denotes the incremental change in M due to the incremental change in G.

* $\partial M/M$ & $\partial G/G$ denote the percentage change in M and G.

$$\boxed{S_G^M = \frac{\partial M}{\partial G} \times \frac{G}{M} = \frac{1}{1+GH}.}$$

* Sensitivity function can be made arbitrarily small by ↑ng GH, Provided that the system remains stable.

* In an open-loop system, the gain of the sys will respond in one-to-one fashion to the variation in G.

* In general, the sensitivity of the system gain of a feedback system to parameter variations depends on where the parameter is located.

① Find the sensitivity of the system For the transfer function

$$T = \frac{1+2K}{3+4K} ?$$

Sol

$$T = \frac{1+2K}{3+4K}.$$

$$\frac{\partial}{\partial K}(T) \cdot \frac{K}{T}$$

$$S_K^T = \frac{\partial T}{\partial K} \cdot \frac{K}{T} = \frac{(3+4K)(2) - (1+2K)(4)}{(3+4K)^2} \times \frac{K}{T.} = \frac{6+\cancel{8K}-4-\cancel{8K}}{(3+4K)^2} \times \frac{K}{\frac{1+2K}{(3+\cancel{4K})}}.$$

$$= \frac{2K}{3+8K^2+10K.}$$

② $G = (10)(10)(10)$, $\frac{C}{R}$ 100 , find the feedback factor H?

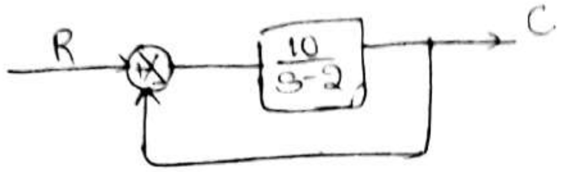
Sol Given $\frac{C}{R} = 100$

$$\frac{C}{R} = \frac{G}{1+GH}$$

$$100 = \frac{(10)(10)(10)}{1+10^3 H}$$

$$100+10^5 H = 10^3$$

$$10^5 H = 900 \Rightarrow \boxed{H = 9\times10^{-3}}$$

③ 

Find the TF?

Sol Given data $G = \frac{10}{S-2}$, $H = 1$.

$CLTF = \frac{G}{1+G}$ → for unity feedback sys.

$$CLTF = \frac{10/S-2}{1+\frac{10}{S-2}} = \frac{10}{\cancel{S-2}} \times \frac{\cancel{S-2}}{S-2+10}$$

$$\boxed{CLTF = \frac{10}{S+8}}$$

④ CLTF of a unity feedback system $\frac{4}{S^2+7S+13}$ find the OLTF?

Sol $\boxed{OLTF = \frac{CLTF}{1-CLTF} = G(S).}$

$$CLTF = \frac{4}{S^2+7S+13} \Rightarrow OLTF = G(S) = \frac{\frac{4}{S^2+7S+13}}{1-\frac{4}{S^2+7S+13}}$$

$$= \frac{4}{\cancel{S^2+7S+13}} \times \frac{\cancel{S^2+7S+13}}{S^2+7S+13-4}$$

$$\boxed{OLTF = \frac{4}{S^2+7S+9}}$$

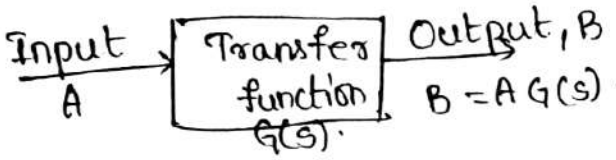
Block Diagrams :-

* In control engineering to show function perform by each Component, we commnly use a diagram called 'Block diagram'.
* A block diagram ~~of~~ a sys is a pictorial representation of functions performed by each component & flow of signals.
* The elements of a block diagram are block, branch point & summing point.

Block :-

In a block diagram all system variables are linked to each other through functional blocks.

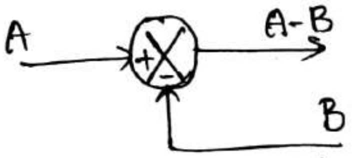
* The functional block or simply block is a symbol for the mathematical operation on the i/p signal to the block that produce o/p.



Functional Block.

Summing point :-

* Summing points are used to add two or more signals in the system.
* A circle with a cross is the symbol that indicates a summi operation.

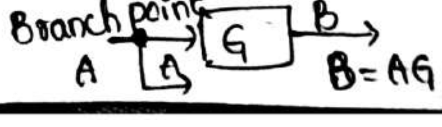


Summing point.

* The plus or minus sign at each arrowhead indicates whethe the signal is to be added or subtracted.

Branch point (or) Take off point :-

* A branch point is a point from which the signal from a blo goes concurrently to other blocks or summing points.



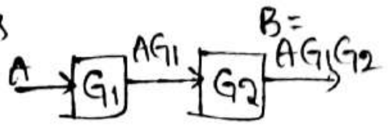
Block diagram Reduction Techniques:-

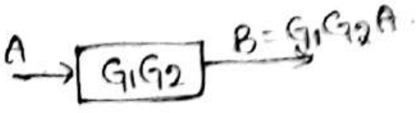
The block diagram can be reduced to find the overall transfer function of the system.

* The following rules can be used for block diagram reduction.

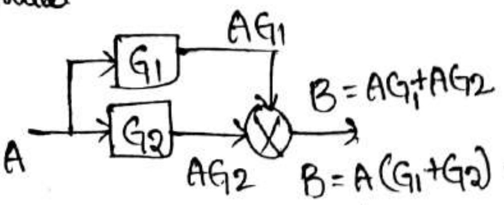
Rules — **Actual B.D** — **Equivalent B.D**

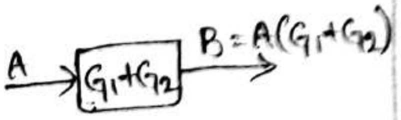
1. Combining the blocks in cascade.



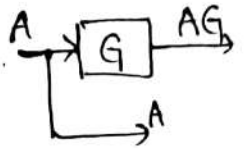


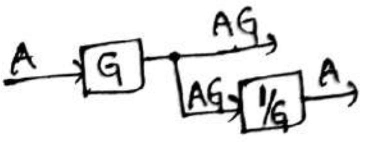
2. Combining the parallel blocks.



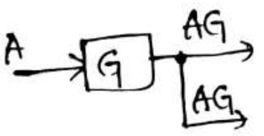


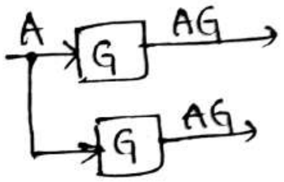
3. Moving the branch point ahead of the block.



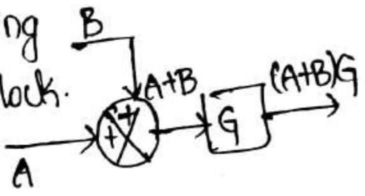


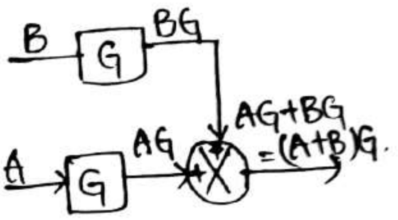
4. Moving the branch point before the block.



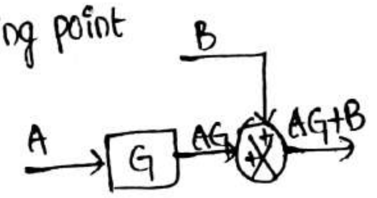


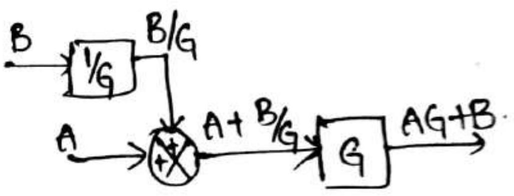
5. Moving the summing point ahead of the block.



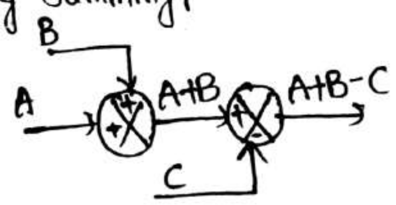


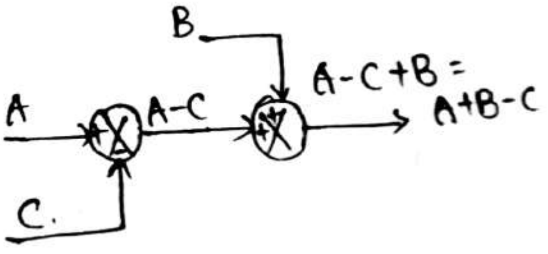
6. Moving the summing point before the block.





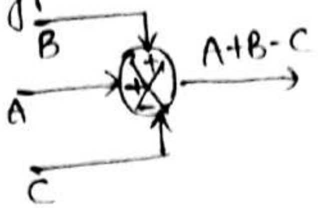
7. Interchanging Summing point

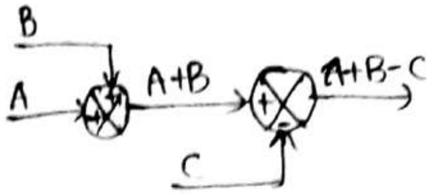




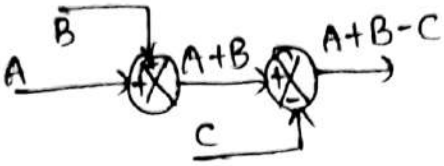
·Rules Actual B.D Equivalent B.D

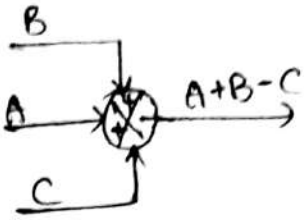
8. Splitting summing point



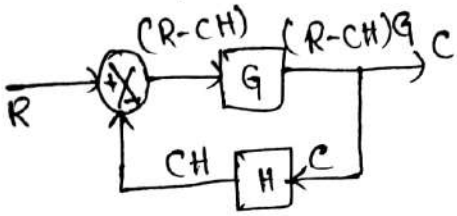


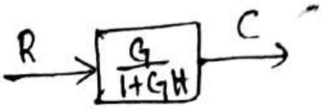
9. Combining summing points.





10. Elimination of (negative) feedback loop.





Proof:

$C = (R-CH)G$

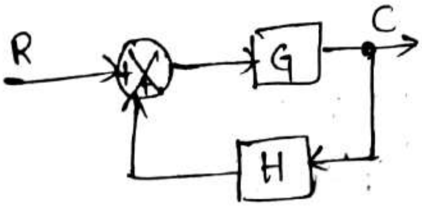
$C = RG - CHG$

$C + CHG = RG$

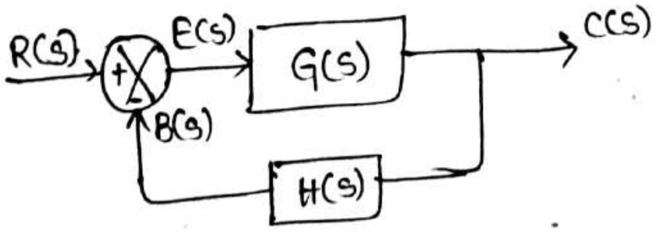
$C(1+HG) = RG$

$\frac{C}{R} = \frac{G}{1+GH}$.

11. Elimination of (positive) feedback loop.



Block diagram of a closed loop system with -ve feedback:-



$E(S) = R(S) - B(S)$.

$C(S) = G(S)E(S) \longrightarrow ①$

$E(S) = R(S) - H(S)C(S) \longrightarrow ②$

$$T(s) = \frac{C(s)}{R(s)} \rightarrow ③$$

$$C(s) = G(s)\left[R(s) - H(s)C(s)\right]$$

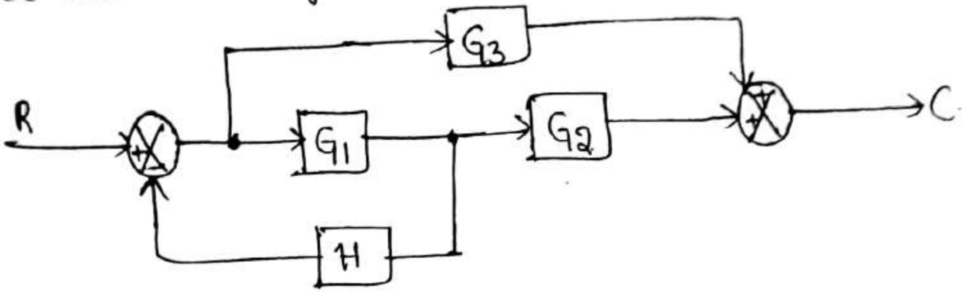
$$C(s) = G(s)R(s) - G(s)H(s)C(s)$$

$$C(s)\left[1 + G(s)H(s)\right] = G(s)R(s)$$

$$C(s) = \frac{G(s)R(s)}{1+G(s)H(s)}$$

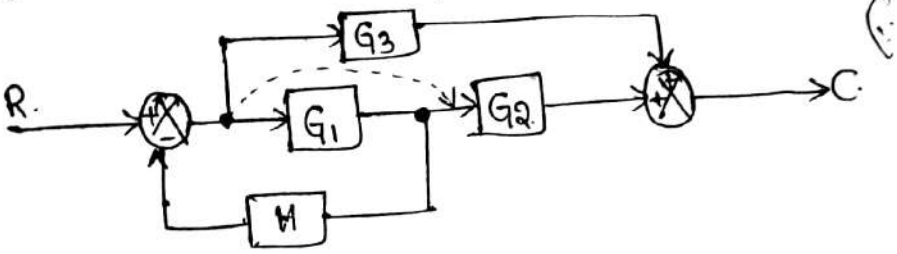
$$\boxed{T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}}$$

(i) Reduce the block diagram shown in fig. and find C/R.



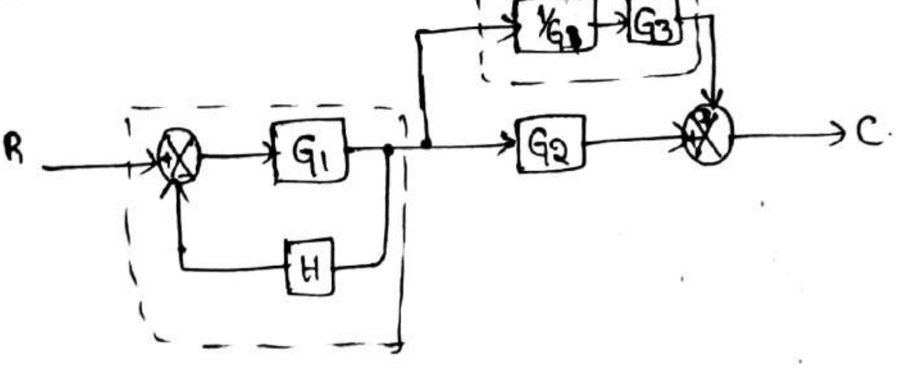
Sol step ①: Move the branch ~~oft~~ point after the block.

(∵ Apply rule ③)



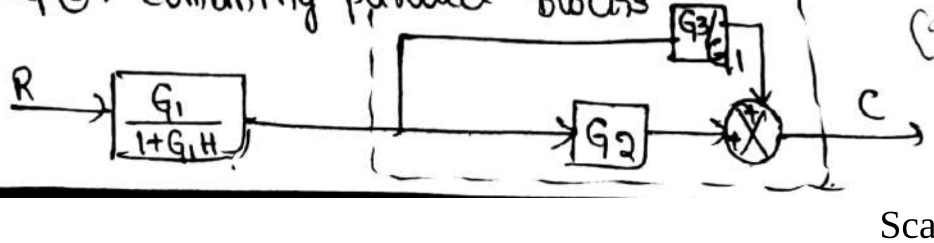
step ②: Eliminate the feedback path & combining blocks in cascade.

(∵ Rule 10 & Rule 1)

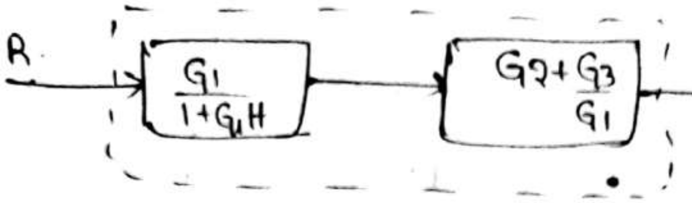


step ③: Combining parallel blocks.

(∵ Rule ②)



step ④ : combining blocks in cascade.

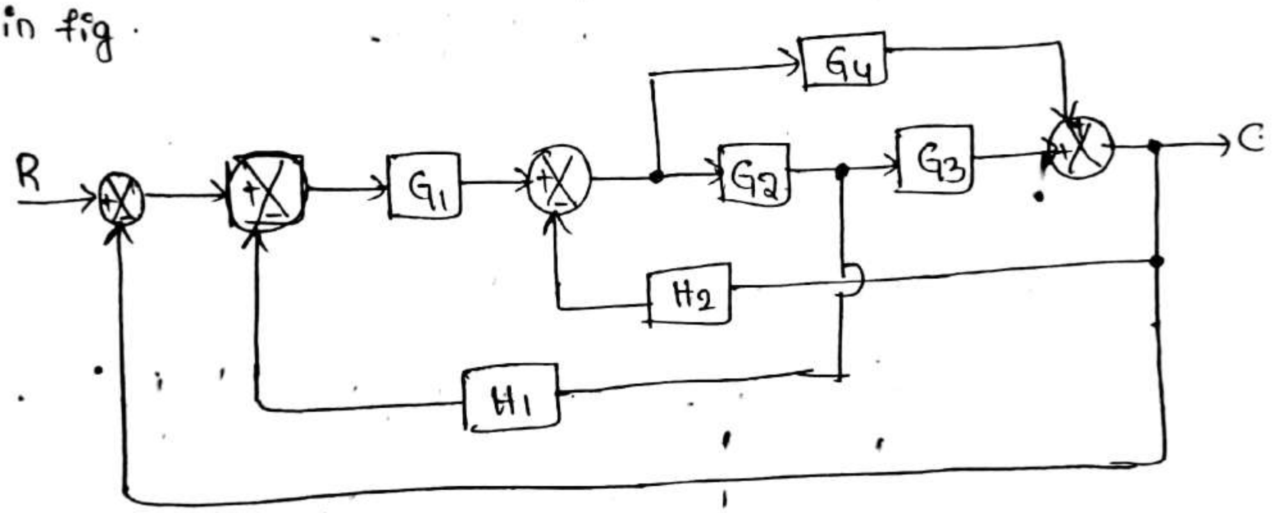


(∵ Rule ①)

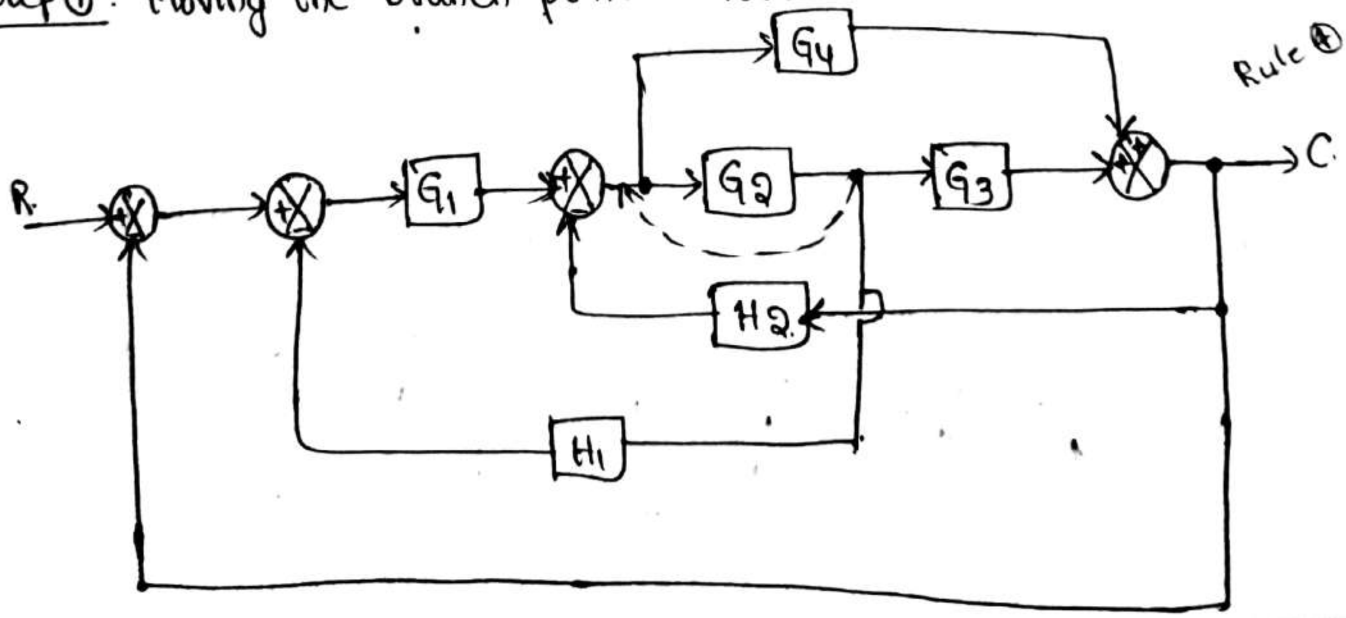
$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right) \cdot = \left(\frac{\cancel{G_1}}{G_1H+1}\right)\left[\frac{G_2G_1+G_3}{\cancel{G_1}}\right]$$

$$\boxed{\frac{C}{R} = \frac{G_3+G_1G_2}{1+G_1H}}$$

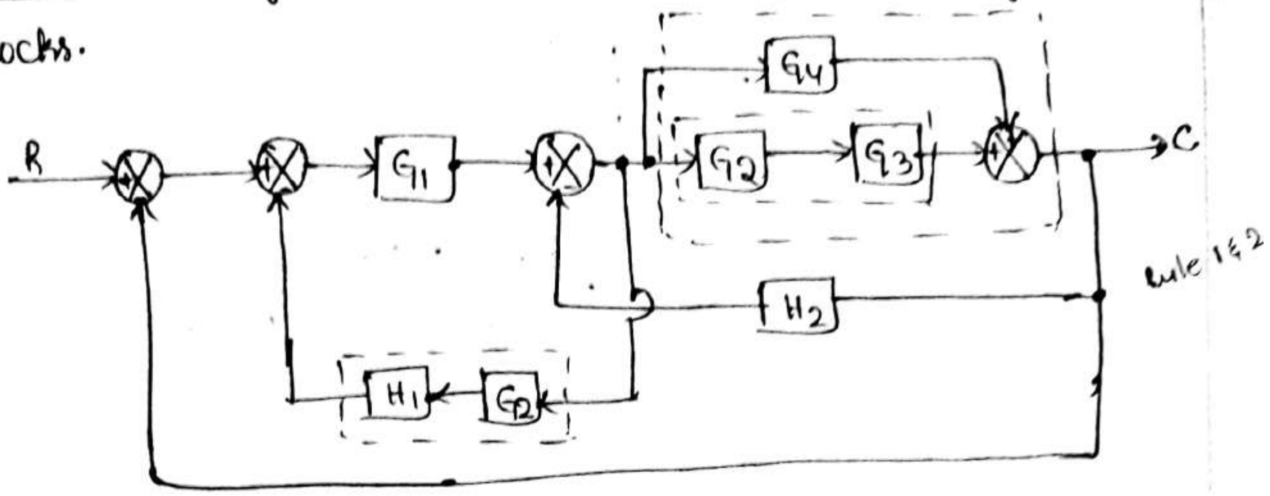
② Using block diagram reduction techniques find closed loop transfer function & the system whose block diagram is shown in fig.



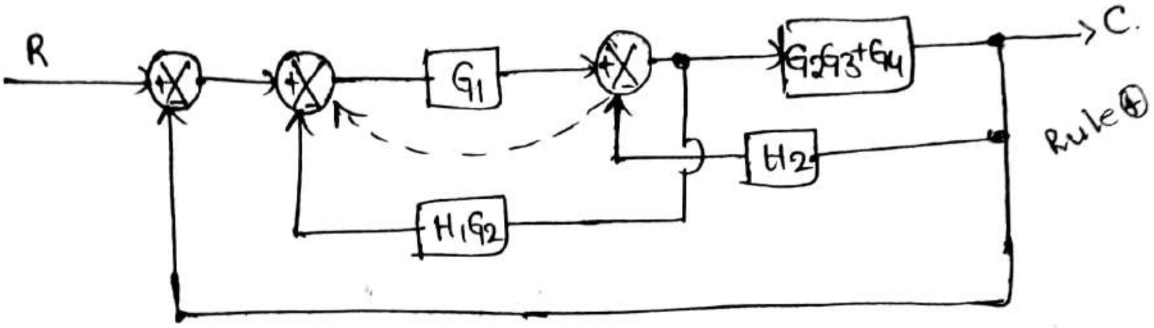
step ① : Moving the branch point before the block.



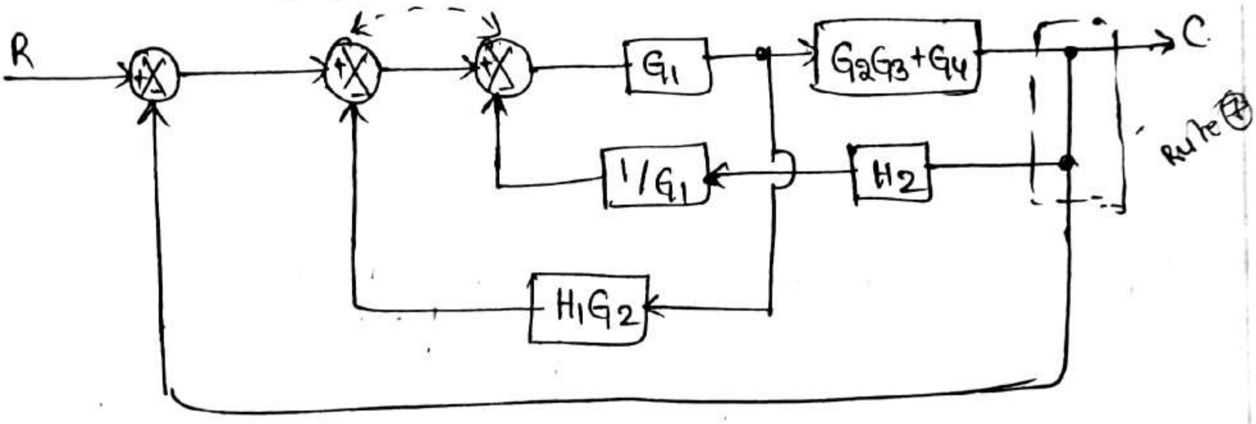
step ②: Combining the blocks in cascade & eliminating parallel blocks.



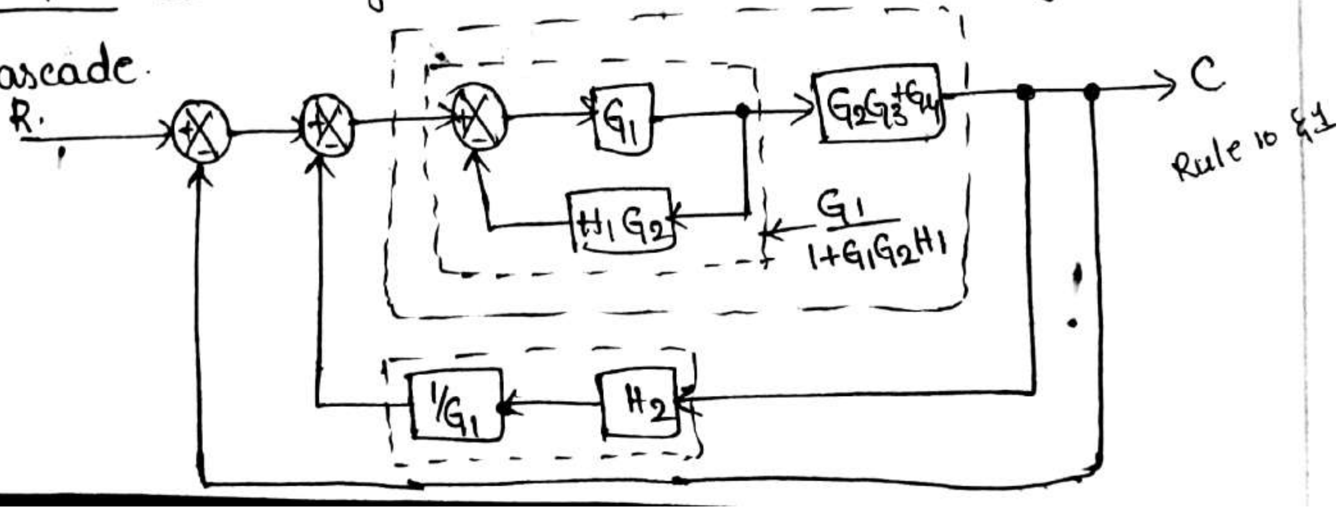
step ③: Moving summing point before the block.



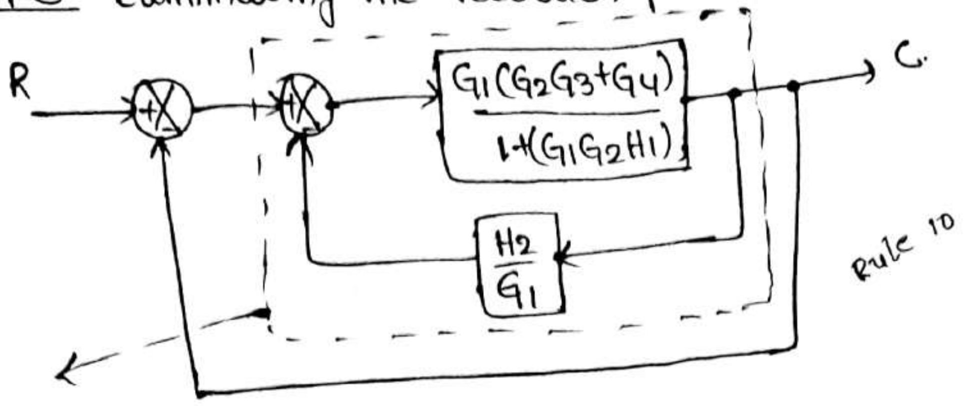
step ④: Interchanging summing points & modifying branch points.



step 5: Eliminating the feedback path & combining blocks in cascade.



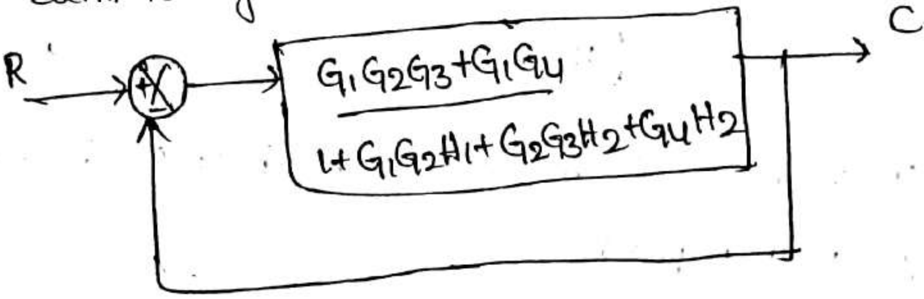
Step ⑥: Eliminating the feedback path.



$$\frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\times\frac{H_2}{\cancel{G_1}}} = \frac{\frac{G_1G_2G_3+G_1G_4}{1+\cancel{G_1G_2H_1}}}{\frac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{\cancel{1+G_1G_2H_1}}}$$

$$= \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

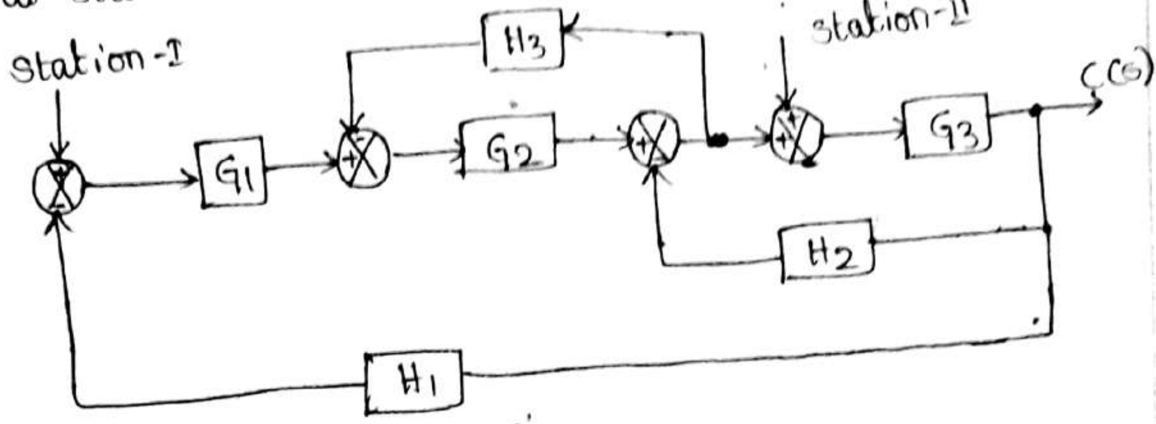
Step ⑦: Eliminating the feedback path.



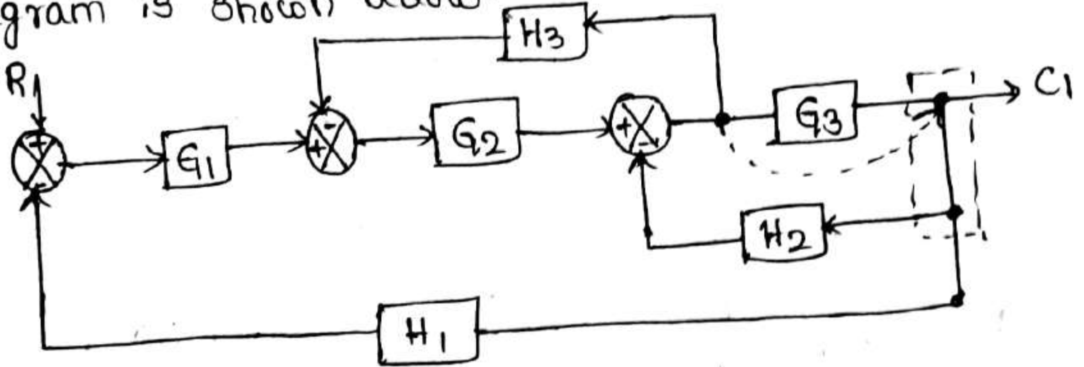
$$\frac{C}{R} = \frac{\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}$$

$$\boxed{\frac{C}{R} = \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}}$$

③ For the system represented by the block diagram shown in fig. Evaluate the closed loop transfer function when the input R is (i) at station-I (ii) station-II

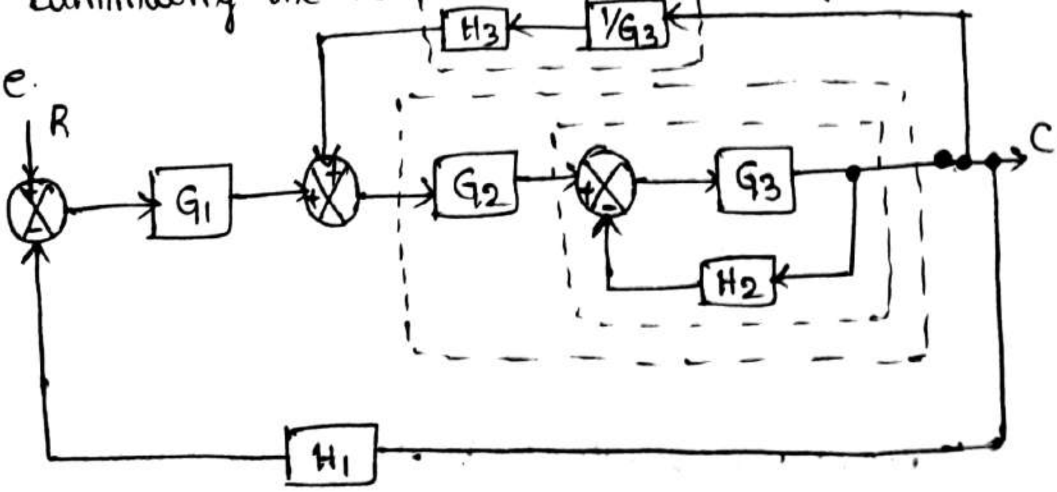


Sol) Case i): Consider the i/p R is at station-I & so the i/p at station-II is made zero. Let the output be C_1. So theres is no i/p at station-II that summing point can be removed & resulting block diagram is shown below.

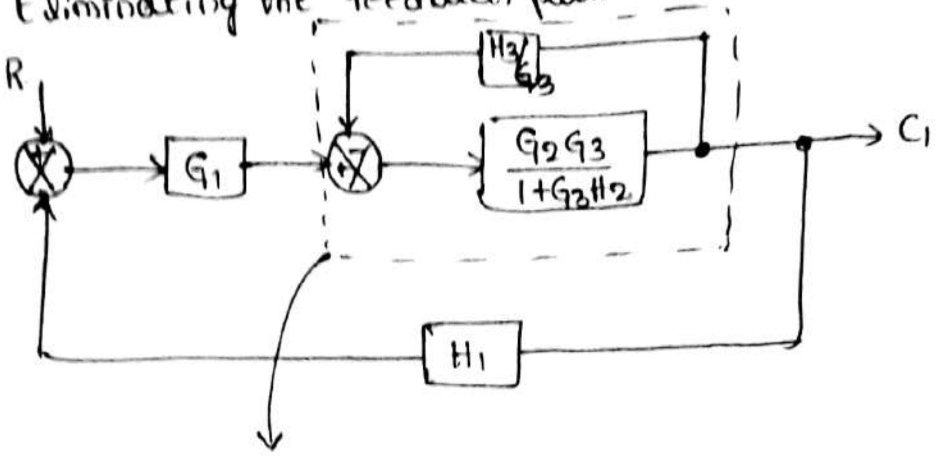


Step ①: shift the take off point of feedback H_3 beyond G_3 & rearrange branch points.

Step ②: Eliminating the feedback H_2 & combining the blocks in cascade.

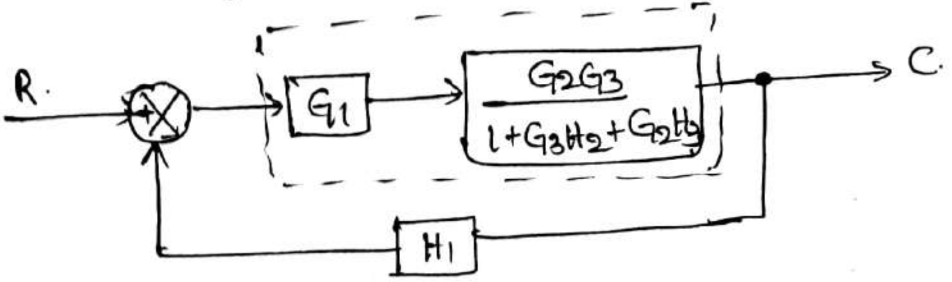


step ③: Eliminating the feedback path.

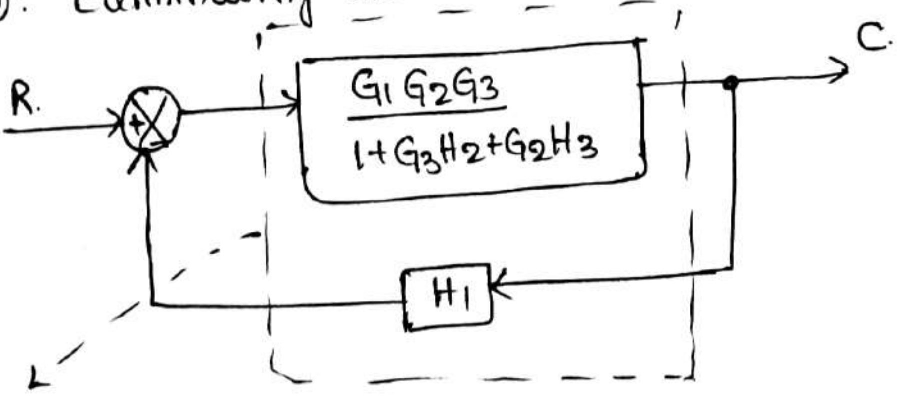


$$\frac{\frac{G_2G_3}{1+G_3H_2}}{1+\frac{G_2\cancel{G_3}}{1+G_3H_2}\times\frac{H_3}{\cancel{G_3}}} = \frac{\frac{G_2G_3}{\cancel{1+G_3H_2}}}{\frac{1+G_3H_2+G_2H_3}{\cancel{1+G_3H_2}}} = \frac{G_2G_3}{1+G_3H_2+G_2H_3}$$

step④: Combining the blocks in cascade.



step⑤: Eliminating the feedback path. H₁.



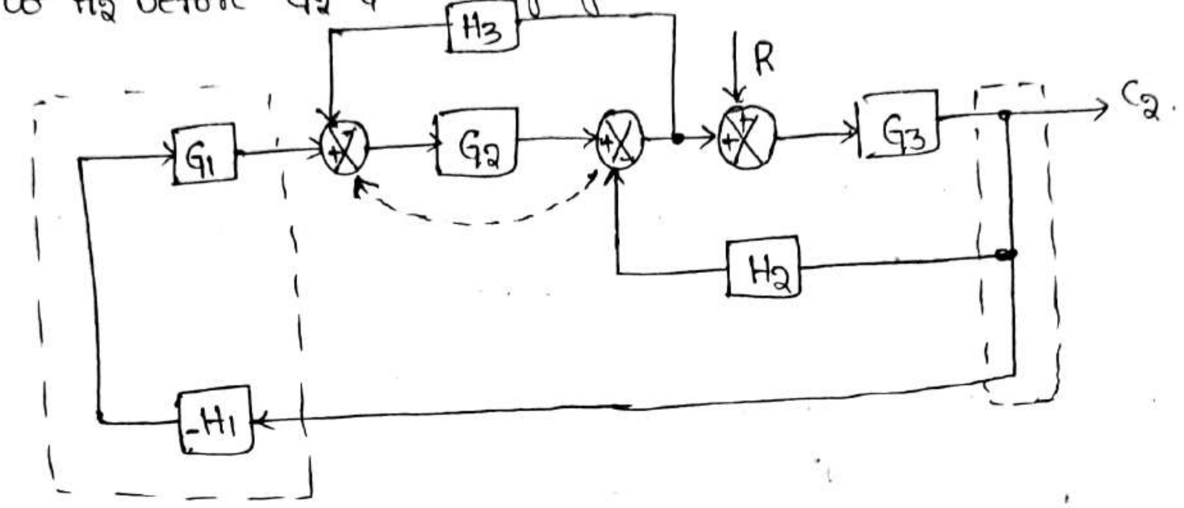
$$\frac{\frac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\frac{G_1G_2G_3}{1+G_3H_2+G_2H_3}\times H_1.} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1.}$$

$$\Rightarrow \boxed{\frac{C_1(S)}{R(S)} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}}$$

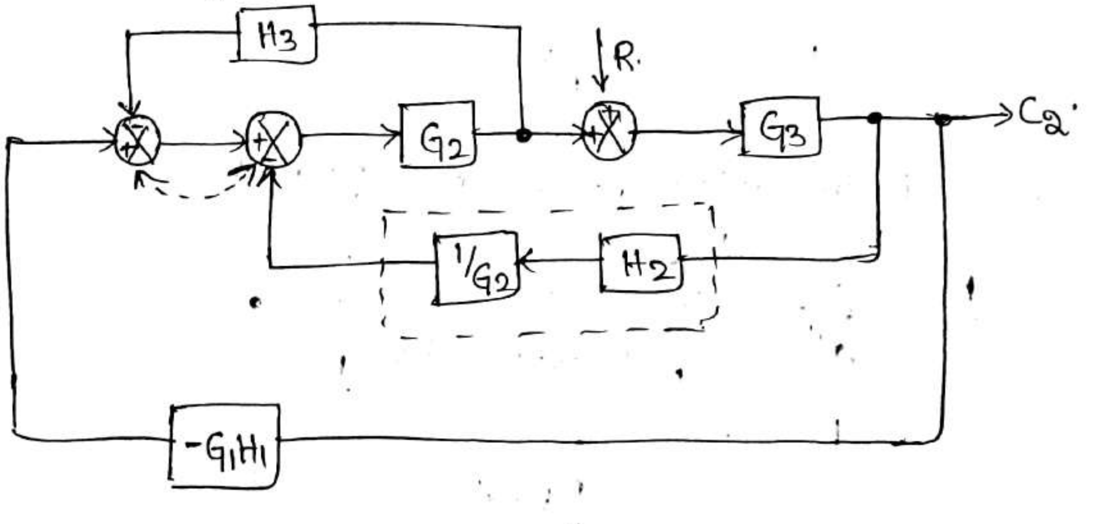
Case(ii):

(ii) Consider the input R at station-II, the input at station-I is made zero. Let output be C_2. Since there is no input in station-I that corresponding summing point can be removed & a negative sign can be attached to the feedback path gain H_1.

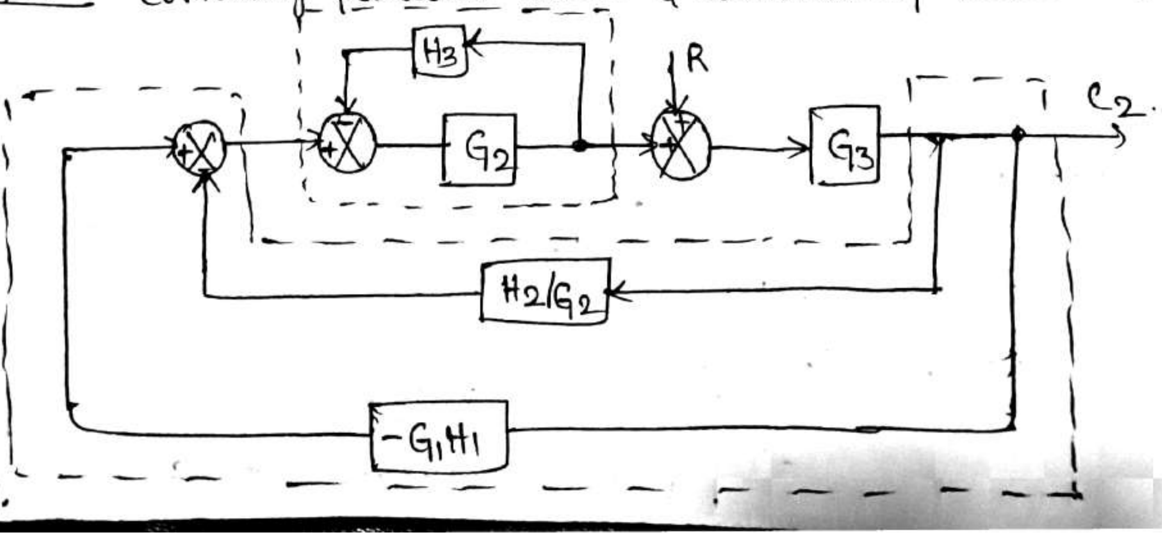
Step(i): Combining the blocks in cascade, shifting the summing point of H_2 before G_2 & rearranging the branch points.



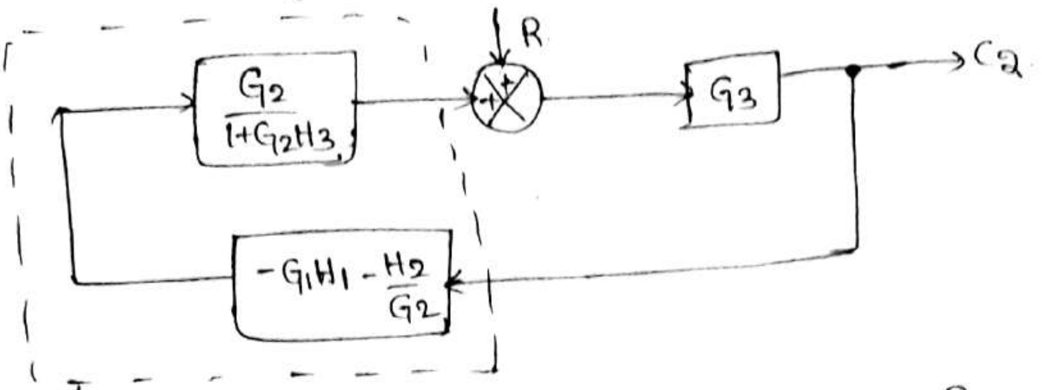
Step②: Interchanging summing points & combining blocks in cascade.



Step③:- Combining parallel blocks & eliminating feedback path.



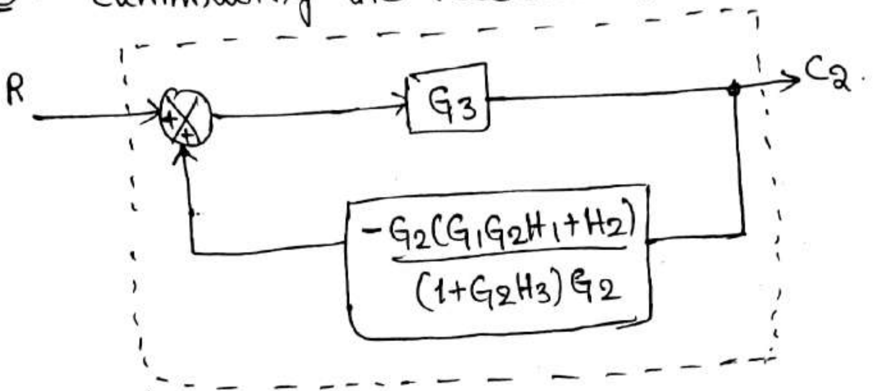
Step④: Combining the blocks in cascade



$$\left(\frac{G_2}{1+G_2H_3}\right)\left[-G_1H_1-\frac{H_2}{G_2}\right] = \left(\frac{G_2}{1+G_2H_3}\right)\left[\frac{-G_1H_1G_2-H_2}{G_2}\right]$$

$$= \frac{-G_2(G_1H_1G_2+H_2]}{(1+G_2H_3)(G_2]}$$

Step⑤ :- Eliminating the feedback path.



$$\frac{G_3}{1-\left[-\frac{\cancel{G_2}(G_1G_2H_1+H_2)}{(1+G_2H_3)\cancel{G_2}}\right]G_3} = \frac{G_3}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

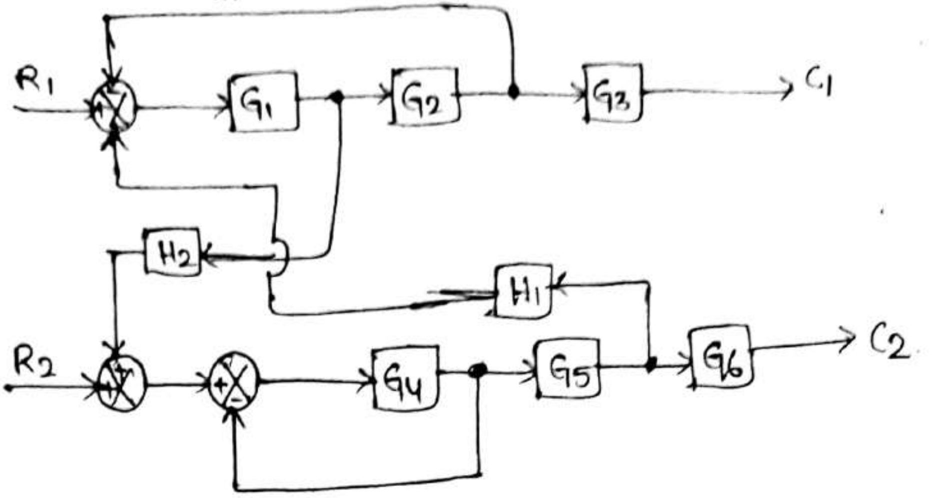
$$\boxed{\frac{C_2}{R} = \frac{G_3}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}}$$

Result: The transfer function of the system with input at station-I is & station-II is

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

$$\frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_2(G_1G_2H_1+H_2)}$$

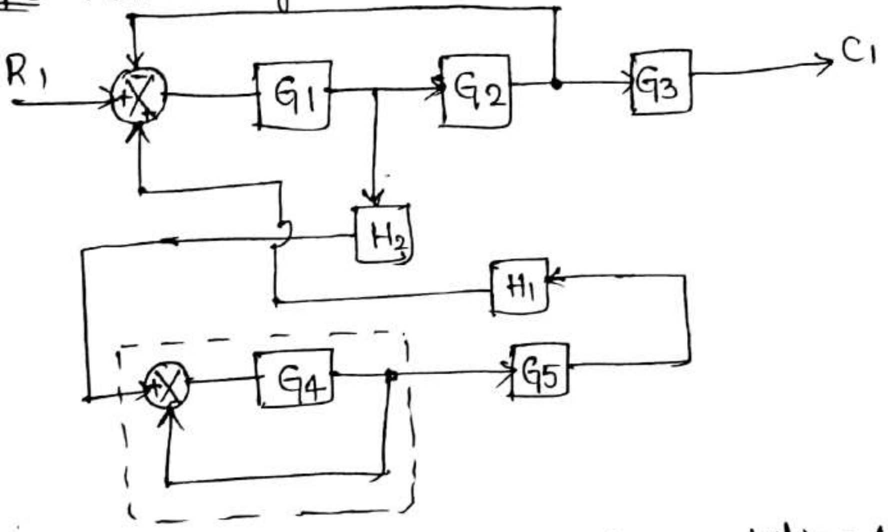
④ For the system represented by the block diagram shown in fig. determine $\frac{C_1}{R_1}$ & $\frac{C_2}{R_1}$.



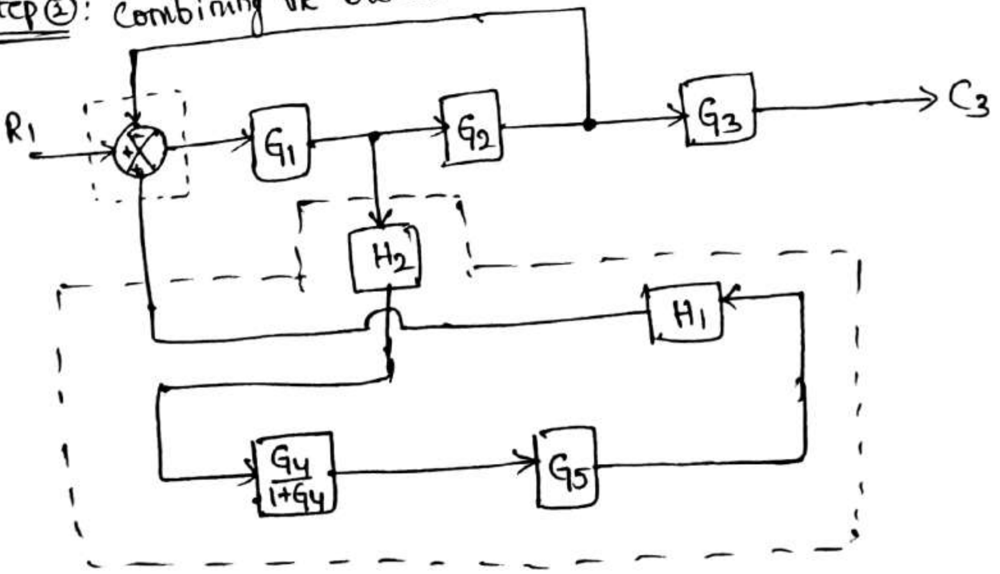
Case (i): To find $\frac{C_1}{R_1}$.

Set $R_2=0$ & only one o/p C_1. Remove the summing point which adds R_2 & need not consider G_6, since G_6 is open path.

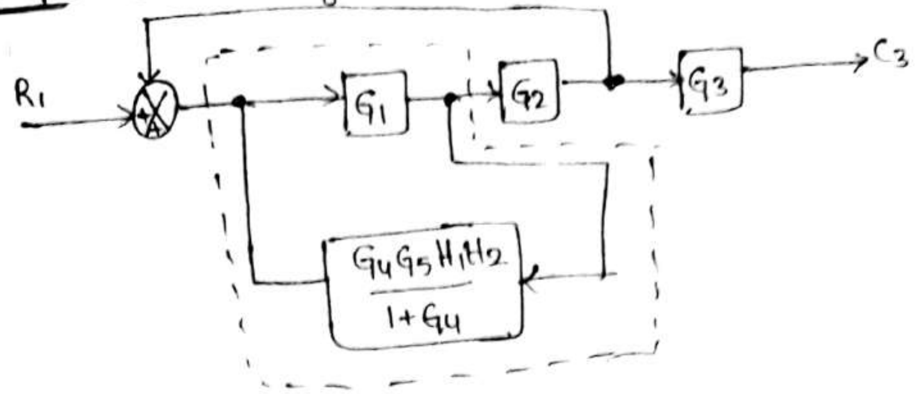
Step①: Eliminating the feedback path.



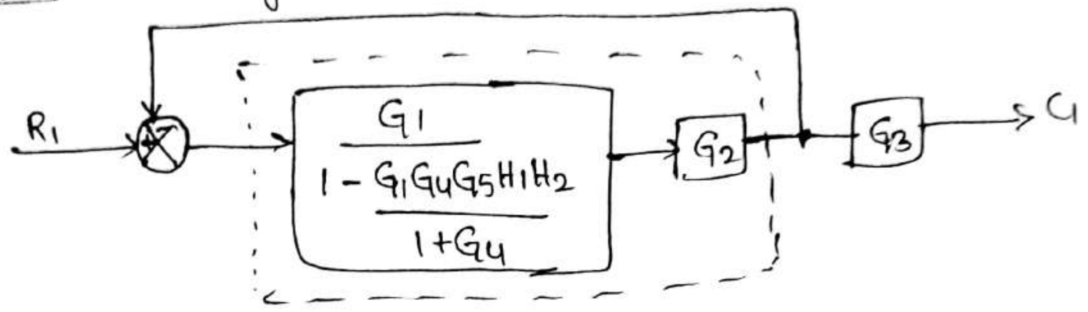
Step②: Combining the blocks in cascade & splitting the summing point.



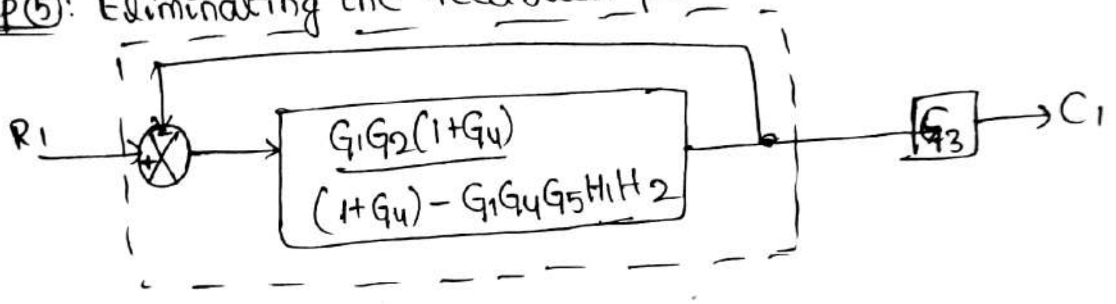
step ③: Eliminating the feedback path.

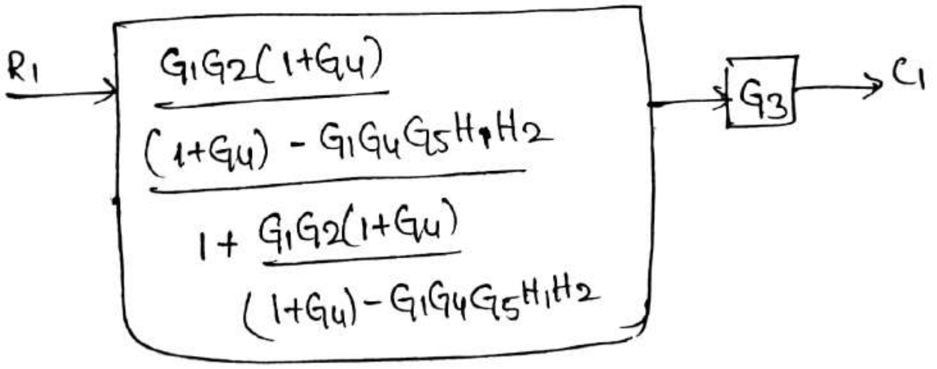


step ④: Combining the blocks in cascade.

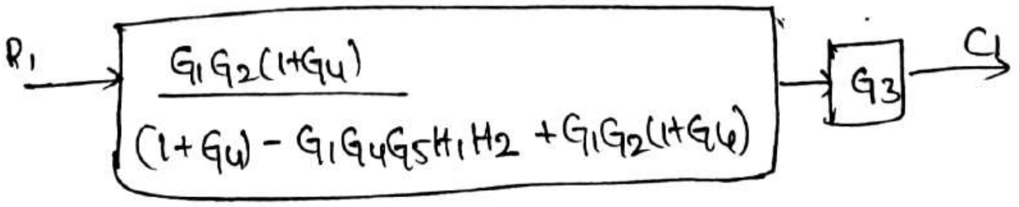


step ⑤: Eliminating the feedback path.





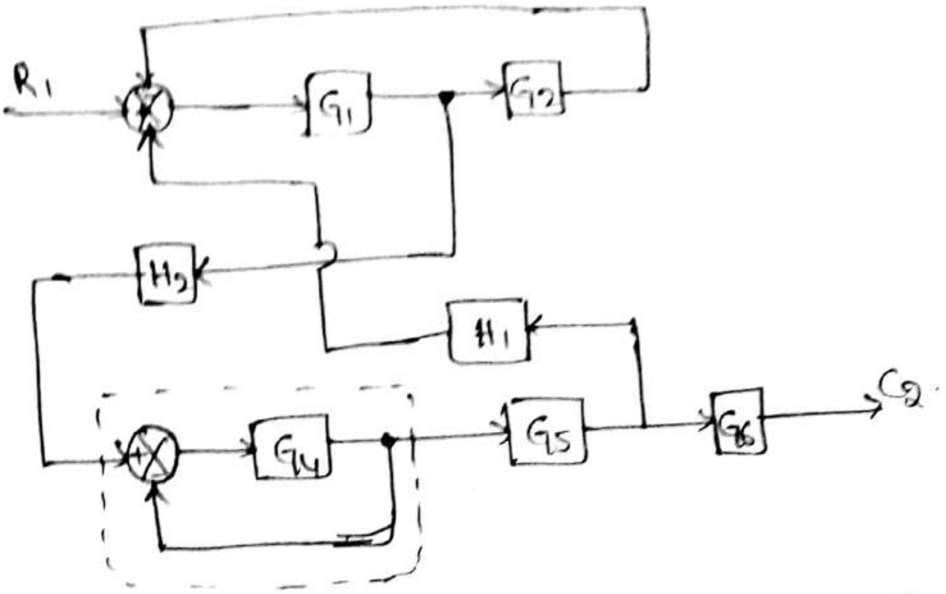
step ⑥: combining the blocks in cascade.



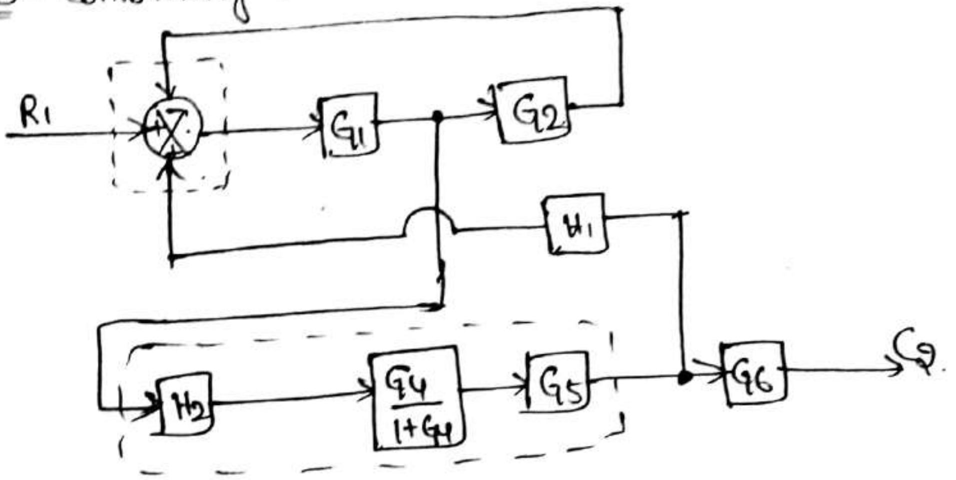
$$\frac{C_1}{R_1} = \frac{G_1G_2G_3(1+G_4)}{(1+G_4) - G_1G_4G_5H_1H_2 + G_1G_2(1+G_4)}.$$

Case ②: set $R_2 = 0$. only o/p C_2. We can remove summing point which adds R_2 & need not consider G_3, G_3 is open path.

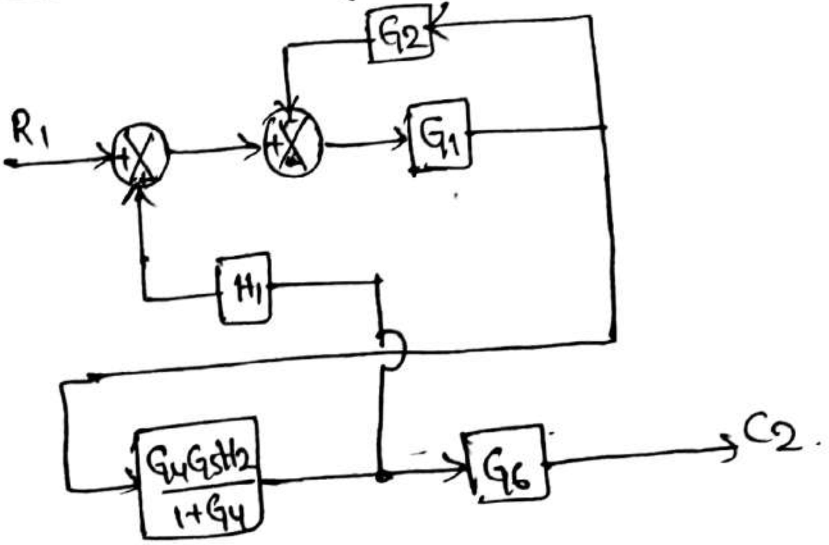
Step ① Eliminate the feedback path.



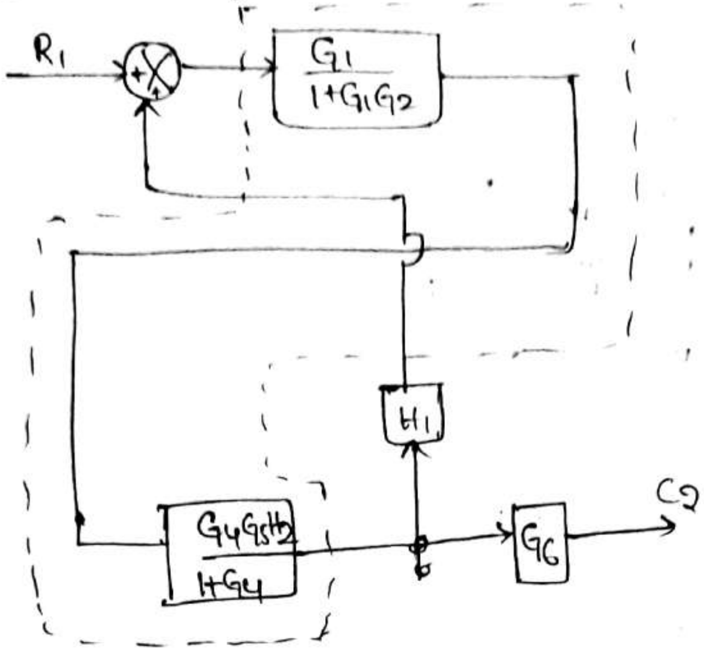
Step ②: Combining blocks in cascade & splitting the summing point



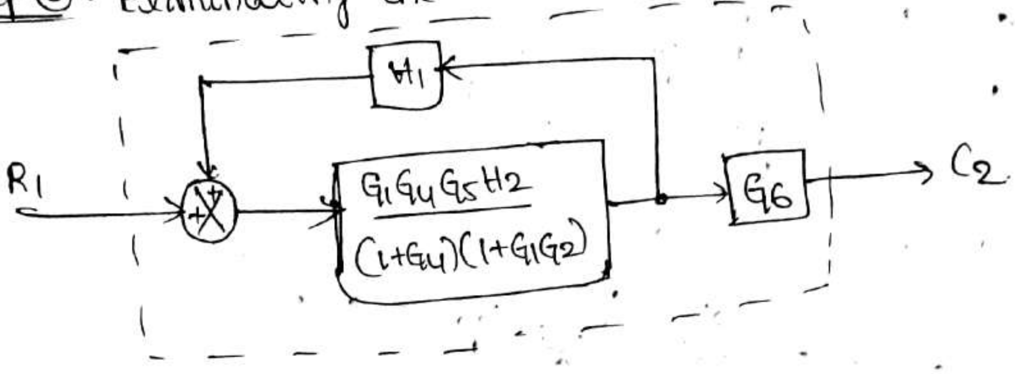
Step ③: Eliminating the feedback path.

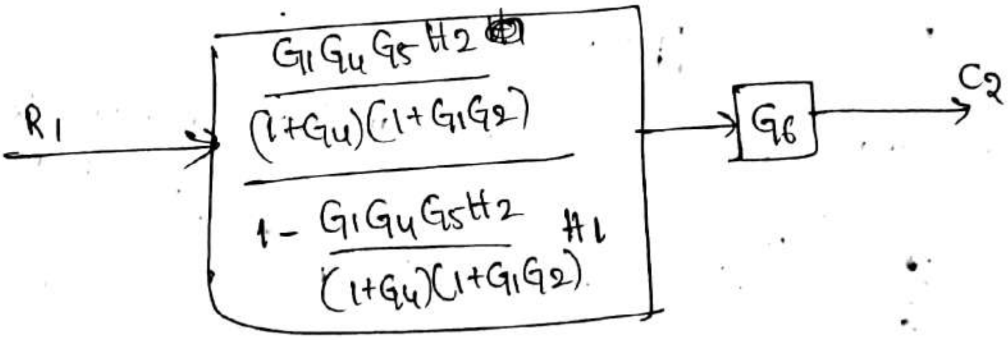


step ④: Combining the blocks in cascade

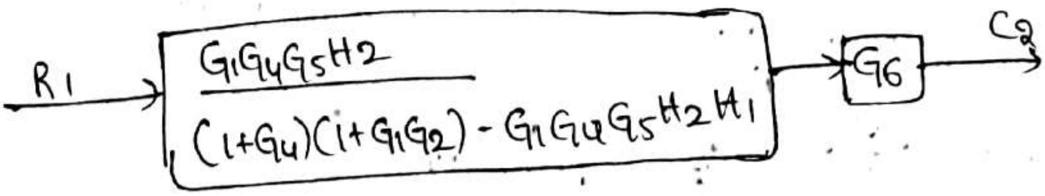


step ⑤: Eliminating the feedback path.



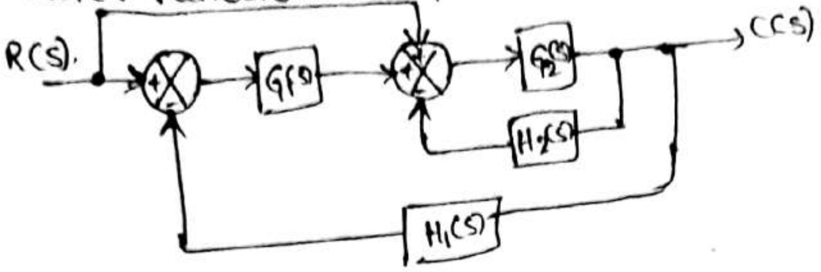


step ⑥: Combining the blocks in cascade.



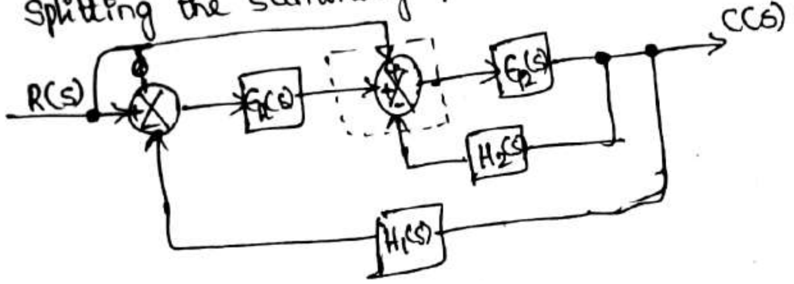
$$\boxed{\frac{C_2}{R_1} = \frac{G_1G_4G_5H_2G_6}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_2H_1}}$$

① The block diagram of a closed loop system is shown. Using the block diagram reduction technique determine the closed loop transfer function C(S)/R(S).

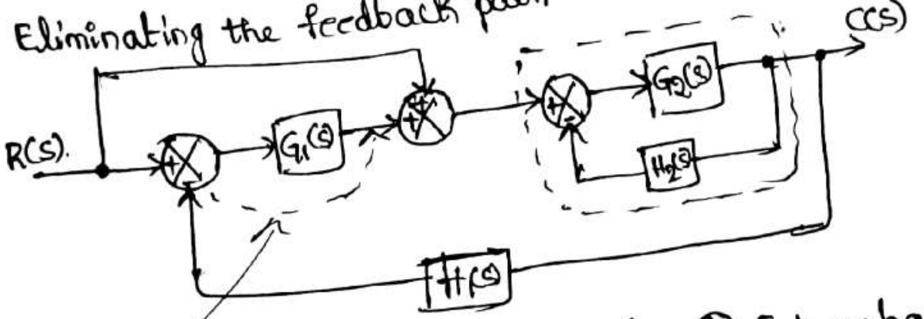


T.f ⇒ $\frac{C(S)}{R(S)} = \frac{G_2(S)[G_1(S)+1]}{1+G_2(S)H_2(S)+G_1(S)G_2(S)H_1(S)}$

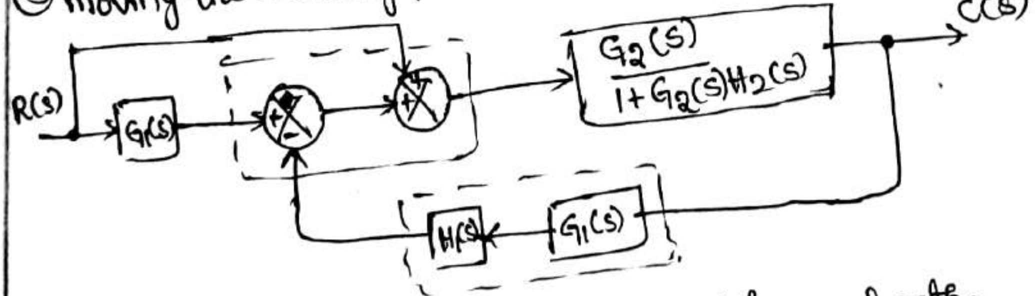
① Splitting the summing point.



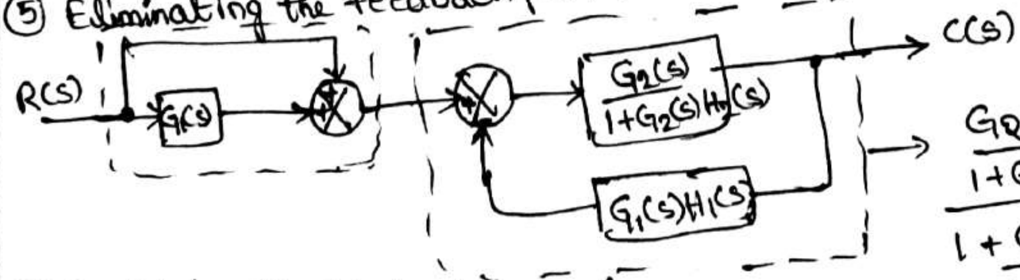
② Eliminating the feedback path.



③ moving the summing point after the block. ④ Interchanging the summing points & combing the blocks in cascade.

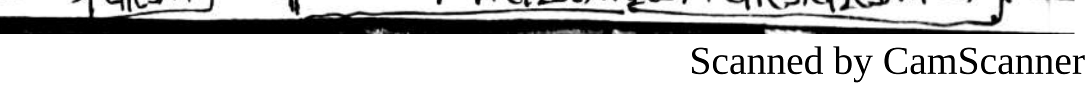


⑤ Eliminating the feedback path & feed forward path.



$\rightarrow \frac{\frac{G_2(S)}{1+G_2(S)H_2(S)}}{1+\frac{G_2(S)}{1+G_2(S)H_2(S)} G_1(S)H_1(S)}$

⑥ Combining the blocks in cascade.



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① Signal Flow Graph :- (SFG)

The signal flow graph is used to represent the control system graphically & it was developed by S.J.Mason

* Signal flow graph approach & block diagram approach yield the same information.

* A SFG consists of a network in which nodes are connected by directed branches.

* Each node represents a system variable & each branch is connected b/w two nodes act as a signal multiplier.

* In a SFG, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch & the gain is indicated along the branch.

=> The following terms are used in the SFG method.

Node :- A node is a point or junction representing a variable or signal.

Branch :- A branch is line segment joining two nodes.

arrow → branch direction (transmittance – gain of branch)

Transmittance :- The gain acquired by the signal when it travels from one node to another is called transmittance. (real or complex)

Input node (or) **Source node** :- It is a node ~~that~~ which has only outgoing branches.

Output node (or) **Sink node** :- It is a node ~~that~~ which has only incoming branches.

Mixed node :- It is a node which has both incoming & outgoing branches.

Path :- It is a traversal of connected branches in the direction of the branch arrows. path should not cross a node more than once.

Open path :- It is a path which starts at ~~a~~ node and ends at another node

Closed path :- closed path starts & ends ~~with~~ at same node.

Forward path :- It is a path from an input node to an output node that does not cross any node more than once.

Forward path gain :- It is the product of the branch gains of a forward path.

Individual loop :- It is a closed path which starts & ends at same node.

Loop gain :- It is the product of the branch gains of a loop.

Non-touching loops :- The loop which has no common path or common node.

Properties of signal flow graph :-

(i) The algebraic eqn's which are used to construct SFG must be in the form of cause & effect relationship.

(ii) It is applicable to linear systems only.

(iii) A node in the SFG represents a variable or s/l.

(iv) A node adds the s/l's of all incoming branches. & transmitts the sum to all outgoing branches.

(v) A Mixed node which has both incoming & outgoing s/l's.

(vi) A branch indicates functional dependence of one s/l on the other.

(vii) The s/l travel along branches only ~~and~~ in the marked direction.

(viii) The signal flow graph is not unique.

MASON'S GAIN FORMULA :-

The Mason's gain formula is used to determine th transfer function of the system from the signal flow graph of the system.

Let R(s) = Input to the system

C(s) = Output of the system.

Transfer function of the system, $T(s) = \frac{C(s)}{R(s)}$.

Mason's formula states the overall gain of the system as foll

Overall gain $\boxed{T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K.}$

Where,

T = T(s) = Transfer function of the system.

P_K = Forward path gain of K^{th} forward path.

K = Number of forward paths in the signal flow graph.

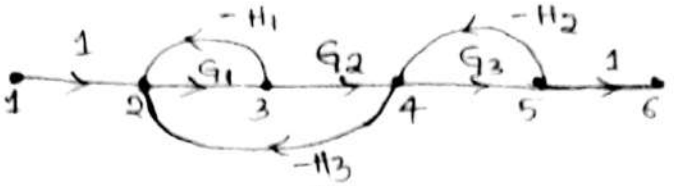
Δ = 1-(Sum of individual loop gains) +

(Sum of gain products of all possible combinations of two non-touching loops) -

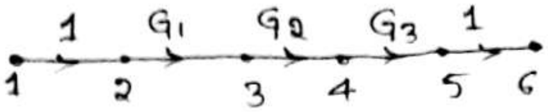
(Sum of gain products of all possible combinatio of three-non-touching loops) + ---

Δ_K = Δ for that part of the graph which is not touchin K^{th} forward path.

① Find the overall transfer function of system which is shown in fig.



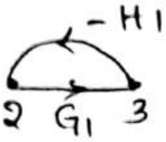
Sol No. of forward paths$^{(K)}$ = 1

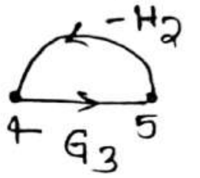


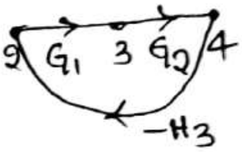
Gain of forward path $P_K = P_1 = G_1G_2G_3$

$$\Delta_1 = 1.$$

No. of individual loops = 3

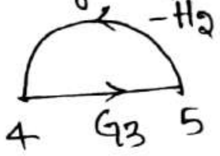






$$\sum_{m=1}^{3} P_{11}+P_{12}+P_{13} = -G_1H_1 - G_3H_2 - G_1G_2H_3.$$

Sum of two non-touching loops



$$\sum_{m=1}^{n} P_{2m} = P_{21} = G_1G_3H_1H_2.$$

$$\Delta = 1 = 1 - \Sigma P_{1m} + \sum_{m=1}^{n} P_{2m}$$

$$= 1 - (-G_1H_1 - G_3H_2 - G_2G_1H_3) + G_1G_3H_1H_2$$

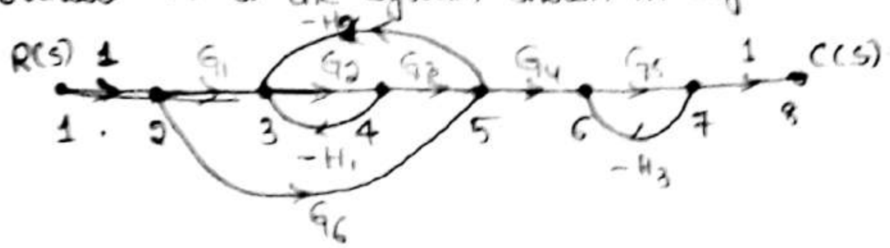
$$= 1 + G_1H_1 + G_3H_2 + G_1G_2H_3 + G_1G_3H_1H_2.$$

$$T = \frac{1}{\Delta}\sum_{K} P_K\Delta_K$$

$$= \frac{1}{\Delta}\sum_{1} P_1\Delta_1$$

$$T = \frac{G_1G_2G_3(1)}{1+G_1H_1+G_3H_2+G_1G_2H_3+G_1G_3H_1H_2}.$$

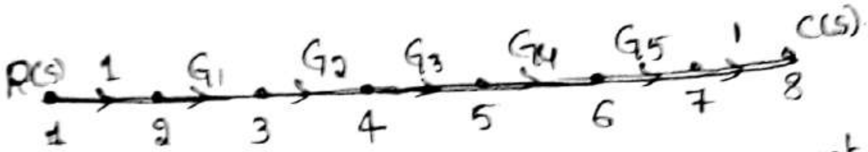
② Find overall T.F of the system shown in fig.



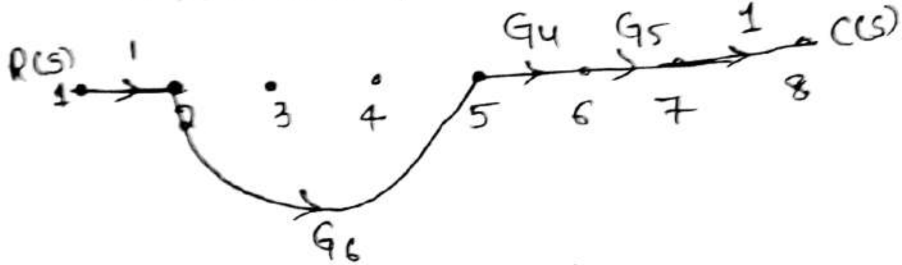
Sol

1. Forward path gain

no of forward paths(K) = 2. i.e P_1, P_2.



$P_1 = G_1G_2G_3G_4G_5$, $\Delta_1 = 1$. (∵ there is no part of graph which is not touching first forward path).



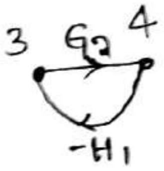
$P_2 = G_4G_5G_6$; $\Delta_2 = 1+G_2H_1$.

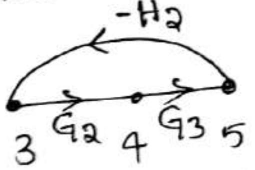
↓

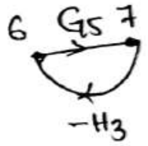
Gain of forward path-2.

2. Individual loop gain

There are three individual loops ~~gain~~ loop gains be P_{11}, P_{21}, P_{31}





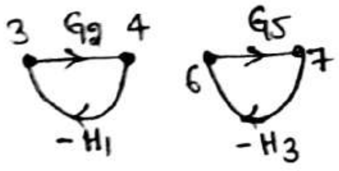


loop gain of individual loop-1 $P_{11} = -G_2H_1$

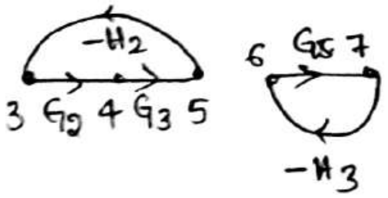
$P_{21} = -G_2G_3H_2$

$P_{31} = -G_5H_3$.

3. Gain product of two non-touching loops.



$P_{12} = P_{11}P_{31} = (-G_2H_1)(-G_5H_3) = G_2G_5H_1H_3$



$P_{22} = (-G_2G_3H_2)(-G_5H_3) = G_2G_3G_5H_2H_3$.

4. Calculation of Δ

$$\Delta = 1-(P_{11}+P_{21}+P_{31})+(P_{12}+P_{22}).$$

$$= 1-(-G_2H_1-G_2G_3H_2-G_5H_3)+(G_2G_5H_1H_3+G_2G_3G_5H_2H_3)$$

$$= 1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3.$$

5. Transfer function.

By Mason's gain formula the transfer function, T is given by.

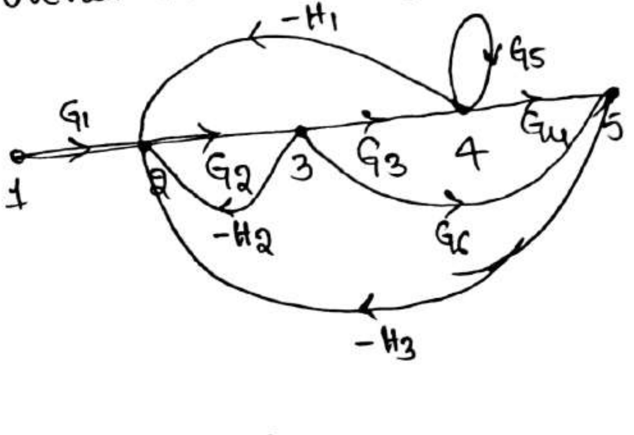
$$T=\frac{1}{\Delta}\sum_{K}P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1+P_2\Delta_2).$$

$$T = \frac{[G_1G_2G_3G_4G_5(1)]+[G_4G_5G_6(1+G_2H_1)]}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

$$= \frac{G_1G_2G_3G_4G_5+G_4G_5G_6+G_2G_4G_5G_6H_1}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

$$T = \frac{G_2G_4G_5(G_1G_3+G_6/G_2+G_6H_1)}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}.$$

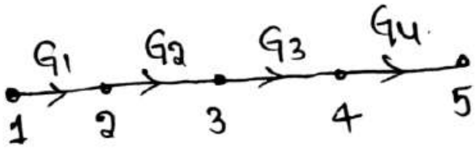
③ Find Overall T.F for the system shown in fig.

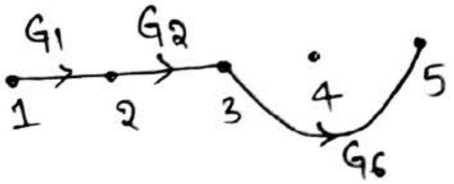


Sol

1. forward path gain

no. of forward paths k = 2.



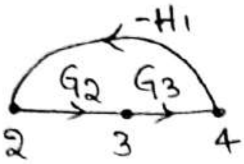


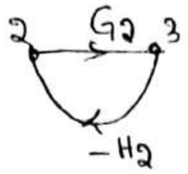
Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

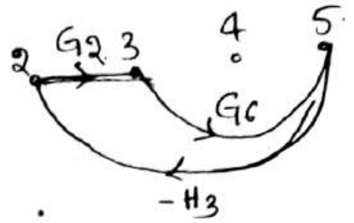
$P_2 = G_1G_2G_6$

2. Individual loop gain :-

no. of individual loops = 5.



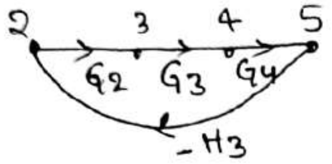




loop gain $P_{11} = -G_2G_3H_1$

$P_{21} = -H_2G_2$.

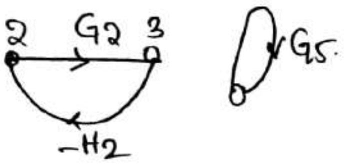
$P_{31} = -G_2G_6H_3$.



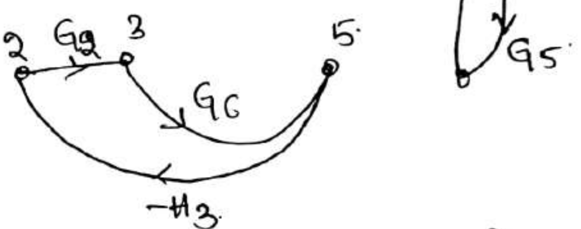
$P_{41} = -G_2G_3G_4H_3$

$P_{51} = G_5$

3. Gain products of two non-touching loops.



$P_{12} = (-G_2H_2)(G_5) = -G_2G_5H_2$



$P_{22} = (-G_2G_6H_3)(G_5)$.

$= -G_2G_5G_6H_3$.

4. Calculation of Δ & Δ_K.

$$\Delta = 1 - [-G_2G_3H_1 - G_2H_2 - G_2G_6H_3 - G_2G_3G_4H_3 + G_5] + (-G_2G_5H_2 - G_2G_5G_6H$$

$$= 1 + G_2G_3H_1 + G_2H_2 + G_2G_6H_3 + G_2G_3G_4H_3 - G_5 - G_2G_5H_2 - G_2G_5G_6H_3$$

$\Delta_1 = 1$ (There is no non-touching loops).

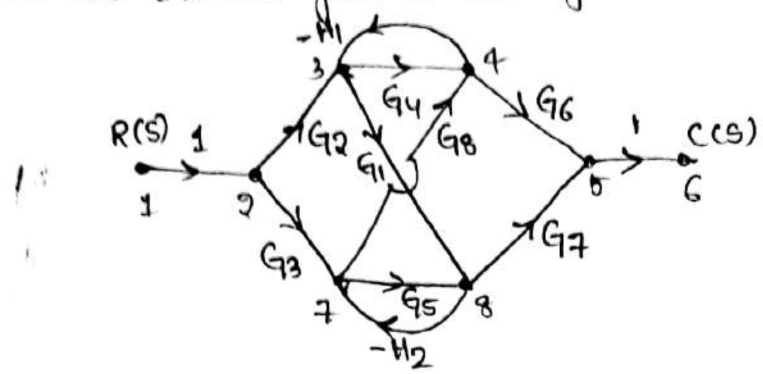
$\Delta_2 = 1 - G_5$ (one non touching loop exist)

5. Transfer function

$$T = \frac{1}{\Delta}\sum_{K} P_K \Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3G_4(1) + G_1G_2G_6(1-G_5)}{1 + G_2G_3H_1 + G_2H_2 + G_2G_6H_3 + G_2G_3G_4H_3 - G_5 - G_2H_2G_5 - G_2G_5G_6H_3}$$

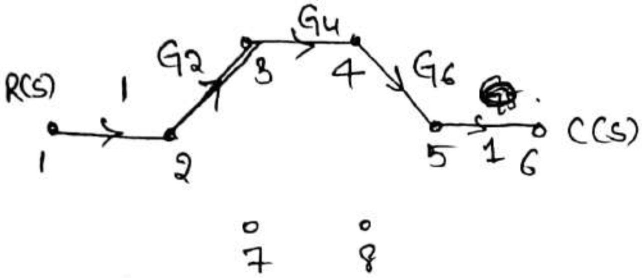
④ Find the overall gain of the system whose system is shown in fig.



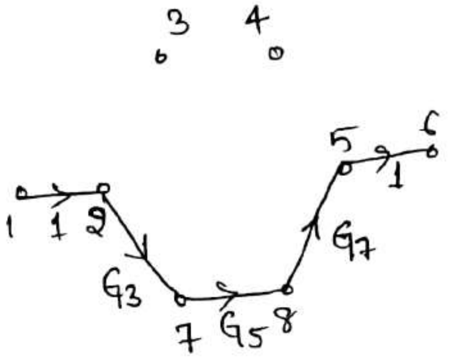
Sol Let us consider the nodes.

1. Forward path gains

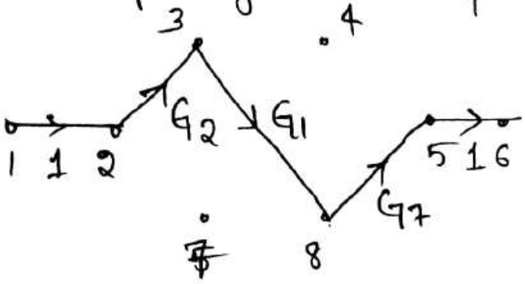
no. of forward paths K=6.



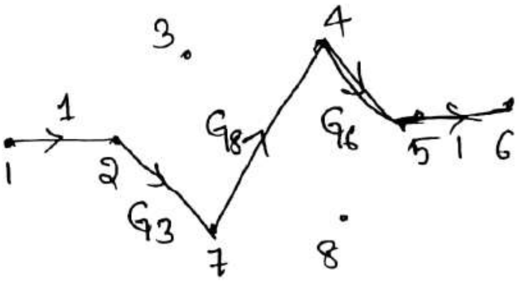
Forward path gain $P_1 = G_2G_4G_6$



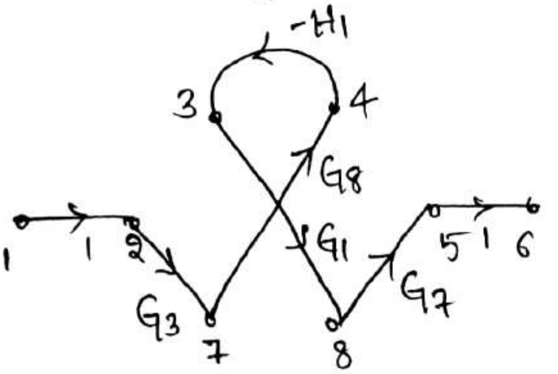
$P_2 = G_3G_5G_7$



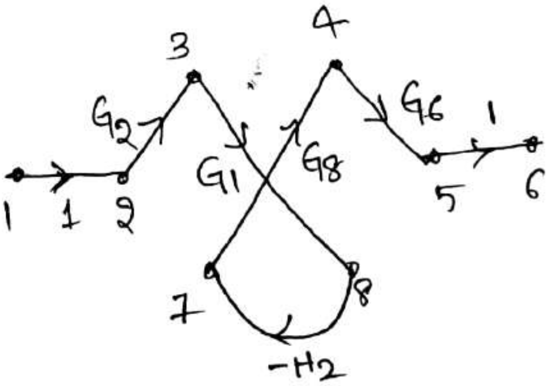
$P_3 = G_1G_2G_7$



$P_4 = G_3G_6G_8$



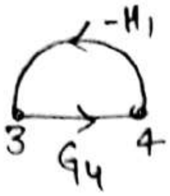
$P_5 = -G_1G_3G_7G_8H_1$



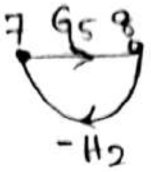
$P_6 = -G_1G_2G_6G_8H_2$

2. Individual loop gain.

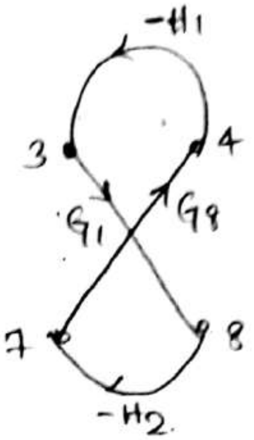
no. of individual loops: 3.



$P_{11} = -G_4H_1$

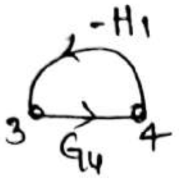


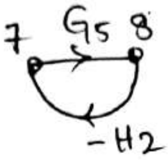
$-G_5H_2 = P_{21}$



$P_{31} = G_1G_8H_1H_2$

3. Gain products of two non-touching loops.





$P_{12} = (-G_4H_1)(-G_5H_2) = G_4G_5H_1H_2$

4. Calculation of Δ & Δ_K.

$\Delta = 1-(-G_4H_1-G_5H_2+G_1G_8H_1H_2)+(G_4G_5H_1H_2)$

$= 1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2$

$\Delta_1 = 1-(-G_5H_2) = 1+G_5H_2$ ($\because$ one non-touching loop exist)

$\Delta_2 = 1-(-G_4H_1) = 1+G_4H_1$ ($\because$ one non-touching loop exist)

$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$ (no non-touching loops).

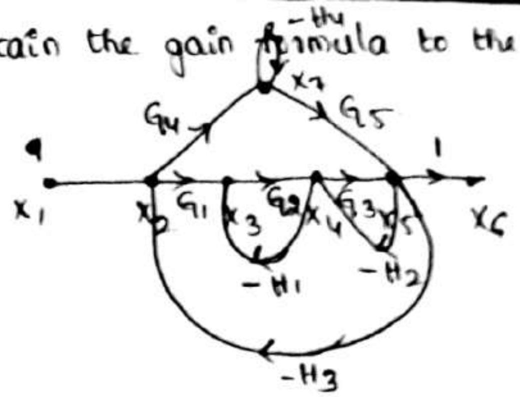
5. Transfer function

$T = \frac{1}{\Delta}\left[\sum_K P_K\Delta_K\right] = \frac{1}{\Delta}\left[P_1\Delta_1+P_2\Delta_2+P_3\Delta_3+P_4\Delta_4+P_5\Delta_5+P_6\Delta_6\right)$

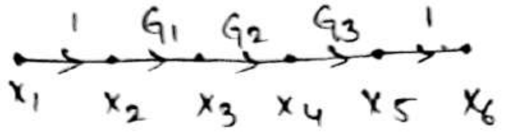
$$T = \frac{G_2G_4G_6(1+G_5H_2)+G_3G_5G_7(1+G_4H_1)+G_1G_2G_7+G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_2}{1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2}$$

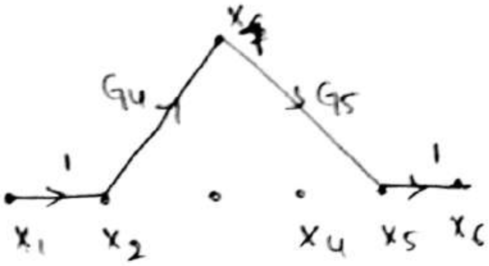
Scanned by CamScanner

⑤ Obtain the gain formula to the system shown in fig.



sol





no. of forward paths = 2.

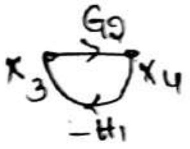
$P_1 = G_1G_2G_3$

$\Delta_1 = 1-(-H_4) = 1+H_4$

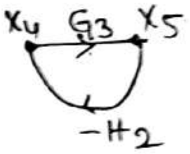
$P_2 = G_4G_5$

$\Delta_2 = 1-(G_2H_1) = 1+G_2H_1$

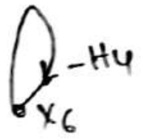
No. of individual loops:



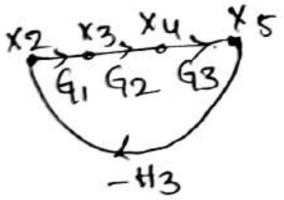
$P_{11} = -G_2H_1$



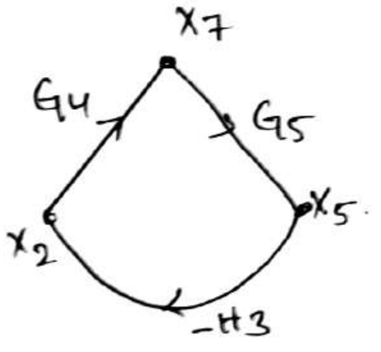
$P_{12} = -G_3H_2$



$P_{13} = -H_4$



$P_{14} = -G_1G_2G_3H_3$.



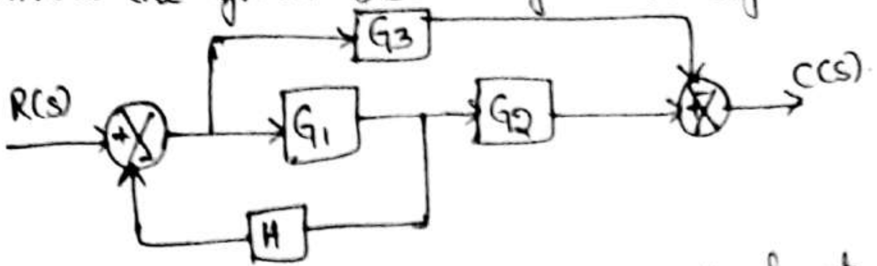
$P_{15} = -G_4G_5H_3$.

$$T = \frac{1}{\Delta}\sum_{K=1}^{n} P_K\Delta_K = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3(1+H_4) + G_4G_5(1+G_2H_1)}{1+G_2H_1+G_3H_2+H_4+G_1G_2G_3H_3+G_4G_5H_3+H_1H_4G_2+ (H_2H_4G_3 + (H_4H_3G_1G_2G_3) + G_5G_2G_4H_3H)}$$

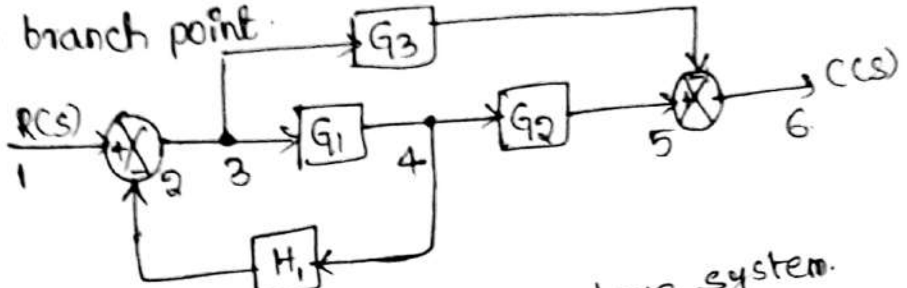
[no three non touching loops]

Convert the given block diagram to signal flow graph & det. $\frac{C(}{R}$

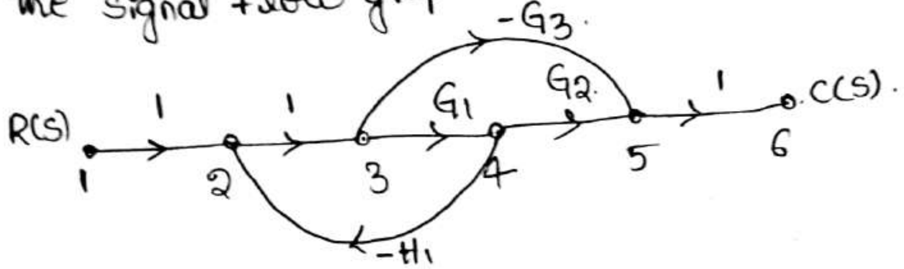


The nodes are arranged at input, output, at every summing po

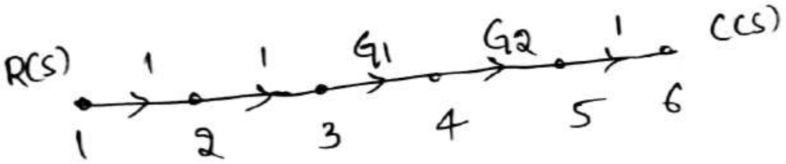
& branch point.



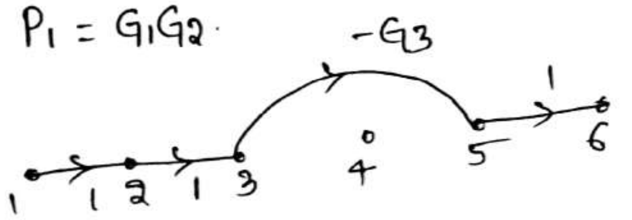
The signal flow graph of the above system.



Forward path gains $k=2$.

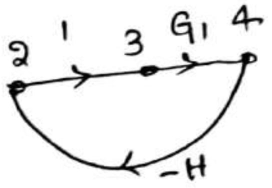


$P_1 = G_1G_2$.



$P_2 = -G_3$.

Individual loop gain.



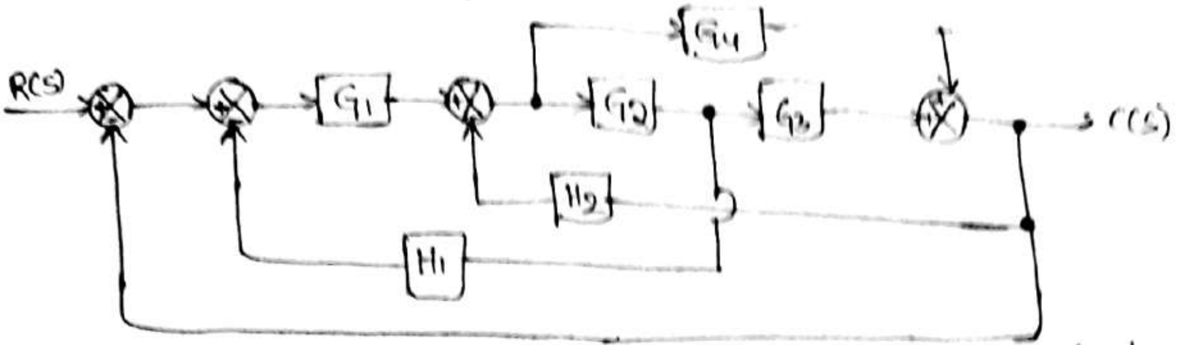
$P_{11} = -G_1H$.

No two-non-touching loops.

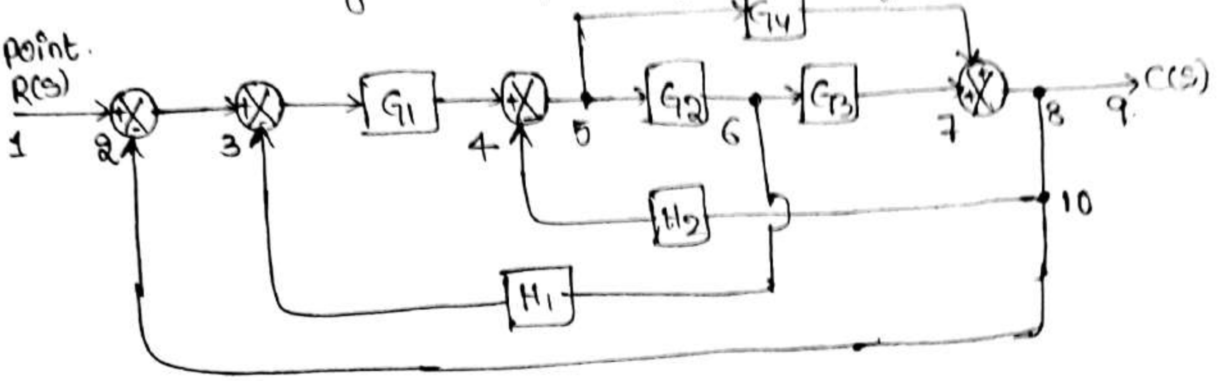
Calculation of Δ & Δ_K => $\Delta = 1+G_1H$, $\Delta_1 = \Delta_2 = 1$

Transfer function $= \frac{1}{\Delta}\sum_{K} P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) = \frac{G_1G_2 - G_3}{1+G_1H}$.

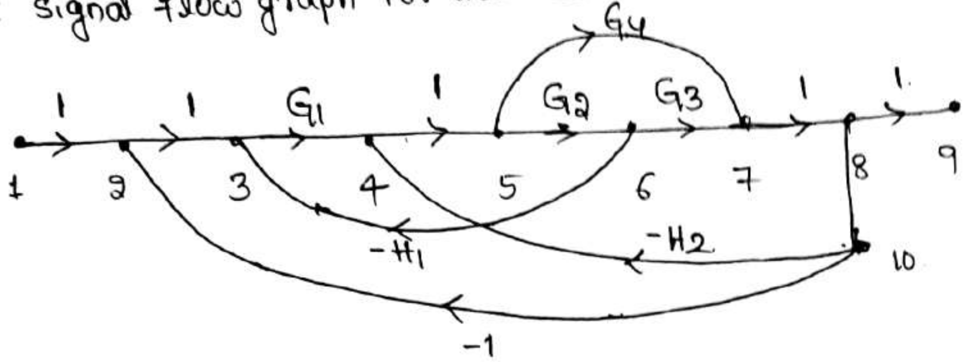
⑧ Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula



Sol The nodes are assigned at input, output, at every summing point & branch point.

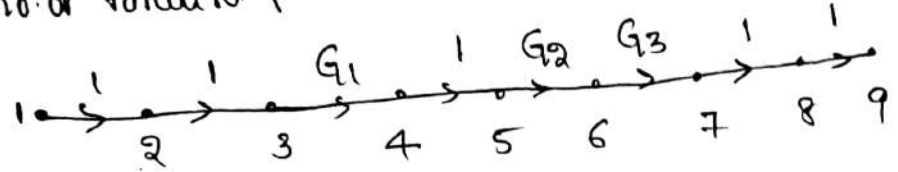


The signal flow graph for the above ckt is

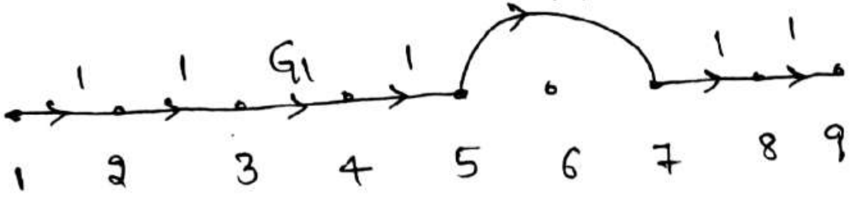


1. Forward path gain.

no. of forward paths K = 2.

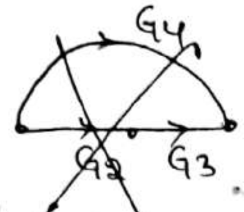


$P_1 = G_1G_2G_3$. $\Delta_1 = 1$

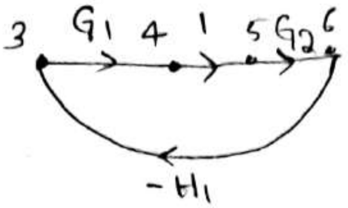


$P_2 = G_1G_4$. $\Delta_2 = 1$

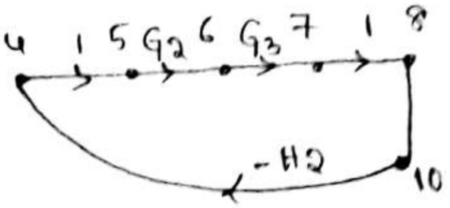
No. of individual loop:



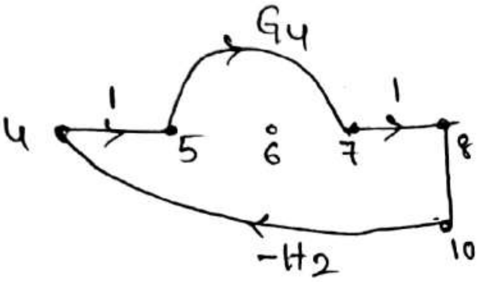
~~$P_{11} = G_2G_3G_4$~~



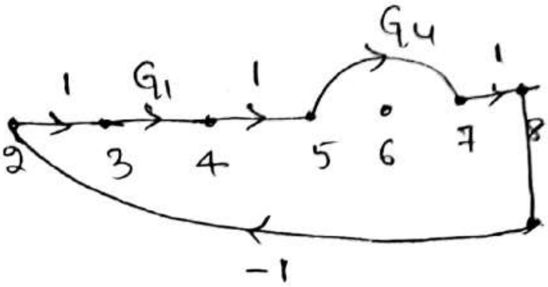
$P_{11} = -G_1G_2H_1$



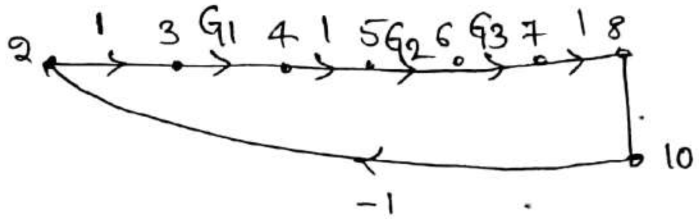
$P_{21} = -G_2G_3H_2$



$P_{31} = -G_4H_2$



$P_{14} = -G_1G_4$



$P_{15} = -G_1G_2G_3$

3. There are no possible combinations of two non-touching loops, three non-touching loops.

4. Calculation of Δ & Δ_K

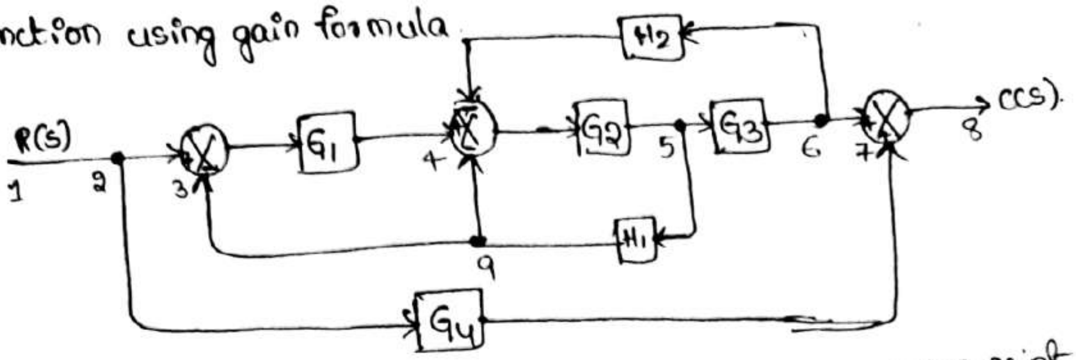
$$\Delta = 1 - [P_{11} + P_{21} + P_{31} + P_{41} + P_{51}] = 1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_4 + G_1G_2$$

5. Transfer function

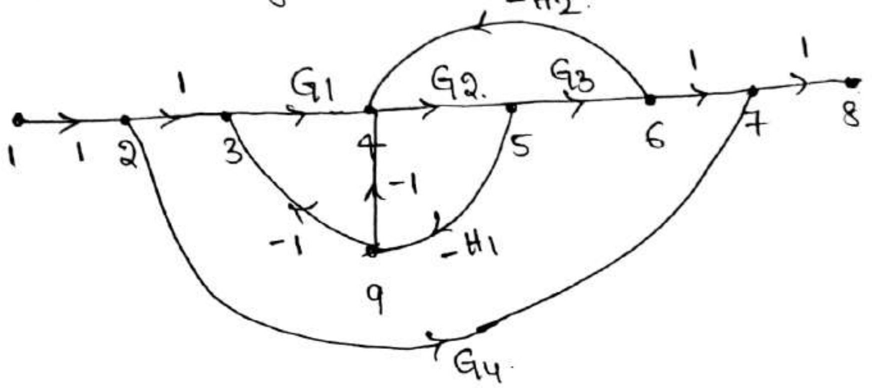
$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$T = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_4 + G_1G_2G_3}$$

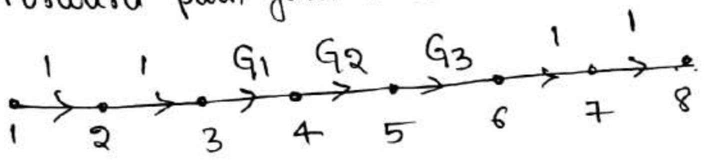
⑧ Convert the block diagram to signal flow graph & determine the transfer function using gain formula.



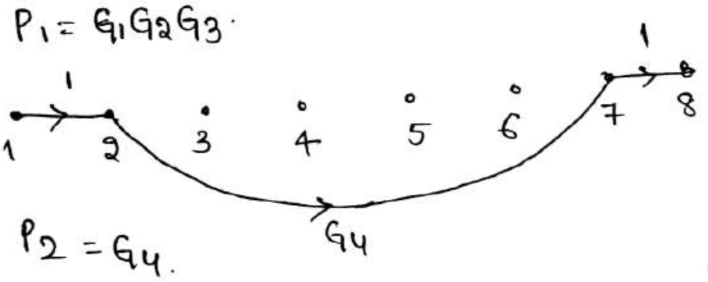
Sol The nodes are assigned at input, output, every summing point & branch pt.



1. Forward path gain K=2.



$P_1 = G_1G_2G_3$.



$P_2 = G_4$.

3. There are no possible combinations or two- non touching loops, three non-touching loops.

4. Calculation or Δ & Δ_K.

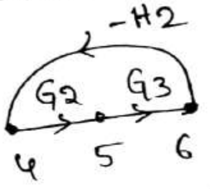
$\Delta = 1 + G_2G_3H_2 - G_1G_2H_1 - G_2H_1$.

$\Delta_1 = 1$, $\Delta_2 = 1 - (-G_1G_2H_1 - G_2G_3H2 - G_2H_1)$.

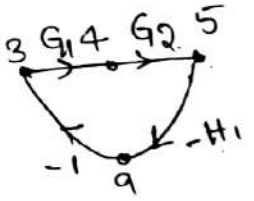
5. Transfer function.

$$T = \frac{1}{\Delta} \sum_K \Delta_K P_K = \frac{G_1G_2G_3 + G_4(1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1)}{1 + G_2G_3H_1 - G_1G_2H_1 - G_2H_1}$$

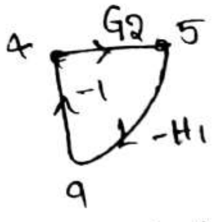
2. Individual loop gain.



$P_{11} = -G_2G_3H_2$.



$P_{12} = +G_1G_2H_1$.



$P_{13} = +G_2H_1$

Differences Between the block diagram & Signal flow graph

Block diagram	Signal flow graph
1. Basic importance given to the elements & their transfer funct^n.	1. Basic importance is given to the variables of the system.
2. Each element is represent by a block	2. Each variable is represented by a separate node.
3. T.F of the element is shown inside the corresponding block.	3. T.F is shown along the branches connecting the nodes.
4. Summing & take off points are separate.	4. Summing & take off are absent. Any node can have any no. of incoming & outgoing branches.
5. feedback path is present from o/p to i/p	5. Instead of feedback path, various feedback loops are considered for the analysis.
6. A minor feedback loop present. the formula $\frac{G}{1\pm GH}$ can be used	6. Gain of various forward paths & feedback loops are just the product of associative branch gains.
7. Block diagram reduction techniques can be used to obtain the resultant T.F.	7. The masons gain formula is available which can be used directly to get the resultant T.F.
8. Method is slightly complicated & time consuming.	8. No need to draw signal flow graph again & again.
9. Concept of self loop is not existing in Block diagram approach	9. self loop can exist in signal flow graph ~~ga~~ approach.
10. Applicable only to linear time invarient system.	10. Applicable to linear time varient & Linear time invarient system.

Transfer function of DC servo motor:

Servomotors: The motors that are used in automatic control systems are called servomotors. When the objective of the sys is to control the position of an object then the system is called servomechanism.

- The servomotors are used to convert an electrical signal to angular displacement of the shaft.
- There are variety of servomotors available for control system appl

Features of servomotors:

i) linear relationship b/w the speed & electric control signal

ii) steady state stability.

iii) Wide range of speed control.

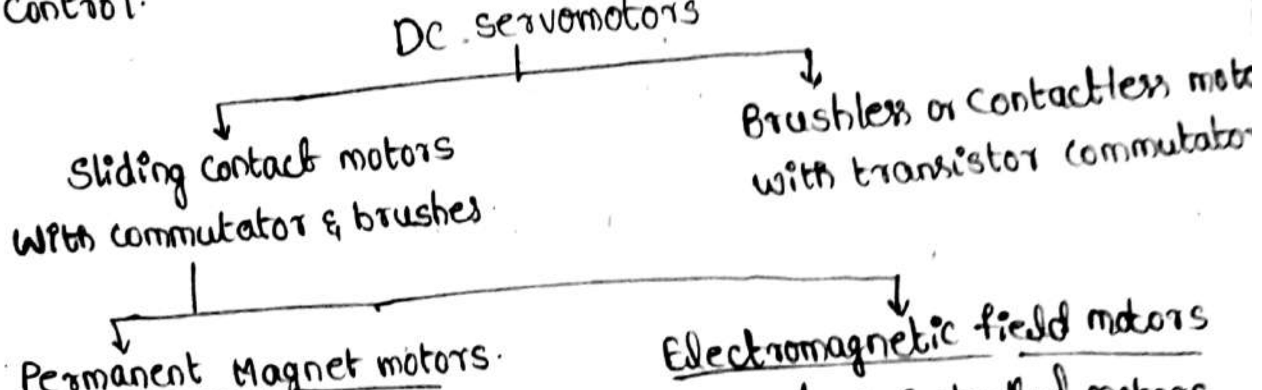
iv) Low mechanical & electrical inertia.

v) Fast response.

Depending on the supply required to run the motor, they are broadly classified as DC servomotors & AC servomotors.

→ DC motors are expensive than AC motors.

→ DC servomotors have linear characteristics & it is easier to control.



Permanent Magnet motors.

1. Cylindrical armature motors
2. DISC armature motors
3. Moving coil motors

Electromagnetic field motors

1. Armature Controlled motors
2. Field controlled motors.
3. Series motor
4. Split field motor.

Scanned by CamScanner

Transfer function of AC servomotor:

Let. T_m = Torque developed by servomotor

θ = Angular displacement of rotor

$\omega = \frac{d\theta}{dt}$ = Angular speed

T_l = Torque required by the load

J = Moment of inertia of load & the rotor

B = Viscous-frictional coefficient of load & the rotor

K_1 = slope of control phase voltage Vs Torque characteristics.

K_2 = slope of speed-torque characteristic

Torque developed by motor, $T_m = K_1 e_c - K_2 \frac{d\theta}{dt}$.

The rotating part of motor & the load can be modelled

Load torque, $T_l = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$.

At equilibrium the motor torque is equal to load torque.

$$\therefore J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_1 e_c - K_2\frac{d\theta}{dt}.$$

Apply Laplace transform with zero initial conditions.

$$J s^2\theta(s) + Bs\,\theta(s) = K_1 E_c(s) - K_2 s\,\theta(s).$$

$$\left[Js^2 + Bs + K_2 s\right]\theta(s) = K_1 E_c(s).$$

$$\therefore \frac{\theta(s)}{E_c(s)} = \frac{K_1}{s(Js+B+K_2)} = \frac{K_1/(B+K_2)}{s\left(\frac{J}{B+K_2}s+1\right)} = \frac{K_m}{s(\tau_m s+1)} \longrightarrow ①$$

where $K_m = \frac{K_1}{B+K_2}$ = Motor gain constant

$\tau_m = \frac{J}{B+K_2}$ = Motor time constant.

eq① is called transfer function of ac servomotor.

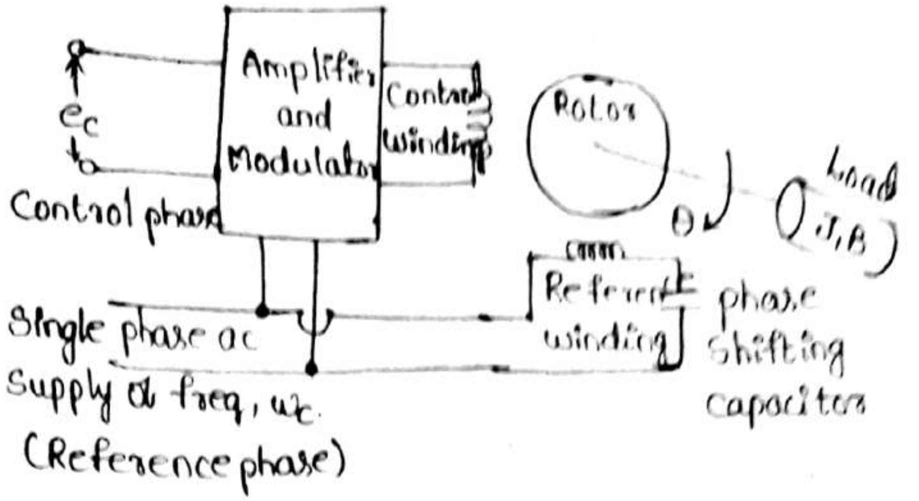
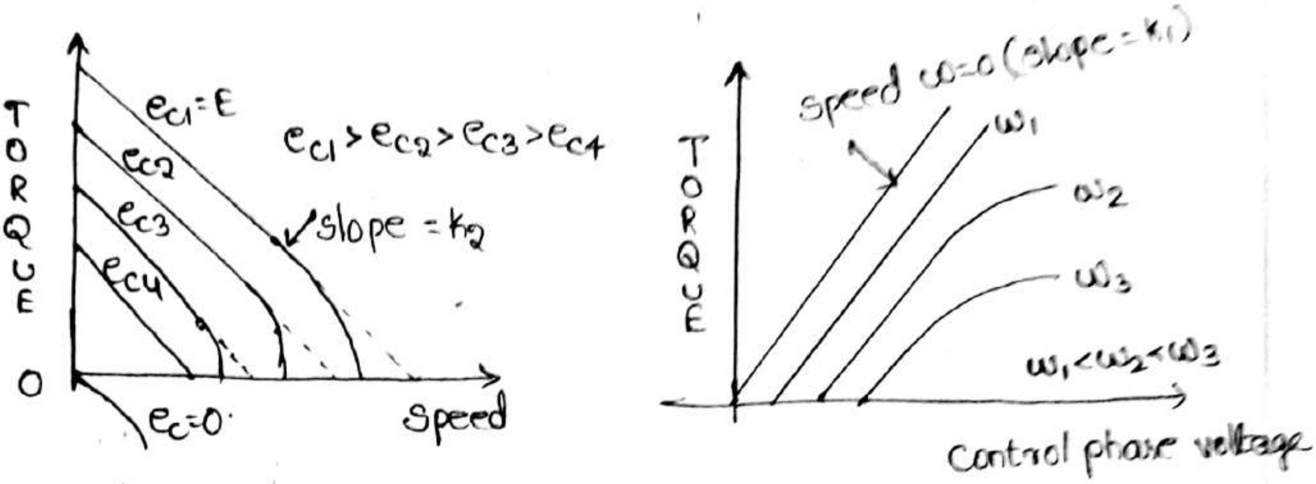


Fig: Symbolic representation of an ac servomotor

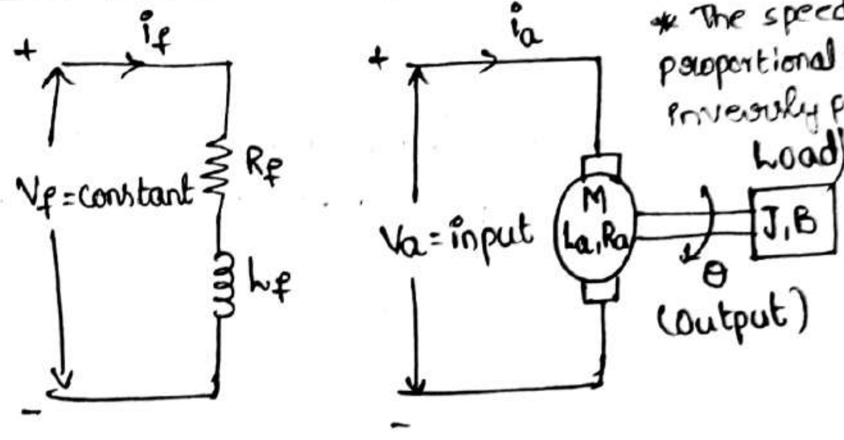


Speed-torque curves of an ac servomotor.

Control voltage Vs Torque curves of an ac servomotor.

Fig: Characteristics of ac servomotor.

Transfer function of armature Controlled Dc motor :-



* The speed of Dc motor is directly proportional to armature voltage & inversely proportional to flux in field winding.

* In armature controlled Dc motor the desired speed is obtained by varying the armature voltage.

* This speed control system is an electro-mechanical control sys.

Armature Controlled Dc motor

* The electrical system consists of the armature & field ckt but for analysis armature ckt is considered becoz the field is excited by a constant voltage.

* The mechanical system consist of rotating part of the motor & load connected to the shaft

Let R_a : Armature resistance, Ω

L_a : Armature inductance, H.

i_a : Armature current, A

V_a : Armature voltage, V

e_b : Back emf, V

K_t : Torque constant, N-m/A.

T : Torque developed by motor, N-m.

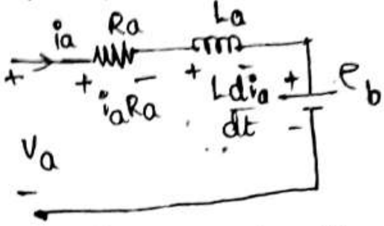
θ : Angular displacement of shaft, rad.

J : Moment of inertia of motor & load, kg-m² / rad.

B : Frictional coefficient of motor & load, N-m/(rad/sec)

K_b : Back emf constant of V/(rad/sec).

The equivalent ckt of armature is

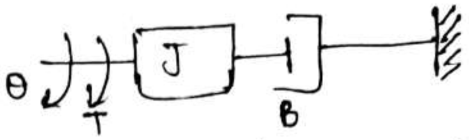


By KVL, $i_aR_a + L_a\frac{di_a}{dt} + e_b = V_a$.

Torque of DC motor is proportional to the product of flux & curre

$T \propto i_a$

∴ Torque, $T = K_t i_a$.



The differential eqⁿ governing the mechanical system of motor

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

The back emf of DC machine is proportional to speed of shaft

∴ $e_b \propto \frac{d\theta}{dt}$ or Back emf, $e_b = K_b\frac{d\theta}{dt}$.

L.T of various time domain signals involved in the system,

$L\{V_a\} = V_a(s)$; $L\{e_b\} = E_b(s)$, $L[T] = T(s)$ $L\{i_a\} = I_a(s)$, $L(\theta) =$

The differential eqⁿs governing the armature controlled DC m speed control system are,

$$i_aR_a + L_a\frac{di_a}{dt} + e_b = V_a \; ; \; T = K_t i_a \; ; \; J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T, \; e_b = K_b\frac{d\theta}{dt}$$

taking Laplace transform with zero initial conditions

$$I_a(S)R_a + L_a S I_a(S) + E_b(S) = V_a(S) \longrightarrow (i)$$

$$T(S) = K_t I_a(S) \longrightarrow ①$$

$$JS^2\theta(S) + BS\theta(S) = T(S) \longrightarrow ②$$

$$E_b(S) = K_b S\theta(S)$$

equating (1) & (2) $K_t I_a(S) = (JS^2+BS)\theta(S)$

$$I_a(S) = \frac{(JS^2+BS)}{K_t}\theta(S)$$

(i) can be written as, $(R_a+SL_a)I_a(S) + E_b(S) = V_a(S)$.

$$(R_a+SL_a)\left[\frac{JS^2+BS}{K_t}\right]\theta(S) + K_b\theta(S) = V_a(S)$$

$$\left[\frac{(R_a+SL_a)(JS^2+BS)+K_bK_tS}{K_t}\right]\theta(S) = V_a(S)$$

Required transfer function is $\frac{\theta(S)}{V_a(S)}$

$$\therefore \frac{\theta(S)}{V_a(S)} = \frac{K_t}{(R_a+SL_a)(JS^2+BS)+K_bK_tS}$$

$$= \frac{K_t}{R_aJS^2+R_aBS+L_aJS^3+L_aBS^2+K_bK_tS}$$

$$= \frac{K_t}{S[JL_aS^2+(JR_a+BL_a)S+(BR_a+K_bK_t)]}$$

$$= \frac{K_t/JL_a}{S\left[S^2+\left(\frac{JR_a+BL_a}{JL_a}\right)S+\left(\frac{BR_a+K_bK_t}{JL_a}\right)\right]}$$

The transfer function of armature controlled dc motor can be expressed in another standard form

$$\frac{\theta(S)}{V_a(S)} = \frac{K_t}{(R_a+SL_a)(JS^2+BS)+K_bK_tS}$$

$$= \frac{K_t}{R_a\left(\frac{sL_a}{R_a}+1\right)Bs\left(1+\frac{Js^2}{Bs}\right)+K_bK_t s}$$

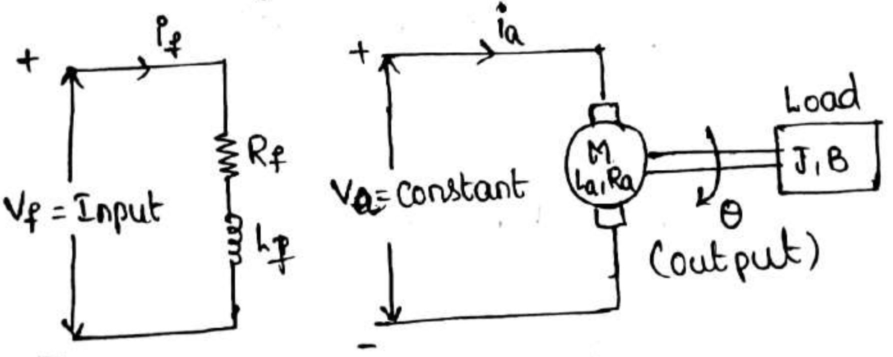
$$\boxed{\frac{\theta(s)}{V_a(s)} = \frac{K_t/R_aB}{s\left[(1+sT_a)(1+sT_m)+\frac{K_bK_t}{R_aB}\right]}}$$

$\frac{L_a}{R_a} = T_a$ = Electrical time constant.

$\frac{J}{B} = T_m$ = Mechanical time constant

Transfer function of Field controlled DC motor:-

- The speed of a DC motor is directly proportional to armature voltage & inversely proportional to flux.
- In field controlled DC motor the armature voltage is kept constant & speed is varied by varying the flux of the machine.
- flux is directly proportional to field current, the flux is varied by varying field current.
- The speed control system is an electromechanical control system.



Field controlled DC motor.

Let R_f = field resistance, Ω.

L_f = field inductance, H

i_f = field current, A.

V_f = field voltage, V.

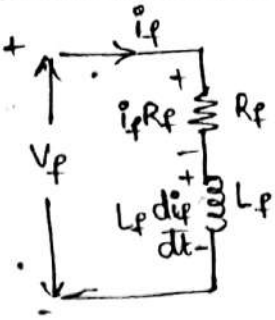
T = Torque developed by motor, N-m.

K_{tf} = Torque constant, N-m/A.

J = Moment of inertia of rotor & load, Kg-m^2/rad.

B = Frictional coefficient of rotor & load, N-m(rad/sec).

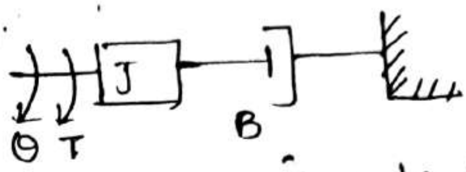
· The equivalent circuit of field is



By KVL, $R_f i_f + L_f \frac{di_f}{dt} = V_f$.

* The torque of DC motor is proportional . to product of flux & armature current.

I_a is constant, the torque is proportional to flux alone, flux is proportional to field current.



$T \alpha i_f$, ∴ Torque $= K_{tf} i_f$.

Differential eqn governing the mechanical system of motor is

$$J\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}=T.$$

The Laplace transform of various time domain signals involved in the system is,

$L\{i_f\}=I_f(s)$, $L[T]=T(s)$, $L[V_f]=V_f(s)$, $L(\theta)=\theta(s)$.

Differential eqns governing the field controlled DC motor are,

$$K_f i_f + L_f\frac{di_f}{dt} = V_f \; ; \; T = K_{tf} i_f \; ; \; J\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}=T.$$

taking Laplace transform with zero initial condition we get,

$$R_f I_f(s) + L_f s I_f(s) = V_f(s) \longrightarrow (i)$$

$$T(s) = K_{tf} I_f(s) \longrightarrow ②$$

$$Js^2\theta(s)+Bs\theta(s)=T(s) \longrightarrow ③$$

$$K_{tf}I_f(s) = Js^2\theta(s)+Bs\theta(s).$$

$$I_f(s) = s\frac{(Js+B)}{K_{tf}}\theta(s).$$

$$(R_f+sL_f)I_f(s) = V_f(s). \longrightarrow ④$$

$$(R_f + SL_f)\, S\, \frac{(JS+B)}{K_{tf}}\, \theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{S(R_f + SL_f)(B + SJ)}$$

$$= \frac{K_{tf}}{SR_f\left(1 + \frac{SL_f}{R_f}\right) B \left(1 + \frac{SJ}{B}\right)}$$

$$\boxed{\frac{\theta(s)}{V_f(s)} = \frac{K_m}{S(1+ST_f)(1+ST_m)}}$$

Where $K_m = \frac{K_{tf}}{R_f B}$ = Motor gain constant.

$T_f = \frac{L_f}{R_f}$ = Field time constant.

$T_m = \frac{J}{B}$ = Mechanical time constant.

Comparision between the AC & DC servomotors

Dc seervomotor	Ac Servomotor
1. Higher power output	1. Relatively lesser power output than DC servomotor of same size.
2. characteristics are linear.	2. characteristics are non-linear.
3. fast response due to low electrical & mechanical time constant.	3. The response is relatively slower than DC servomotors due to higher values of time constants.
4. suitable for large power applications.	4. Suitable for low power applications.
⑤ Efficiency of these motors is high	⑤ η of these motors are very less i.e 5% - 20%.

· **Synchros:-** Synchro is a device used to convert an angular ~~position~~ motion to an electrical signal or viceversa.

((Synchro is a generic name of inductive devices) which works on the principle of a rotating transformer (Induction motor). [The trade names for selsyn, Autosyn, & Telesyn]

* A synchro system is formed by interconnection of devices called synchro transmitter & synchro control transformer. It is also called synchro pair.
* synchro pair measures & compares two angular displacements & o/p voltage is approximately linear with angular difference of the axis of both shafts.

(i) To control the angular position of load from a remote place.

② For automatic correction of changes due to disturbance in the angular position of the load.

Synchro Transmitter:-

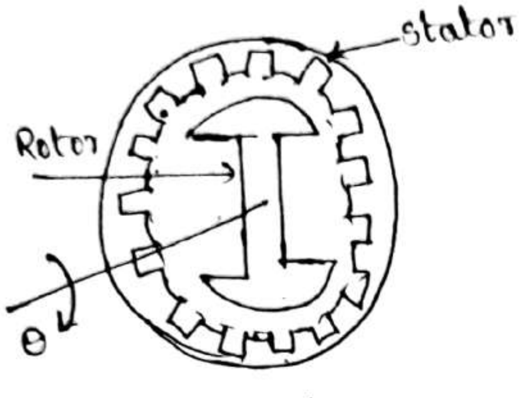
Construction:-

* It consists of two parts, they are stator & rotor. The stator is identical to the stator of 3-ϕ alternator.
* It is made up of laminated silicon steel & slotted on the inner periphery to accomdate a balanced 3-ϕ winding.
* The stator winding is concentric type with the axis of 3-coils 120° apart & stator is star connected (Y-connection)
* The rotor is dumb bell construction with a single winding. The ends of rotor windings are terminated on two slip rings.
* A single phase ac excitation voltage is applied to rotor through slip rings.

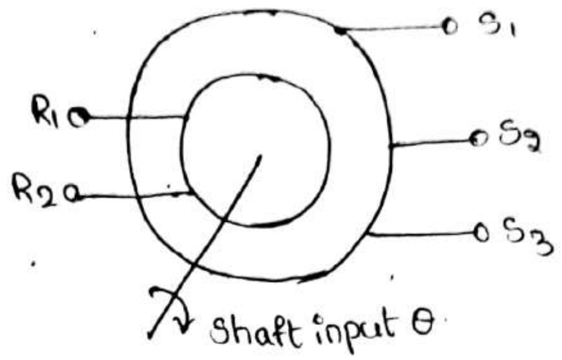
Working principle:-

When rotor is excited by ac voltage, the rotor current flows & a magnetic field is produced.

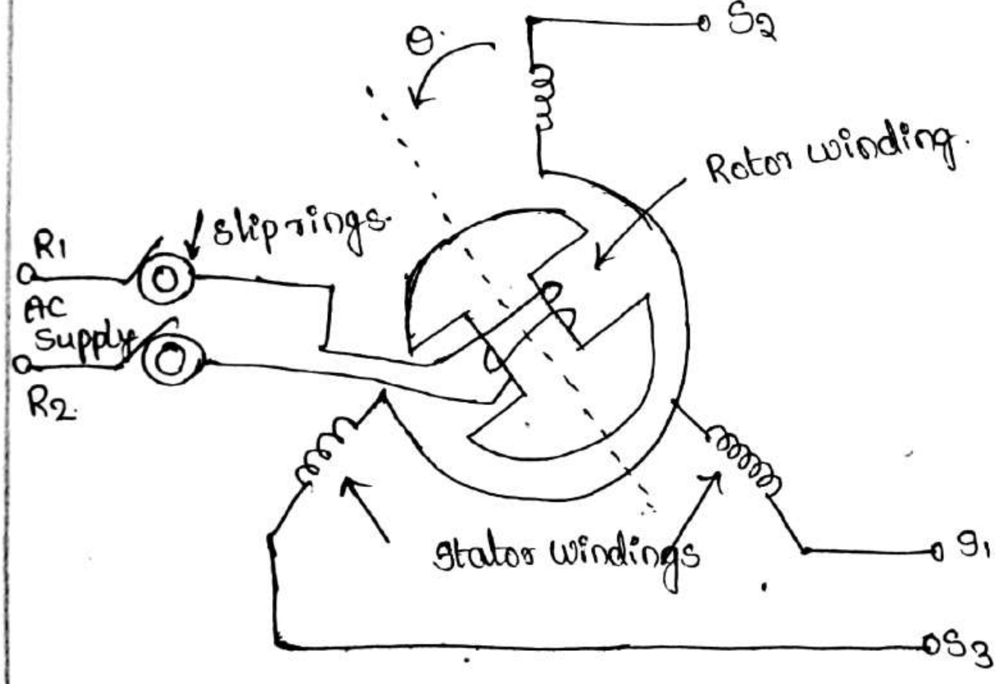
- The rotor magnetic field induces an emf in the stator coils by a transformer action.
- The effective voltage induced in any stator coil depends upon the angular position of the coil axis w.r.t rotor axis.



Constructional features.



Schematic symbol of a synchro transmitter.



Electrical Circuit
Synchro Transmitter.

e_r = Instantaneous value of ac voltage applied to rotor.

e_{S1}, e_{S2}, e_{S3} = Instantaneous value of emf induced in stator coils S1, S2, S3 w.r.t neutral respectively.

E_r = Max. value of rotor excitation voltage.

ω = Angular frequency of rotor excitation voltage.

- K_t = Turns ratio of stator & rotor windings.

K_c = coupling coefficient

θ = Angular displacement of rotor w.r.t reference.

Instantaneous value of rotor excitation voltage, $e_r = E_r \sin\omega t$

Induced emf in stator coil $= K_t K_c K_r \sin\omega t$.

$\therefore$ Coupling coefficient, K_c for coil- $S_2 = K_1 \cos\theta$.

coil - $S_3 = K_1 \cos(\theta - 120^\circ)$

coil - $S_1 = K_1 \cos(\theta - 240^\circ)$

emfs of stator coils w.r.t neutral

$$e_{S2} = K_t K_1 \cos\theta \, E_r \sin\omega t = K E_r \cos\theta \sin\omega t$$

$$e_{S3} = K_t K_1 \cos(\theta - 120^\circ) E_r \sin\omega t = K E_r \cos(\theta - \overset{120}{\cancel{240}}) \sin\omega t.$$

$$e_{S1} = K_t K_1 \cos(\theta - 240^\circ) E_r \sin\omega t = K E_r \cos(\theta - 240^\circ) \sin\omega t.$$

$$e_{S1S2} = e_{S1} - e_{S2} = \sqrt{3} K E_r \sin(\theta + 240^\circ) \sin\omega t.$$

$$e_{S2S3} = e_{S2} - e_{S3} = \sqrt{3}. K E_r \sin(\theta + 120^\circ) \sin\omega t.$$

$$e_{S3}e_{S1} = e_{S3} - e_{S1} = \sqrt{3} K E_r \sin\theta \sin\omega t.$$

$\cos(A-B) = \cos A \cos B + \sin A \sin B.$

$$e_{S1S2} = K E_r \cos(\theta - 240^\circ) \sin\omega t - K E_r \cos\theta \sin\omega t.$$

$$= K E_r [\cos\theta \cos 240^\circ + \underline{\sin\theta \sin 240^\circ}) - \cos\theta] \sin\omega t$$

$$= K E_r (\cos\theta(-0.5) + \sin\theta\left(-\frac{\sqrt{3}}{2}\right) - \cos\theta] \sin\omega t.$$

$$= K E_r \left(\sin\theta \left(\frac{-\sqrt{3}}{2}\right) + \cos\theta \left[\frac{3}{2}\right] \sin\omega t\right)$$

$$= K E_r (\sqrt{3}) \left[\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(-\frac{\sqrt{3}}{2}\right)\right] \sin\omega t.$$

$$= \sqrt{3} K E_r [\sin\theta \cos 240^\circ + \cos\theta \sin 240^\circ] \sin\omega t.$$

$$= \sqrt{3} K E_r \sin(\theta + 240^\circ) \sin\omega t.$$

$$e_{S2S3} = e_{S2} - e_{S3} = K E_r \cos\theta \sin\omega t - K E_r \cos(\theta - 120^\circ) \sin\omega t$$

$$= K E_r (\cos\theta - \cos\theta \cos 120^\circ - \sin\theta \sin 120^\circ) \sin\omega t.$$

$$= K E_r [\cos\theta - \cos\theta(-0.5) - \sin\theta\left(\frac{\sqrt{3}}{2}\right)] \sin\omega t.$$

$$= \sqrt{3} K E_r \left[\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(\frac{\sqrt{3}}{2}\right)\right] \sin\omega t$$

$$= \sqrt{3} K E_r [\sin\theta \cos 120^\circ + \cos\theta \sin 120^\circ]$$

$$= \sqrt{3} K E_r [\sin(\theta + 120^\circ)] \sin\omega t$$

$$e_{S_3S_1} = e_{S_3} - e_{S_1} = KE_r\cos(\theta-120^\circ)\sin\omega t - KE_r\cos(\theta-240^\circ)\sin\omega t$$

$$= KE_r[\cos\theta\cos120^\circ + \sin\theta\sin120^\circ - \cos\theta\cos240^\circ - \sin\theta\sin240^\circ]\sin\omega t$$

$$= KE_r[\cos\theta(-0.5) + \sin\theta(\tfrac{\sqrt{3}}{2}) - \cos\theta(-0.5) - \sin\theta(-\tfrac{\sqrt{3}}{2})]\sin\omega t$$

$$= \sqrt{3}KE_r\sin\theta\sin\omega t.$$

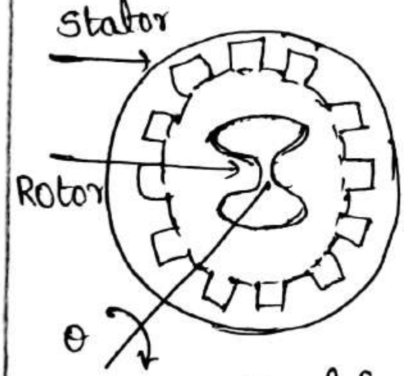
Synchro Control Transformer:-

Construction:-

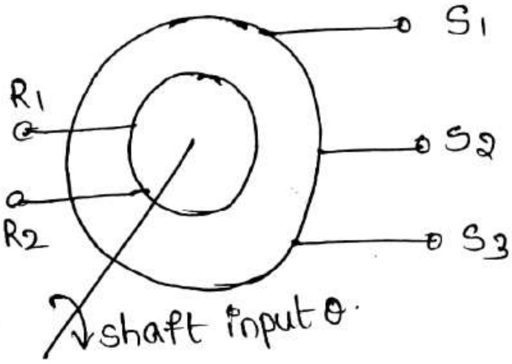
* Constructional features of synchro control transformer is similar to that of synchro transmitter except the shape of rotor.

* The rotor is made up of cylindrical so that air gap is practically uniform.

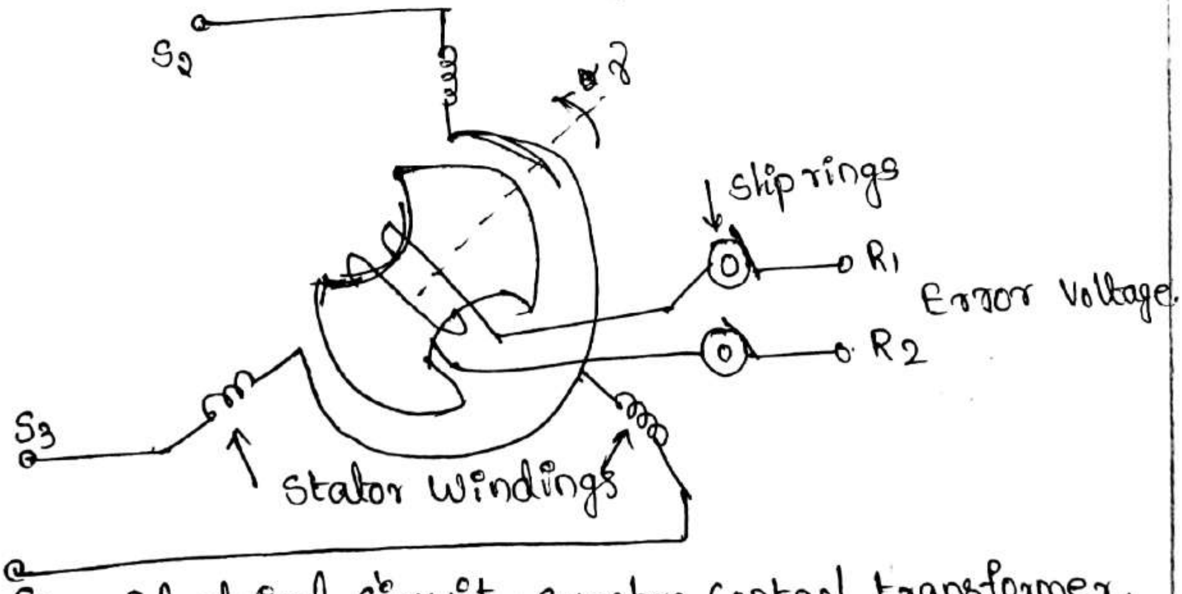
* It minimizes the changes in rotor impedance with the rotation of the shaft.



Constructional features.



Schematic symbol of a synchro Control transformer.



Electrical circuit, synchro control transformer.

· Working:-

* The generated emf of the synchro transmitter is applied as input to the stator coils of control transformer.
* The rotor shaft is connected to the load whose position has to be maintained at desired value.
* Depending on current position of the rotor & applied emf on the stator, an emf is induced in the rotor winding.
* emf can be measured & to drive a motor so that the position of the load is corrected.

Transfer function for synchro Transmitter & Receiver :-

Let θ = Angle of rotation of transmitter from electrical zero position.

γ = Angle of rotation of the receiver from the electrical zero position.

Then the torque produced by synchro control transformer is given by $T_\gamma(t) = K[\theta(t) - \gamma(t)] \rightarrow ①$.

Apply L.T $\Rightarrow T_\gamma(s) = K[\theta(s) - \gamma(s)] \rightarrow ②$

where K = sensitivity of synchro error detector.

The torque developed/produced by synchro transmitter is

$$T_\gamma(t) = J \cdot \frac{d^2\gamma}{dt^2} + B\frac{d\gamma}{dt} \rightarrow ③$$

Apply L.T $T_\gamma(s) = J \cdot s^2\gamma(s) + Bs\gamma(s) \rightarrow ④$

compare (2) & (4) $\Rightarrow K[\theta(s) - \gamma(s)] = Js^2\gamma(s) + Bs\gamma(s)$

$$T.F \Rightarrow \boxed{\frac{\gamma(s)}{\theta(s)} = \frac{K}{Js^2 + Bs + K}}$$

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Mathematical Models of control system (3 types)

1. Mechanical translational system ② Mechanical rotational system
3. Electrical system

1. Mechanical translational system: (linear displacement)

Any model of mechanical system consist of 3 basic inputs

① Mass ② Dashpot ③ spring
↓ ↓ (piston element) ↓
force friction elasticity.

2. Mechanical Rotational (angular displacement)

Moment of inertia friction elasticity.

3. Electrical – R,L,C.

Symbols used in mechanical translational system:

x = displacement (Meters)

V = velocity $\left(\frac{dx}{dt}\right)$ (m/sec)

a = acceleration $\left(\frac{dv}{dt} = \frac{d^2x}{dt^2}\right)$ $(m/s^2]$

f : Applied force (N)

f_m = opposing force offered by mass of the body, N

f_K = opposing force offered by elasticity of the body (spring), N

f_b = opposing force offered by friction of body (dashpot).

M = Mass (kg)

K : stiffness of spring. (N/m)

B = Viscous friction coefficient. (N-sec/m).

Force balanced equations of ideal elements:

1)

Ideal mass element $f \propto \frac{d^2x}{dt^2} \Rightarrow f = M\frac{d^2x}{dt^2}$

f_m → opposing force offered by mass of the body.

$$f = f_m = M\frac{d^2x}{dt^2}.$$

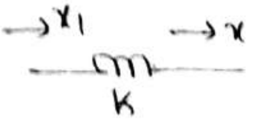
2) 

$f_b = B\frac{d}{dt}(x_1 - x)$.

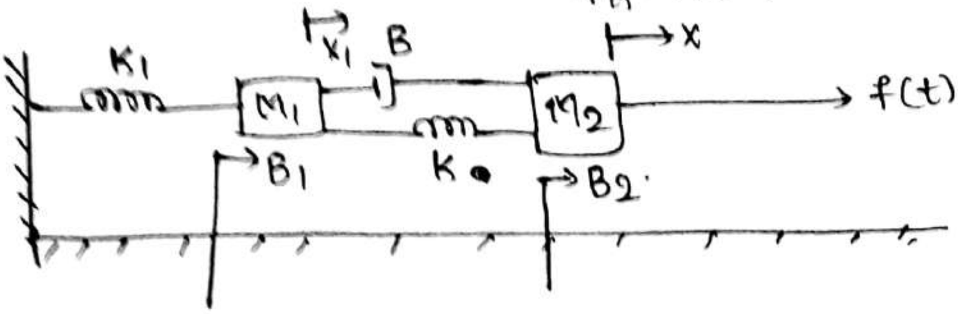
$f \propto \frac{dx}{dt} \Rightarrow f = B\frac{dx}{dt}$, $f_b = B\frac{dx}{dt}$ → opposing force offered by the friction of the body.

3) 

$f \propto x \Rightarrow f_K = Kx$



$f_K = K(x_1 - x)$.

ex: 

Nodes indicate the mass or element

Det. the transfer function of a given system:-

1) Identify the no. of nodes (mass elements) in a given system.

- Draw the freebody diagram of each body separately.
- Apply Newtons second law $\Sigma F = 0$
- Apply laplace transform for the equation
- Rearrange the differential eq^n to obtain a real function.

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Mathematical Models :-

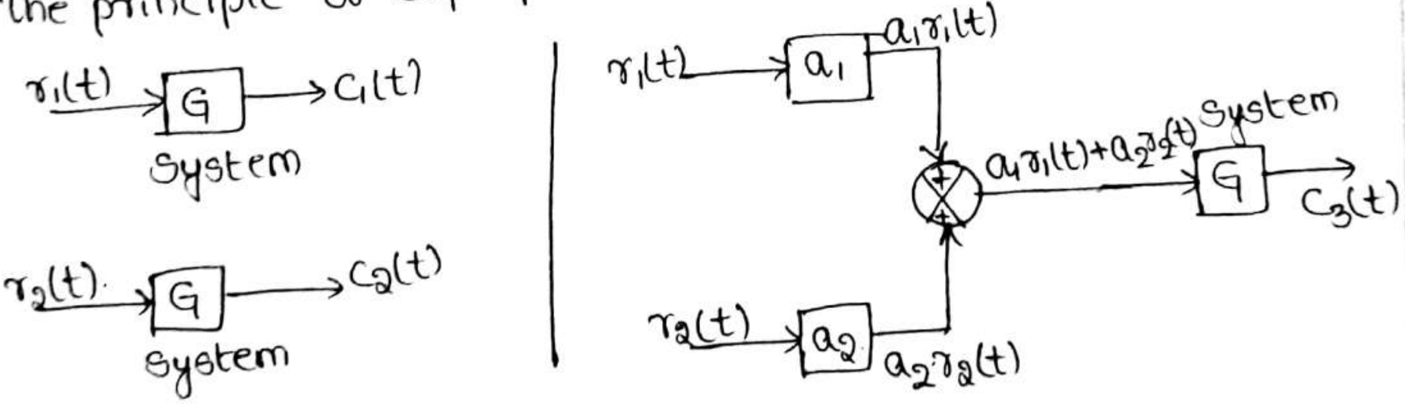
A control system is a collection of physical objects connected together to serve an objective The input output relations of various physical components of a system are governed by differential equations.

* The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

* The mathematical model of a system is linear if it obeys the principle of superposition & homogenity.

* The principle implies that if a system model has responses $y_1(t)$ & $y_2(t)$ to any inputs $x_1(t)$ & $x_2(t)$

The principle of superposition can be explained diagramatically.

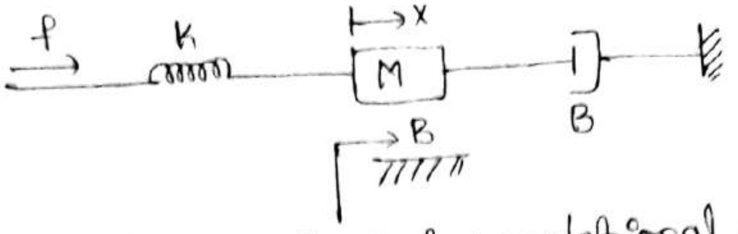


Principle of linearity & superposition.

If $\boxed{C_3(t) = a_1C_1(t) + a_2c_2(t)}$ then system G is linear.

$$\text{Transfer function} = \left.\frac{\text{Laplace Transform of O/P}}{\text{Laplace Transform of I/P}}\right|_{\text{with zero initial conditions}}$$

Mechanical translational system:-



The model of mechanical translational system can be obtained by using three basic elements.

① Mass ② Spring or elasticity ③ Dashpot or friction.

* The weight of the mechanical system is represented by the element "mass" & it is assumed to be concentrated at the center of the body.

* The elastic deformation of the body can be represented by a "spring". The friction existing in rotating mechanical system can be represented by a "dash-pot" (or) "friction".

* The force acting on a mechanical body are governed by "Newton's Second law of motion".

* For translational systems it states that the sum of forces acting on a body is zero.

List of symbols used in mechanical translational system:-

x = Displacement (meters).

v = velocity ($\frac{dx}{dt}$) (m/sec)

a = acceleration = $\frac{dv}{dt} = \frac{d^2x}{dt^2}$ (m/s^2).

f = Applied force, N (Newtons).

f_m = Opposing force offered by mass of the body, N.

f_b = opposing force offered by the friction of the body (dash-pot)

f_K = Opposing force offered by the elasticity of the body (spring)

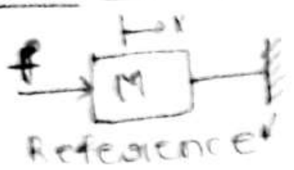
M = Mass, (Kgs)

K = stifness of spring (N/m)

B = Viscous friction coefficient, N-sec/m.

Force balance equations of idealized elements :-

Ideal Mass element :



Consider an ideal mass element which has negligible friction & elasticity.

$f \propto a$

$f_m = Ma$

$f_m = M\left(\frac{d^2x}{dt^2}\right)$

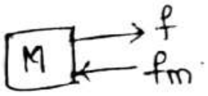
Let a force be applied as it the mass will offer opposing force which is proportional to acceleration of the body.

Let 'f' be the applied force.

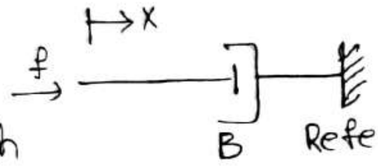
'f_m' opposing force offered by mass of the body.

$f_m \propto a$; $f_m = Ma$; $f_m = M\frac{d^2x}{dt^2}$

$f = f_m = M\frac{d^2x}{dt^2}$ $\quad f - f_m = 0$



Ideal dashpot element :-



Ideal dashpot with one end fixed to reference.

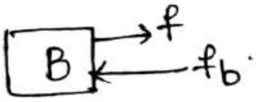
Consider an ideal dashpot element shown in fig. which has negligible mass & elasticity.

Let a force be applied on it. the friction will offer opposing force which is proportional to velocity of the body.

Let 'f' be the applied force.

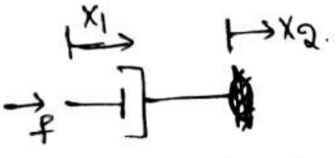
f_b = opposing force offered by the friction of the body.

$f_b \propto v$; $f_b = B\frac{dx}{dt}$



$f - f_b = 0$

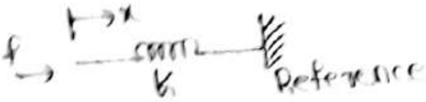
$f = f_b = B\frac{dx}{dt} = B.\frac{d(x_1 - x_2)}{dt}$.



Ideal dashpot with displacement at both ends.

3. Ideal spring element :

Consider an ideal spring element shown in fig. which has negligible mass & friction



Let a force applied on it the elasticity Ideal spring with one end will offer opposing force which is proportional fixed to reference to displacement of the body.

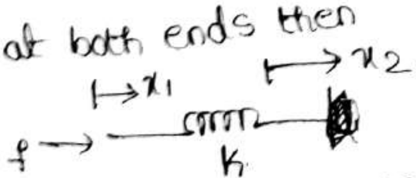
Let f = applied force.

f_K = opposing force due to elasticity.

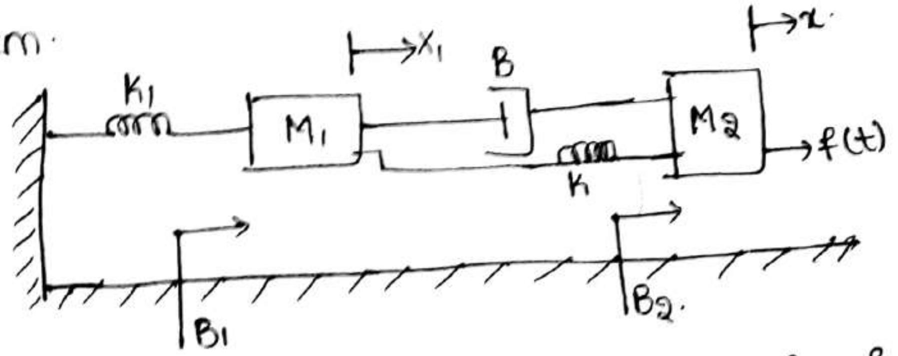
$f_K \alpha x$ (or) $f_K = Kx$. K = stiffness of the spring.

When the spring has displacement at both ends then

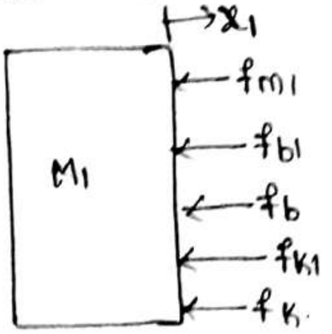
$f = f_K = K(x_1 - x_2)$.



① Write the differential equations governing the mechanical system shown in fig. & det. the transfer function of the system.



Sol The freebody diagram of mass M_1. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{K1}, f_K.



$f_{m1} = M_1 \frac{d^2x_1}{dt^2}$

$f_b = B\frac{d}{dt}(x_1 - x)$

$f_{b1} = B_1 \frac{dx_1}{dt}$

$f_K = K(x_1 - x)$.

$f_{K1} = K_1 x_1$

Freebody diagram of mass M_1.
(node 1).

By newtons second law

$f_{m1} + f_{b1} + f_b + f_{K1} + f_K = 0$.

$$\therefore M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B\frac{d}{dt}(x_1-x) + K_1x_1 + K(x_1-x) = 0$$

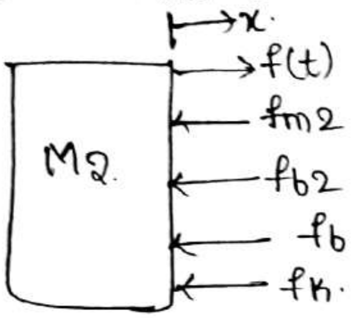
apply L.T with zero initial conditions

$$M_1s^2X_1(s) + B_1sX_1(s) + Bs[X_1(s)-X(s)] + K_1X_1(s) + K[X_1(s)-X(s)] = 0$$

$$X_1(s)[M_1s^2+(B_1+B)s+(K_1+K)] - X(s)[Bs+K] = 0$$

$$\therefore X_1(s) = X(s)\frac{[Bs+K]}{M_1s^2+(B_1+B)s+(K_1+K)} \longrightarrow ①$$

The freebody diagram of mass M_2 ~~&~~ & opposing forces acting on M_2 are f_{m2}, f_{b2}, f_b & f_K.



$$f_K = K(x-x_1)$$

$$f_{m2} = M_2\frac{d^2x}{dt^2}$$

$$f_{b2} = B_2\frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x-x_1)$$

Freebody diagram of mass M_2(node 2).

$$f_{m2} + f_{b2} + f_b + f_K = f(t)$$

$$M_2\frac{d^2x}{dt^2} + B_2\frac{dx}{dt} + B\frac{d}{dt}(x-x_1) + K(x-x_1) = f(t)$$

Apply L.T with zero initial conditions.

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s)-X_1(s)] + K[X(s)-X_1(s)] = F(s)$$

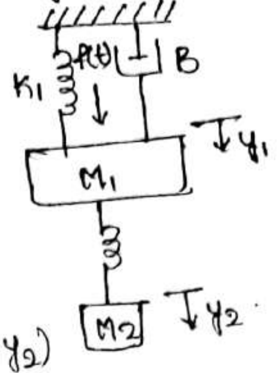
$$X(s)[M_2s^2+(B_2+B)s+K] - X_1(s)[Bs+K] = F(s)$$

$$X(s)[M_2s^2+(B_2+B)s+K] - X(s)\left(\frac{(Bs+K)}{M_1s^2+(B_1+B)s+(K_1+K)}\right)(Bs+K) = F(s)$$

$$X(s)\left(\frac{[M_2s^2+(B_2+B)s+K][M_1s^2+(B_1+B)s+(K_1+K)] - (Bs+K)^2]}{M_1s^2+(B_1+B)s+(K_1+K)}\right) = F(s)$$

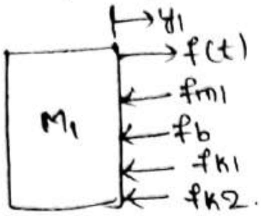
$$\boxed{\therefore \frac{X(s)}{F(s)} = \frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)][M_2s^2+(B_2+B)s+K] - (Bs+K)^2}}$$

② Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system.



Sol The freebody diagram of Mass M_1.

The opposing forces are $f_{m1}, f_b, f_{k1}, f_{k2}$.



$f_{m1} = M_1\frac{d^2y_1}{dt^2}$ $f_{k1} = K_1y_1$

$f_b = B\frac{dy_1}{dt}$ $f_{k2} = K_2(y_1-y_2)$

By newtons second law, $f_{m1}+f_b+f_{k1}+f_{k2} = f(t)$.

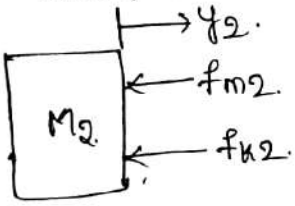
$\therefore M_1\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1y_1 + K_2(y_1-y_2) = f(t)$.

Apply L.T with zero initial conditions.

$M_1s^2Y_1(s) + BsY_1(s) + K_1Y_1(s) + K_2(Y_1(s) - Y_2(s)] = F(s)$.

$Y_1(s)\{M_1s^2+Bs+(K_1+K_2)] - Y_2(s)K_2 = F(s)$. $\longrightarrow$ ①

The freebody diagram of mass M_2. opposing forces acting on M_2 are f_{m2} & f_{k2}.



$f_{m2} = M_2\frac{d^2y_2}{dt^2}$

$f_{k2} = K_2(y_2-y_1)$.

By N.S.L (Newtons Second law). $f_{m2}+f_{k2} = 0$.

$\therefore M_2\frac{d^2y_2}{dt^2} + K_2(y_2-y_1) = 0$.

Apply L.T with I.C = 0 ⇒ $M_2s^2Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$.

$Y_2(s)\{Ms^2+K_2] - Y_1(s)K_2 = 0$.

$$\therefore \boxed{Y_1(s) = Y_2(s)\frac{M_2s^2+K_2}{K_2}} \longrightarrow ②$$

Sub ② in eq ①

$$Y_2(s)\frac{M_2s^2+K_2}{K_2}[M_1s^2+Bs+(K_1+K_2)] - Y_2(s)K_2 = F(s).$$

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{[M_1s^2+Bs+(K_1+K_2)][M_2s^2+K_2] - K_2^2}.$$

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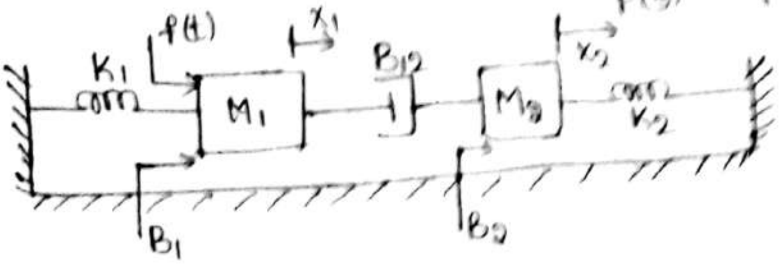
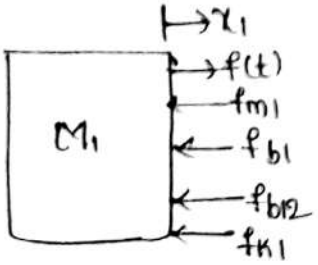
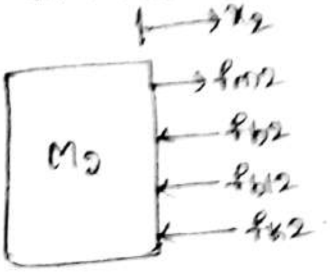
③ Determine the transfer function, $\frac{X_1(s)}{F(s)}$ & $\frac{X_2(s)}{F(s)}$ for the system.



Sol

The freebody diagrams of M1 & M2 & opposing forces are marked as $f_{m1}, f_{b1}, f_{B12}, f_{K1}$ & f_{m2}, f_{b2}, f_{b12} & f_{K2}.





Newtons second law $f_{m1}+f_{b1}+f_{b12}+f_{K1} = f(t)$.

$$M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d(x_1-x_2)}{dt}+K_1x_1 = f(t).$$

Apply L.T. $M_1s^2X_1(s)+B_1sX_1(s)+B_{12}s[X_1(s)-X_2(s)]+K_1X_1(s) = F(s)$

$$X_1(s)[M_1s^2+(B_1+B_{12})s+K_1] - B_{12}sX_2(s) = F(s). \longrightarrow ①$$

$f_{m2}+f_{b2}+f_{b12}+f_{K2} = 0$.

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+B_{12}\frac{d(x_2-x_1)}{dt}+K_2x_2 = 0 \longrightarrow 2$$

$$M_2s^2X_2(s)+B_2sX_2(s)+B_{12}s[X_2(s)-X_1(s)]+K_2X_2(s) = 0.$$

$$X_2(s)[M_2s^2+(B_2+B_{12})s+K_2] - B_{12}sX_1(s) = 0.$$

$$X_2(s) = \frac{B_{12}sX_1(s)}{[M_2s^2+(B_2+B_{12})s+K_2]} \longrightarrow (2)$$

From (1). $$X_1(s)\{M_1s^2+(B_1+B_{12})s+K_1) - \frac{(B_{12}s)^2X_1(s)}{(M_2s^2+(B_2+B_{12})s+K_2)} = F(s).$$

$$X_1(s)\left[\frac{[M_1s^2+(B_1+B_{12})s+K_1][M_2s^2+(B_2+B_{12})s+K_2]-(B_{12}s)^2}{(M_2s^2+(B_2+B_{12})s+K_2)}\right] = F(s)$$

$$\left[\frac{X_1(s)}{F(s)} = \frac{M_2s^2+(B_2+B_{12})s+K_2}{[M_1s^2+(B_1+B_{12})s+K_1][M_2s^2+(B_2+B_{12})s+K_2]-(B_{12}s)^2}\right]$$

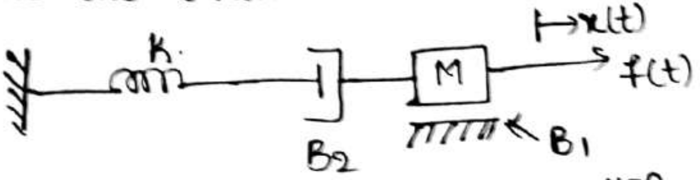
From(i) $X_1(s) = \frac{[M_2s^2+(B_2+B_{12})s+K_2]X_2(s)}{B_{12}s}$

$$\frac{X_2(s)[M_2s^2+(B_2+B_{12})s+K_2]}{B_{12}s}[M_1s^2+(B_1+B_{12})s+K_1] - B_{12}sX_2(s) = F(s)$$

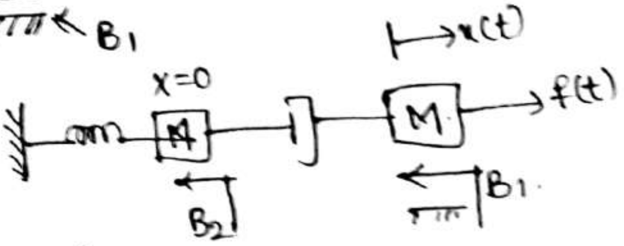
$$X_2(s)\left[\frac{[M_2s^2+(B_2+B_{12})s+K_2][M_1s^2+(B_1+B_{12})s+K_1]-(B_{12}s)^2}{B_{12}s}\right] = F(s).$$

$$\boxed{\frac{X_2(s)}{F(s)} = \frac{B_{12}s}{[M_2s^2+(B_2+B_{12})s+K_2][M_1s^2+(B_1+B_{12})s+K_1]-(B_{12}s)^2}}$$

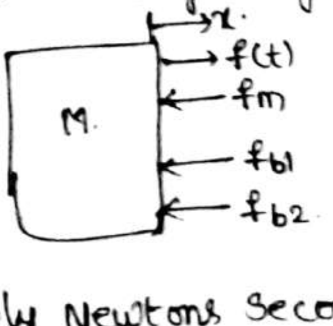
④ Write the equations of motion in s-domain for the system. Determine the transfer function of the system.



Sol By rearranging the system.



The freebody diagram of mass M is & opposing forces are f_m, f_{b1}, f_{b2}.



$$f_m = M\frac{d^2x}{dt^2},\ f_{b1} = B_1\frac{dx}{dt},\ f_{b2} = B_2\frac{d}{dx}(x-x_1).$$

Apply Newtons second law. $f_m+f_{b1}+f_{b2} = f(t)$.

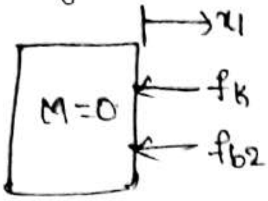
$$\therefore M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x-x_1) = f(t).$$

L.T. $\Rightarrow Ms^2X(s) + B_1sX(s) + B_2s[X(s)-X_1(s)] = F(s).$

$$[Ms^2+(B_1+B_2)s]X(s) - B_2sX_1(s) = F(s).$$

The freebody diagram at the meeting point of spring & dashpot.

opposing forces are f_K & f_{b2}.



$$f_{b2}+f_K=0.$$

$$B_2\frac{d}{dt}(x_1-x)+Kx_1=0.$$

Apply L.T. $B_2s[X_1(s)-X(s)]+KX_1(s)=0$

$$[B_2s+K]X_1(s)-B_2sX(s)=0.$$

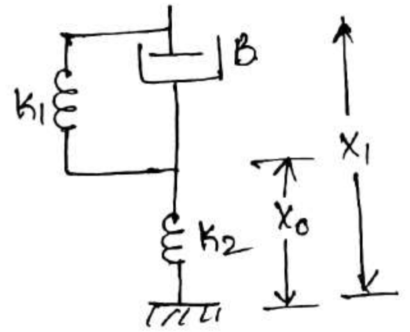
$$\therefore X_1(s)=\frac{B_2s}{B_2s+K}X(s).$$

$$\Rightarrow [Ms^2+(B_1+B_2)s]X(s)-B_2(s)\left[\frac{B_2s}{B_2s+K}\right]X(s)=F(s).$$

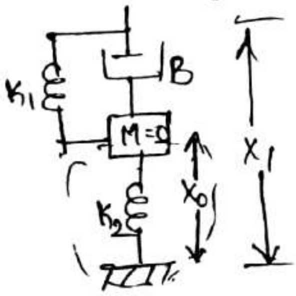
$$X(s)\left[\frac{[Ms^2+(B_1+B_2)s][B_2s+K]-(B_2s)^2}{(B_2s+K)}\right]=F(s).$$

$$\therefore \boxed{\frac{X(s)}{F(s)}=\frac{(B_2s+K)}{[Ms^2+(B_1+B_2)s](B_2s+K)-(B_2s)^2}}$$

⑤ Obtain the T.F of $\frac{X_0(s)}{X_1(s)}$ of the following system shown in fig.



sol By rearranging the system,



$$B\frac{d}{dt}(x_0-x_1)+k_1(x_0-x_1)+k_2x_0=0.$$

$$Bs(X_0(s)-X_1(s)]+k_1(X_0(s)-X_1(s)]+k_2X_0(s)=0.$$

$$X_0(s)[sB+k_1+k_2]-X_1(s)[Bs+k_1]=0.$$

$$\boxed{\frac{X_0(s)}{X_1(s)}=\frac{sB+k_1}{sB+k_1+k_2}.}$$

Mechanical Rotational system:

* The model of rotational mechanical systems can be obtained by using three elements, moment of inertia (J) & mass, dash pot with rotational frictional coefficient (B), torsional spring with stiffness (K).

List of symbols used in Mechanical rotational system:

θ = Angular displacement, rad.

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec.

$\frac{d\theta}{dt^2}$ = Angular acceleration, rad/sec^2.

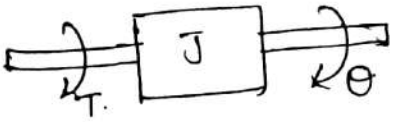
T = Applied torque, N-m.

J = Moment of inertia, Kg-m^2/rad.

B = Rotational frictional coefficient; (N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad.

Torque Balance Equations of idealised elements:-



Ideal rotational mass element.

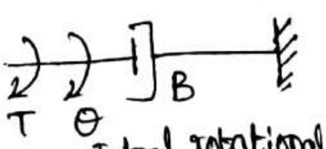
The opposing torque due to moment of inertia is proportional to the angular acceleration.

T = Applied force.

T_j = opposing torque due to moment of inertia of the body.

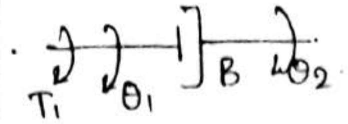
$T_j \propto \frac{d^2\theta}{dt^2}$; $T_j = J\frac{d^2\theta}{dt^2}$.

$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}}$ ⟶ Newton's Second law.



$T_b \propto \frac{d\theta}{dt}$, $T_b = B\frac{d\theta}{dt}$. $\Rightarrow$ $\boxed{T = T_b = B\frac{d\theta}{dt}}$

Ideal rotational dash-pot with one end fixed to reference.

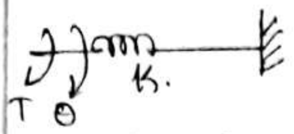


Ideal dash-pot with angular displacement at both ends.

$T_b \alpha \frac{d}{dt}(\theta_1-\theta_2)$

$T_b = B\frac{d}{dt}(\theta_1-\theta_2)$

$\boxed{T = T_b = B\frac{d}{dt}(\theta_1-\theta_2)} \longrightarrow$ Newtons Second law (N.S.L).



Ideal spring with one end fixed to reference.

$T_K \alpha \theta$ or $T_K = K\theta$

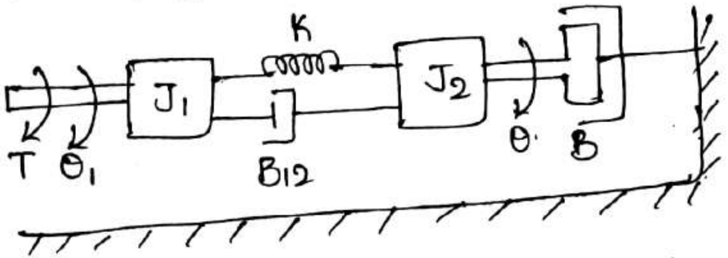
$\boxed{T = T_K = K\theta} \longrightarrow$ N.S.L.

Ideal spring with angular displacement with at both ends

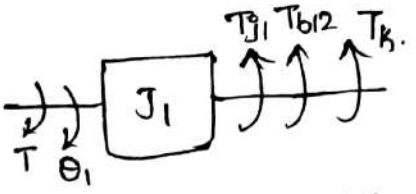
$T_K \alpha (\theta_1-\theta_2)$ (or) $T_K = K(\theta_1-\theta_2)$

$\boxed{T = T_K = K(\theta_1-\theta_2)}$

① Obtain the T.F of the mechanical rotational system shown in fig.



Sol) The freebody diagram of J_1 is the opposing torques acting on J_1 are marked as T_{J1}, T_{b12} & T_K.



$T_{J1} = J_1\frac{d^2\theta_1}{dt^2}$; $T_{b12} = B_{12}\frac{d}{dt}(\theta_1-\theta)$

$T_K = K(\theta_1-\theta)$.

By Newtons Second law.

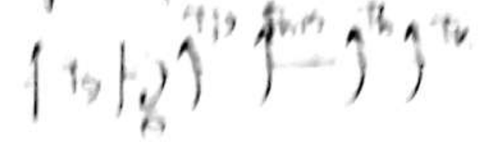
$T_{J1} + T_{b12} + T_K = T. \Rightarrow J_1\frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1-\theta) + K(\theta_1-\theta) = T.$

Apply L.T with zero initial conditions.

$J_1 s^2\theta_1(s) + sB_{12}[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s).$

$\theta_1(s)\left[J_1 s^2 + sB_{12} + K\right] - \theta(s)\left[sB_{12} + K\right] = T(s) \longrightarrow ①$

The free body diagram of J_2, the opposing torques are T_{j2}, T_{B12}, T_B & T_K



$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$, $T_{B12} = B_{12}\frac{d}{dt}(\theta-\theta_1)$

$T_B = B\frac{d\theta}{dt}$, $T_K = K(\theta-\theta_1)$

By Newtons second law, $T_{j2}+T_{B12}+T_B+T_K = 0$

$$J_2\frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta-\theta_1) + B\frac{d\theta}{dt} + K(\theta-\theta_1) = 0$$

Apply L.T with zero initial conditions

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + B_{12}s\theta(s) + B\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + [B_{12}+B]s\theta(s) + K\theta(s) - K\theta(s) = 0$$

$$\theta(s)[J_2 s^2 + s(B_{12}+B) + K] - \theta_1(s)[sB_{12}+K] = 0$$

$$\theta_1(s) = \frac{[s^2 J_2 + s(B_{12}+B)+K]}{[sB_{12}+K]}\theta(s) \longrightarrow ③$$

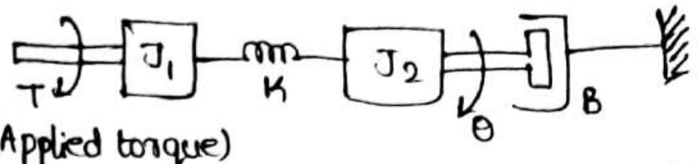
Sub. eq ③ in eq ①

$$[J_1 s^2 + sB_{12} + K]\left\{\frac{[J_2 s^2 + s(B_{12}+B)+K]\theta(s)}{(sB_{12}+K)} - (sB_{12}+K)\right\}\theta(s) = T(s)$$

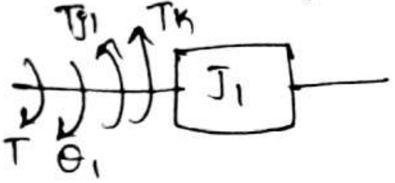
$$\left\{\frac{(J_1 s^2 + sB_{12} + K)(J_2 s^2 + s(B_{12}+B)+K)\theta(s) - (sB_{12}+K)^2}{(sB_{12}+K)}\right\}\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12}+K)}{[J_1 s^2 + sB_{12}+K][J_2 s^2 + s(B_{12}+B)+K] - (sB_{12}+K)^2}$$

② Write the differential equations governing the mechanical rotational system. Obtain the transfer function ~~of~~ the system.



Sol The freebody diagram of J_1 & opposing torques acting on J_1 are T_{j1} & T_K



$T_{j1} = J_1\frac{d^2\theta_1}{dt^2}$, $T_K = K(\theta_1-\theta)$

Apply N.S.L (Newtons second law) $T_{j1}+T_K = \cancel{0}T$.

$$J_1\frac{d^2\theta_1}{dt^2}+K(\theta_1-\theta)=T.$$

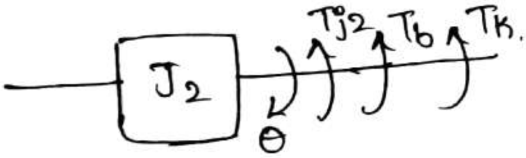
$$J_1\frac{d^2\theta_1}{dt^2}+K\theta_1-K\theta=T. \rightarrow \cancel{①}$$

Apply L.T $J_1s^2\theta_1(s)+K\theta_1(s)-K\theta(s)=T(s)$

$$(J_1s^2+K)\theta_1(s)-K\theta(s)=T(s)\rightarrow ①$$

Freebody diagram of mass ~~m2~~ with moment of inertia J_2. opposing torques acting on J_2 are T_{j2}, T_b & T_K.

$T_{j2}=J_2\frac{d^2\theta}{dt^2}$, $T_b=B\frac{d\theta}{dt}$, $T_K=K(\theta-\theta_1)$



By N.S.L $T_{j2}+T_b+T_K=0. \Rightarrow J_2\frac{d^2\theta}{dt^2}+B\frac{d\theta}{dt}+K(\theta-\theta_1)=0.$

Apply L.T $J_2s^2\theta(s)+Bs\theta(s)+K\theta(s)-K\theta_1(s)=0.$

$$\theta_1(s)=\frac{(J_2s^2+Bs+K)}{K}\theta(s)\rightarrow ②.$$

Sub eq ② in eq ①

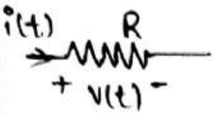
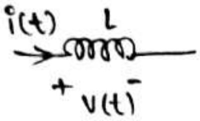
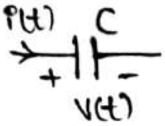
$$(J_1s^2+K)\left[\frac{J_2s^2+Bs+K}{K}\right]\theta(s)-K\theta(s)=T(s).$$

$$\boxed{\frac{\theta(s)}{T(s)}=\frac{K}{(J_1s^2+K)(J_2s^2+Bs+K)-K^2}}$$

Electrical Systems :-

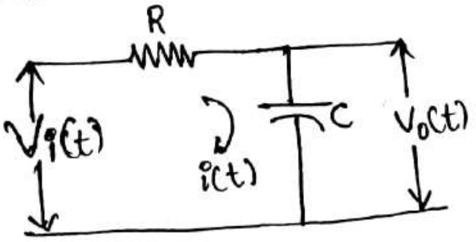
The models of electrical systems can be obtained by using resistor, capacitor & inductor. The current-voltage relation of resistor, inductor & capacitor.

Current - Voltage relation of R, L, C :-

Element	voltage	Current
	$v(t)=Ri(t)$	$i(t)=\frac{v(t)}{R}$.
	$v(t)=L\frac{d}{dt}i(t)$	$i(t)=\frac{1}{L}\int v(t)dt$.
	$v(t)=\frac{1}{C}\int i(t)dt$	$i(t)=C\frac{dv(t)}{dt}$.

* The differential eq governing the electrical system can be formed by writting KVL or KCL.

① Obtain the transfer function of the electrical network shown in fi



$$V_i(t) = R\,i(t) + \frac{1}{C}\int i(t)dt. \longrightarrow ①$$

$$V_o(t) = \frac{1}{C}\int i(t)dt. \longrightarrow ②$$

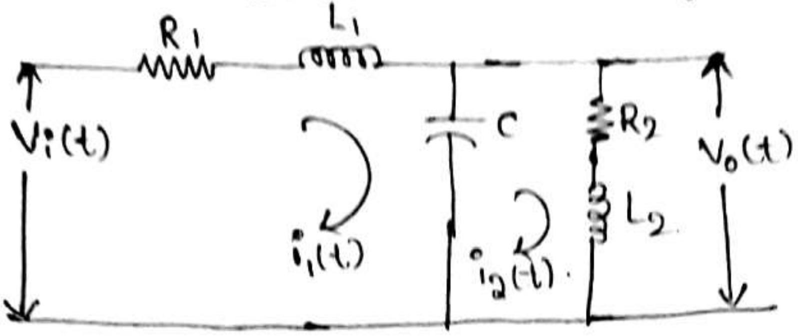
$$V_I(s) = R\,I(s) + \frac{1}{Cs}I(s).$$

$$V_o(s) = \frac{1}{Cs}I(s).$$

$$T.F = \frac{V_o(s)}{V_I(s)} = \frac{\frac{1}{Cs}I(s)}{I(s)R + \frac{1}{Cs}I(s)} = \frac{\frac{1}{\cancel{Cs}}\left[\cancel{I(s)}\right]}{\frac{1}{\cancel{Cs}}\left[RCs+1\right]\cancel{I(s)}}$$

$$\boxed{T.F = \frac{1}{1+RCs}}$$

② Obtain the T.F for the electrical system shown in fig.



Sol KVL

$$V_i(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + \frac{1}{C}\left[\int [i_1(t) - i_2(t)]\, dt\right] \longrightarrow ①$$

$$0 = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} + \frac{1}{C}\int [i_2(t) - i_1(t)]\, dt \longrightarrow ②$$

$$V_0(t) = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} \longrightarrow ③$$

Apply L.T ⇒ $V_I(s) = R_1 I_1(s) + L_1 s I_1(s) + \frac{1}{Cs}[I_1(s) - I_2(s)]$ ---(i)

$$0 = R_2 I_2(s) + L_2 s \cdot I_2(s) + \frac{1}{Cs}[I_2(s) - I_1(s)] \text{ ----(ii)}$$

$$V_0(s) = R_2 I_2(s) + L_2 s \cdot I_2(s) \text{ ----(iii)}$$

$$I_2(s)\left[R_2 + sL_2 + \frac{1}{Cs}\right] = I_1(s)\frac{1}{Cs}$$

$$I_1(s) = Cs\left[R_2 + sL_2 + \frac{1}{Cs}\right] I_2(s) \rightarrow ④$$

eq④ in eq(i)

$$V_I(s) = \left[R_1 + L_1 s + \frac{1}{Cs}\right]\left[\left(R_2 + sL_2 + \frac{1}{Cs}\right)Cs\right] I_2(s) - \frac{1}{Cs} I_2(s)$$

$$V_I(s) = \left[\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(R_2 + sL_2 + \frac{1}{Cs}\right)Cs - \frac{1}{Cs}\right] I_2(s) \rightarrow ⑤$$

eq⑤ in eq(iii):

$$V_0(s) = (R_2 + L_2 s)\frac{V_I(s)}{\left[\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(R_2 + sL_2 + \frac{1}{Cs}\right)Cs - \frac{1}{Cs}\right]}$$

$$\frac{V_0(s)}{V_I(s)} = \left[\frac{(R_2 + L_2 s)}{\left(R_1 + L_1 s + \frac{1}{Cs}\right)\left(1 + R_2 Cs + s^2 C L_2\right) - \frac{1}{Cs}}\right]$$

Electrical Analogous of mechanical translational system or Mechanical rotational system:-

* System remains as analogous as long as the differential eq's governing the system are in identical form.

* The electrical analogous of other ~~hand~~ kind of systems is of greater importance, since it is easier to construct electrical models & analyse them.

* The input force in mechanical system is analogous either voltage (or) current source of electrical system.

* Since the electrical system has two types of i/p's either voltage or current source. There are two types of analysis.

1. Force voltage analogy.
2. Force current analogy.

	TMS	RMS	Electrical Parameters (F-V, T-V)	Electrical Parameters. (F-I, T-I)
Elements:	Mass(M)	Moment of inertia(J)	Inductance(L)	Capacitance (C)
	Friction(B)	Friction(B)	Resistance(R)	Resistance($\frac{1}{R}$)
	Spring(K)	Spring(K)	Capacitance($\frac{1}{C}$)	Inductance($\frac{1}{L}$)
Variables:	Displacement(x)	Angulare displacement (θ)	Charge(q)	flux linkage(ϕ)
	velocity(v)	Angular velocity ($\frac{d\theta}{dt}$)	loop current(i)	Node voltag
	Force (f)	Torque(T)	Source voltage (V)	source curren (i).

Mechanical system: Input : Force , Output : Velocity.

Electrical System : Input - Voltage Source, Output - Current through element.

Electrical systems.

Force -Voltage analogy:

Mechanical system

Input : Force

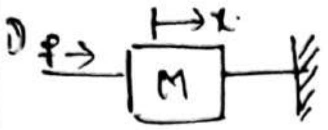
Output : Velocity

Electrical System

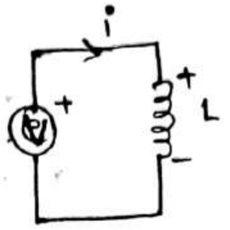
Input : voltage source

Output : Current through Element

1)

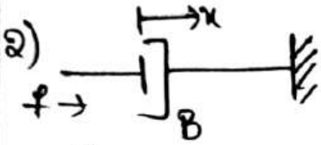


$f = M\frac{dx^2}{dt} = M\frac{dV}{dt}$.

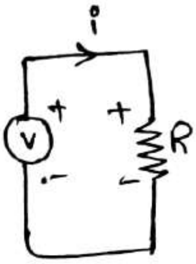


$V = L\frac{di}{dt}$.

2)

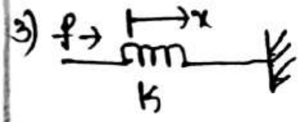


$f = B\frac{dx}{dt} = Bv$

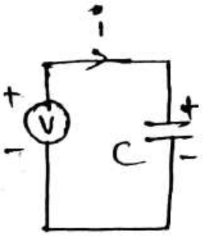


$V = iR$

3)



$f = Kx = K\int v\,dt$.



$e = \frac{1}{C}\int i\,dt$.

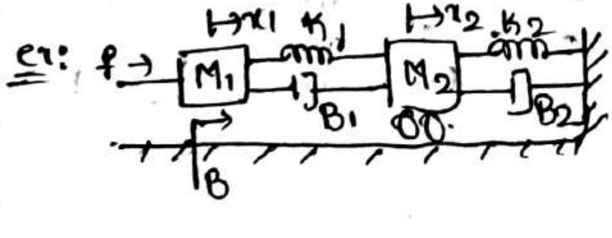
① In mechanical systems the elements having same velocity are said to be in series, similarly in electrical system the elements in series will have same current.

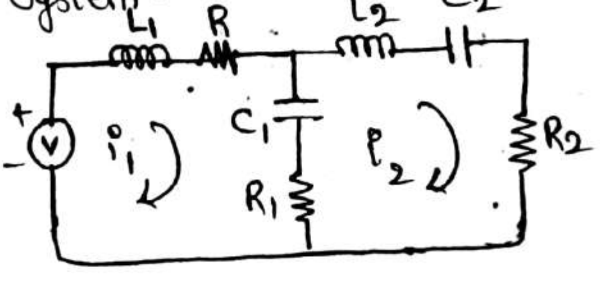
② Each node (mass) in the mechanical system corresponds to a closed loop in electrical system.

③ Number of meshes in electrical system is equal to Number of masses in mechanical system.

④ The element connected between two masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

eg:





Force - Current Analogy:

Mechanical System

Input : force

Output : velocity

Electrical system

Input : Current source

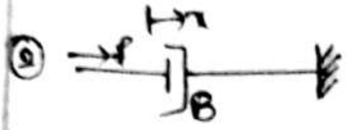
Output : Voltage across the through element

1

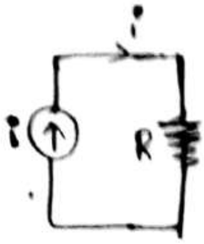
$f = M\frac{dx^2}{dt} = M\frac{dv}{dt}$.

$i = C\frac{dv}{dt}$

②

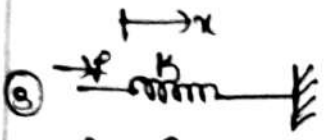


$f = B\frac{dx}{dt} = BV$

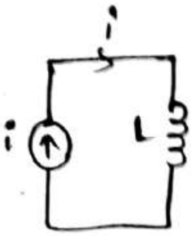


$i = \frac{V}{R}$

③



$f = K\int x = K\int vdt$



$i = \frac{1}{L}\int vdt$.

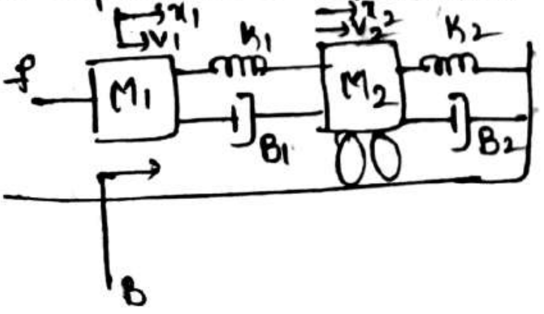
Procedure :-

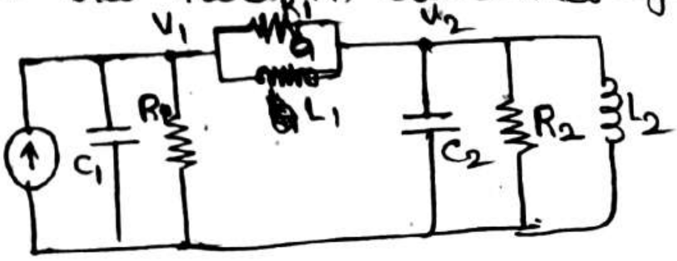
1) In mechanical system the elements in parallel will have same force similarly in electrical system parallel elements will have same voltage.

② Each node (Mass) in mechanical system corresponds to a node in electrical system.

③ Number of nodes in electrical system is equal to the number of nodes in mechanical system.

④ The elements connected b/w two nodes in mechanical system is represented as common element b/w nodes in electrical syster





Torque - voltage analogy:

Mechanical Rotational system

Input : Torque

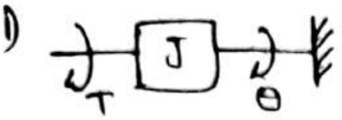
Output : angular velocity

Electrical system

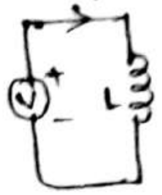
Input : voltage

Output : Current through element

1)

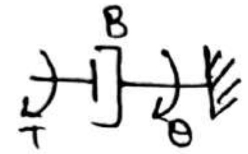


$$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}.$$



$$v = L\frac{di}{dt}$$

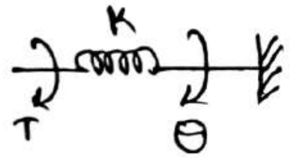
2)



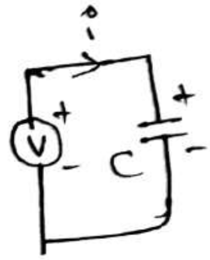
$$T = B\frac{d\theta}{dt} = B\omega$$

$$v = iR.$$

3)

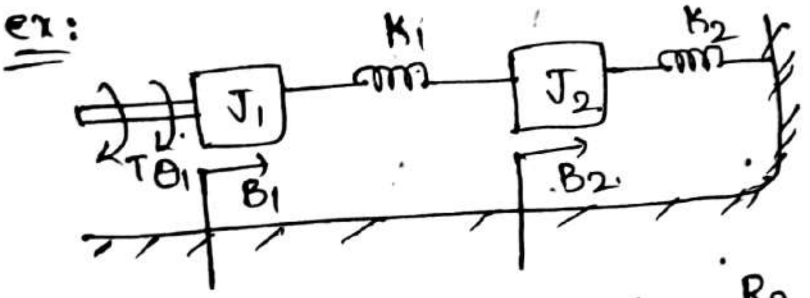


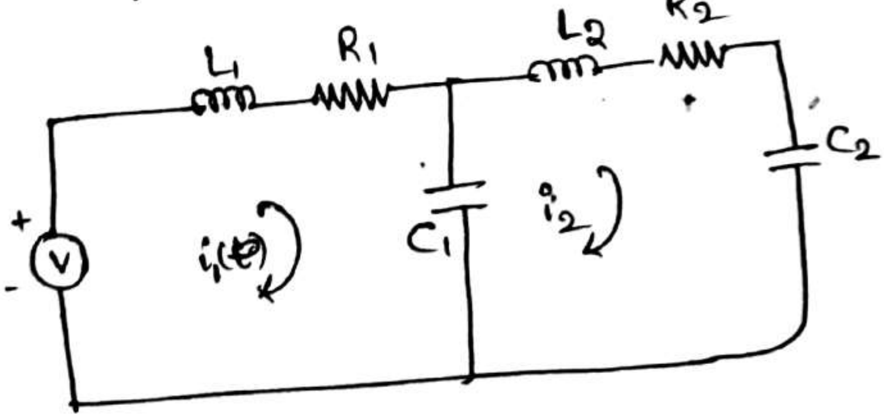
$$T = K\theta = K\int \omega dt.$$



$$v = \frac{1}{C}\int i dt.$$

ex:





$$v = L_1\frac{di_1}{dt} + i_1R_1 + \frac{1}{C_1}\int(i_1 - i_2)dt.$$

$$0 = L_2\frac{di_2}{dt} + i_2R_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int(i_2 - i_1)dt.$$

Torque - current analogy:

Mechanical Rotational system

Input : Torque

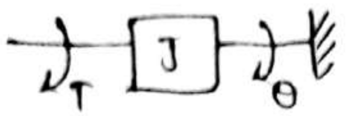
Output : angular velocity

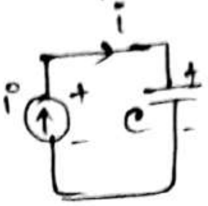
Electrical system

Input : current source

output : voltage across the element

1)

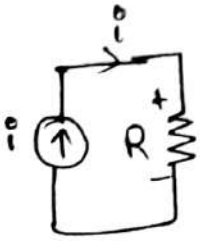




$i = C\frac{dv}{dt}$

$T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$

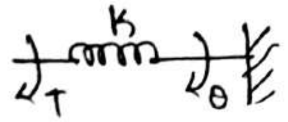
2)

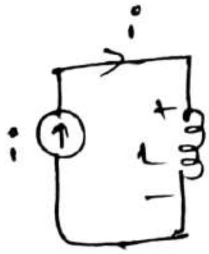


$i = \frac{V}{R}.$

$T = B\frac{d\theta}{dt} = B\omega$

3)

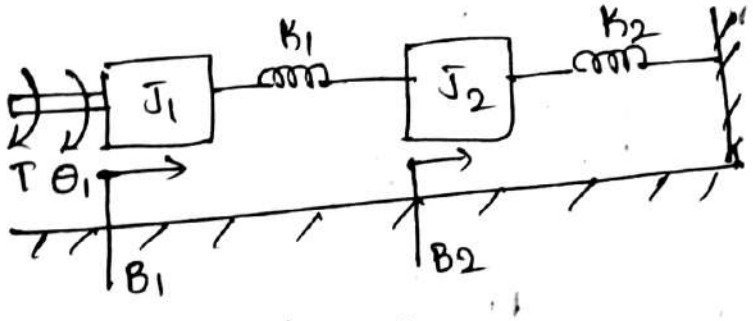


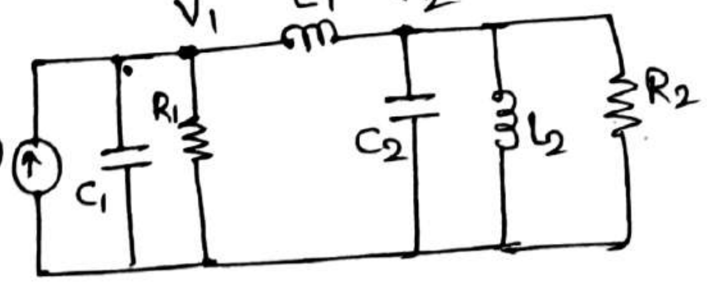


$i = \frac{1}{L}\int v dt.$

$T = K\theta = K\int \omega dt.$

ex:



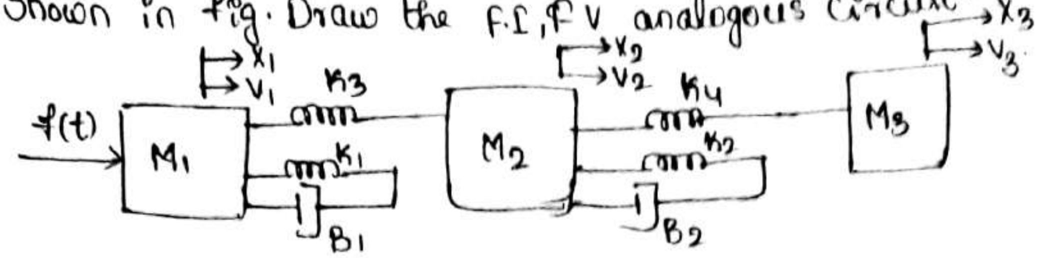


KCL

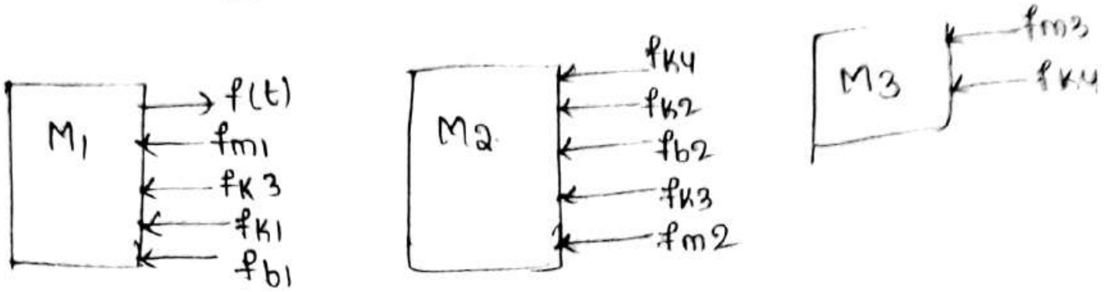
$$i(t) = C_1\frac{dv_1}{dt} + \frac{V_1}{R_1} + \frac{1}{L_1}\int(V_1 - V_2)dt.$$

$$0 = C_2\frac{dV_2}{dt} + \frac{1}{L_2}\int V_2 dt + \frac{V_2}{R_2} + \frac{1}{L_1}\int(V_2 - V_1)dt.$$

① Write the differential equations governing the mechanical system shown in fig. Draw the F.I, F V analogous circuit.



Sol



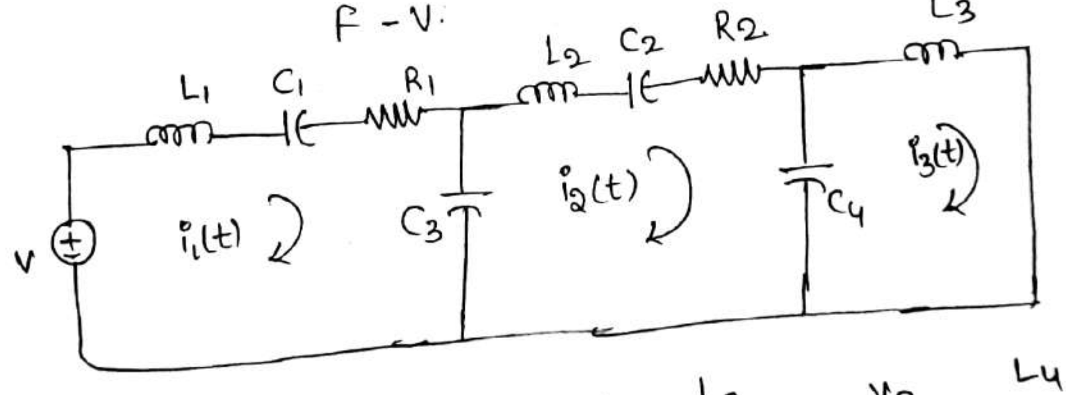
$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_1x_1 + K_3(x_1-x_2) = f(t).$$

$$M_1\frac{dv_1}{dt} + B_1v_1 + K_1\int v_1\,dt + K_3\left[\int(v_1-v_2)dt = f(t) \rightarrow ①\right.$$

$$M_2\frac{dv_2}{dt} + B_2v_2 + K_2\int v_2\,dt + K_3\left[\int(v_2dt - v_1dt) + K_4 + \left[\int v_2dt - \int v_3dt)\right]\right] = 0 \rightarrow ②$$

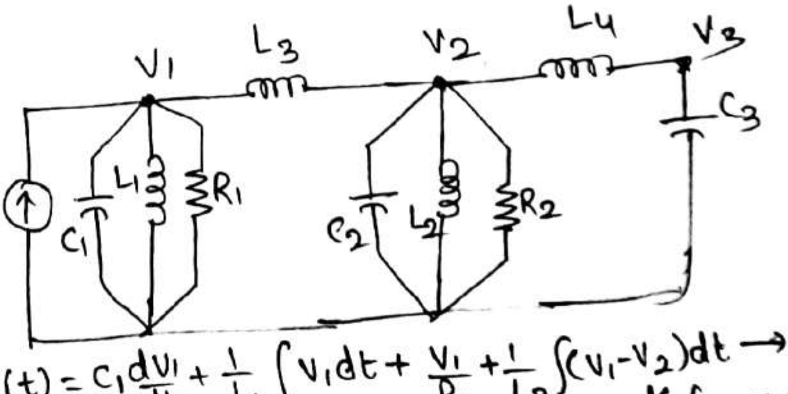
$$M_3\frac{dv_3}{dt} + K_4\left[\int v_3dt - \int v_2dt\right] = 0 \rightarrow ③.$$

f-v analogy:
$M_1 - L_1$
$K_1 - 1/C_1$ (series)
$B_1 - R_1$
$F - V$.



F-I analogy:

$M - C$
$K - 1/L$
$B - 1/R$ (parallel).
$F - I$



$$I(t) = C_1\frac{dv_1}{dt} + \frac{1}{L_1}\int v_1dt + \frac{v_1}{R_1} + \frac{1}{L_3}\int(v_1-v_2)dt \rightarrow ④$$

$$0 = C_2 dv_2/dt + \frac{1}{L_2}\int v_2dt + v_2/R_2 + 1/L_3\int(v_2-v_1)dt + 1/L_4\int(v_2-v_3)dt \rightarrow ⑤.$$ $$0 = 1/L_4\int(v_3-v_2)dt + C_3\,dv_3/dt$$

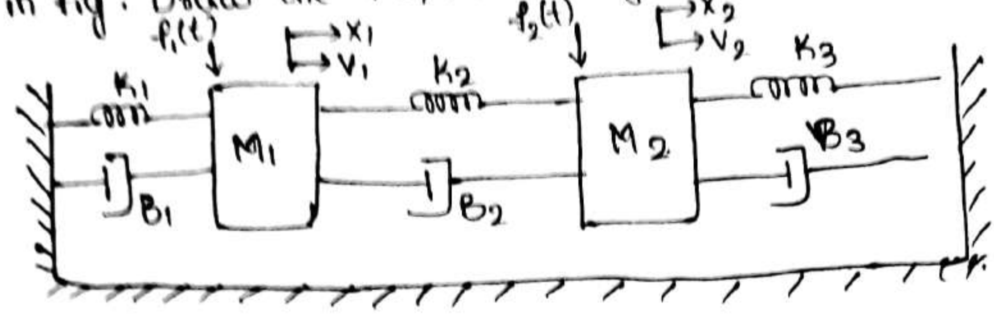
$$v(t) = L_1\frac{di_1(t)}{dt} + \frac{1}{C_1}\int i_1(t)dt + i_1(t)R_1 + \frac{1}{C_2}\int(i_1(t) - i_2(t))dt$$

$$0 = \frac{1}{C_3}\int((i_2(t) - i_1(t)dt) + L_2\frac{di_2(t)}{dt} + \frac{1}{C_2}\int i_2(t)dt + i_2(t)R_2$$

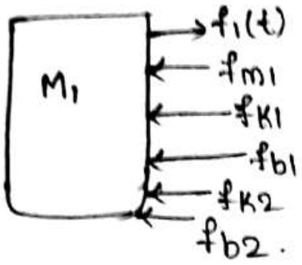
$$+ \frac{1}{C_4}\int(i_2(t) - i_3(t))dt$$

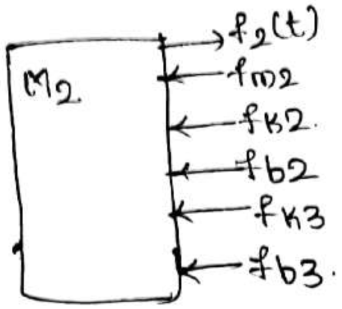
$$0 = \frac{1}{C_4}\int(i_3(t) - i_2(t))dt + L_3\frac{di_3(t)}{dt}$$

② Write the differential equations governing the mechanical shown in fig. Draw the FI, FV analogous circuit.



Sol





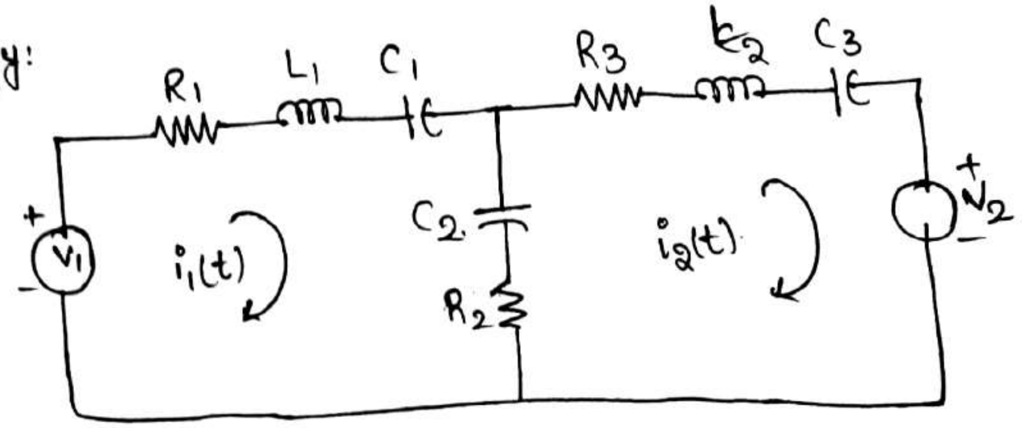
$$M_1\frac{dV_1}{dt} + B_1V_1 + K_1\int V_1dt + B_2(V_1 - V_2) + K_2\int(V_1 - V_2)dt = f_1(t) \longrightarrow ①$$

$$+ K_3\int V_2dt + B_3V_2 = f_2(t).$$

$$M_2\frac{dV_2}{dt} + B_2(V_2 - V_1) + K_2\int(V_2 - V_1)dt$$

F-V analogy:

M - L
B - R
K - 1/C
F - V



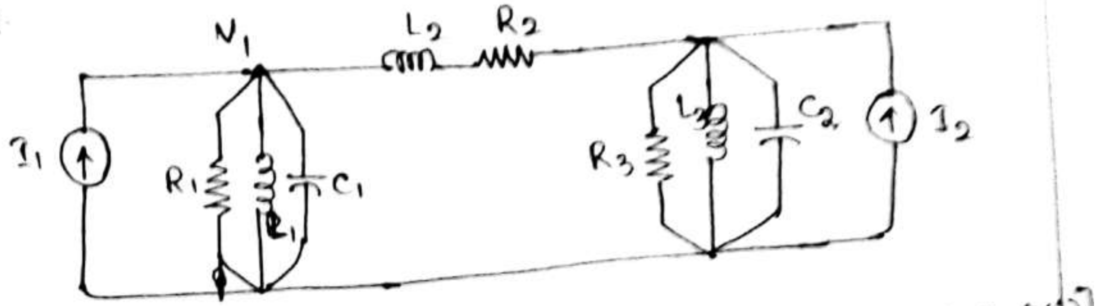
F-I analogy:

M - C

B - 1/R

K - 1/L

f - I



$$V_1(t) = i_1(t)R_1 + L_1\frac{di_1(t)}{dt} + \frac{1}{C_1}\int i_1(t)dt + \frac{1}{C_2}\int(i_1(t) - i_2(t))dt + R_2(i_1(t) - i_2(t))$$

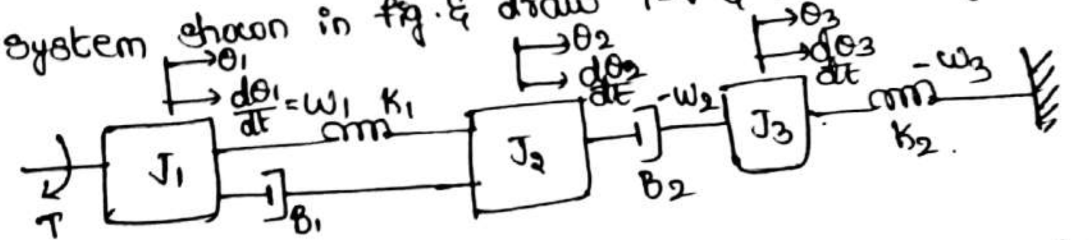
$$V_2(t) = \frac{1}{C_2}\int(i_2(t) - i_1(t))dt + R_2[i_2(t) - i_1(t)] + R_3 i_2(t) + L_2\frac{di_2(t)}{dt} +$$

$$\frac{1}{C_3}\int i_2(t)dt. \quad + R_2 \rightarrow ⑤$$

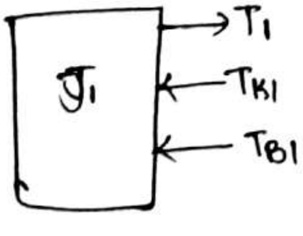
$$I_1 = \frac{V_1(t)}{R_1} + \frac{1}{L_1}\int V_1 dt + C_1\frac{dV_1(t)}{dt} + \frac{1}{L_2}\int(V_1(t) - V_2(t))dt$$

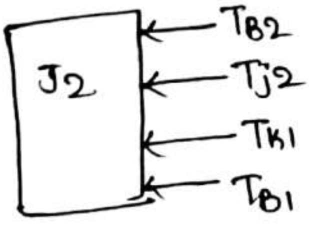
$$I_2(t) = \frac{1}{L_2}\int(V_2(t) - V_1(t))dt + \frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{1}{L_3}\int V_2 dt + C_2\frac{dV_2(t)}{dt} \rightarrow ⑥$$

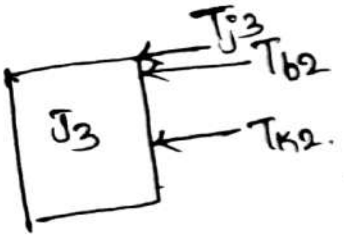
③ Write the differential equations governing the mechanical rotation system shown in fig. & draw T-V & T-I analogous circuit.



sol







sol

$$J = T_{j1} + T_{K1} + T_{B1}$$

$$J_1\frac{d^2\theta_1}{dt^2} + K_1(\theta_1 - \theta_2) + B_1\frac{d\theta_1}{dt} = T \rightarrow ①$$

$$J_2\frac{d^2\theta_1}{dt^2} + B_2\frac{d(\theta_2 - \theta_3)}{dt} + K_1(\theta_2 - \theta_1) + B_1 d(\theta_2 - \theta_1) = 0 \rightarrow ②$$

$$J_3\frac{d^2\theta_3}{dt^2} + K_2\theta_3 + B_2(\theta_3 - \theta_2) = 0 \rightarrow ③$$

$$J_1 s^2 \Theta_1(S) + K_1 (\Theta_1(S) - \Theta_2(S)] + B s \Theta_1(S) = T(S) \longrightarrow ④$$

$$J_2 s^2 \Theta_2(S) + B_2(s)[\Theta_2(s) - \Theta_3(S)] + K_1(\Theta_2(s) - \Theta_1(s)] + B_1 s[\Theta_2(s) - \Theta_1(S)] = 0. \longrightarrow ⑤$$

$$J_3 s^2 \Theta_3(S) + K_2 \Theta_3(S) + B_2 S[\Theta_3(s) - \Theta_2(s)] = 0 \longrightarrow ⑥.$$

$$\Theta_3(S)[J_3 S^2 + K_2 + B_2 S] = \Theta_2(S)$$

$$\Theta_3(S) = \frac{\Theta_2(S)}{J_3 S^2 + K_2 + B_2 S}$$

$$(J_2 s^2 + B_2 S + K_1 + B_1 s)\Theta_2(S) - B_2 s(\Theta_3(s) - \Theta_1(S)](1+K_1) = 0.$$

$$J_2 s^2 + B_2 S + K_1 + B_1(s)\,\Theta_2(s) - B_2(S)\,\frac{\Theta_2(S)}{J_3 S^2 + K_2 + B_2 S} - \Theta_1(s)(1+K_1) = 0.$$

$$\Theta_2(S)\left[J_2 s^2 + B_2 s + K_1 + B_1 S\right](J_3 S^2 + K_2 + B_2 s] - B_2(s)] = \Theta_1(s)(1+K)[J_3 S^2 + K_2 + B_2 S]$$

$$\Theta_2(S) = \frac{\Theta_1(S)(1+K)(J_3 S^2 + K_2 + B_2 S)}{(J_2 s^2 + B_2 s + K_1 + B_1 S)(J_3 S^2 + K_2 + B_2 S) - B_2 S}$$

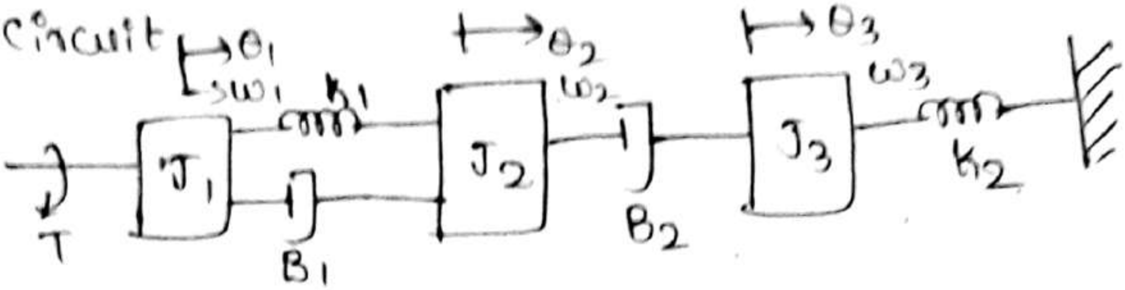
$$J_1 s^2 \Theta_1(S) + K_1(\Theta_1(S) - \Theta_2(S)] + B s \Theta_1(S) = T(S).$$

$$\Theta_1(S)[J_1 s^2 + K_1 + B_1 s] - \frac{(1+K)(J_3 s^2 + K_2 + B_2 s)}{(J_2 S^2 + B_2 S + K_1 + B_1 s)(J_3 S^2 + K_2 + B_2 S)} = T(S)$$

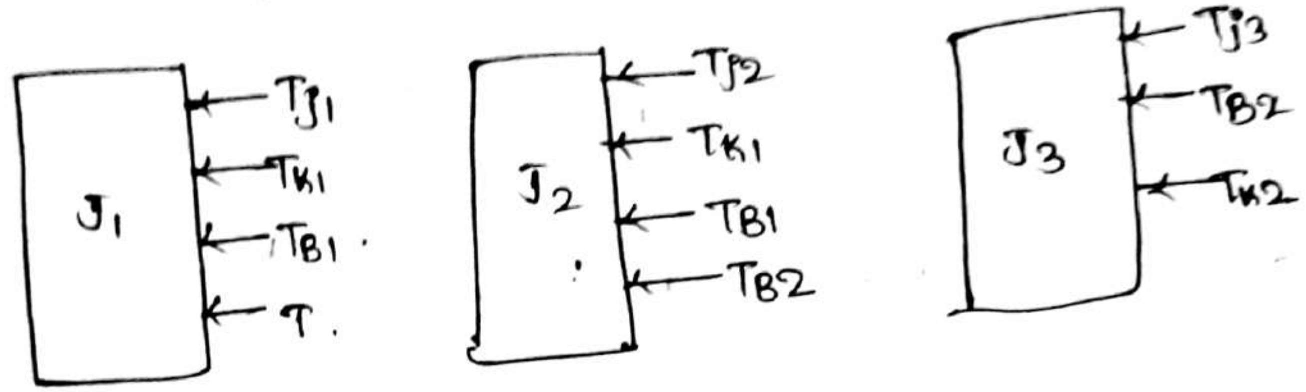
$$\frac{\Theta_1(S)}{T(S)} = \frac{(J_1 s^2 + K_1 + B_1 S)(J_2 S^2 + B_2 S + K_1 + B_1 S)(J_3 S^2 + K_2 + B_2 S) - B_2 s - (1+K_1)(J_3 s^2 + K_2 + B_2 S)}{(J_2 S^2 + B_2 s + K_1 + B_1 S)(J_3 S^2 + K_2 B_2 S) - B_2(S).}$$

T-V

③ Write the differential equations governing the mechanical rotation system shown in fig. Draw. T-V & T-I analogous circuit



Sol



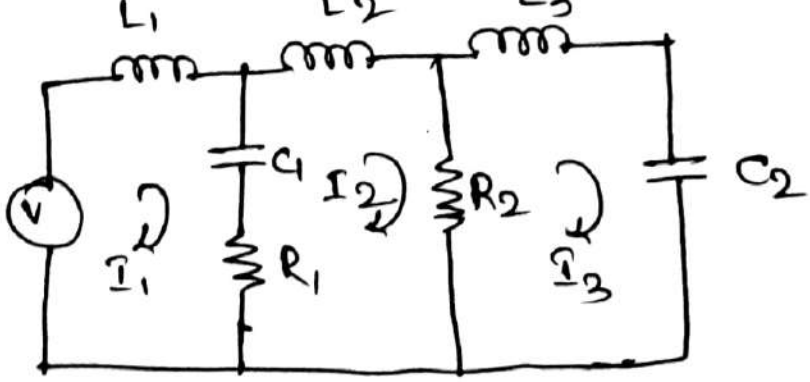
$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1-\omega_2) + K_1\int(\omega_1-\omega_2)\,dt = T \longrightarrow ①$$

$$J_2 \frac{d\omega_2}{dt} + B_1(\omega_2-\omega_1) + B_2(\omega_2-\omega_3) + K_1\int(\omega_2-\omega_1)\,dt = 0 \longrightarrow ②$$

$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3-\omega_2) + K_2\int\omega_3\,dt = 0.$$

T-V

T - V
J - L
B - R
K - 1/C.



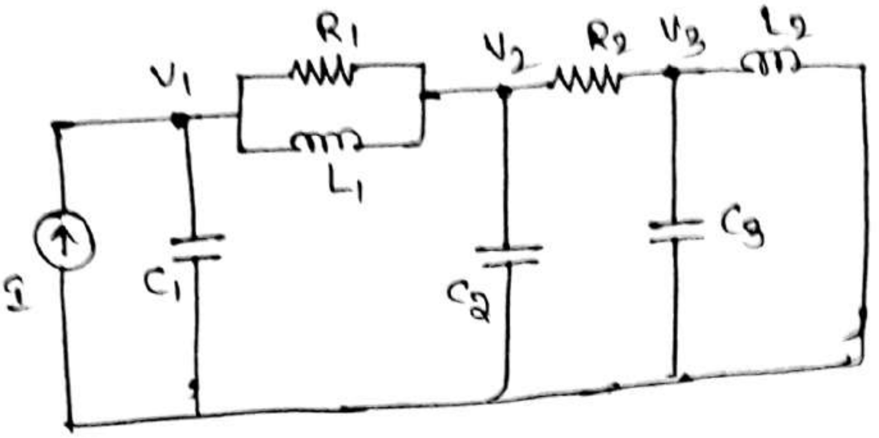
T-I

J → L

K → $\frac{1}{C}$

B → $\frac{1}{R}$

T → I



$$V = L_1\frac{d\theta}{dt} + \frac{1}{C_1}\int(\theta_1 - \theta_2)dt + R_1(\theta_1 - \theta_2)t = 0 \longrightarrow ①$$

$$0 = L_2\frac{d\theta_2(t)}{dt} + \frac{1}{C_1}(\theta_2 - \theta_1)dt + R_1(\theta_2 - \theta_1)t + R_2(\theta_2 - \theta_3) = 0 \rightarrow ②$$

$$0 = L_3\frac{d\theta_3(t)}{dt} + R_2(\theta_2 - \theta_3)t + \frac{1}{C_3}\int\theta_3 dt = 0.$$

Assignment

①

16/02/23

2. TIME RESPONSE ANALYSIS

Time response:

→ The time Response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, their response c(t) can be obtained from the transfer function of the system and the input to the system.

→ The closed loop transfer function.

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\cdot H(S)} = M(S)$$

$$\frac{C(S)}{R(S)} = M(S)$$

$$C(S) = R(S)\cdot M(S)$$

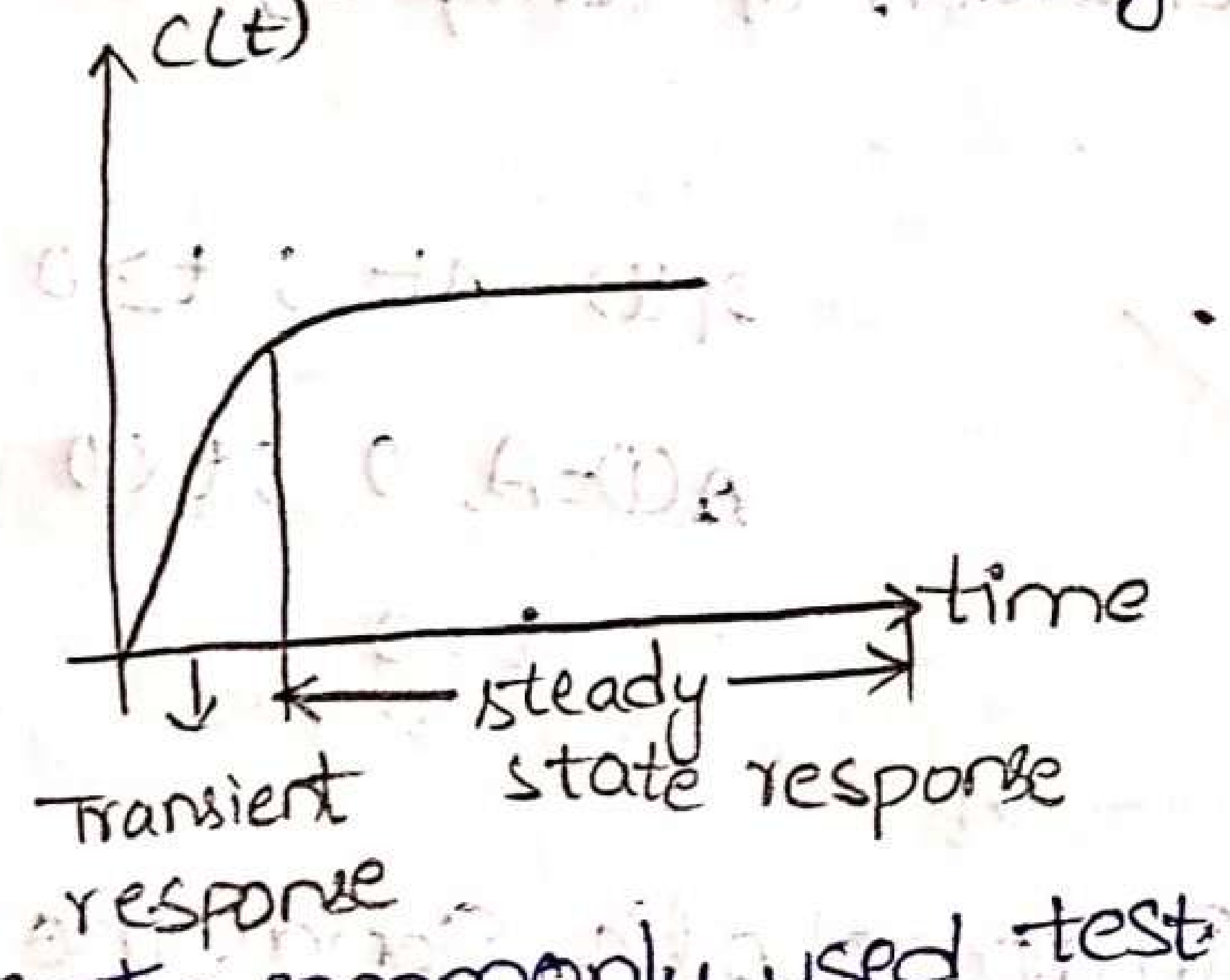
$$C(t) = L^{-1}\{R(S)\cdot M(S)\}$$

where $M(S) = \frac{G(S)}{1+G(S)\cdot H(S)}$

Time response analysis can be devided into i) Transient response
ii) steady state response.

i) **Transient response:** If output are response of the any system is varied with the time then it is called transient response.

2) **steady state response:** If output of any system is not varied with the time then it is called steady state response.



Test signals: The commonly used test input signals are impulse, step, ramp signal, acceleration, and synusoidal signals.

→ The standard test signals are ① a) step signal
b) unit step signal
② a) Ramp signal
b) unit Ramp signal
③ a) perabolic signal
b) unit parabolic signal.
④ impulse signal
⑤ synusoidal signal.

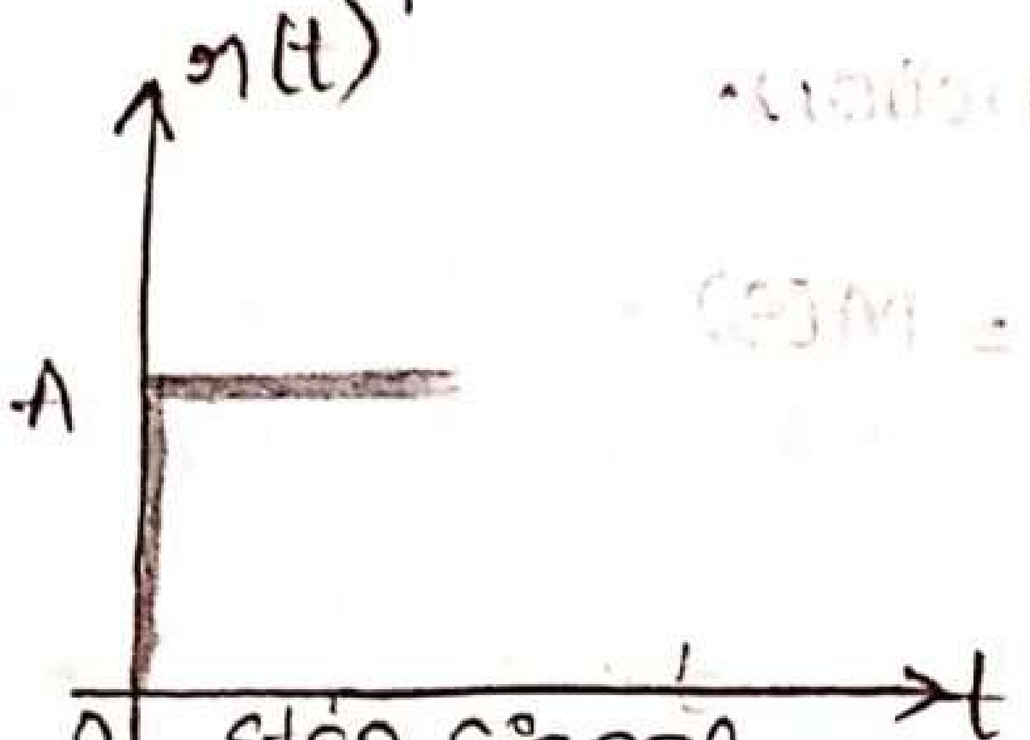
1) a) step signal:

→ The step signal is a signal whose values changes from 0 to A at t=0 and remains constant at A for t>0.

→ The step signal recemble an actual study input to a signal

→ A special case of step signal is unit step in which 'A' is unity.

→ The mathematical representation of a step signal is,



step signal.

$r(t) = 1 \; ; t \geq 0$

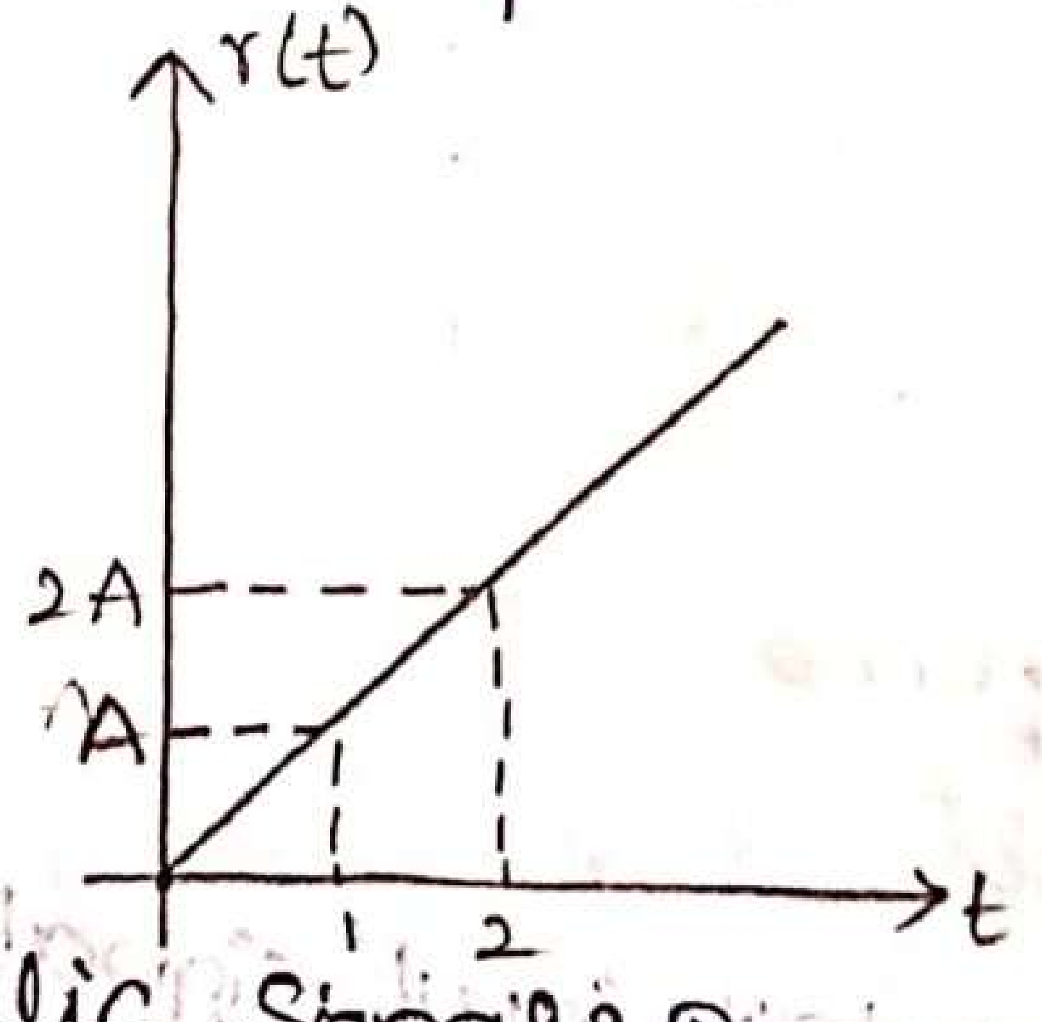
$= 0 \; ; t < 0$

2) a) Ramp Signal: The ramp signal is a signal whose value increases ~~from~~ linearly with time from initial value of 0 at t=0.

→ The ramp signal recembles a constant very input to the system.

→ A special case of ramp signal is unit ramp signal in which the value of A is unity.

→ The mathamatical representation of Ramp signal.



$r(t) = At \; ; t \geq 0$

$A(t) = A. \; 0 \; ; t < 0$

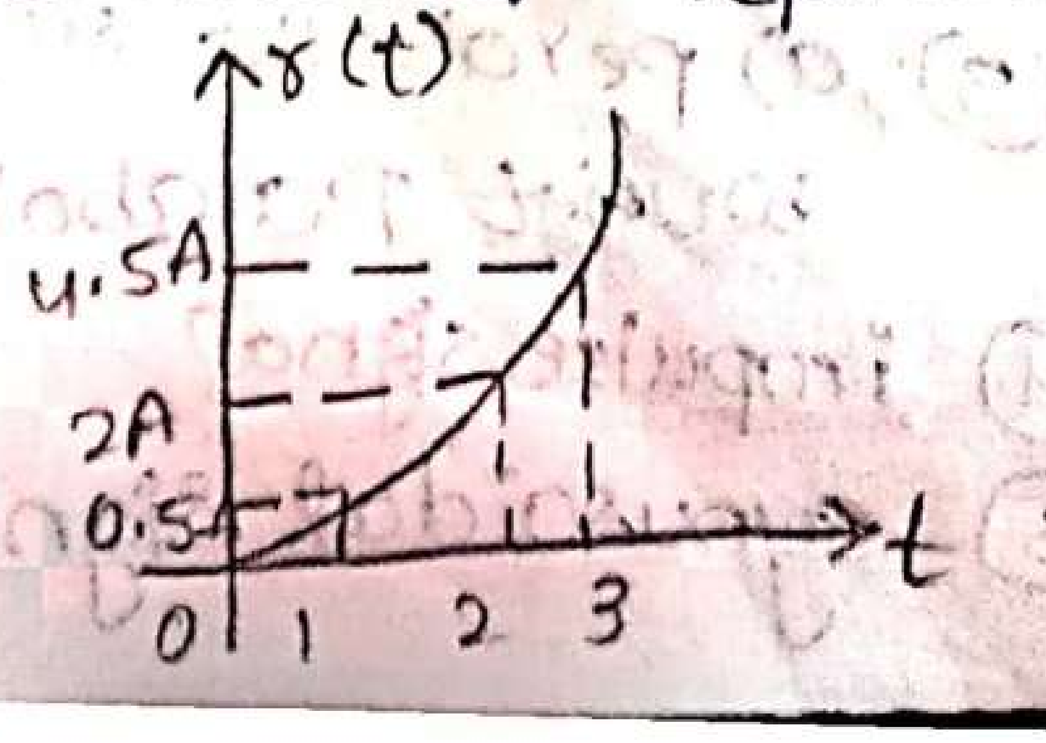
$t = 0$

3) a) parabolic Signal: In a parabolic Signal, the instantaneous value varies as square of the time from the initial value of zero at t=0.

→ The parabolic signal recembles the constant accelration input to the system.

→ The special signal case of parabolic signal is unit parabolic signal in which 'A' is unity.

→ The mathematical representation of the parabolic signal is,



$r(t) = \frac{At^2}{2} \; ; t \geq 0$

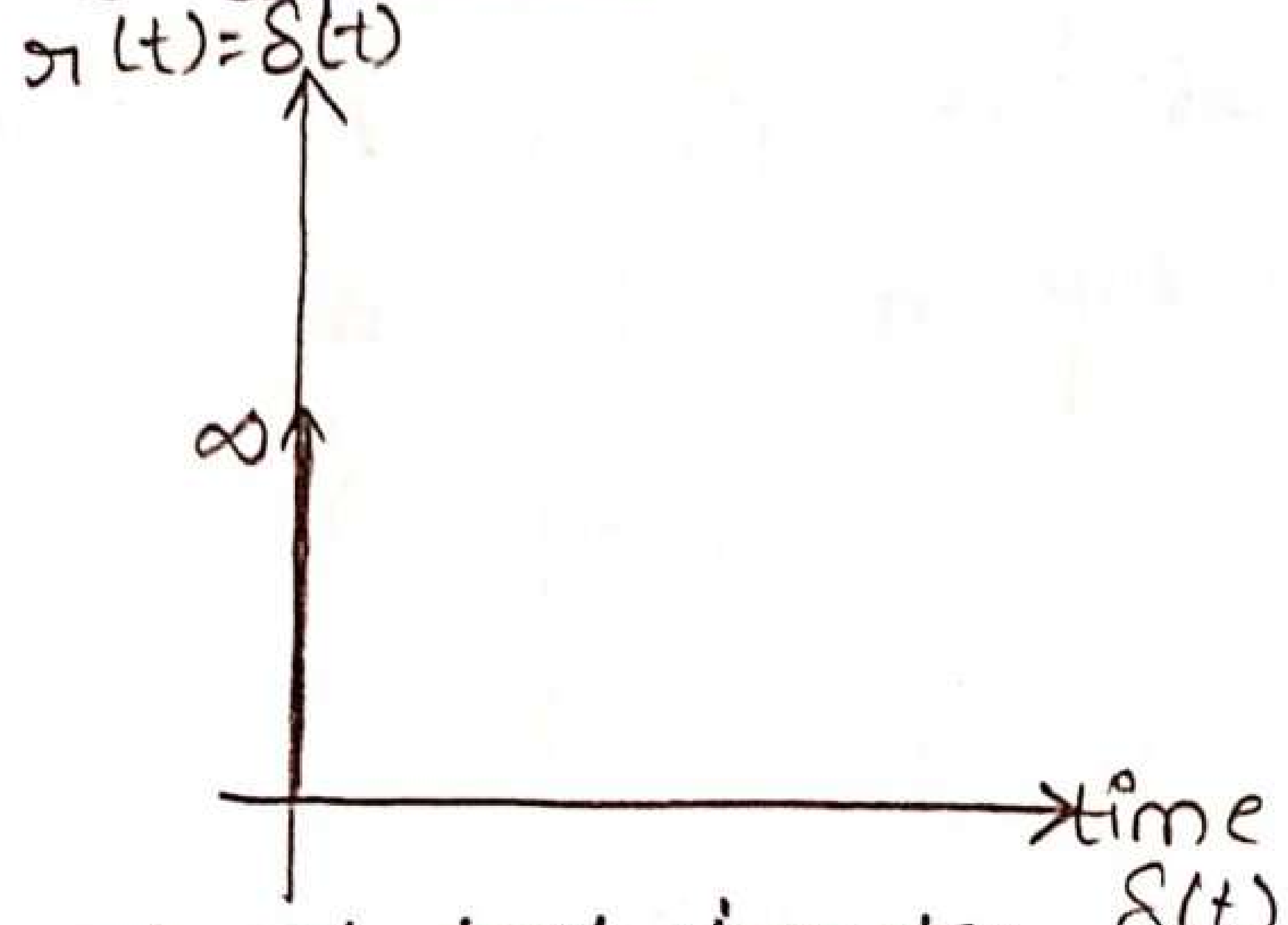
$0 \; ; t < 0$

4) Impulse Signal:

→ A signal of very large magnitude which is available for very short duration is called impulse signal.

→ Ideal Impulse signal is a signal with infinite magnitude and zero duration with an area of A. The unit impulse signal is a special case in which 'A' is unity.

→ The impulse signal is denoted by $\delta(t)$ and mathamatically, it is expressed as



$\delta(t)$

$$r(t) = 1 \text{ (or) } \infty \; ; t=0$$
$$0 \; ; t \neq 0 .$$

Standard test signals:

Name of the Signal	Time domain in eqn of signal, $r(t)$	L.T of the signal
step ~~unit~~	A	A/s
unit step	1	$1/s$
Ramp	At	A/s^2
unit ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
unit parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

Order of the system:

→ The order of the system given by the order of the differential equation governing the system.

→ If the system is governed by n^{th} order differential equation then the system is called n^{th} order system.

→ transfer function
$$T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{m-1} s + a_n}$$

where $P(s)$ = numerator polynomial.
$Q(s)$ = denominator polynomial.

Imp point

→ The order of the system is given by the maximum power of "s" in the denominator polynomial, $Q(s)$

→ Here $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$.

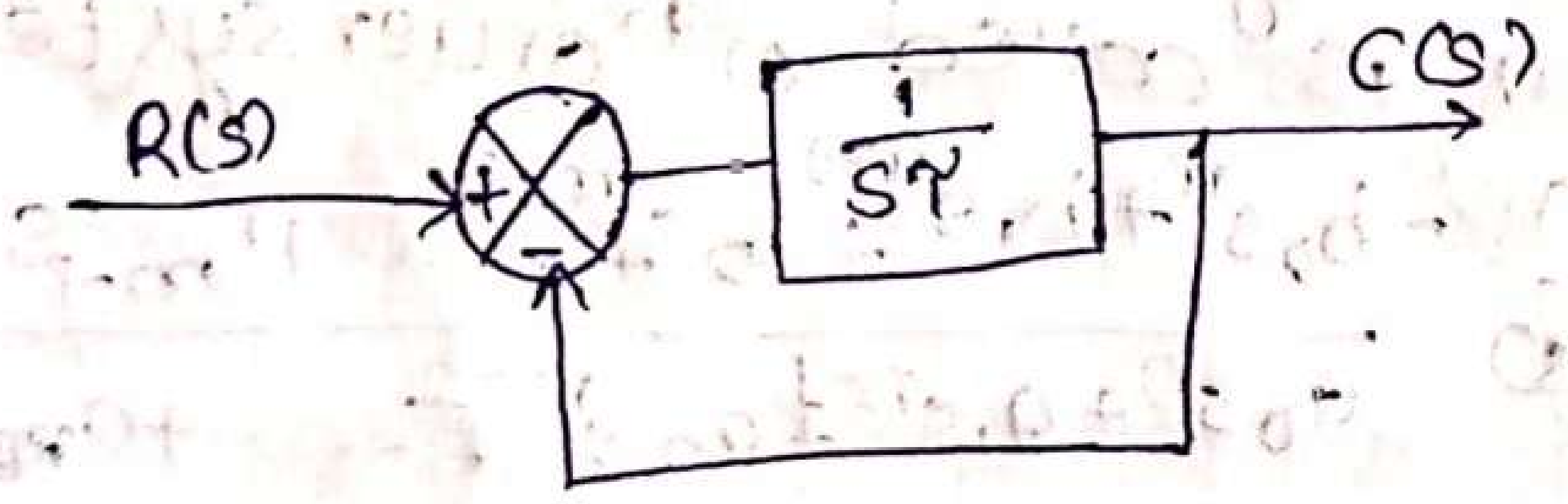
→ now n is the order of the system ~~where~~

→ when $n=0$, the system is zero order system

→ when $n=1$, the system is first order system.

when $n=2$, the system is Second order System.

Response of first order system for unit step input: $R(t)=1$



$$G(S) = \frac{C(S)}{R(S)} = \frac{\frac{1}{S\tau}}{1+\frac{1}{S\tau}} = \frac{1}{1+S\tau}$$

$$\therefore C(S) = R(S) = \frac{1}{1+S\tau}$$

$$r(t) = u(t) = 1 \;;\; t \geq 0$$

$$R(S) = L\{r(t)\} = \frac{1}{S}$$

$$C(S) = \frac{1}{S\tau\left[\frac{1}{\tau}+S\right]}$$

$$C(S) = \frac{\frac{1}{\tau}}{s\left(s+\frac{1}{\tau}\right)}$$

By partial fraction expansion

$$c(S) = \frac{\frac{1}{\tau}}{s\left(s+\frac{1}{\tau}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{\tau}\right)} \rightarrow ①$$

A is obtained by multiplying c(s) by s and letting s=0

$$A = C(S) \times S\Big|_{S=0}$$

$$= \frac{1}{\cancel{s}}\left[\frac{\frac{1}{\tau}}{s+\frac{1}{\tau}}\right]\cancel{s} = \frac{\frac{1}{\tau}}{s+\frac{1}{\tau}}\Big|_{s=0}$$

$$= \frac{\frac{1}{\tau}}{\frac{1}{\tau}} = 1 \Rightarrow A = 1$$

B is obtained when C(s) is multiplied with $\left(s+\frac{1}{\tau}\right)$ and letting $s = -\frac{1}{\tau}$

$$B = C(S) \times \left(S+\frac{1}{\tau}\right)\Big|_{S=-\frac{1}{\tau}}$$

$$= \frac{1}{S} \cdot \frac{\frac{1}{\tau}}{\cancel{S+\frac{1}{\tau}}} \times \cancel{\left(S+\frac{1}{\tau}\right)}$$

$$= \frac{1}{S} \cdot \frac{1}{\tau}\Big|_{S=-\frac{1}{\tau}}$$

$$= \frac{1}{-\frac{1}{\tau}} \times \frac{1}{\tau} = \frac{\frac{1}{\tau}}{-\frac{1}{\tau}} = -1$$

$$B = -1$$

sub A and B value in eqn ①

$$C(S) = \frac{1}{S} - \frac{1}{S+\frac{1}{\tau}}$$

Apply I.L.T on both sides.

$$L^{-1}\{C(S)\} = L^{-1}\left\{\frac{1}{S} - \frac{1}{S+\frac{1}{\tau}}\right\}$$

$$c(t) = L^{-1}\left\{\frac{1}{S}\right\} - L^{-1}\left\{\frac{1}{S+\frac{1}{\tau}}\right\}$$

$$c(t) = 1 - e^{-\frac{1}{\tau}t}$$

$$c(t) = 1 - e^{-t/\tau}$$

∴ closed loop first order system for unit step response,

$$c(t) = 1 - e^{-t/\tau}$$

for step response $c(t) = A(1-e^{-t/\tau})$

$c(s) = 1-e^{-t/\tau}$

$t=0 \Rightarrow 1-e^{-0/\tau} = 1-e^{-1} = 0$

$t=\tau \Rightarrow 1-e^{-\tau/\tau} = 1-e^{-1} = 0.632$

$t=2\tau \Rightarrow 1-e^{-\frac{2\tau}{\tau}} = 1-e^{-2} = 0.864$

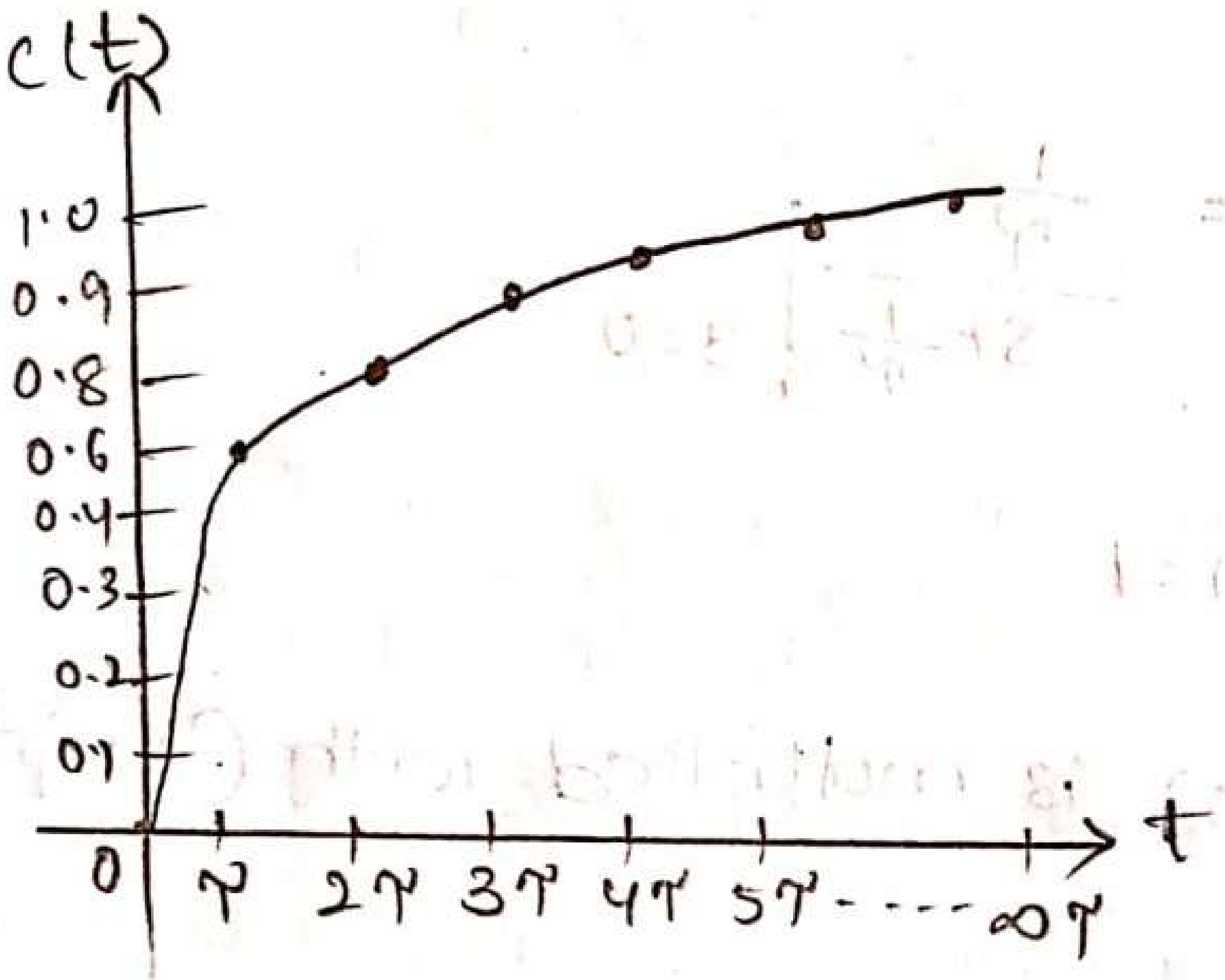
$t=3\tau \Rightarrow 1-e^{-3\tau/\tau} = 1-e^{-3} = 0.950$

$t=4\tau \Rightarrow 1-e^{-4\tau/\tau} = 1-e^{-4} = 0.981$

$t=5\tau = 1-e^{-5\tau/\tau} = 1-e^{-5} = 0.993$

$\vdots$

$t=\infty = 1-e^{-\infty\tau/\tau} = 1-e^{-\infty} = 1-0 = 1$



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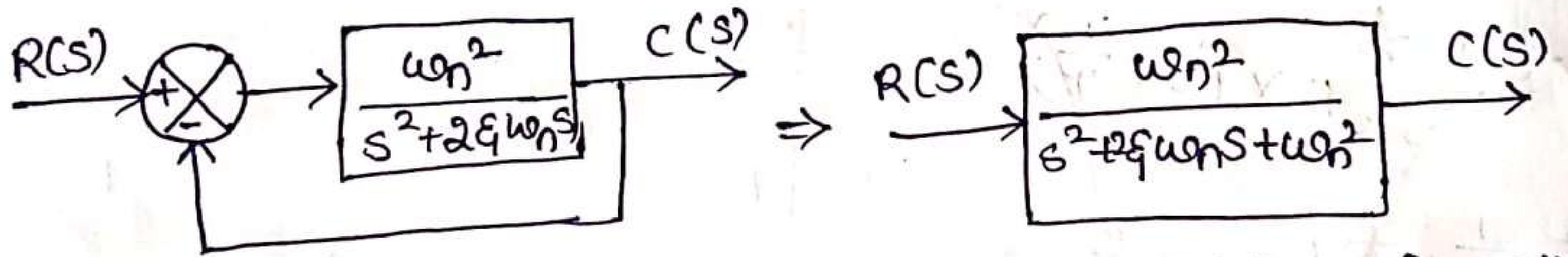
Second order System:

The closed loop second order system is,



The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

ω_n = undamped natural frequency, rad/sec

ξ = Damping ratio.

Damping ratio is defined as the ratio of actual damping to critical damping. The response c(t) of second order system depends on values of damping ratio.

Depending on the values of ξ the system can be classified into the following 4 cases.

case:1 - undamped system, $\xi=0$

case:2 - underdamped system, $0<\xi<1$

case:3 - critically damped system, $\xi=1$

case:4 - overdamped system, $\xi>1$

The characteristic equation of second order system is,

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

It is a quadratic equation and roots of this Equation is given by,

$$s_1,s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2-4\omega_n^2}}{2} = \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2-1)}}{2}$$

$$= \frac{\cancel{2}\left[-\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2-1)}\right]}{\cancel{2}} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

when $\xi=0$, $s_1,s_2 = \pm j\omega_n$; $\left\{\begin{array}{l}\text{roots are purely imaginary}\\ \text{and system is undamped}\end{array}\right.$

when $\xi=1$, $s_1,s_2 = -\omega_n$; $\left\{\begin{array}{l}\text{roots are real and equal}\\ \text{the system is critically damped}\end{array}\right.$

when $\xi>1$, $s_1,s_2 = -\xi\omega_n + \omega_n\sqrt{\xi^2-1}$; $\left\{\begin{array}{l}\text{roots are real and unEqual}\\ \text{the system is overdamped}\end{array}\right.$

when $0<\xi<1$, $s_1,s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$

$$= -\xi\omega_n \pm \omega_n\sqrt{-1}\sqrt{1-\xi^2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$= -\xi\omega_n \pm j\omega_d$; $\left\{\begin{array}{l}\text{roots are complex conjugate}\\ \text{the system is underdamped}\end{array}\right.$

where, $\omega_d = \omega_n\sqrt{1-\xi^2}$

ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

Response of undamped second order system for unit step input:

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Apply partial fractions,

as for undamped system, $\delta=0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

when the input is unit step, $r(t)=1$ and $R(s)=\frac{1}{s}$

the response in s-domain, $C(s) = R(s)\cdot\frac{\omega_n^2}{s^2+\omega_n^2} = \frac{1}{s}\cdot\frac{\omega_n^2}{s^2+\omega_n^2}$

By paratial fractions,

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} \rightarrow ①$$

$$\Rightarrow \omega_n^2 = A(s^2+\omega_n^2) + B(s)$$

$$\omega_n^2 = As^2 + A\omega_n^2 + Bs$$

put $s=0$

$\omega n^2 = A\omega n^2$

$A=1$

compare s^2 terms

put $s^2=-\omega n^2$ then $s=-\omega n$

$\omega n^2 = B(-\omega n)$

$B=-\omega n$

$B=-s$

put A and B values in eqn ①

$$C(s) = \frac{1}{s} + \frac{-s}{s^2+\omega^2}$$

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega^2}$$

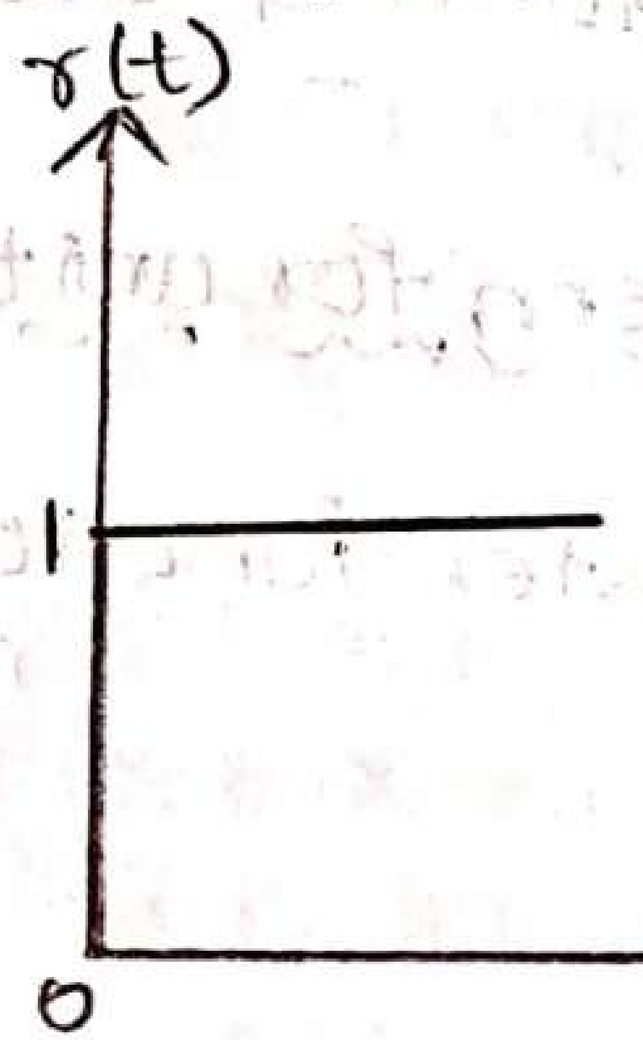
Apply inverse laplace transform,

$$c(t) = 1-\cos\omega nt$$

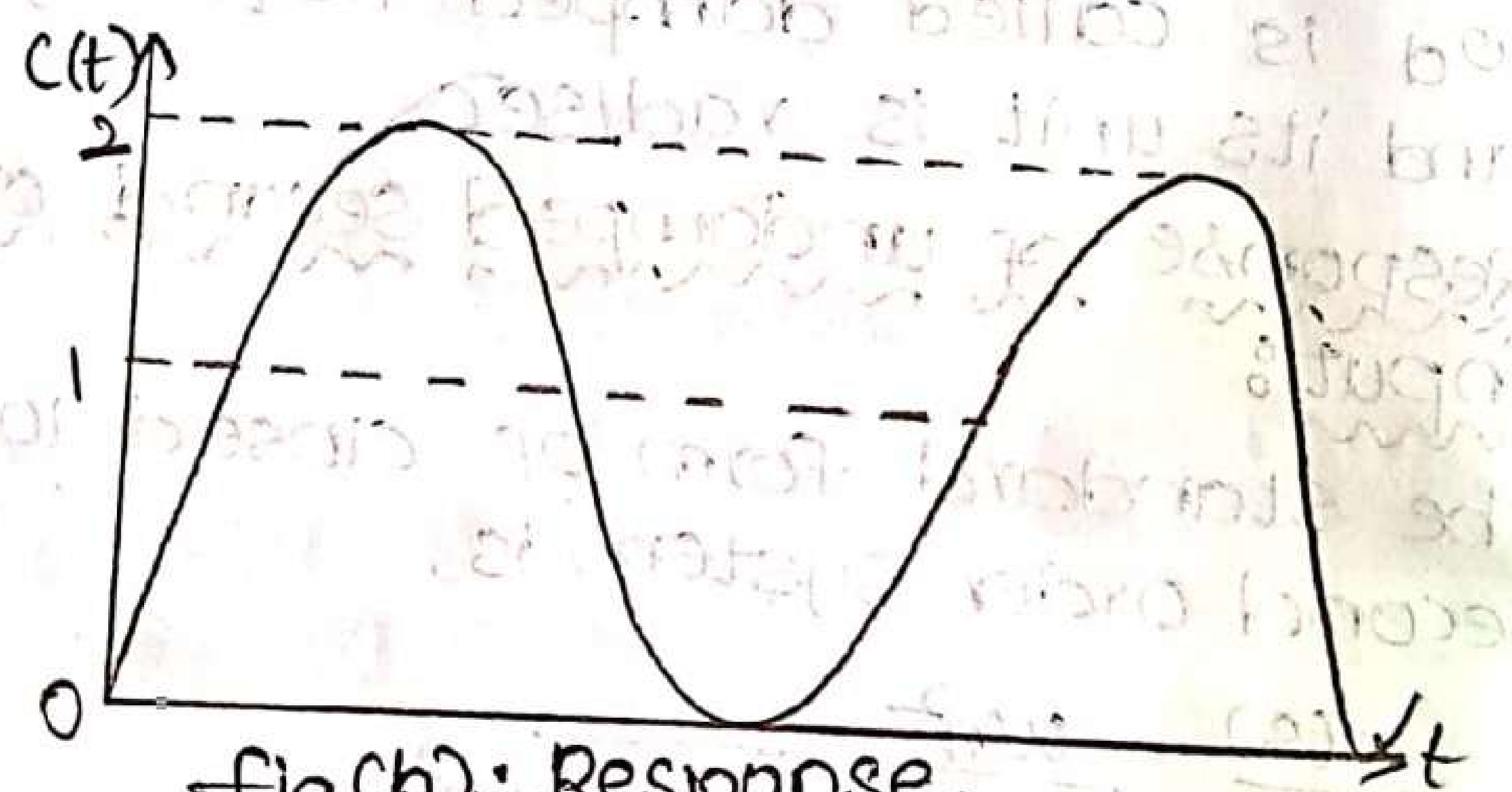
for closed loop undamped second order system,

unit step response $=1-\cos\omega nt$

Step response $=A(1-\cos\omega_n t)$



fig(a): Input



fig(b): Response.

Response of underdamped second order system for unit step Input:

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2+2\xi\omega_n s+\omega n^2}$$

for underdamped system, $0<\xi<1$

The roots of denominator are, $s=-\xi\omega n \pm \sqrt{\xi^2-1}$

since $\xi<1$, ξ^2 is also less then 1, and so, $1-\xi^2$ is always positive.

$$\therefore s = -\xi\omega n \pm \omega n\sqrt{(-1)(1-\xi^2)}$$

$$= -\xi\omega n \pm j\omega n\sqrt{1-\xi^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$s = -\xi\omega_n \pm j\omega_d$$

The response in s-domain, $C(S) = R(S) \cdot \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

for unit step input, $r(t)=1$ and $R(S) = \frac{1}{s}$

$$\therefore\ C(S) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

Apply partial fractions,

$$C(S) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{BS+C}{s^2+2\xi\omega_n s+\omega_n^2} \rightarrow ①$$

~~put s=0~~

$$\omega_n^2 = A(s^2+2\xi\omega_n s+\omega_n^2\cancel{s}) + (BS+C)(s)$$

$$\omega_n^2 = As^2+2\xi\omega_n sA + A\omega_n^2 + Bs^2 + Cs$$

put $S=0$,

$$\omega_n^2 = A\omega_n^2$$

$$A = 1$$

compare s^2 terms,

$$A+B=0$$

$$1+B=0$$

$$B=-1$$

compare s terms,

$$C+2\xi\omega_n=0$$

$$C=-2\xi\omega_n$$

Substitute A, B, C in eqn ①

$$C(S) = \frac{1}{s} + \frac{(-1)s-(2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$C(S) = \frac{1}{s} - \frac{s+2\omega_n\xi}{s^2+2\xi\omega_n s+\omega_n^2}$$

add and subtract $\xi^2\omega_n^2$ to the denominator

$$C(S) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2+\xi^2\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s^2+2\xi\omega_n s+\xi^2\omega_n^2)+\omega_n^2(1-\xi^2)}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s^2+2\xi\omega_n s+\xi^2\omega_n^2)+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

multiply and divide by ω_d in 3rd term of the above equation.

$$\therefore C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\cancel{\xi}\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

The response in time domain is given by;

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}\cdot e^{-\xi\omega_n t}\sin\omega_d t$$

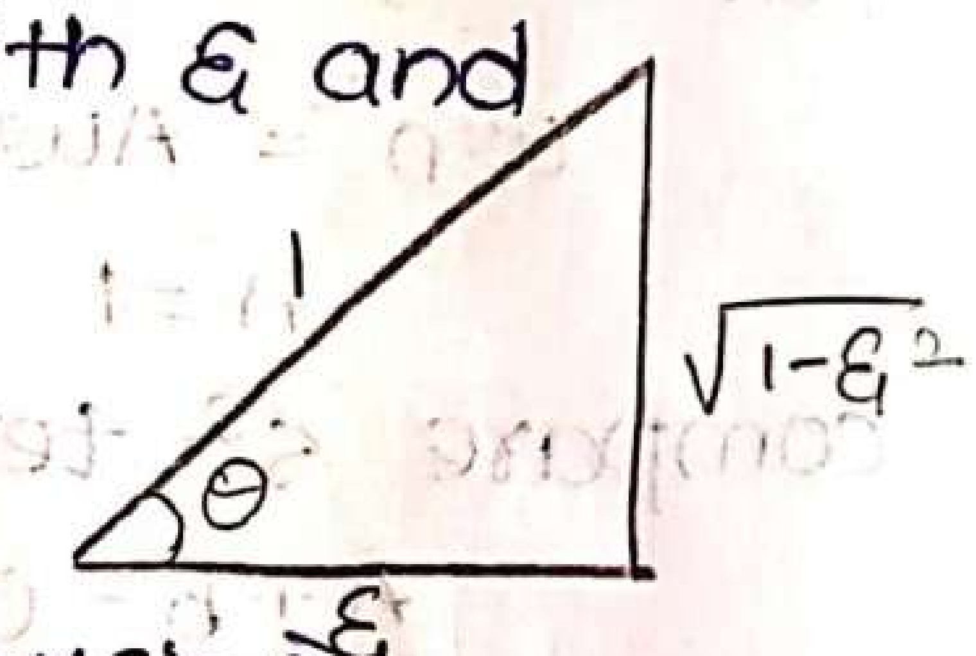
$$\left[\because L\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2} \text{ and } L\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}\right]$$

$$= 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$= 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

on constructing right angle triangle with ξ and $\sqrt{1-\xi^2}$, we get,



$$\sin\theta = \sqrt{1-\xi^2},\ \cos\theta = \xi,\ \tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta\right]$$

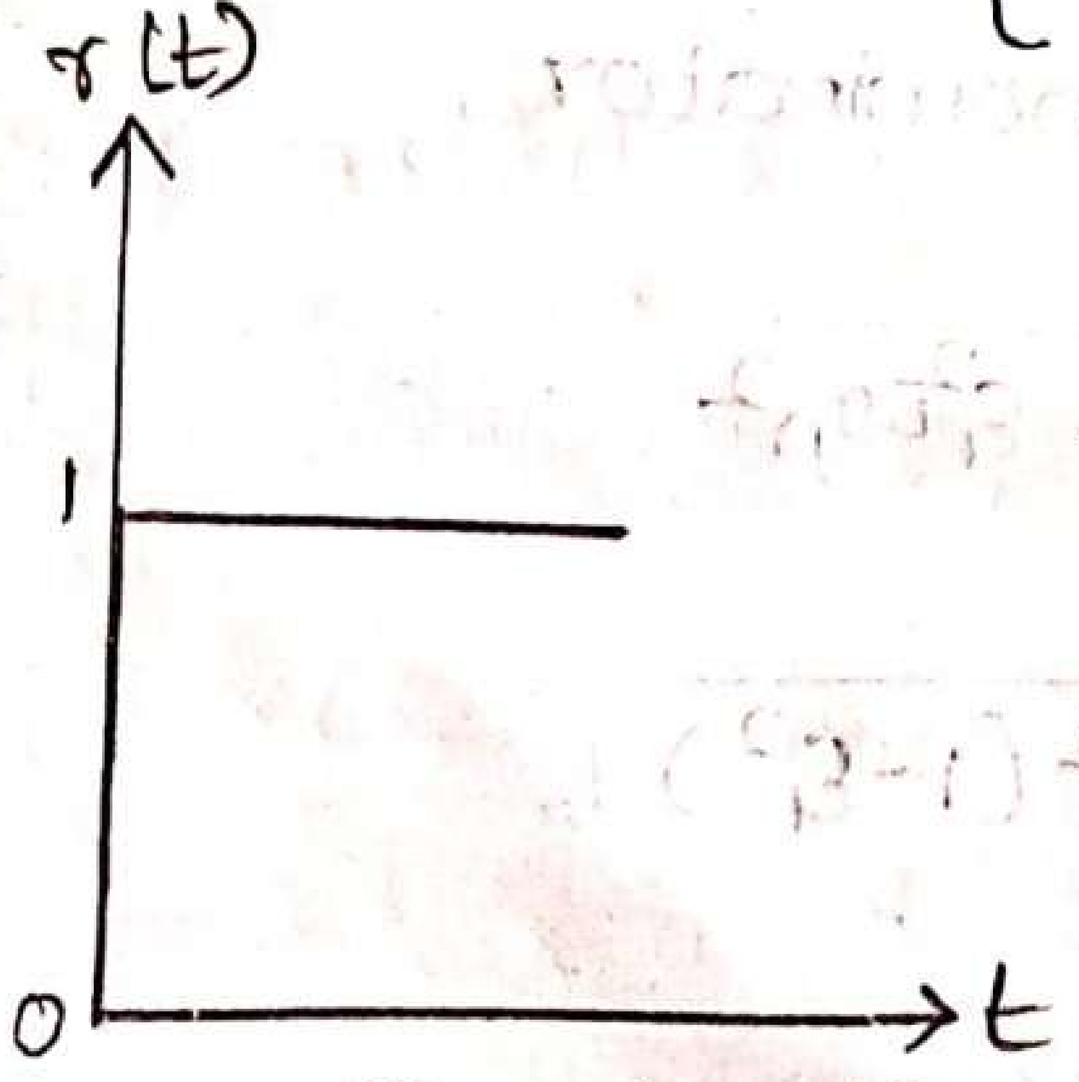
$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t+\theta)\right]$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

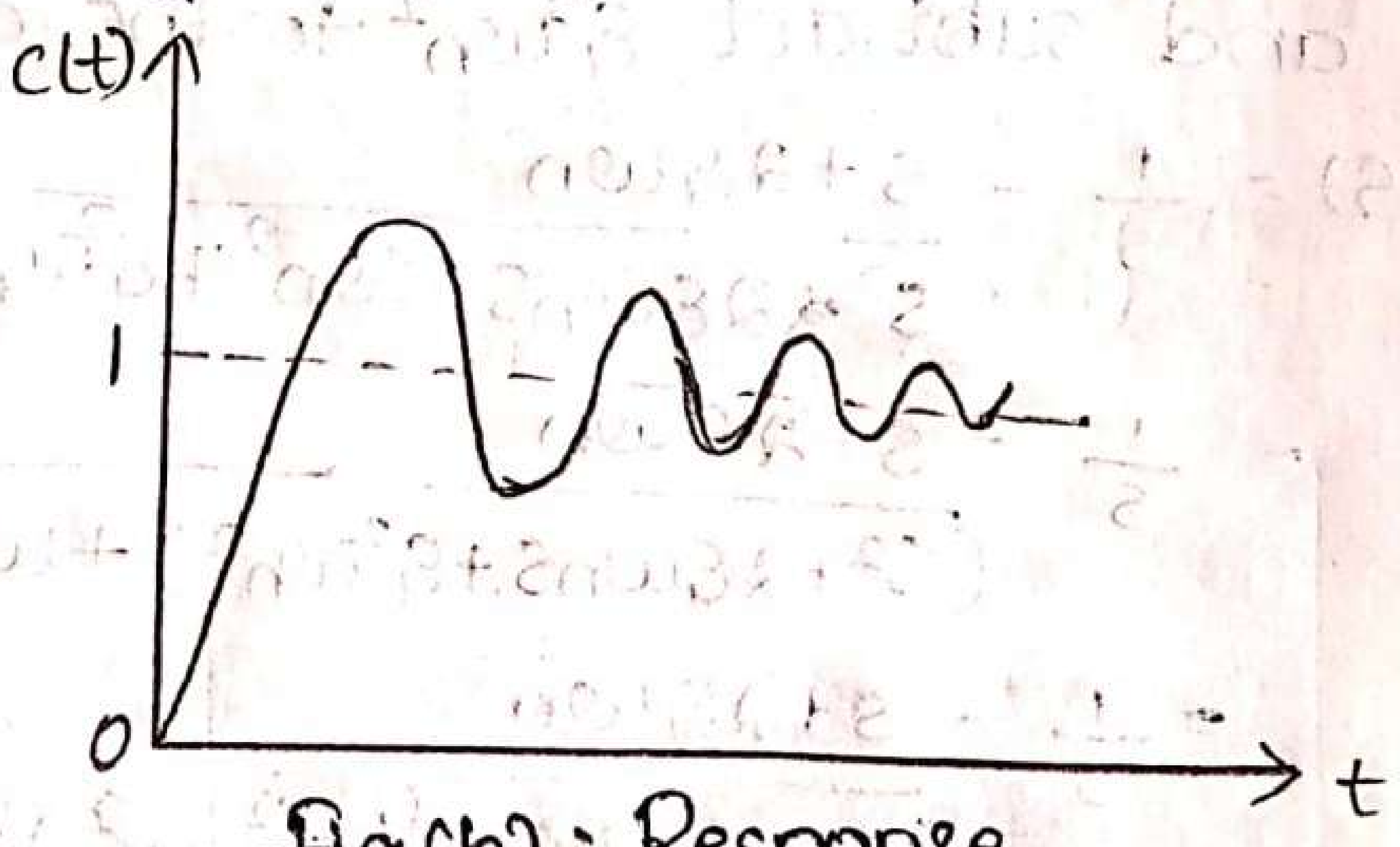
for closed loop under damped second order system,

unit step response $= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)$; $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

step response $= A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)\right]$; $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$



fig(a): Input



fig(b): Response.

Response of critically damped Damped second order system for unit step input:

The standard form of closed loop transfer function of second order system is,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

for critically damped, $\xi=1$

$$\therefore \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} = \frac{\omega_n^2}{(S+\omega_n)^2}$$

when input is unit step, $r(t)=1$ and $R(S)=1/S$

$\therefore$ The response in s-domain,

$$C(S) = R(S)\cdot\frac{\omega_n^2}{(S+\omega_n)^2} = \frac{1}{S}\frac{\omega_n^2}{(S+\omega_n)^2} = \frac{\omega_n^2}{S(S+\omega_n)^2}$$

Apply partial fractions,

$$C(S) = \frac{\omega_n^2}{S(S+\omega_n)^2} = \frac{A}{S}+\frac{B}{(S+\omega_n)^2}+\frac{C}{S+\omega_n} \rightarrow ①$$

$$\omega_n^2 = A(S+\omega_n)^2+BS+C(S+\omega_n)(S)$$

Put $S=0$,

$$\omega_n^2 = A(\omega_n)^2$$

$$A=1$$

$$* \omega_n^2 = A(S^2+\omega_n^2+2S\omega_n)+BS+CS^2+C\omega_n S$$

$$A+C=0$$

$$C=-1$$

Compare S^2 terms,

$$A+C=0$$

$$C=-1$$

$$* 2\omega_n+B+C\omega_n=0$$

$$B+2\omega_n-\omega_n=0$$

$$B+\omega_n=0$$

$$B=-\omega_n$$

Compare S terms,

$$2\omega_n+B+C\omega_n=0$$

$$2\omega_n+B-\omega_n=0$$

$$B=-\omega_n$$

Substitute A, B, C in ①st eqn

$$\frac{1}{S}-\frac{\omega_n}{(S+\omega_n)^2}-\frac{1}{S+\omega_n} = C(S)$$

Apply inverse laplace transform

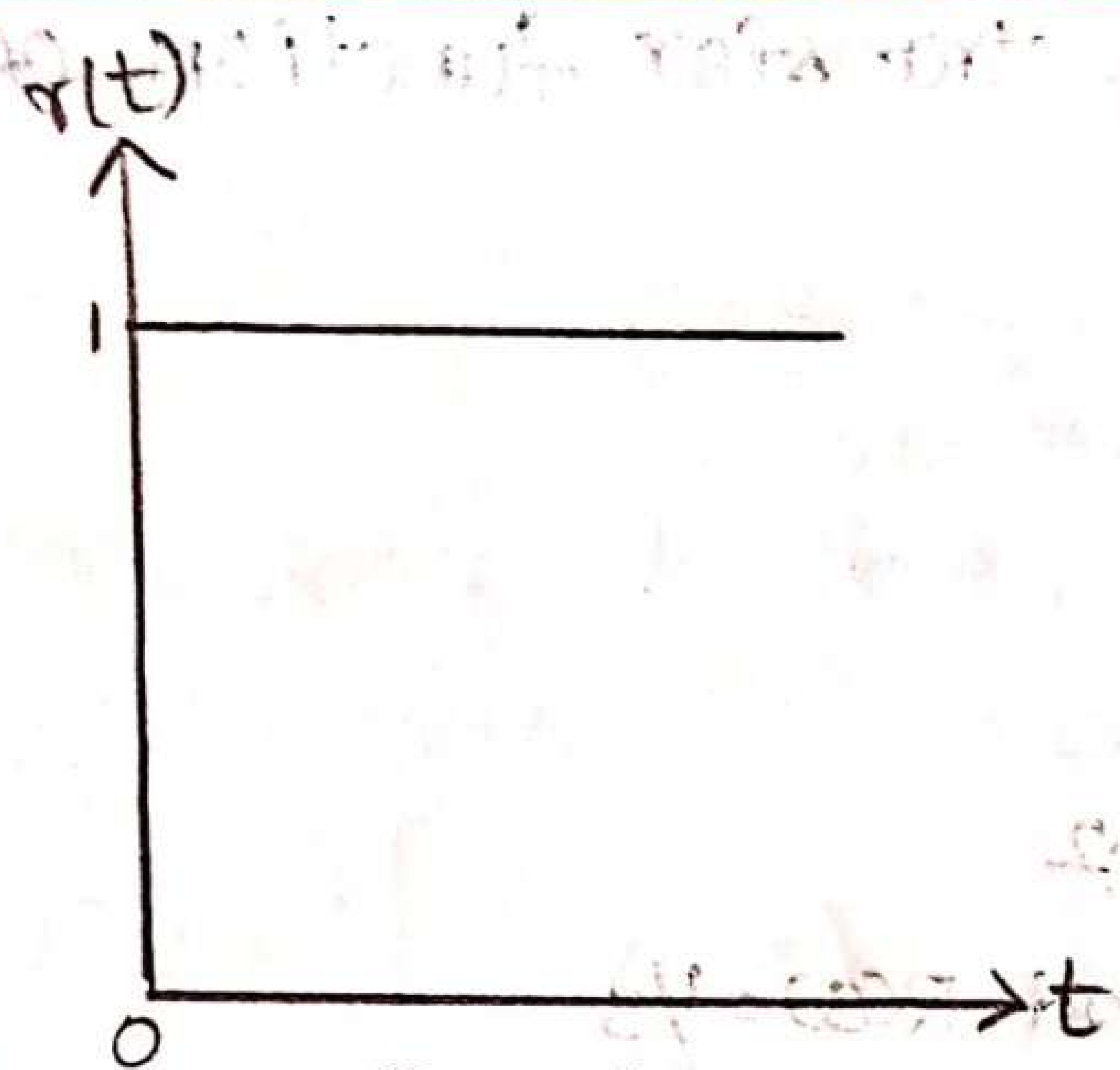
$$c(t) = 1-\omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

$$\left[\because L\{1\}=\frac{1}{S},\ L\{te^{-at}\}=\frac{1}{(S+a)^2},\ L\{e^{-at}\}=\frac{1}{S+a}\right]$$

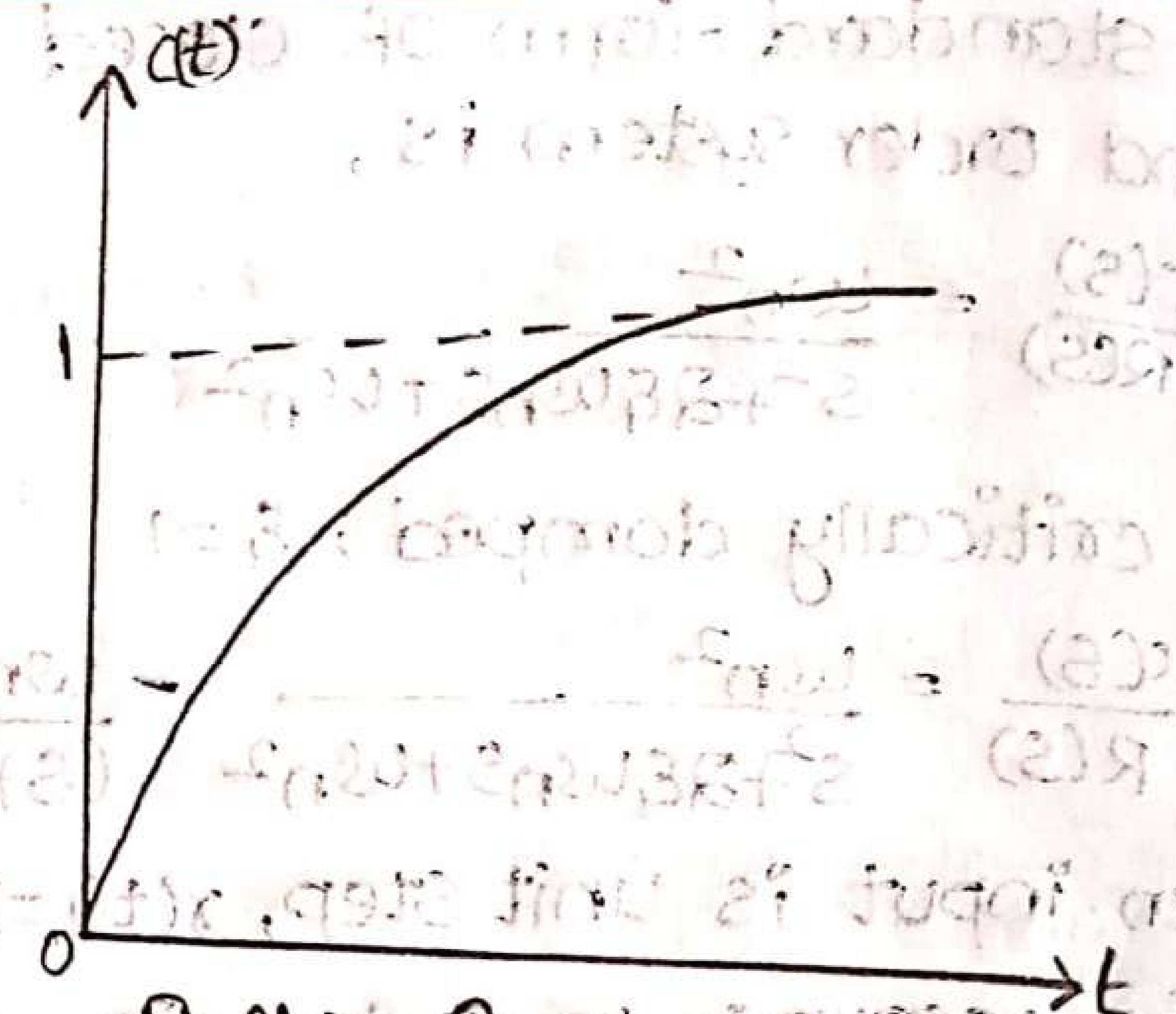
$$c(t) = 1-e^{-\omega_n t}(1+\omega_n t)$$

For closed loop critically damped second order system, unit step response $=1-e^{-\omega_n t}(1+\omega_n t)$

$\therefore$ step response $= A\left[1-e^{-\omega_n t}(1+\omega_n t)\right]$



Fig(a): Input



fig(b): Response.

21/09/23

Response of over damped Second order System for unit step input:

The standard form of closed loop transfer function of Second order System is,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

for overdamped system $\xi > 1$,
let the roots are S_a, S_b.

$$S_a, S_b = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\left[\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}\right]$$

let $S_1 = -S_a$, $S_2 = -S_b$ $\quad \therefore S_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$

$$S_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

The closed loop transfer function can be written in terms of S_1 and S_2 as shown below.

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} = \frac{\omega_n^2}{(S+S_1)(S+S_2)}$$

for unit step input $r(t)=1$ and $R(S)=1/S$

$$\therefore C(S) = R(S)\cdot\frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} = \frac{\omega_n^2}{S(S+S_1)(S+S_2)}$$

Apply partial fractions,

$$C(S) = \frac{\omega_n^2}{S(S+S_1)(S+S_2)} = \frac{A}{S} + \frac{B}{S+S_1} + \frac{C}{S+S_2} \rightarrow ①$$

$$A = S\times C(S)\Big|_{S=0}$$

$$= S\times\frac{\omega_n^2}{S(S+S_1)(S+S_2)}\Bigg|_{S=0} = \frac{\omega_n^2}{S_1S_2}$$

$$= \frac{\omega_n^2}{\left[\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right]\left[\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right]} = \frac{\omega_n^2}{\xi^2\omega_n^2-\omega_n^2(\xi^2-1)}$$

$$= \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (S+S_1) \times C(S)\Big|_{S=-S_1}$$

$$= \frac{\omega_n^2}{S(S+S_2)}\Big|_{S=-S_1} = \frac{\omega_n^2}{-S(-S_1+S_2)}$$

$$= \frac{-\omega_n^2}{S_1[-\varepsilon_1\omega_n + \omega_n\sqrt{\varepsilon_1^2-1} + \varepsilon_1\omega_n + \omega_n\sqrt{\varepsilon_1^2-1}]} = \frac{-\omega_n^2}{[2\omega_n\sqrt{\varepsilon_1^2-1}]S_1}$$

$$= \frac{-\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_1}$$

$$C = (S+S_2) \times C(S)\Big|_{S=-S_2}$$

$$= \frac{\omega_n^2}{S(S+S_1)}\Big|_{S=-S_2} = \frac{\omega_n^2}{-S_2(-S_2+S_1)}$$

$$= \frac{\omega_n^2}{-S_2[-\varepsilon_1\omega_n - \omega_n\sqrt{\varepsilon_1^2-1} + \varepsilon_1\omega_n - \omega_n\sqrt{\varepsilon_1^2-1}]} = \frac{\omega_n^2}{[2\omega_n\sqrt{\varepsilon_1^2-1}]S_2}$$

$$= \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_2}$$

The response in time domain c(t) is given by,

$$c(t) = L^{-1}\left\{\frac{1}{S} - \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S}\frac{1}{(S+S_1)} + \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_2} \cdot \frac{1}{(S+S_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_1} e^{-S_1 t} + \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \frac{1}{S_2} \cdot e^{-S_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}}\left(\frac{e^{-S_1 t}}{S_1} - \frac{e^{-S_2 t}}{S_2}\right)$$

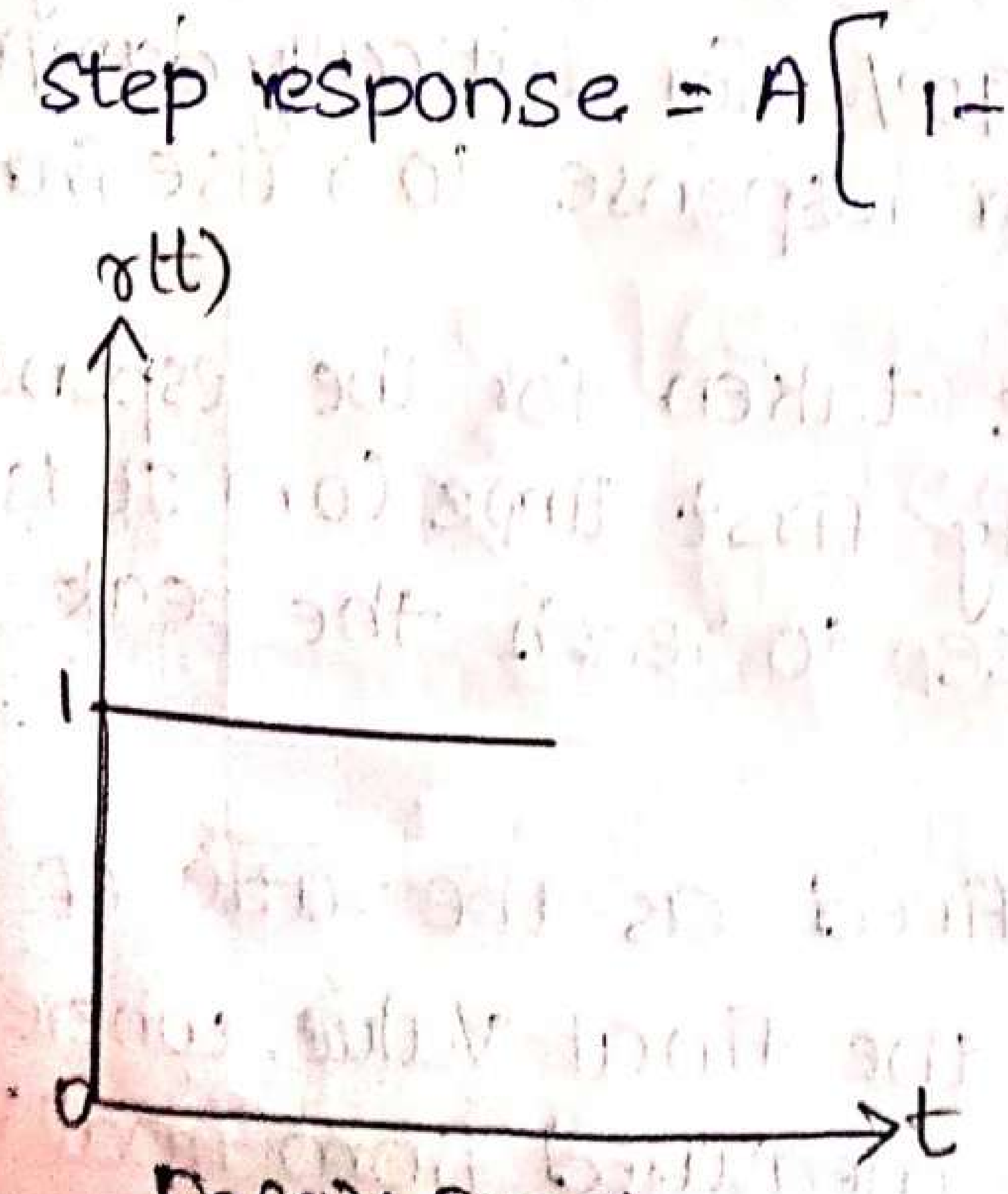
where, $S_1 = \varepsilon_1\omega_n - \omega_n\sqrt{\varepsilon_1^2-1}$

$S_2 = \varepsilon_1\omega_n + \omega_n\sqrt{\varepsilon_1^2-1}$

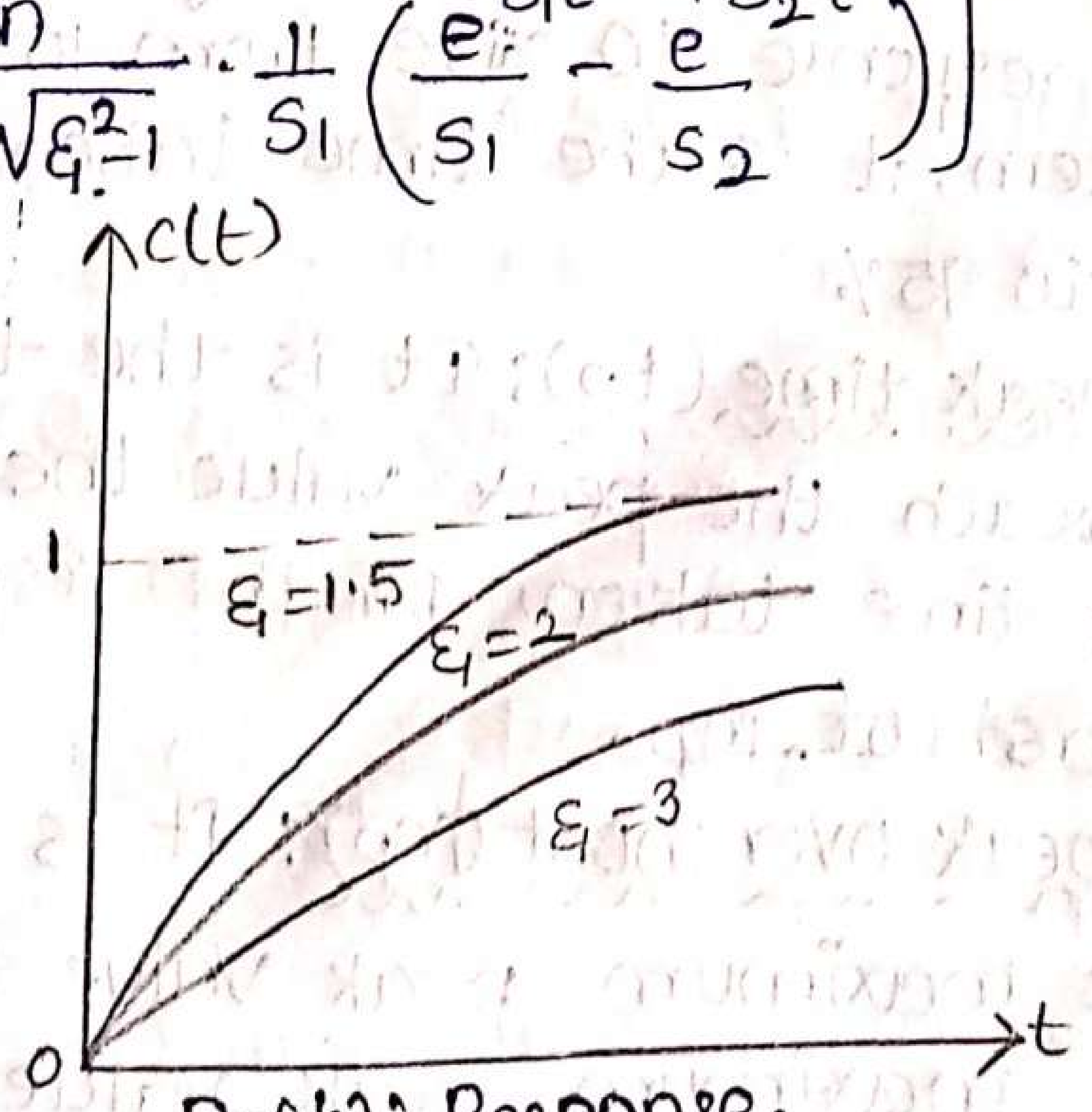
For closed loop over damped second order system,

$$\text{unit step response} = 1 - \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_1}\left(\frac{e^{-S_1 t}}{S_1} - \frac{e^{-S_2 t}}{S_2}\right)$$

$$\text{step response} = A\left[1 + \frac{\omega_n}{2\sqrt{\varepsilon_1^2-1}} \cdot \frac{1}{S_1}\left(\frac{e^{-S_1 t}}{S_1} - \frac{e^{-S_2 t}}{S_2}\right)\right]$$



Fig(a): Input



Fig(b): Response.

22/09/23 **Time domain specifications:**

→ The desired performance characteristics of control systems are specified in terms of time domain specifications.

→ A typical damped oscillation response of a system is shown in the figure below.

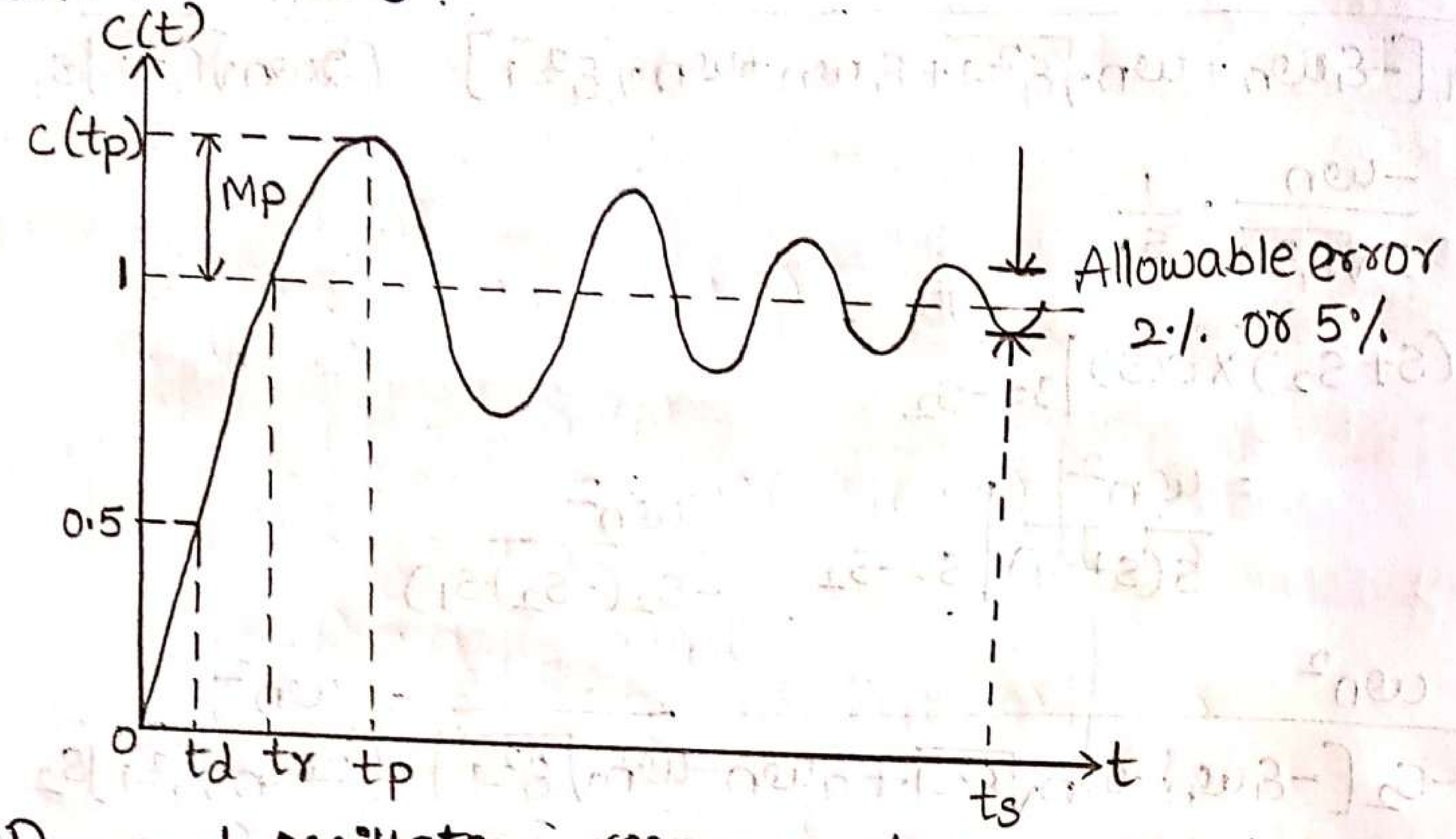


fig: Damped oscillatory response of second order system for unit step input.

→ The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

i) Delay time (td)
ii) Rise time (tr)
iii) peak time (tp)
iv) maximum overshoot (mp)
v) Settling time (ts)

i) Delay time (td): It is the time taken for response to reach 50% of the final value, for the very first time.

ii) Rise time (tr): It is the time taken for response to raise from 0 to 100% for very first time. for underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to rise from 10% - 90%. for critically damped System, it is the time taken for response to raise from 5% to 95%.

iii) peak time (tp): It is the time taken for the response to reach the peak value the very first time (or) It is the time taken for the response to reach the peak overshoot, Mp.

iv) peak overshoot (Mp): It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value

Let $c(\infty)$ = final value of $c(t)$

$c(t_p)$ = Maximum Value of $c(t)$

Now peak overshoot, $M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$

% peak overshoot, $\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

vi) Settling time (t_s): It is defined as the time taken by the response to reach and stay within a Specified error. It is usually Expressed as % of final Value. The usual tolerable error is 2% to 5% of the final value.

* Expressions for time domain Specifications:

i) Rise time (t_r):

The unit step response of second order system for under damped case is given by,

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

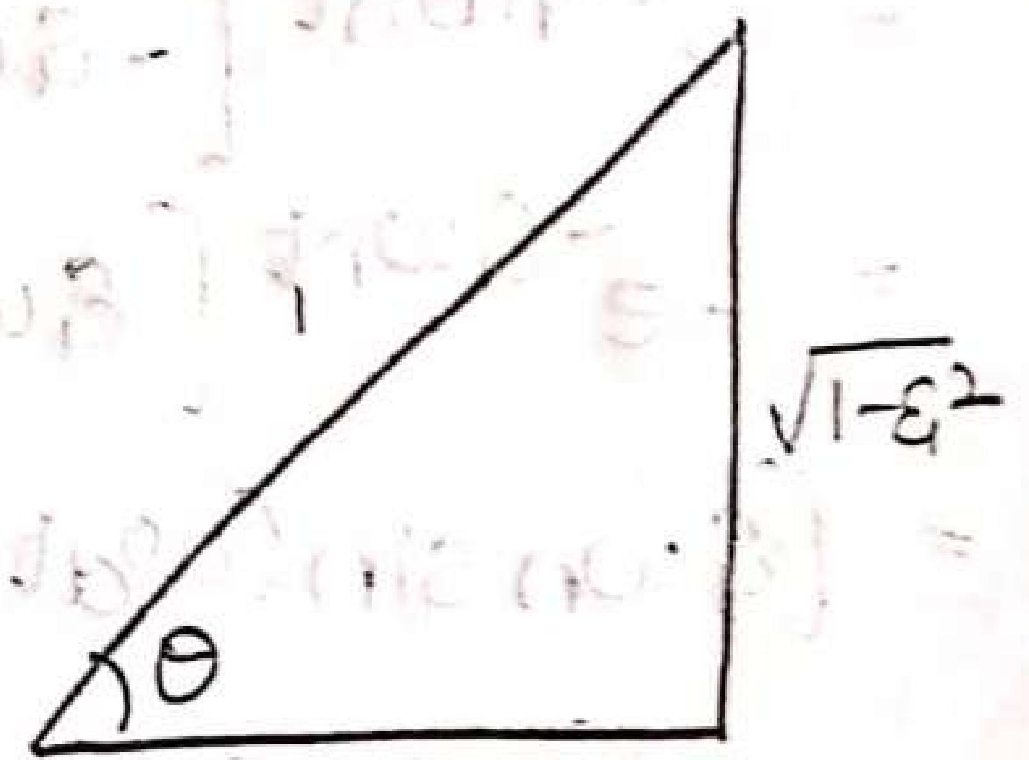
At $t = t_r$, $c(t) = c(t_r) = 1$

$$c(t_r) = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$1 = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$\frac{-e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

$$\sin(\omega_d t_r + \theta) = 0$$



$\sin\pi = \sin 2\pi = 0$

$\sin \omega_d t_r + \theta = \sin \pi$

$\omega_d t_r + \theta = \pi$

$\omega_d t_r = \pi - \theta$

$t_r = \frac{\pi - \theta}{\omega_d}$

$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$

$\therefore$ Rise time, $t_r = \frac{\pi - \theta}{\omega_d}$

Here $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$ and should be measured in radians

ω_d = damped frequency of oscillations $= \omega_n\sqrt{1-\xi^2}$

$\therefore$ Rise time, $t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$ in sec.

23/09/23 (ii) **peak time (t_p):**

To find the expression for peak time, t_p, differentiate $c(t)$ with respect to t and equate to zero.

$$\frac{d}{dt}c(t)\Big|_{t=t_p}=0$$

The unit step response of underdamped second order system is given by,

$$c(t)=1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cdot\sin(\omega_d t+\theta)$$

$$\frac{d}{dt}[c(t)]=\frac{d}{dt}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cdot\sin(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=\left[\frac{d}{dt}(1)-\frac{d}{dt}\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cdot\sin(\omega_d t+\theta)\right]\right]_{t=t_p}=0$$

$$=\left[0-\frac{1}{\sqrt{1-\xi^2}}\frac{d}{dt}\left[e^{-\xi\omega_n t}\cdot\sin(\omega_d t+\theta)\right]\right]_{t=t_p}=0$$

$$=-\frac{1}{\sqrt{1-\xi^2}}\frac{d}{dt}\left[e^{-\xi\omega_n t}\cdot\sin(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=\frac{d}{dt}\left[\overset{u}{\underline{e^{-\xi\omega_n t}}}\cdot\overset{v}{\underline{\sin(\omega_d t+\theta)}}\right]_{t=t_p}=0 \qquad \frac{d}{dt}(u\cdot v)=u'v+v'u$$

$$=e^{-\xi\omega_n t}\cdot(-\xi\omega_n)\sin(\omega_d t+\theta)+\cos(\omega_d t+\theta)\cdot\omega_d e^{-\xi\omega_n t}$$

$$=e^{-\xi\omega_n t}\left[-\xi\omega_n\sin(\omega_d t+\theta)+\omega_d\cos(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=-e^{-\xi\omega_n t}\left[\xi\omega_n\sin(\omega_d t+\theta)-\omega_d\cos(\omega_d t+\theta)\right]_{t=t_p}=0 \qquad \boxed{\omega_d=\omega_n\sqrt{1-\xi^2}}$$

$$=\left[\xi\omega_n\sin(\omega_d t+\theta)-\omega_n\sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=\omega_n\left[\xi\sin(\omega_d t+\theta)-\sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=\left[\xi\sin(\omega_d t+\theta)-\sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right]_{t=t_p}=0$$

construct a triangle using ξ and $\sqrt{1-\xi^2}$

$$=\left[\cos\theta\sin(\omega_d t+\theta)-\sin\theta\cos(\omega_d t+\theta)\right]_{t=t_p}=0$$

$$=\left[\sin(\underset{\cos\theta}{\omega_d t+\theta})-\cos(\omega_d t+\theta)\sin\theta\right]_{t=t_p}=0$$

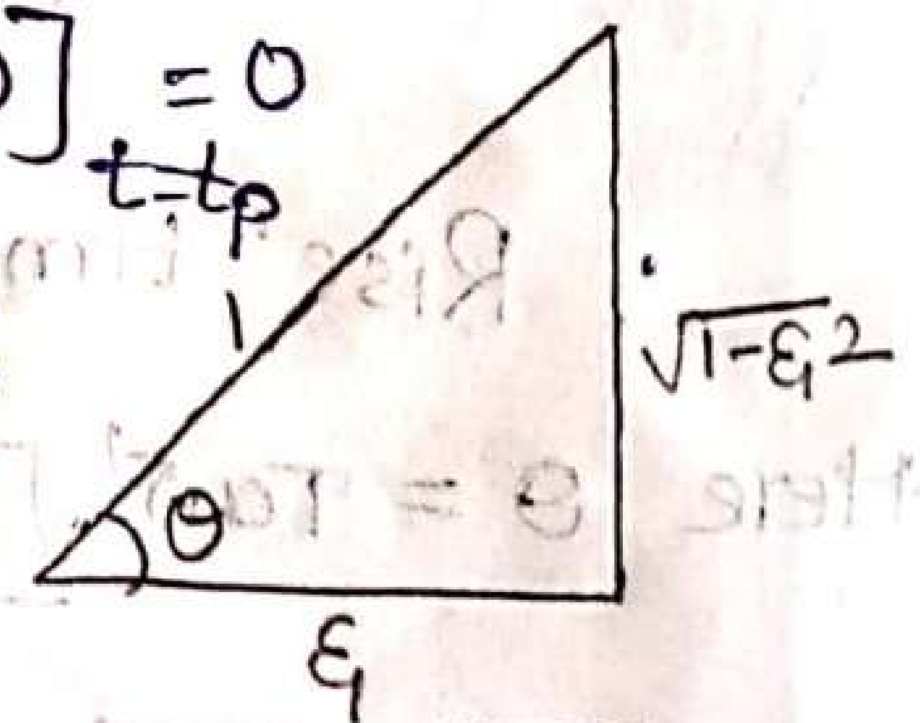
$$\left[\because\sin A\cos B-\cos A\sin B\right]=\left[\sin(A-B)\right]$$

$$=\left[\sin(\omega_d t+\underset{-\theta}{\theta})\right]_{t=t_p}=0$$

$$\sin\pi=\sin 2\pi=0$$

$$=\sin(\omega_d t_p-\cancel{\theta\pm\theta})=\sin\pi$$

$$=\omega_d t_p-\cancel{\theta\pm\theta}=\pi$$



$\sin\theta=\sqrt{1-\xi^2}$

$\cos\theta=\xi$.

$\tan\theta=\dfrac{\sqrt{1-\xi^2}}{\xi}$

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d}$$

The damped frequency of oscillations is, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

iii) Peak overshoot (Mp):

$$\%\text{ peak overshoot}(M_p) = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

$c(t_p)$ = peak value at $t = t_p$

$c(\infty)$ = final steady state value.

The unit step response of underdamped second order system is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t + \theta) \rightarrow ①$$

case(i): put $t = \infty$

$$c(\infty) = 1 - \frac{e^{-\xi\omega_n(\infty)}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d(\infty) + \theta)$$

$$[\because e^{-\infty} = 0]$$

$$c(\infty) = 1 - 0 = 1$$

case(ii): put $t = t_p$

$$c(t_p) = 1 - \frac{e^{-\xi\omega_n(t_p)}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t_p + \theta)$$

$$\left[\because t_p = \frac{\pi}{\omega_d}\right]$$

$$c(t_p) = 1 - \frac{e^{-\xi\omega_n \cdot \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}} \cdot \sin\left(\cancel{\omega_d} \cdot \frac{\pi}{\cancel{\omega_d}} + \theta\right)$$

$$c(t_p) = 1 - \frac{e^{-\xi\cancel{\omega_n} \cdot \frac{\pi}{\cancel{\omega_n}\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin(\pi + \theta) \quad \left[\omega_d = \omega_n\sqrt{1-\xi^2}\right]$$

$$c(t_p) = 1 - \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot (-\sin\theta) \quad [\because \sin(\pi+\theta) = -\sin\theta]$$

$$c(t_p) = 1 + \frac{e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin\theta$$

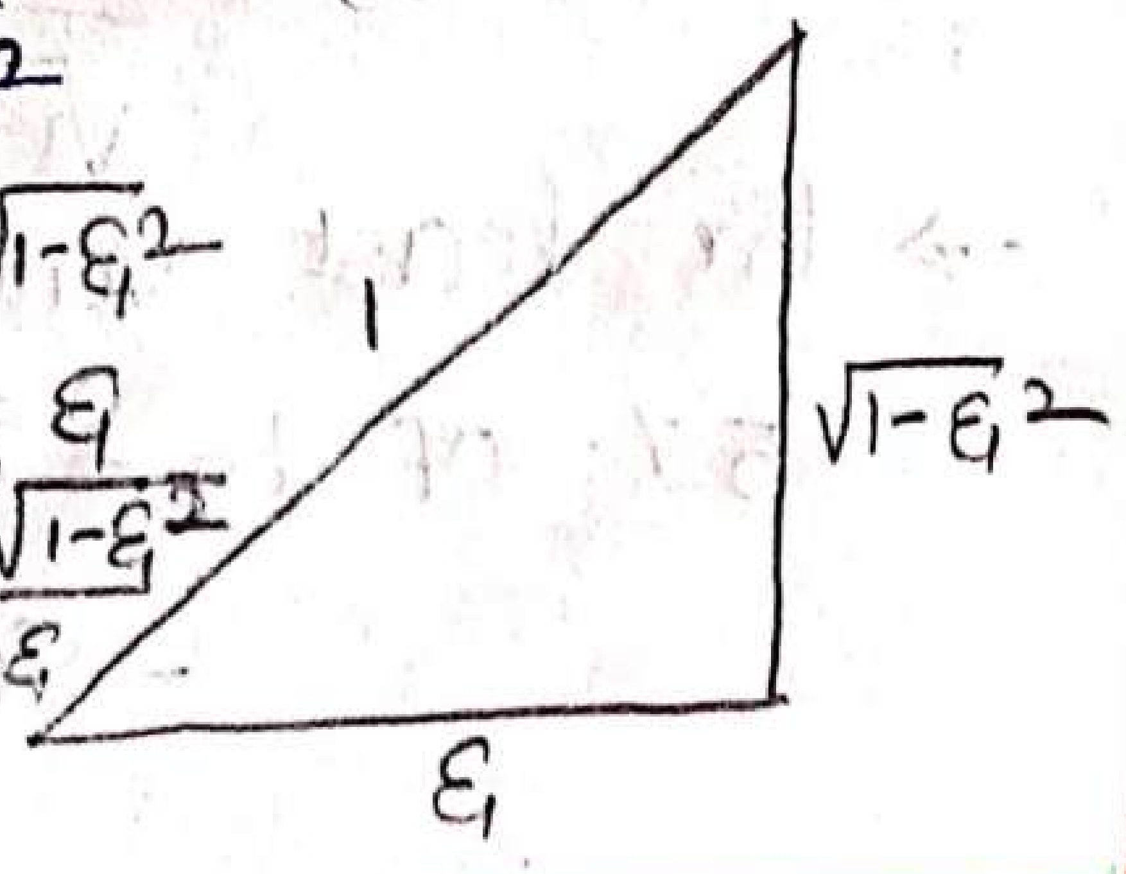
construct a triangle using ξ and $\sqrt{1-\xi^2}$

$$c(t_p) = 1 + \frac{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}{\cancel{\sqrt{1-\xi^2}}} \cdot \cancel{\sqrt{1-\xi^2}}$$

$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$



$$c(tp) = 1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

Substitute $c(\infty)$ and $c(tp)$ in $\% Mp$

$$Mp = \frac{\cancel{1} + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} - \cancel{1}}{1}$$

$$Mp = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\% Mp = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

(iv) Settling time (ts):

The response of second order system has two components

i) Decaying exponential component, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

ii) Sinusoidal component, $\sin(\omega_d t + \theta)$

→ In this the decaying exponential term reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component.

→ The settling time can be found out by equating the exponential component to percentage tolerance errors.

→ for 2% of $ts = \frac{e^{-\xi\omega_n ts}}{\sqrt{1-\xi^2}} = 2\%$

→ for least value of ξ,

2% of $ts = e^{-\xi\omega_n ts} = 0.02$

$$-\xi\omega_n ts = \ln(0.02)$$

$$-\xi\omega_n ts = -4$$

$$\xi\omega_n ts = 4$$

$$ts = \frac{4}{\xi\omega_n}$$

$$ts = 4 \times \frac{1}{\xi\omega_n}$$

where $T = \frac{1}{\xi\omega_n}$

$\therefore ts = 4T$

Here T = time constant.

→ 5% of $ts = \frac{e^{-\xi\omega_n ts}}{\sqrt{1-\xi^2}} = 5\%$

→ for least value of ξ,

5% of $ts = e^{-\xi\omega_n ts} = 0.05$

$$-\xi\omega_n ts = \ln(0.05)$$

$$-\xi \omega_n t_s = -3$$

$$\xi \omega_n t_s = 3$$

$$t_s = \frac{3}{\xi \omega_n}$$

$$t_s = 3 \times \frac{1}{\xi \omega_n}$$

$$t_s = 3T \quad \left[\because T = \frac{1}{\xi \omega_n}\right]$$

Here T = time constant.

problems:

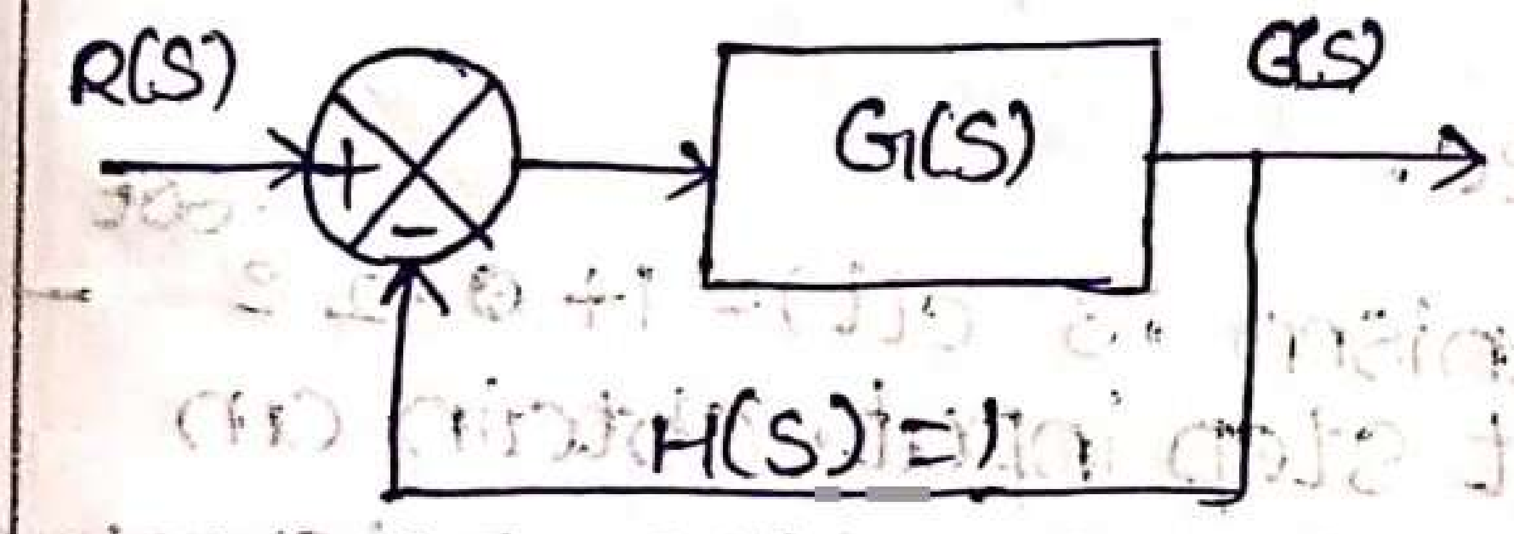
1) The unity feedback system is characterised by an open loop transfer function $G(S) = \frac{K}{S(S+10)}$. Determine the gain 'K', so that the system will have a damping ratio of 0.5 for this value of K. Determine peak time, peak overshoot for the unit step input. and also find settling time.

Given that,

$G(S) = \frac{K}{S(S+10)}$; $H(S) = 1$

time at peak over shoot (or) peak time

damping ratio $= \xi = 0.5$



$$T(S) = \frac{G(S)}{1+G(S)\cdot H(S)}$$

$$T(S) = \frac{\frac{K}{S(S+10)}}{1+\frac{K}{S(S+10)}\cdot(1)} = \frac{\frac{K}{\cancel{S(S+10)}}}{\frac{S(S+10)+K}{\cancel{S(S+10)}}} = \frac{K}{S(S+10)+K}$$

$$T(S) = \frac{K}{S^2+10S+K} \longrightarrow ①$$

for 2nd order system standard second order eqn is,

$$T(S) = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} \longrightarrow ②$$

comparing eqn ① and ②

$2\xi\omega_n = 10$

$2 \times 0.5 \times \omega_n = 10$

$\omega_n = 10$ rad/sec

$K = \omega_n^2$

$K = 10^2$

$K = 100$

i) peak time(t_p):

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \quad [\because \omega_d = \omega_n\sqrt{1-\xi^2}]$$

$$t_p = \frac{3.14}{10\sqrt{1-0.5^2}} = 0.36 \text{ sec}$$

ii) %Mp:

$$\%M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

$$= e^{\frac{-0.5\times3.14}{\sqrt{1-0.5^2}}} \times 100$$

$$\%M_p = 16.3\%$$

iii) settling time(t_s):

$$2\% \text{ of } t_s = 4T$$

$$T = \frac{1}{\xi\omega_n} = \frac{1}{0.5\times10} = 0.2$$

$$2\% \text{ of } t_s = 4\times0.2 = 0.8 \text{ sec}$$

$$5\% \text{ of } t_s = 3T = 3\times0.2 = 0.6 \text{ sec.}$$

2) The response of a servo mechanism is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

Given that

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

we know that $\frac{C(S)}{R(S)} = T(S)$

For unit step input $r(t) = 1$ and $R(S) = 1/s$

Apply laplace for c(t)

$$L\{c(t)\} = L\{1 + 0.2e^{-60t} - 1.2e^{-10t}\}$$

$$C(S) = \frac{1}{s} + 0.2\cdot\frac{1}{S+60} - 1.2\cdot\frac{1}{S+10}$$

$$C(S) = \frac{1}{s} + \frac{0.2}{S+60} - \frac{1.2}{S+10}$$

$$= \frac{1(S+60)(S+10) + 0.2(s)(S+10) - 1.2(S)(S+60)}{S(S+60)(S+10)}$$

$$= \frac{s^2 + 10s + 60s + 600 + 0.2s^2 + 2s - 1.2s^2 - 72s}{s(S+60)(S+10)}$$

$$C(S) = \frac{600}{s(s+60)(s+10)} = \frac{1}{s} \cdot \frac{600}{(s+60)(s+10)}$$

$$C(S) = R(S) \cdot \frac{600}{(s+60)(s+10)} \quad [\because R(S) = 1/s]$$

$$\frac{C(S)}{R(S)} = T(S) = \frac{600}{s^2+70s+600} \rightarrow ①$$

Standard Transfer function $= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \rightarrow ②$

compare eqn ① and ②

$2\xi\omega_n = 70$

$2\times\xi\times 24.49 = 70$

$\xi = 1.42$

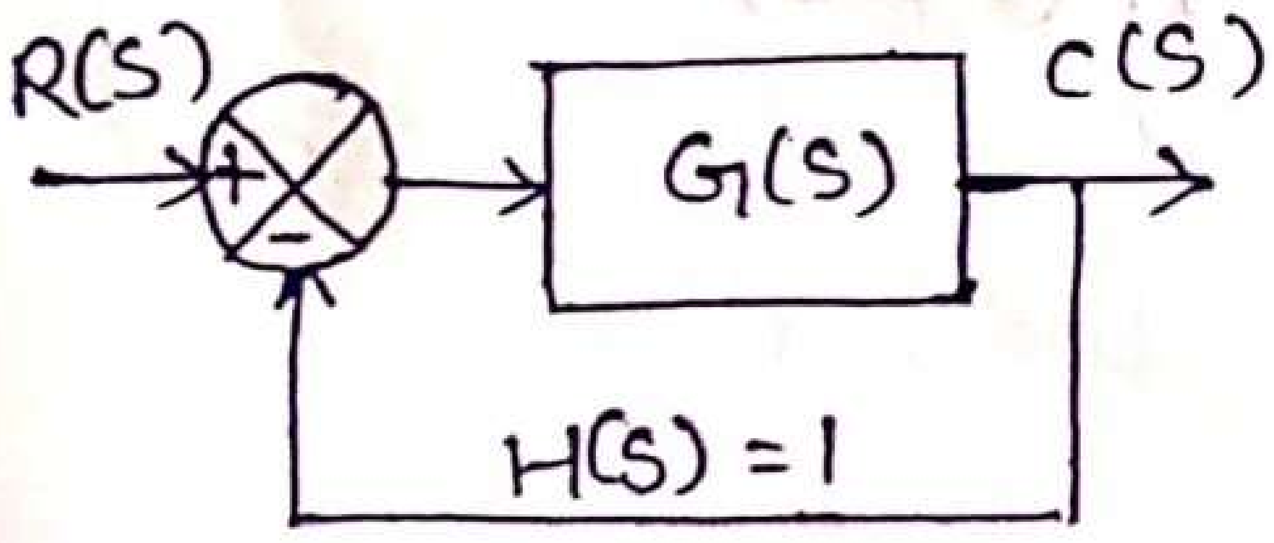
$\omega_n^2 = 600$

$\omega_n = 24.49$ rad/sec.

3) A unity feed back system has an open loop transfer function $G(S) = \frac{10}{S(S+2)}$. find rise time, percent overshoot, peak time and settling time for a step input of 12 min.

Given that,

$G(S) = \frac{10}{S(S+2)}$; $H(S) = 1$



$$T(S) = \frac{G(S)}{1+G(S)H(S)}$$

$$= \frac{\frac{10}{S(S+2)}}{1+\frac{10}{S(S+2)}\cdot(1)} = \frac{\frac{10}{S(S+2)}}{\frac{S(S+2)+10}{S(S+2)}} = \frac{10}{S^2+2S+10}$$

$$T(S) = \frac{10}{S^2+2S+10} \rightarrow ①$$

for second order system, the standard equation is,

$$T(S) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \rightarrow ②$$

compare Eqn ① and ②

$2\xi\omega_n = 2$

$2\times 3.16\times\xi = 2$

$\xi = \frac{2}{2\times 3.16} = 0.316$

$\omega_n^2 = 10$

$\omega_n = \sqrt{10}$

$\omega_n = 3.16$

i) Rise time (t_r):

$$t_r = \frac{\pi-\theta}{\omega d}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \quad ; \omega d = \omega n\sqrt{1-\xi^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-(0.316)^2}}{0.316}\right)$$

$$\theta = 71.57°$$

$$\theta = 71.57 \times \frac{\pi}{180}$$

$$\theta = 1.24 \text{ rad}$$

$$\omega d = 3.16\sqrt{1-(0.316)^2}$$

$$= 2.99 \text{ rad/sec}$$

$$t_r = \frac{3.14-1.24}{2.99}$$

$$t_r = 0.635 \text{ sec.}$$

ii) peak time (t_p):

$$t_p = \frac{\pi}{\omega d} = \frac{\pi}{\omega n\sqrt{1-\xi^2}} = \frac{3.14}{3.16\sqrt{1-(0.316)^2}}$$

$$= 1.047 \text{ sec}$$

iii) settling time (t_s):

$$T = \frac{1}{\xi\omega n} = \frac{1}{0.316 \times 3.16} = 1$$

for 2% = 4T = 4sec

for 5% = 3T = 3sec

iv) percentage overshoot (%Mp):

$$\%Mp = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

$$= e^{\frac{-0.316 \times 3.14}{\sqrt{1-(0.316)^2}}} \times 100$$

$$\%Mp = 35.92\,\%$$

$$= \frac{35.92}{100} \times 12$$

$$= 4.31 \text{ units}$$

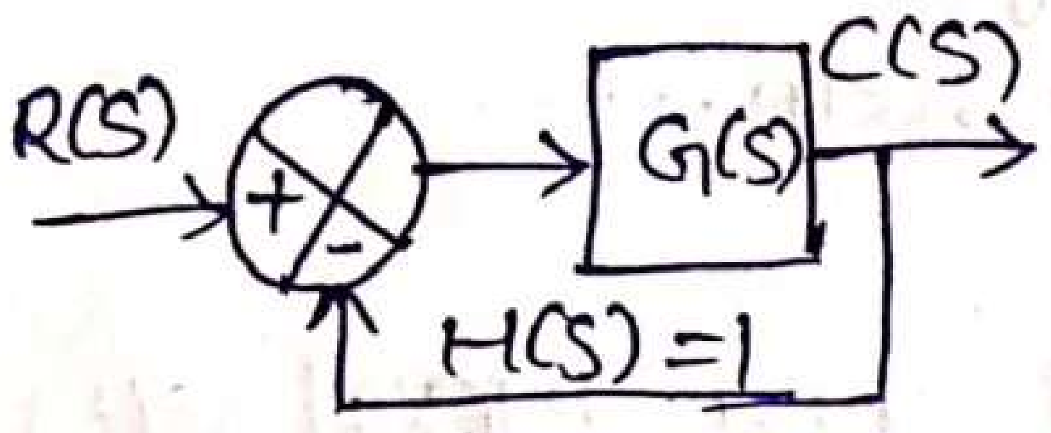
Homework problem:

1) The open loop Transfer function of ~~f~~ unity feedback control system is given by $G(S) = \frac{9}{s(s+3)}$. find the

natural frequency of the response, damping ratio, damped frequency and Time constant.

Given that,

$$G(S) = \frac{9}{S(S+3)} \; ; \; H(S) = 1$$



$$T(S) = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)(H)(S)}$$

$$= \frac{\frac{9}{S(S+3)}}{1+\frac{9}{S(S+3)}\cdot(1)} = \frac{\frac{9}{S(S+3)}}{\frac{S(S+3)+9}{S(S+3)}} = \frac{9}{S(S+3)+9}$$

$$\therefore T(S) = \frac{9}{S^2+3S+9} \rightarrow ①$$

for second order system the standard eqn is,

$$T(S) = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} \rightarrow ②$$

compare ① and ②

$2\xi\omega_n = 3$

$2\times3\times\xi = 3$

$6\xi = 3$

$\xi = \frac{3}{6}$

$\xi = 0.5$

$\omega_n^2 = 9$

$\omega_n = \sqrt{9}$

$\omega_n = 3 r/s$

$\omega_n = 3 \text{ rad/sec}$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$= 3\sqrt{1-0.5^2}$

$\omega_d = 2.59 \text{ rad/sec}$

$T = \frac{1}{\xi\omega_n} = \frac{1}{0.5\times3} = 0.6 \text{ Sec}$

25/09/23

Type Number of control system:

- → The type number is specified for the loop transfer function $G(S).H(S)$.
- → The no. of poles of the loop transfer function lying at the origin decides the type number of the system.
- → In general, if N is the number of poles at the origin then the type number is N.
- → The loop transfer function can be expressed as a ratio of two polynomials in S

$$G(S).H(S) = K\cdot\frac{P(S)}{Q(S)} = K\cdot\frac{(S+z_1)(S+z_2)(S+z_3)\cdots}{S^N(S+P_1)(S+P_2)(S+P_3)\cdots}$$

where, $z_1, z_2, z_3 \ldots$ are zeros of transfer function

$P_1, P_2, P_3 \ldots$ are poles of transfer function

K = constant, N = Number of poles at the origin.

→ The value of 'N' in the denominator polynomial of loop transfer function shown in below; decides the type number of the system.

→ If N=0, then the system is type-0 System.

→ If N=1, then the System is type-1 System.

→ If N=2, then the system is type-2 system....

07/10/23

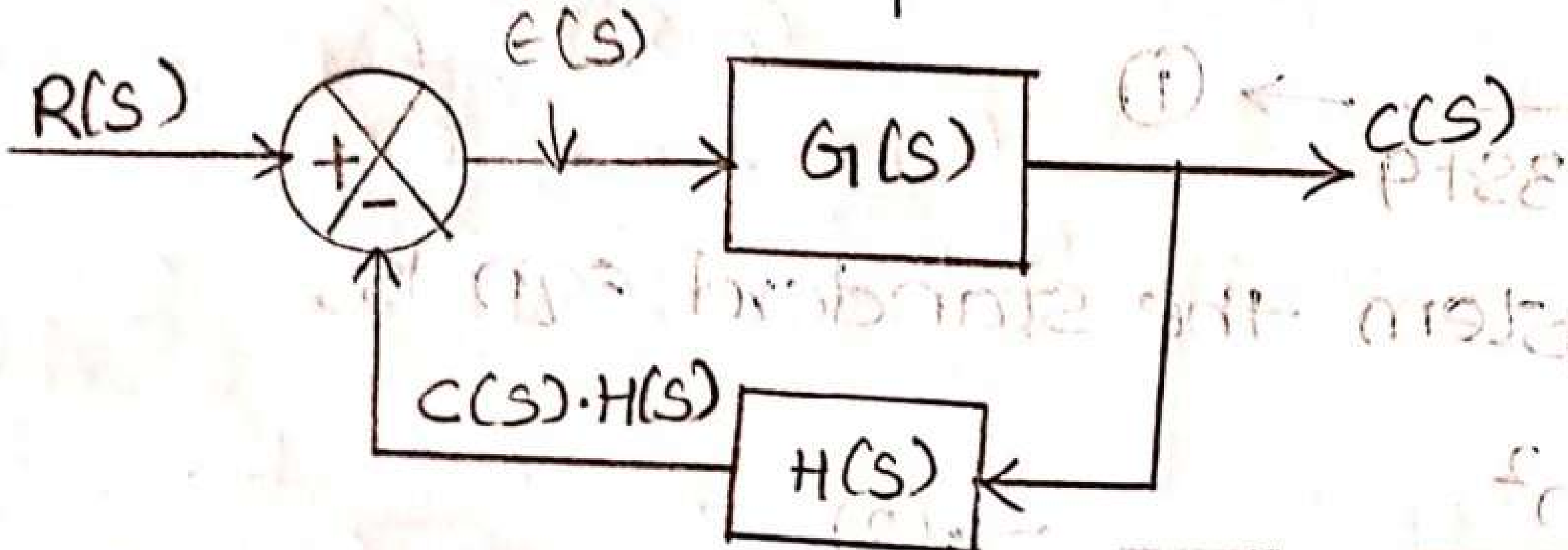
Steady state Error:

→ The steady state error is the value of error signal e(t) when $t \to \infty$.

→ The steady state error is a measure of system accuracy.

→ These errors arise from the nature of inputs, type of System and from non-linearity of the system components.

→ This Steady state performance of a stable control system is generally judged by ~~th~~ its steady state error to step, ramp and parabolic inputs.



$$e(s) = R(s) - C(s)\cdot H(s)$$
$$C(S) = G(S)\cdot E(S)$$

$$E(S) = R(S) - G(S)\cdot E(S)\cdot H(S)$$
$$E(S) = R(S) - G(S)\cdot E(S)\cdot H(S)$$
$$E(S) + G(S)\cdot E(S)\cdot H(S) = R(S)$$
$$E(S)\left[1 + G(S)\cdot H(S)\right] = R(S)$$
$$E(S) = \frac{R(S)}{1+G(S)\cdot H(S)}$$

* Final Value Theorem: $f(s) = \lim_{t\to\infty} f(t)$; then $\lim_{s\to 0} s\cdot f(s)$

let e(t) = error signal in time domain.

$$e(t) = L^{-1}\{E(S)\}$$

ess = steady state error

$$e_{ss} = \lim_{t\to\infty} e(t) \qquad f(s) = L\{f(t)\}$$

From final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot f(s)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\, E(s)$$

$$e_{ss} = \lim_{s\to 0} s\left[\frac{R(S)}{1+G(S)\cdot H(S)}\right]$$

$$e_{ss} = \underset{S\to 0}{lt} \frac{S\cdot R(S)}{1+G(S)\cdot H(S)}$$

Static Error Constants:

→ When a Control System Excited with standard i/p signal, the steady state error may be zero, constant (or) infinity.

→ The value of steady state error will depends on the type number and the i/p signal.

→ Type-0 system will have a constant steady state error when the input is step signal.

→ Type-1 will have the constant steady state error when the input is ramp signal or velocity signal.

→ Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal

→ for the 3-cases mentioned above the steady state error is associate with one of the constants defined as follows.

positional error constant, $K_p = \underset{S\to 0}{lt}\; G(S)\cdot H(S)$

velocity error constant, $K_v = \underset{S\to 0}{lt}\; S\cdot G(S)\cdot H(S)$

Acceleration error constant, $K_a = \underset{S\to 0}{lt}\; S^2\, G(S)\cdot H(S)$

→ The Kp, Ka, Kv are ingeneral called static error constant

Steady state error when the input is unit step signal:

$$e_{ss} = \underset{S\to 0}{lt} \frac{S\cdot R(S)}{1+G(S)\cdot H(S)}$$

for unit signal $= R(S) = \frac{1}{S}$

$$e_{ss} = \underset{S\to 0}{lt} \frac{\cancel{S}\times\frac{1}{\cancel{S}}}{1+G(S)\cdot H(S)} = \underset{S\to 0}{lt} \frac{1}{1+G(S)\cdot H(S)}$$

$$e_{ss} = \frac{1}{1+\underset{S\to 0}{lt}\, G(S)\cdot H(S)} = \frac{1}{1+K_p}$$

$[\because K_p = \text{positional error constant} = \underset{S\to 0}{lt}\; G(S)\cdot H(S)]$

Type-0 System:

$$K_p = \underset{S\to 0}{lt}\; G(S)\cdot H(S)$$

$$K_p = \underset{S\to 0}{lt}\; K \frac{(S+Z_1)(S+Z_2)----}{S^N(S+P_1)(S+P_2)---}$$

$$K_p = \underset{S\to 0}{lt}\; K \frac{\cancel{K}(S+Z_1)(S+Z_2)---}{(S+P_1)(S+P_2)----}$$

$$K_p = K.\frac{Z_1Z_2Z_3----}{P_1P_2P_3----} = \text{Constants}$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\text{constant}} = \text{Constant}.$$

Type-1 system:

$$K_p = \underset{S\to 0}{lt}\ G(S)\cdot H(S)$$

$$K_p = \underset{S\to 0}{lt}\ K\cdot \frac{(S+z_1)(S+z_2)\div ---}{S^N (S+P_1)(S+P_2)---}$$

$$K_p = \underset{S\to 0}{lt}\ K.\frac{(S+z_1)(S+z_2)---}{S(S+P_1)(S+P_2)--}$$

$$K_p = \cancel{lt}\ K\cdot \frac{z_1 z_2 z_3 ----}{0}$$

$$K_p = K\cdot\infty = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0$$

Steady state error when input is unit ramp signal:

$$e_{ss} = \underset{S\to 0}{lt}\ \frac{S\cdot R(S)}{1+G(S)\cdot H(S)}$$

$$\text{Ramp signal} = R(S) = \frac{1}{S^2}$$

$$e_{ss} = \underset{S\to 0}{lt}\ \frac{\cancel{S}\cdot\frac{1}{S^{\cancel{2}}}}{1+G(S)\cdot H(S)}$$

$$e_{ss} = \underset{S\to 0}{lt}\ \frac{1}{S(1+G(S)\cdot H(S))} = \underset{S\to 0}{lt}\ \frac{1}{S+S(G(S)\cdot H(S))}$$

$$= \cancel{lt}\ \frac{1}{0+\underset{S\to 0}{lt}\ S\cdot G(S)\cdot H(S)}$$

$$e_{ss} = \frac{1}{K_v}$$

Type-0 system:

$$K_v = \underset{S\to 0}{lt}\ S\cdot G(S)\cdot H(S)$$

$$K_v = \underset{S\to 0}{lt}\ S\cdot K\cdot \frac{(S+z_1)(S+z_2)----}{S^N(S+P_1)(S+P_2)--}$$

$$K_v = 0\cdot K\cdot \frac{z_1 z_2 --}{p_1 P_2 --}$$

$$K_v = 0$$

$$e_{ss} = \frac{1}{0} = \infty$$

Type-1 system:

$$K_v = \underset{S\to 0}{lt}\ S\cdot G(S)\cdot H(S)$$

$$K_v = \underset{S\to 0}{Lt}\ S\cdot K\cdot \frac{(S+Z_1)(S+Z_2)---}{S^N(S+P_1)(S+P_2)----}$$

$$K_v = \underset{S\to 0}{Lt}\ \cancel{S}\cdot K\cdot \frac{(S+Z_1)(S+Z_2)---}{\cancel{S}^1(S+P_1)(S+P_2)--}$$

$$K_v = K\cdot \frac{Z_1Z_2---}{P_1P_2---}$$

$$\therefore\ e_{ss} = \frac{1}{K_v} = \text{constant}.$$

Type-2 system:

$$K_v = \underset{S\to 0}{Lt}\ S\cdot G(S)\cdot H(S)$$

$$K_v = \underset{S\to 0}{Lt}\ S\cdot K \frac{\cancel{S}\cdot\cancel{K}(S+Z_1)(S+Z_2)----}{S^N(S+P_1)(S+P_2)----}$$

$$K_v = \underset{S\to 0}{Lt}\ \cancel{S}\cdot K\cdot \frac{(S+Z_1)(S+Z_2)---}{S^{\cancel{2}}(S+P_1)(S+P_2)---}$$

$$K_v = \underset{S\to 0}{Lt}\ K\cdot \frac{(S+Z_1)(S+Z_2)--}{S(S+P_1)(S+P_2)----}$$

$$K_v = \underset{S\to 0}{Lt}\ K\cdot \frac{Z_1Z_2----}{0}$$

$$K_v = K\cdot\frac{Z_1Z_2}{0} = \infty$$

$$K_v = \infty$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

Steady state error when input is unit parabolic signal:

$$e_{ss} = \underset{S\to 0}{Lt}\ \frac{S\cdot R(S)}{1+G(S)\cdot H(S)} = \underset{S\to 0}{Lt}\ \frac{S\cdot\frac{1}{S^3}}{1+G(S)\cdot H(S)}$$

$$= \underset{S\to 0}{Lt}\ \frac{1}{S^2(1+G(S)\cdot H(S))} = \underset{S\to 0}{Lt}\ \frac{1}{S^2 + S^2(G(S)\cdot H(S))}$$

$$= \frac{1}{0+\underset{S\to 0}{Lt}\ S^2(G(S)\cdot H(S))}$$

$$e_{ss} = \frac{1}{K_a}$$

Type-0 system:

$$K_a = \underset{S\to 0}{Lt}\ S^2\cdot G(S)\cdot H(S)$$

$$K_a = \underset{S\to 0}{Lt}\ S^2\cdot K \frac{(S+Z_1)(S+Z_2)---}{S^N(S+P_1)(S+P_2)---}$$

$$K_a = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Type-1 system:

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 \cdot G(S)\cdot H(S)$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 \cdot K \cdot \frac{(S+z_1)(S+z_2)\cdots}{s^N(S+P_1)(S+P_2)\cdots}$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 \cdot \frac{K\cdot(S+z_1)(S+z_2)\cdots}{\cancel{S}(S+P_1)(S+P_2)\cdots}$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; S \cdot \frac{K(S+z_1)(S+z_2)\cdots}{(S+P_1)(S+P_2)\cdots}$$

$$K_a = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Type-2 system:

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 G(S)\cdot H(S)$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 \cdot K \frac{(S+z_1)(S+z_2)\cdots}{S^N(S+P_1)(S+P_2)\cdots}$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; \cancel{s^2} \cdot \frac{K(S+z_1)(S+z_2)\cdots}{\cancel{s^2}(S+P_1)(S+P_2)\cdots}$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; K \cdot \frac{(S+z_1)(S+z_2)\cdots}{(S+P_1)(S+P_2)\cdots}$$

$$K_a = K \cdot \frac{z_1 z_2 \cdots}{P_1 P_2 \cdots}$$

$$K_a = \text{constant}$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{\text{constant}} = \text{constant}.$$

Type-3 System:

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2 G(S) H(S)$$

$$K_a = \underset{S\to 0}{\mathrm{lt}}\; s^2\, K \cdot \frac{(S+z_1)(S+z_2)\cdots}{S^N(S+P_1)(S+P_2)\cdots}$$

$$K_a = \lim_{s\to 0} s^2 \cdot K \cdot \frac{(s+z_1)(s+z_2)----}{s^3(s+P_1)(s+P_2)---}$$

$$K_a = \lim_{s\to 0} \cdot K \cdot \frac{(s+z_1)(s+z_2)---}{s(s+P_1)(s+P_2)---}$$

$$K_a = \frac{K}{0} = \infty$$

$$e_{ss} = \frac{1}{K_a} = 0$$

09/10/23 static error constants for various types of number of systems:

Error constant	Type of the System			
	0	1	2	3
Kp	constant	∞	∞	∞
Kv	0	constant	∞	∞
Ka	0	0	constant	∞

Steady state error for Various types of input:

Input signal	Type number of system			
	0	1	2	3
unit step	$\frac{1}{1+K_p}$	0	0	0
unit ramp	∞	$\frac{1}{K_v}$	0	0
unit parabolic	∞	∞	$\frac{1}{K_a}$	0

* The unity feedback system has a open loop transfer function of $G(s) = \frac{10}{(s+1)(s+2)}$. Determine the steady state error for unit step input.

Given that

unity feedback i.e., $H(s) = 1$

$$G(s) = \frac{10}{(s+1)(s+2)}$$

Steady state error for unit step input $(e_{ss}) = \frac{1}{1+K_p}$

$$K_p = \lim_{s\to 0} G(s)\cdot H(s)$$

$$K_p = \lim_{s\to 0} \frac{10}{(s+1)(s+2)} \cdot (1)$$

$$K_p = \frac{10}{(1)(2)} = \frac{10}{2} = 5$$

$e_{ss} = \frac{1}{1+5} = \frac{1}{6} = 0.16$ //

* A unity feedback system has a open loop transfer function of $G(S) = \frac{24(S+4)}{S(S+2)(S+0.5)}$. Determine the steady state error for unit ramp signal?

Given that

unity feedback $= H(S) = 1$

$$G(S) = \frac{24(S+4)}{S(S+2)(S+0.5)}$$

Steady state error for unit ramp signal $= e_{ss} = \frac{1}{K_v}$

$$K_v = \lim_{S \to 0} S \cdot G(S) \cdot H(S)$$

$$K_v = \lim_{S \to 0} \cancel{S} \cdot \frac{24(S+4)}{\cancel{S}(S+2)(S+0.5)}$$

$$K_v = \lim_{S \to 0} \frac{24(S+4)}{(S+2)(S+0.5)}$$

$$K_v = \frac{24(4)}{(2)(0.5)} = 96$$

$$e_{ss} = \frac{1}{96} = 0.01$$

* The unity feedback system has a open loop transfer function of $G(S) = \frac{20(S+5)}{S^2(S+0.1)(S+3)}$. Determine the steady state error for unit parabolic signal.

Given that

unity feedback $= H(S) = 1$

$$G(S) = \frac{20(S+5)}{S^2(S+0.1)(S+3)}$$

Steady state error for unit parabolic signal $= e_{ss} = \frac{1}{K_a}$

$$K_a = \lim_{S \to 0} S^2 \, G(S) \cdot H(S)$$

$$K_a = \lim_{S \to 0} \cancel{S^2} \, \frac{20(S+5)}{\cancel{S^2}(S+0.1)(S+3)} \cdot (1)$$

$$K_a = \lim_{S \to 0} \frac{20(S+5)}{(S+0.1)(S+3)}$$

$$K_a = \frac{20(5)}{(0.1)(3)} = 333.3$$

$$e_{ss} = \frac{1}{333.3} = 3 \times 10^{-3} = 0.003$$

Consider a unity feedback with closed loop transfer function $\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$. Determine the steady state error for a unit ramp ~~parabolic~~ signal i/p is given by $e_{ss} = \frac{a-K}{b}$.

$$T(S) = \frac{C(S)}{R(S)} = \frac{KS+b}{S^2+aS+b}$$

$$T(S) = \frac{G(S)}{1+G(S)\cdot H(S)}$$

for a unity feedback (∴ H(S)) = 1

$$T(S) = \frac{G(S)}{1+G(S)}$$

$$G(S) = T(S)[1+G(S)]$$

$$G(S) = T(S) + T(S).G(S)$$

$$G(S) - T(S)\cdot G(S) = T(S)$$

$$G(S)[1-T(S)] = T(S)$$

$$G(S) = \frac{T(S)}{1-T(S)}$$

$$G(S) = \frac{\frac{KS+b}{S^2+aS+b}}{1-\frac{KS+b}{S^2+aS+b}} = \frac{\frac{KS+b}{S^2+a\cancel{S}+b}}{\frac{S^2+aS+b-KS-b}{S^2+a\cancel{S}+b}} = \frac{KS+b}{S^2+(a-K)S}$$

$$G(S) = \frac{KS+b}{S(S+a-K)}$$

ess for unit ramp signal is $(e_{ss}) = \frac{1}{K_v}$

$$K_v = \lim_{S\to 0} S\cdot G(S)\cdot H(S) = \lim_{S\to 0} \cancel{S}\cdot\frac{KS+b}{\cancel{S}(S+a-K)}\cdot(1)$$

$$K_v = \lim_{S\to 0}\frac{KS+b}{S+a-K}$$

$$K_v = \frac{b}{a-K}$$

$$e_{ss} = \frac{1}{K_v} = \frac{a-K}{b}$$

$$E_s = \frac{S+a-K}{S(S^2+aS+b)}$$

$$e_{ss} = \lim_{S\to 0} \cancel{S}\frac{S+a-K}{\cancel{S}(S^2+aS+b)}$$

$$e_{ss} = \lim_{S\to 0}\frac{S+a-K}{S^2+aS+b}$$

$$\boxed{e_{ss} = \frac{a-K}{b}}$$

(or) $e_{ss} = \lim_{S\to 0} S\cdot E\cdot(S)$

$E(S) = \frac{R(S)}{1+G(S)\cdot H(S)}$; ramp input:- R(t) = t

$R(S) = \frac{1}{S^2}$

$$E(S) = \frac{\frac{1}{S^2}}{1+\frac{KS+b}{S^2+S(a-K)}} = \frac{\frac{1}{S^2}}{\frac{S^2+S(a-K)+KS+b}{S^2+S(a-K)}}$$

$$= \frac{\frac{1}{S^2}}{\frac{S^2+aS-\cancel{KS}+\cancel{KS}+b}{S^2+S(a-K)}} = \frac{\frac{1}{S^2}}{\frac{S^2+aS+b}{S^2+S(a-K)}}$$

$$= \frac{S^2+S(a-K)}{S^2(S^2+aS+b)} = \frac{S(S+a-K)}{S^2(S^2+aS+b)}$$

UNIT -

STABILITY

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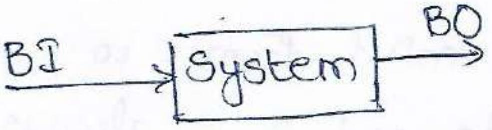
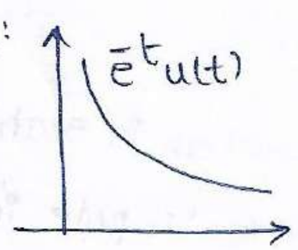
Concept of stability :-

Any system is called as a stable system if the output of the system is bounded for a bounded input.

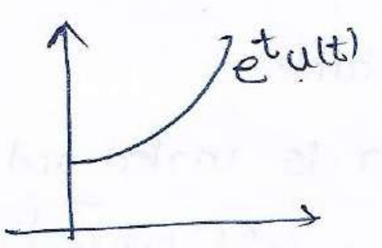
* Any signal is bounded if the max. and min. value are finite.



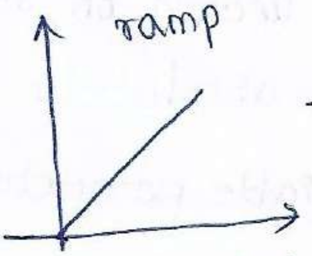
Eg:



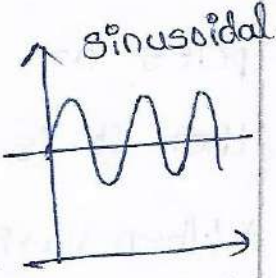
Bounded



unbounded



unbounded



Bounded

* stability of any system depends upon the only location of poles but not on the location of zeros.

* If the poles are located in left side of s-plane, then the system is stable. $\boxed{\lim_{t\to\infty} IR = 0}$

* If the roots are located on imaginary axis which are repeated then the system is unstable. $\boxed{\lim_{t\to\infty} IR = \infty}$

* If the poles are located right half of s-plane, then the system is unstable.

* As pole is approaches origin, stability decreases.

* When roots are located in imaginary axis (Non-repeated

② poles), then the system is marginally ~~un~~stable. $\underset{t\to\infty}{lt}$ IR = fixed

* The poles which are close to origin are called dominant poles.

* The system are classified as

① Absolutely stable system

② Unstable system

③ Conditionally stable system.

* When variable parameter is varied from 0 to ∞, if the poles are located on left side and it is always stable, then it is absolutely stable.

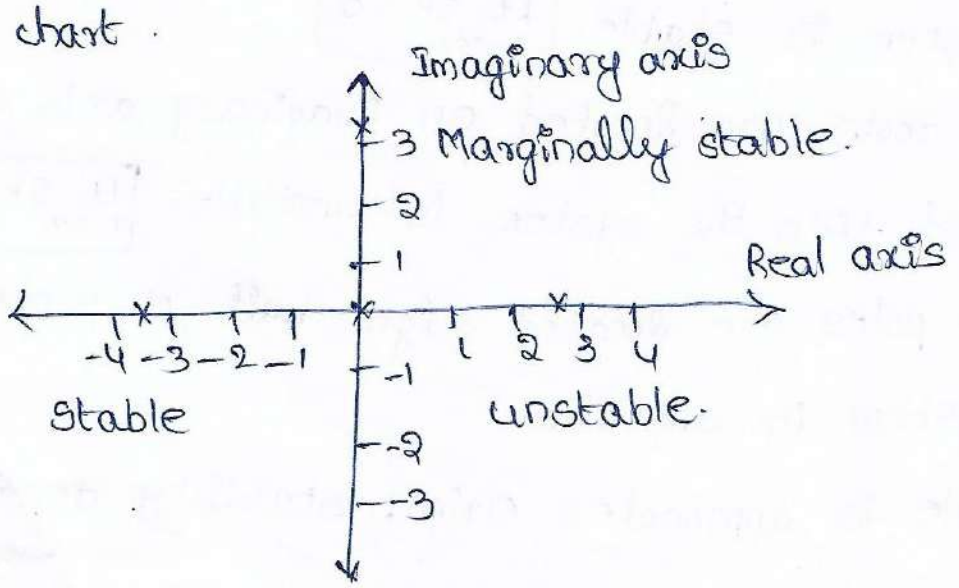
* When variable parameter is varied and a system is stable for values 0 to ∞, at some point onwards there is pole in right side then it is called conditionally stable.

* Techniques used to calculate stability are

(i) RH-criteria ② Root locus

③ Bode plot ④ Nyquist plot.

⑤ Nicholas chart.



③

UNIT-3

STABILITY

Definitions of stability:

Stability:- Stable working condition of a control system. Every working system is designed to be stable.

1)* A system is stable if its output is bounded (finite) for any bounded input. $BI \rightarrow \boxed{System} \rightarrow BO$

2) A system is asymptotically stable, if in the abscence of input the output tends to zero (or equilibrium state] irrespective of initial conditions.

3) A system is stable if for a bounded disturbing input signal the output vanishes ultimately as 't' approaches infinity.

4) A system is unstable if for a bounded disturbing input signal the output is of infinity amplitude and oscillatory.

5) For a bounded input signal, the output has constant amplitude oscillations than the system may be stable or unstable under some limited Constraints ~~conditions~~. such a system is called limitedly or marginally stable system.

6) If the system output is suitable for all variations of its parameters than the system is called absolutely stable system.

7) If the system output is stable for a limited range of variations of its parameters than that system is called conditionally stable system.

⇒* In a stable system, the response or output is predictable, finite and stable for a given input.

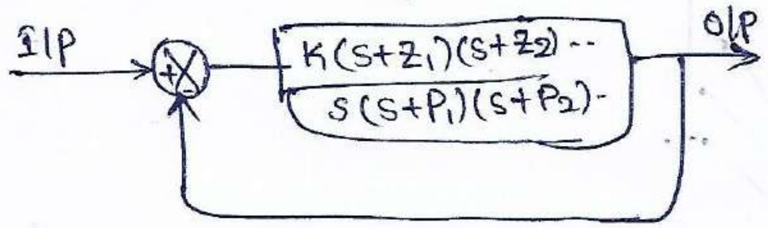
Absolute stability:-

If a system output is stable for all variations of a simple parameters then the system is called absolutely stable system.

* It will provide yes/No information about stability (K-gain information)

Conditional stability:-

* If a system output is stable for a limited range of variations of its parameters then the system is called conditional or limited stable system.



* condition for stable system the value of gain is $K=0$ to ∞.
* Range of K for stability $\boxed{0<K<\infty}$ → always stable ($\because K=$ gain of system)

* $\boxed{10<K<40}$ → Conditionally stable system.

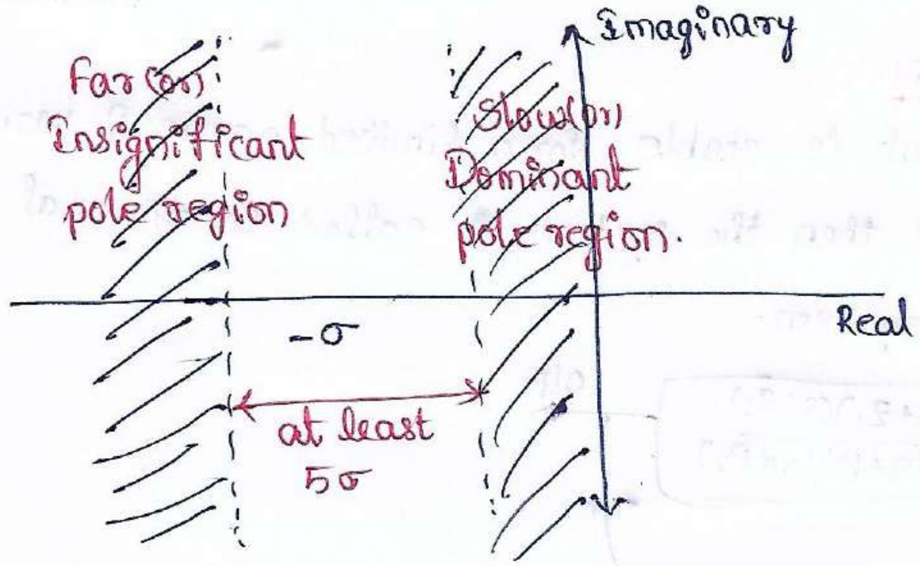
Relative stability:- The degree of stability (or) complete stability information.

*

Near (or) Dominant pole region :-

The poles which are close to the origin are called dominant poles. (or)

The pole lie in the vicinity of the system is called dominant pole.

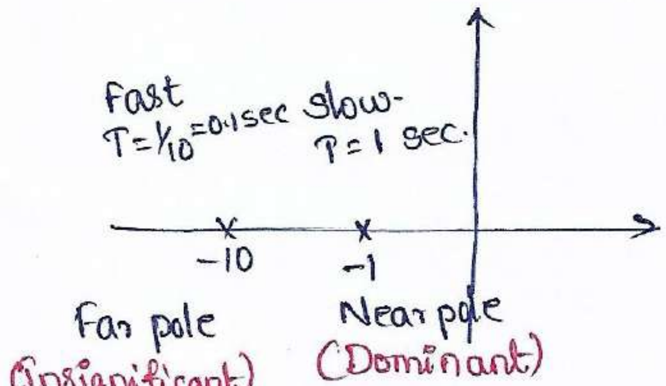


* If the system consists of more than one pole the transient (or) speed of response is dominated by near pole and far pole is ignored.

∴ Near poles are called as dominant poles & far poles are insignificant poles.

Note
* Near pole timeconstant is very large.
* far poles timeconstant is very less.

ex:



Impulse response of a system:-

Let $M(s)$ = Closed loop transfer function of a system

$C(s)$ = Output / Response in s-domain

$R(s)$ = Input in s-domain

Now $M(s) = \frac{C(s)}{R(s)}$.

∴ Response or Output in s-domain $C(s) = M(s)R(s)$.

Now, Response in time domain, $c(t) = L^{-1}[C(s)]$

Input in time domain, $r(t) = L^{-1}[R(s)]$

For an impulse input $r(t) = \delta(t) \Rightarrow \therefore R(s) = L[\delta(t)] = 1$.

$$\text{Impulse response} = L^{-1}[C(s)] = L^{-1}(M(s)R(s)) = L^{-1}(M(s))$$
$$= m(t).$$

⇒ Hence, impulse response of a system is the inverse Laplace transform of system transfer function.

* The importance of impulse response is that, the output of a system for any arbitrary input can be obtained by convolution of input and impulse response.

i.e, Response, $c(t) = m(t) * r(t)$

Where * is the symbol for convolution.

Mathematically convolution operation defined as

$$\boxed{c(t) = \int_{-\infty}^{+\infty} m(\tau)r(t-\tau)\,d\tau.}$$

Where τ is the dummy variable used for integration.

Bounded Input Bounded Output (BIBO) stability:-

A linear relaxed system is said to have BIBO stability ~~for~~ if every bounded (finite) input results in a bounded (finite) output.

* A condition for BIBO stability can be obtained from convolution operation defined by eqn.

$$\text{Response, } c(t) = \int_0^\infty m(\tau)\, r(t-\tau)\, d\tau. \longrightarrow ①$$

* If the input r(t) is bounded there exist a constant A_1, such that $|r(t) \leq A_1 < \infty|$. The condition for bounded ~~o/p~~ output for this bounded input condition can be derived as follows. On taking the absolute value on both sides of eqn.

$$|c(t)| = \left| \int_0^\infty m(\tau)\, r(t-\tau) \right| \longrightarrow ②$$

Since the absolute value of an integral is not greater than the integral of the absolute value of the integrand eq ②

$$|c(t)| \leq \int_0^\infty |m(\tau)\, r(t-\tau)|\, d\tau.$$

$$\Rightarrow |c(t)| \leq \int_0^\infty |m(\tau)|\, |r(t-\tau)|\, |d\tau| \Rightarrow |c(t)| \leq \int_0^\infty |m(\tau)|\, A_1\, d\tau.$$

$$\therefore |c(t)| \leq A_1 \int_0^\infty |m(\tau)\, d\tau.$$

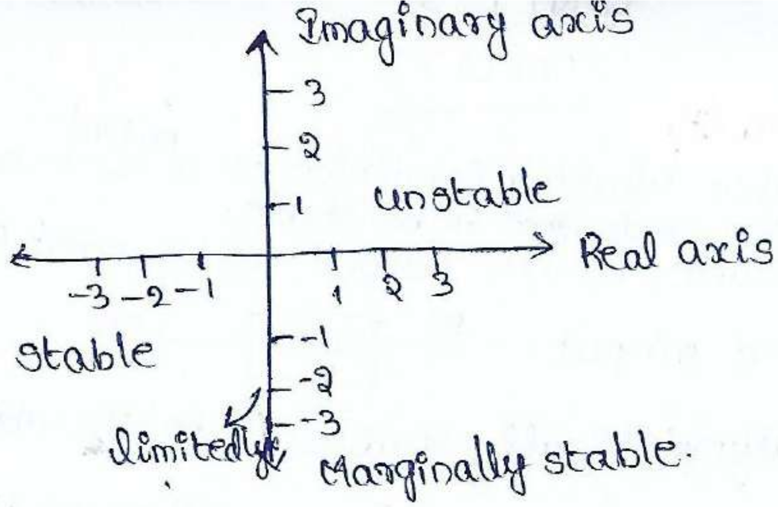
For bounded input, a constant exist such that, $|r(t-\tau)| \leq A_1$

If the o/p c(t) is bounded then there exists a constant A_2, Such that $|c(t)| \leq A_2 < \infty$.

$$\therefore A_1 \int_0^\infty |m(\tau)|\, d\tau \leq A_2 < \infty.$$

condition is satisfied if, $\int_0^\infty |m(\tau) d\tau < \infty$ ($\because \tau$ is dummy variable replaced by t)

Bounded output, $\boxed{\int_0^\infty |m(t)|\, dt < \infty.}$ ⇒ IR m(t) is BIBO stable if and only if the IR is absolutely integrable.



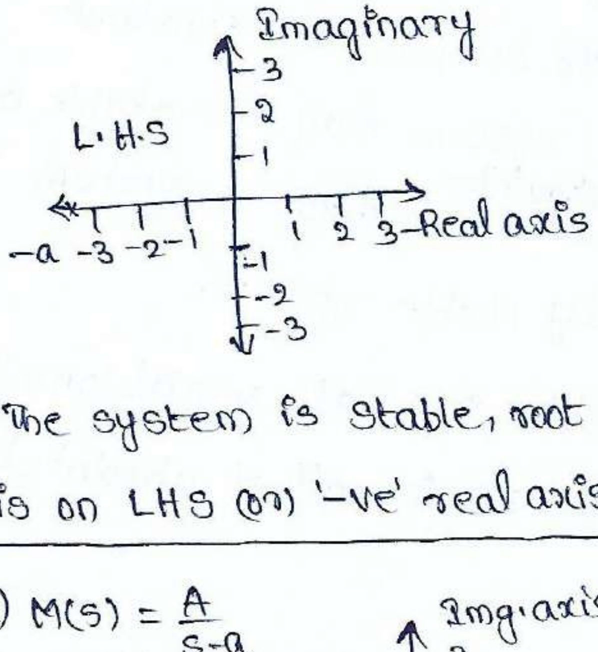
Location of poles on S-plane for stability :-

Transfer function

(i) $M(s) = \frac{A}{s+a}$

$s+a=0$

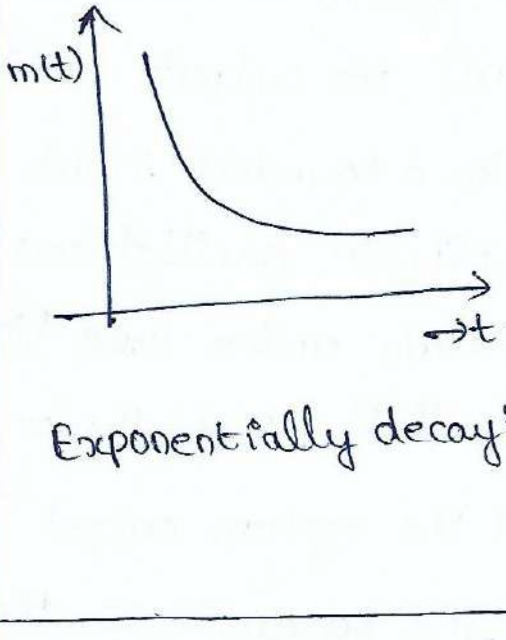
$s=-a$



The system is stable, root is on LHS (or) '-ve' real axis.

Impulse Response

$M(t) = L^{-1}\left(\frac{A}{s+a}\right) = Ae^{-at}$

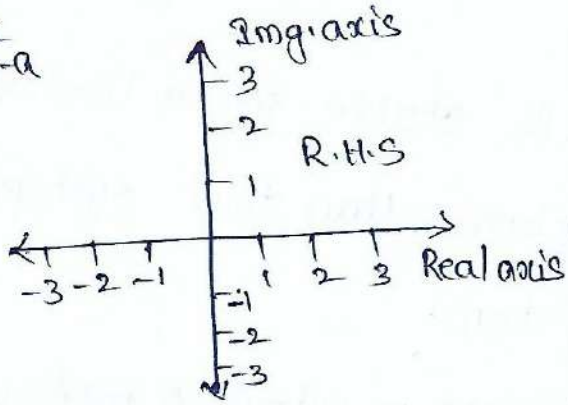


Exponentially decaying.

2) $M(s) = \frac{A}{s-a}$

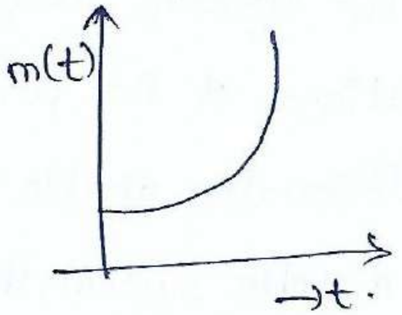
$s-a=0$

$s=a$



The system is unstable, roots is on RHS (or) +ve Real axis.

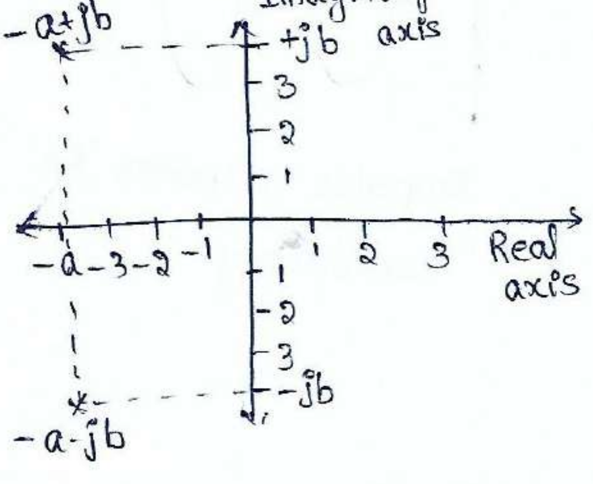
$M(t) = L^{-1}\left[\frac{A}{s-a}\right] = Ae^{at}.$



Exponentially increasing.

⑤ ③ $M(S) = \frac{A}{S+a+jb} + \frac{A^*}{S+a-jb}$

$S = -a \pm jb$



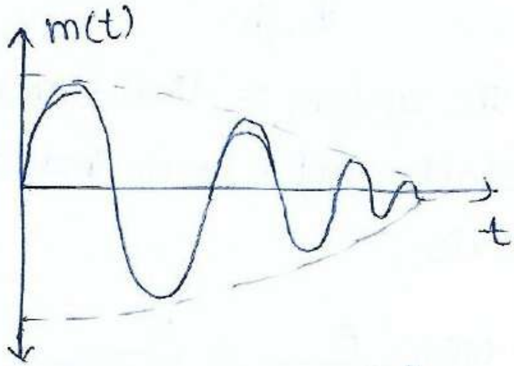
The system is stable, the roots is on LHS (or) '-ve' real axis.

$M(t) = L^{-1}\left(\frac{A}{S+a+jb} + \frac{A}{S+a-jb}\right)$

$M(t) = A e^{-(a+jb)t} + A^* e^{-(a-jb)t}.$

$= A e^{-at} \cos bt$

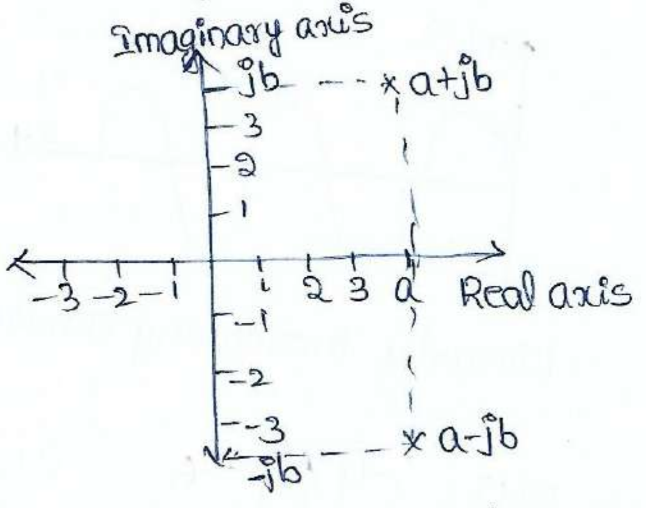
$= 2A e^{-at} \sin(bt+90°)$



Damped Sinusoidal.

④ $M(S) = \frac{A}{S-a+jb} + \frac{A^*}{S-a-jb}$

$S = a \pm jb$



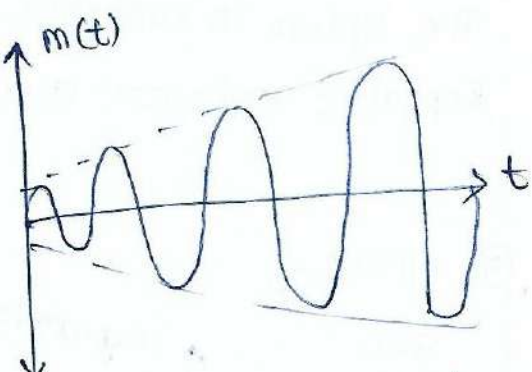
The system is unstable, roots is on LHS (or) '+ve' real axis

$M(t) = L^{-1}\left(\frac{A}{S-a+jb} + \frac{A^*}{S-a-jb}\right]$

$= A e^{(a+jb)t} + A^* e^{(a-jb)t}.$

$= 2A e^{at} \cos bt$

$= 2A e^{at} \sin(bt+90°)$



Damped sinusoidal.

⑤ $M(S) = \frac{A}{S+jb} + \frac{A^*}{S-jb}$

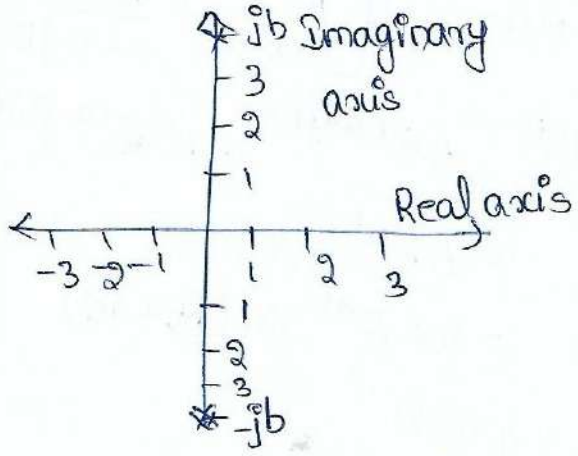
$S = \pm jb$

$M(t) = L^{-1}\left(\frac{A}{S+jb} + \frac{A^*}{S-jb}\right]$

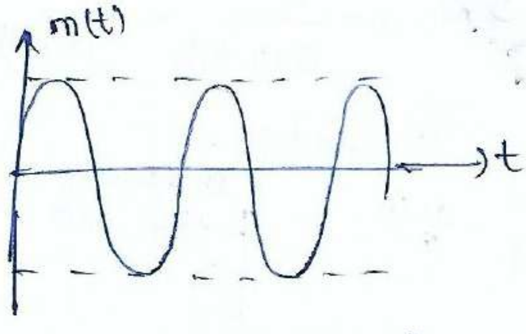
$= A e^{-jbt} + A^* e^{jbt}$

$= 2A\cos bt = 2A\sin(bt+90°)$

⑥



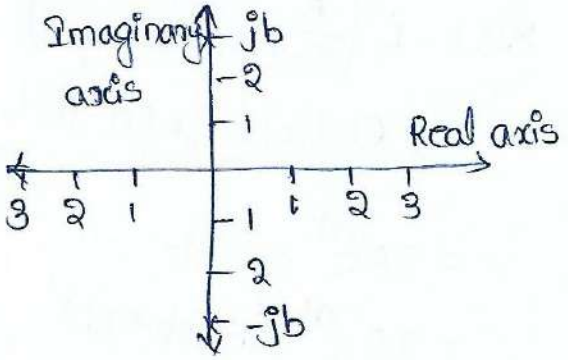
The system is limited (or) marginally stable, roots is on imaginary axis.



Impulse response is oscillatory.

⑥ $M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$

$s = \pm jb, \pm jb.$



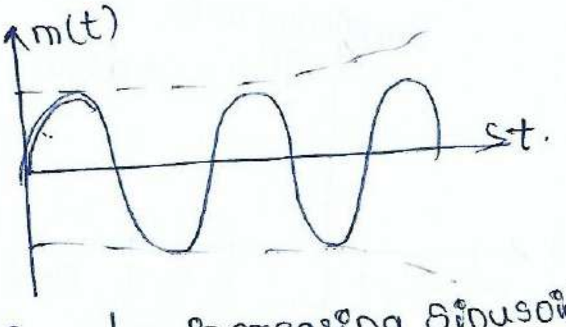
The system is unstable,
Repeated roots are available.

$M(t) = L^{-1}\left(\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right)$

$= Ate^{-jbt} + A^* t e^{+jbt}$

$= 2At\cos bt$

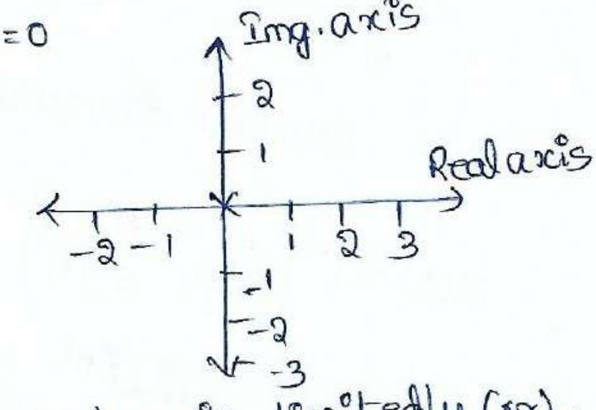
$= 2At\sin(bt+90^\circ)$



Linearly increasing sinusoidal.

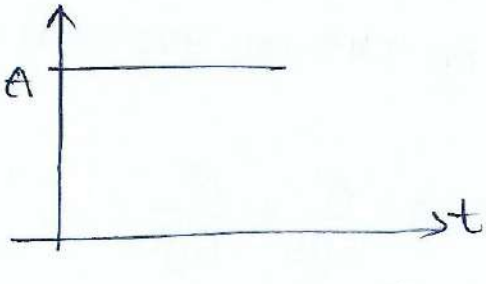
⑦ $M(s) = \frac{A}{s}$

$s = 0$



The system is limitedly (or).
Marginally stable.

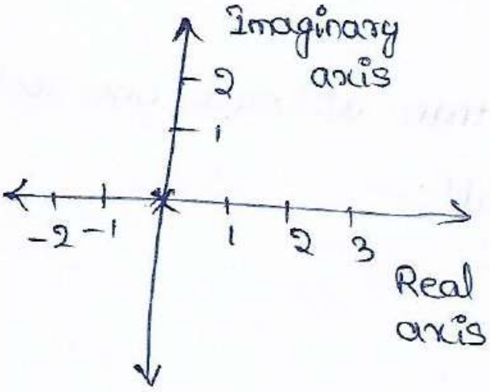
$m(t) = L^{-1}\left[\frac{A}{s}\right] = A$



Impulse response is constant.

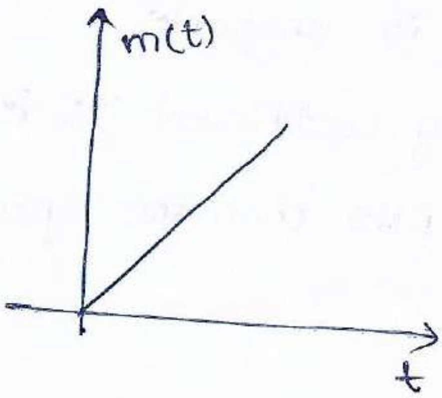
⑦ ⑧ $M(s) = \frac{A}{s^2}$

$s = 0, 0$



System is unstable.

$m(t) = L^{-1}\left[\frac{A}{s^2}\right] = At$



Impulse response is linearly increasing.

Conclusion :-

* If all the roots of characteristic equation has negative real parts than the system is stable system.

* If any roots of the characteristic equation has the positive real parts (or) If there is a repeated roots on the imaginary axis then the system is unstable system.

* If first condition is satisfied except for the presence of one or more repeated roots on the imaginary axis than the system is limitedly or marginally stable system.

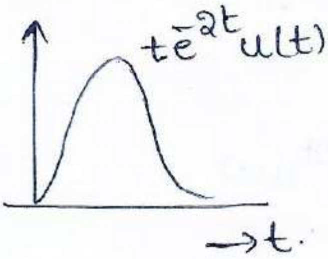
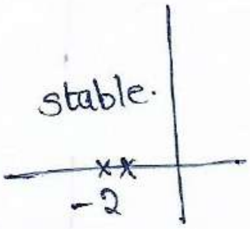
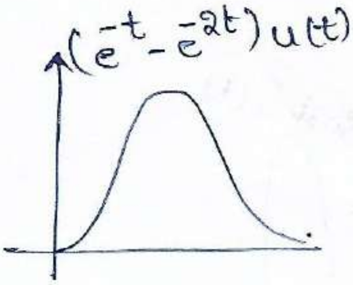
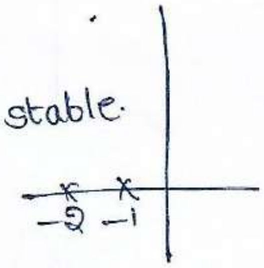
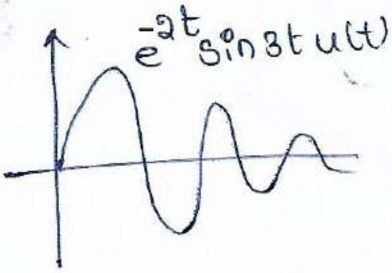
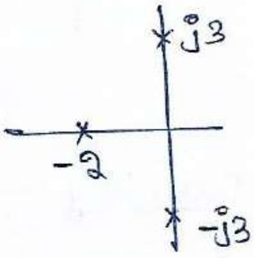
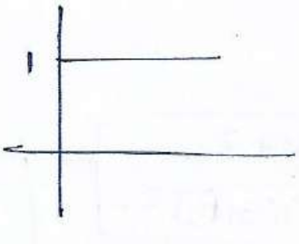
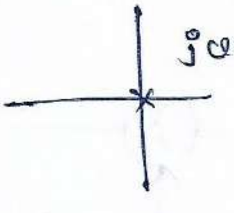
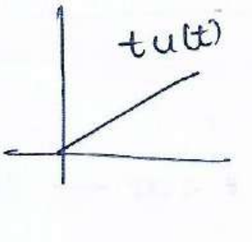
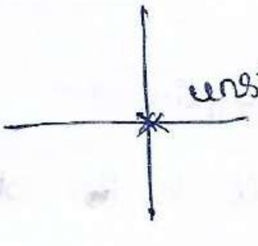
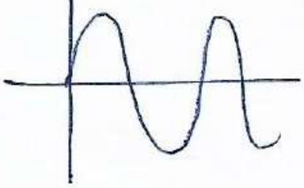
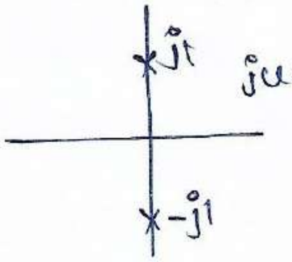
⇒ According to coefficient of characteristic equation

① If all the coefficients are positive & if no. of coefficients is zero than all the roots are on LHS than the system is stable.

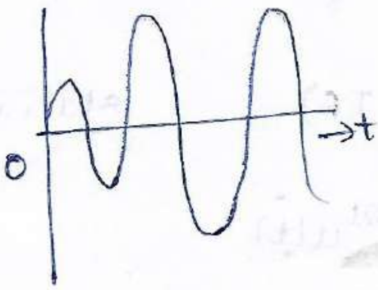
⑧ ② If any coefficient 'a_i' is equal to zero than some of the roots may be on the imaginary axis (or) RHS than the system is unstable.

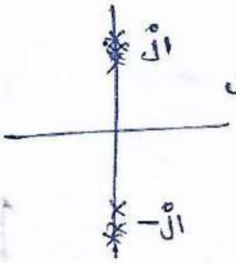
③ If any coefficient 'a_i' is negative than atleast one root is the RHS than the system is unstable.

⑨ H.W

TF	$IR = L^{-1}(TF)$	Stability.
(i) $\frac{1}{(s+2)^2}$	$te^{-2t}u(t)$  $\rightarrow t.$	stable.  -2
2) $\frac{1}{(s+1)(s+2)}$	$(e^{-t}-e^{-2t})u(t)$ 	stable.  -2 -1
3) $\frac{3}{(s+2)^2+3^2}$	$e^{-2t}\sin 3t\, u(t)$ 	 $j3$ -2 $-j3$
④ $\frac{1}{s}$	 1	 just stable.
⑤ $\frac{1}{s^2}$	$tu(t)$ 	 unstable.
⑥ $\frac{1}{s^2+1}$	$\sin t\, u(t)$ 	 $j1$ $-j1$ just stable

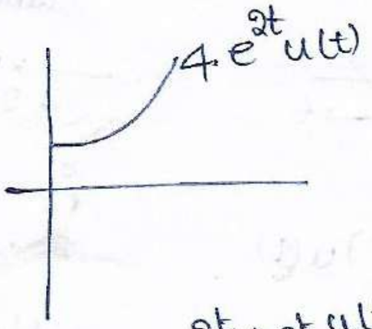
⑦ $\frac{1}{(s^2+1)^2}$

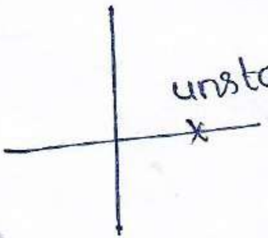




unstable.

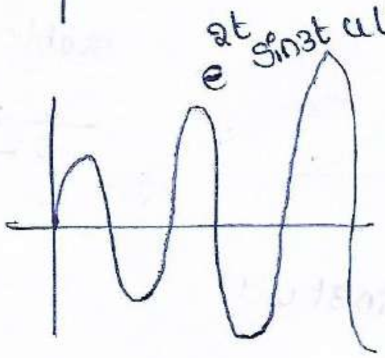
⑧ $\frac{4}{s-2}$

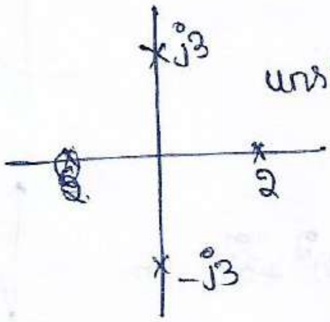




unstable.

⑨ $\frac{3}{(s-2)^2+3^2}$



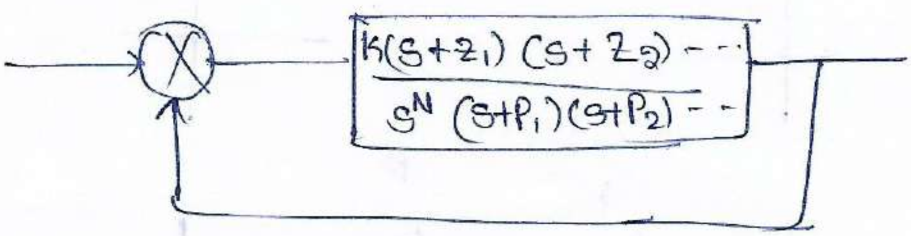


unstable

Conditional stability :-

Condition for stability system the value of gain

$K = 0$ to ∞.



where K = gain

* Range of K for stability $0 < K < \infty$ → always stable.

* $10 < K < 40$ → conditionally stable system.

① Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}.$$

Sol On comparing the quadratic factor in the denominator of G(s) with standard form of quadratic factor we can estimate ξ and ω_n.

$$\therefore s^2+16s+100 = s^2+2\xi\omega_n s+\omega_n^2$$

On comparing we get,

$$\omega_n^2 = 100 \Rightarrow \omega_n = 10$$

$$2\xi\omega_n = 16 \Rightarrow \xi = \frac{16}{2\omega_n} = \frac{16}{2\times10} = 0.8$$

$\therefore$ convert the given s-domain transfer function into bode form or time constant form.

$$\therefore G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{75(1+0.2s)}{100\,s\left[\frac{s^2}{100}+\frac{16s}{100}+1\right]} = \frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

Sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in G(s).

$$\therefore G(j\omega) = \frac{0.75(1+0.2j\omega)}{(j\omega)(1+0.01(j\omega)^2+0.16j\omega)} = \frac{0.75(1+0.2j\omega)}{(j\omega)(1-0.01\omega^2+j0.16\omega)}$$

Magnitude plot:

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec.

$\omega_{c2} = \omega_n = 10$ rad/sec.

Note: for the quadratic factor the corner frequency is ω_n.

Table-1

Term	Corner frequency rad/sec.	slope db/dec	change in slope db/dec.
$\frac{0.75}{j\omega}$	-	-20	
$1+j0.2\omega$	$\omega_{c1} = \frac{1}{0.2} = 5$	20	$-20+20 = 0$.
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2} = \omega_n = 10$	-40	$0-40 = -40$.

choose a low frequency ω_l, such that $\omega_l < \omega_{c1}$

high frequency ω_h, such that $\omega_h > \omega_{c2}$.

$\omega_l = 0.5$ rad/sec.

$\omega_h = 20$ rad/sec.

$A = |G(j\omega)|$ in db.

let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}, \omega_h$.

$\underset{=0.5}{\omega = \omega_l}$, $A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\left|\frac{0.75}{0.5}\right| = 3.5$ db.

$\underset{=5}{\omega = \omega_{c1}}$, $A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\left|\frac{0.75}{5}\right| = -16.5$ db.

$\underset{=10}{\omega = \omega_{c2}}$, $A = \cancel{20}\left(\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\left|\frac{\omega_{c2}}{\omega_{c1}}\right|\right) + A \text{ at } \omega = \omega_{c1}$

$= 0 \times \log\frac{10}{5} + (-16.5) = -16.5$ db.

$\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\left|\frac{\omega_h}{\omega_{c2}}\right|\right] + A \text{ at } \omega = \omega_{c2}$

$= -40 \times \log\left|\frac{20}{10}\right| + (-16.5) = -28.5$ db.

(1 unit = 5db, 0.1 to 100)

Phase plot :-

The phase angle of $G(j\omega)$ as a function of ω is given by

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) \text{ for } \omega \leq \omega_n.$$

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^\circ - \left[\tan^{-1} \frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right] \text{ for } \omega > \omega_n.$$

Table-2

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} \frac{0.16\omega}{1-0.01\omega^2}$	$\phi = \angle G(j\omega)$ (deg)
0.5	5.7	4.6	$-88.9 \simeq -88$
1	11.3	9.2	$-87.9 \simeq -88$
5	45	46.8	$-91.8 \simeq -92$
10	63.4	90	$-116.6 \simeq -92$
20	75.9	$-46.8 + 180 = 133.2$	$-147.3 \simeq -148$
50	84.3	$-18.4 + 180 = 161.6$	$-167.3 \simeq -168$
100	87.1	$-92 + 180 = 170.8$	$-173.7 \simeq -174$

1 unit = 20°. → join the points by a smooth curve.

Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross over frequency, ω_{gc}.

$$\phi_{gc} = 88^\circ.$$

phase margin $= 180^\circ + \phi_{gc} = 180^\circ - 88 = 92^\circ$.

The phase plot crosses -180° at infinity.

The $|G(j\omega)|$ at infinity is $-\infty$ db.

gain margin is $+\infty$.

ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES

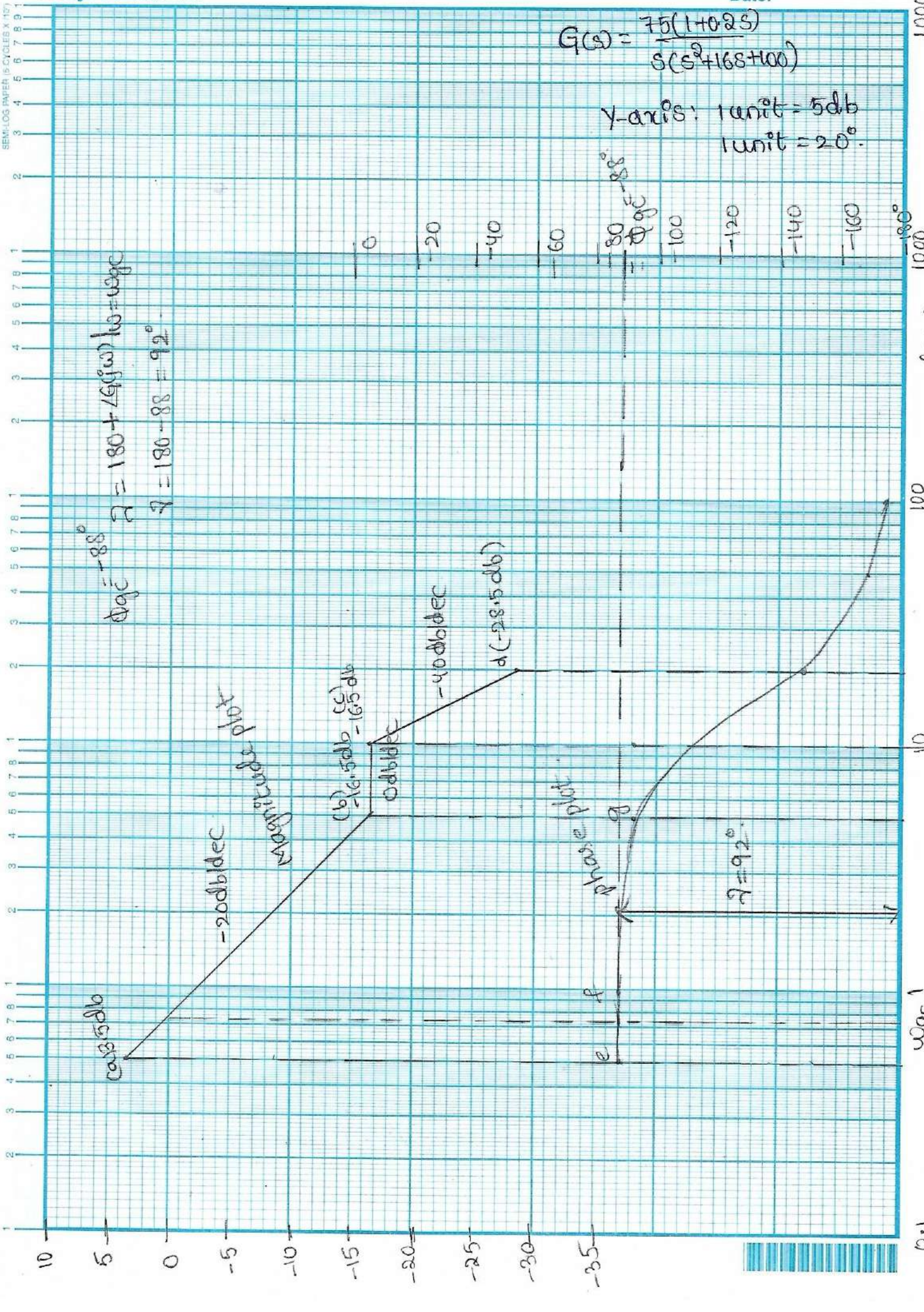
TIRUPATI – 517 520 (A.P.)

Subject Name: **Date:**

$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$

Y-axis: 1unit = 5db

1unit = 20°.



② $G(s) = \frac{Ke^{-0.2s}}{s(s+2)(s+8)}$. Find K so that the system is stable with ⓐ gain margin equal to 2db ⓑ phase margin equal to 45°.

sol Let us take K=1 and convert the given transfer function to time constant form or bode form. put $s=j\omega$.

∴ ~~$G(j\omega) = 0$~~ $G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} = \frac{e^{-0.2s}}{s2\left(1+\frac{s}{2}\right)8\left(1+\frac{s}{8}\right)} = \frac{0.0625e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$

$s=j\omega$

$$\therefore G(j\omega) = \frac{0.0625\, e^{-j0.2\omega}}{(j\omega)(1+j0.5\omega)(1+j0.125\omega)}$$

Note: $|0.0625\, e^{-j0.2\omega}| = 0.0625$

$\angle(0.0625\, e^{-j0.2\omega}) = -0.2\omega$ radians.

Magnitude plot:

Corner frequencies, $\omega_{c1} = \frac{1}{1/2} = 2$ rad/sec.

$\omega_{c2} = \frac{1}{1/8} = 8$ rad/sec.

Table-1

Term	Corner frequency rad/sec	Slope db/dec	change in slope db/dec.
$\frac{0.0625}{j\omega}$	-	-20	-
$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = \frac{1}{0.5} = 2$	-20	-20-20=-40
$\frac{1}{1+j0.125\omega}$	$\omega_{c2} = \frac{1}{0.125} = 8$	-20	-40-20=-60

$\omega_l = 0.5$ rad/sec.

$\omega_h = 50$ rad/sec.

$A = |G(j\omega)|$ in db.

let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}, \omega_h$.

$\omega=\omega_2$, $A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{0.5} = -18\text{db}$

$\omega=\omega_{c1}$, $A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\left|\frac{0.0625}{2}\right| = -30\text{db}$

$\omega=\omega_{c2}$, $A = \left(\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right| + A \text{ at } \omega=\omega_{c1}$

$= -40\times\log\left|\frac{8}{2}\right| + (-30) = -54\text{db}$

$\omega=\omega_h$, $A = \text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}} + A \text{ at } \omega=\omega_{c2}$

$= -60\times\log\left|\frac{50}{8}\right| + (-54) = -102\text{db}.$

1 unit = 10db, join the points by straight line.

phase plot:

The phase angle of G(jω) as a function of ω is given by

$\phi = -0.2\omega\times\frac{180^\circ}{\pi} - 90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.125\omega.$

Table

ω rad/sec	$-0.2\omega(180^\circ/\pi)$ deg.	$\tan^{-1}0.5\omega$ deg	$\tan^{-1}0.125\omega$ deg	$\phi=\angle G(j\omega)$
0.01	-0.1145	0.2864	0.0716	-90.4≃-90
0.1	-1.145	2.862	0.716	-94.7≃-94
0.5	-5.7	14	3.6	-113.3≃-114
1	-11.4	26	7.12	-134.4≃-134
2	-22.9	45	14	-171.9≃-172
3	-34.37	56.30	20.56	-201.2≃-202
4	-45.84	63.43.	26.57	-225.8≃-226.

Note: 1 unit = 20°, join the points by smooth curve.

phase margin $\gamma = 180 + \phi_{gc}$.

ϕ_{gc} is the phase of $G(j\omega)$ at $\omega = \omega_{gc}$.

$\gamma = 45°$, $\phi_{gc} = \gamma - 180° = 45° - 180° = -135°$.

K = 1, the db gain at $\phi = -135°$ ~~at~~ is −24 db.

The gain should be made zero to have PM of 45°.

Hence to every point of magnitude plot a db gain of 24db should be added.

* The corrected magnitude plot is obtained by shifting the plot with K=1 by 24db upwards.

$$20 \log K = 24 \Rightarrow K = 10^{24/20} = 15.84$$

K = 1, the gain margin = −(−32) = 32db. But the required gain margin is 2 db.

* Hence to every point of magnitude plot a db gain of 30db should be added. This addition of 30db shifts the plot upwards.

* The magnitude correction is independent of frequency.

$$20 \log K = 30$$

$$K = 10^{30/20} = 31.62.$$

The magnitude plot with K = 15.84 & 31.62.

③ For the following transfer function draw bode plot and obtain the gain cross over frequency.

$$G(s) = \frac{20}{s(1+3s)(1+4s)}.$$

④ For the function, $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$, draw the bode plot.

Polar plot :-

The polar plot of a Sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity.

* Thus the polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

* The polar plot is also called Nyquist plot.

* It is plotted on the polar graph sheet.

* The polar graph sheet has concentric circles & radial lines.

* The circles represent the magnitude & the radial lines represent the phase angles.

* Each point on the polar graph ~~is~~ has magnitude & phase angle.

* The magnitude of a point is given by the value of the circle passing through that point & phase angle is given by the radial line passing through that point.

* In polar graph sheet a positive phase angle is measured in anticlockwise from the reference axis (0°) and

* negative angle is measured clockwise from the reference axis(0°).

Procedure :-

① plot the polar plot.

② Compute the magnitude & phase of $G(j\omega)$ are computed for various values of ω & tabulate them.

③ The choice of frequencies are usually the corner frequencies & frequencies around corner frequencies.

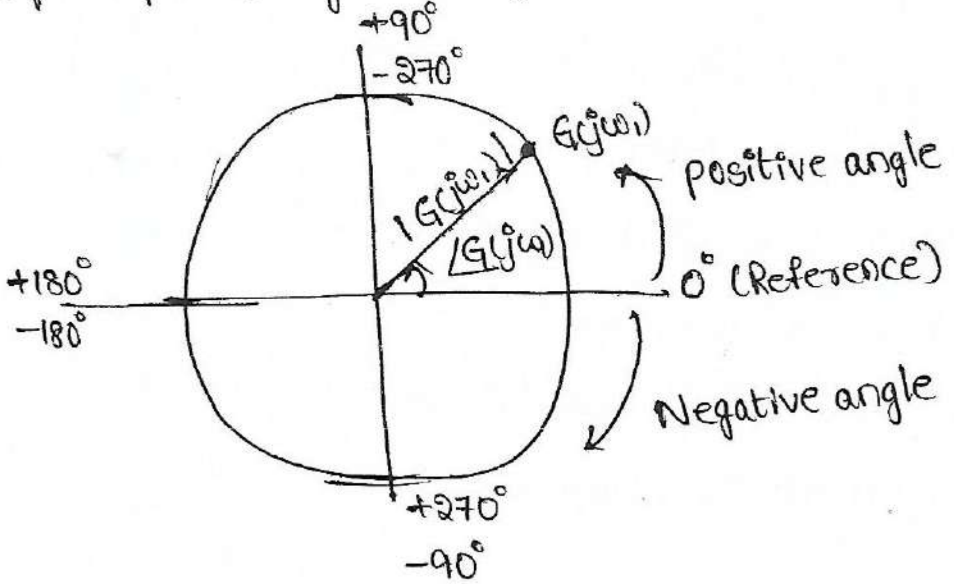
④ Choose proper scale for the magnitude circles.

⑤ Fix all the points on polar graph sheet & join the points by smooth curve.

Note:- To plot the polar plot on the ordinary graph sheet compute the magnitude & phase for different values of ω.

6) Convert the polar to rectangular coordinates using P → R conversion.

7) Sketch the polar plot using rectangular coordinates.



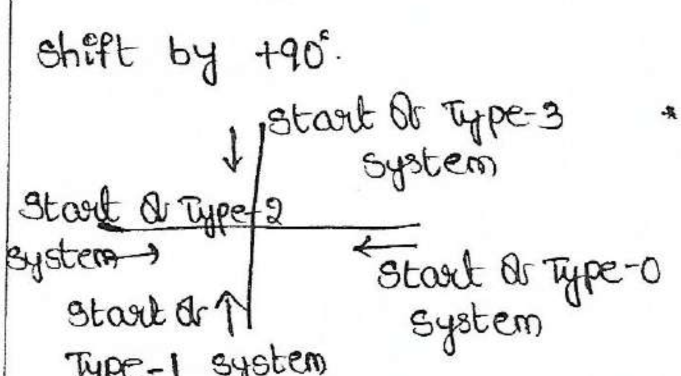
* For minimum phase transfer function with only poles, type number of the system determines the quadrant at which the polar plot starts and the order of the system determines the quadrant at which the polar plot ends.

* The minimum phase systems are systems with all poles & zeros on left half of s-plane.

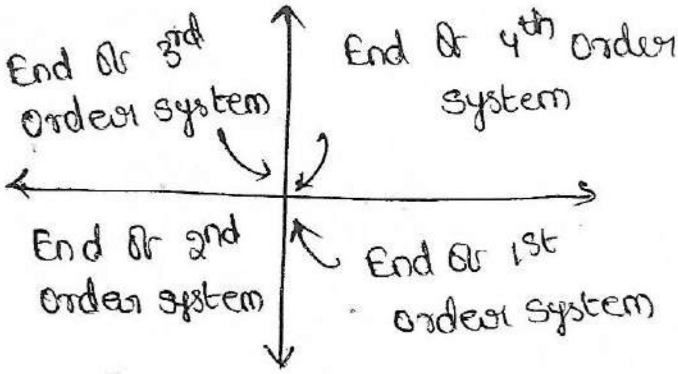
The change in shape of polar plot can be predicted due to addition of pole or zero.

1. When a pole is added to a system, the polar plot end point will shift by -90°.

2. When a zero is added to a system, the polar plot end point will shift by +90°.



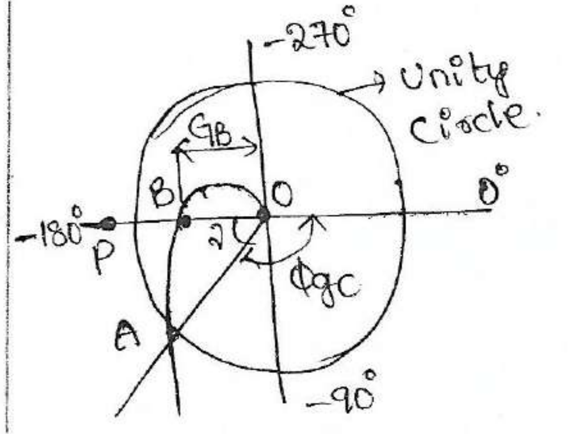
* Start of polar plot of all pole minimum phase system.

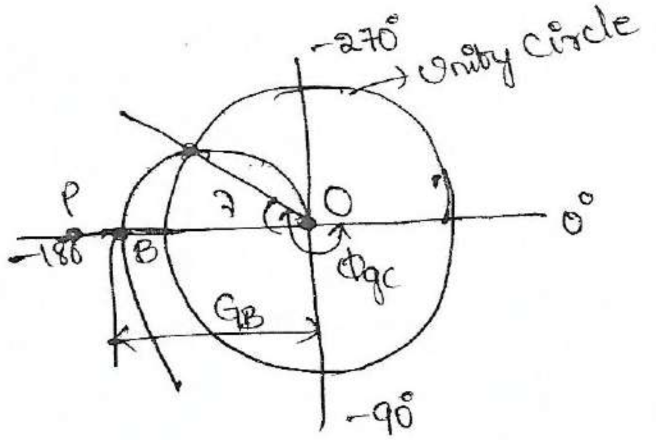


* End of polar plot of all pole minimum phase system.

Determination of Gain Margin & phase Margin from polar plot :-

* The gain margin is defined as the inverse of the magnitude of $G(j\omega)$ at phase crossover frequency. The phase crossover frequency is the frequency at which the phase of $G(j\omega)$ is 180°.
* Let the polar plot cut the 180° axis at a point B and the magnitude circle passing through the point B be G_B. Now the gain margin, $K_g = 1/G_B$.
* If the point B lies ~~on~~ within unity circle, then the gain margin is positive otherwise negative.
* The phase margin is defined as, phase margin $\gamma = 180^\circ + \phi_{gc}$ where ϕ_{gc} is the phase angle of $G(j\omega)$ at gain crossover frequency. The gain crossover frequency is the frequency at which the magnitude of $G(j\omega)$ is unity.
* Let the polar plot cut the unity circle at a point A. Now the phase margin is given by $\angle AOP$. i.e if $\angle AOP$ is below -180° axis then phase margin is positive & if it is above -180° axis then the phase margin is negative.





Gain margin $K_g = \frac{1}{G_B}$

phase margin $\gamma = 180° + \phi_{gc}$

* polar plot showing positive gain margin & phase margin

Gain margin, $K_g = 1/G_B$

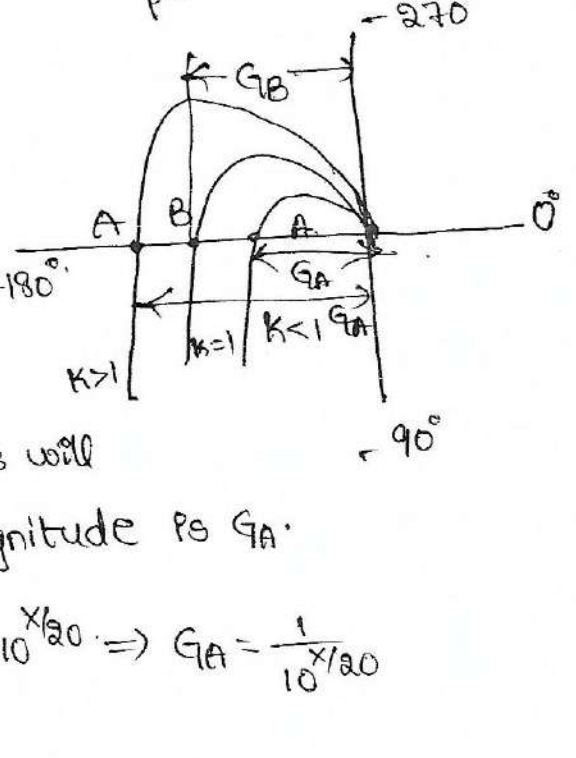
phase margin, $\gamma = 180° + \phi_{gc}$

* polar plot showing negative gain margin and phase margin.

Gain Adjustment Using polar plot :-

To determine K for specified GM :-

polar plot for different values of K.



* Draw the $G(j\omega)$ locus with K=1.

Let it cut the -180° axis at point B corresponding to a gain of G_B.

Let the specified gain margin be 'x' db.

* For this gain margin, the $G(j\omega)$ locus will cut the -180° at point A whose magnitude is G_A.

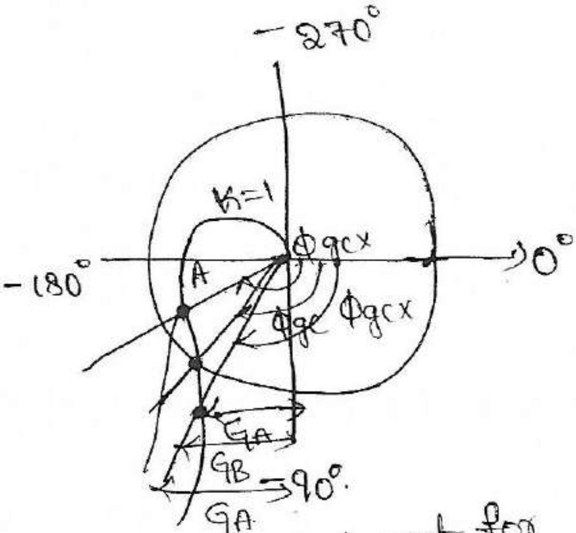
$$20 \log \frac{1}{G_A} = x \Rightarrow \log \frac{1}{G_A} = \frac{x}{20} \Rightarrow \frac{1}{G_A} = 10^{x/20} \Rightarrow G_A = \frac{1}{10^{x/20}}$$

Value of K is $K = \frac{G_A}{G_B}$.

If K>1, then the system gain should be increased.

K<1, then the system gain should be reduced.

To determine K for specified PM:-



Gain adjustment for required phase margin

* Draw the $G(j\omega)$ locus with $K=1$ Let it unity circle at point B.

* Let the specified phase margin be $x°$.

* For the phase margin of $x°$, Let ϕ_{gcx} be the phase angle of $G(j\omega)$ at gain crossover frequency.

$$x° = 180° + \phi_{gcx} \Rightarrow \phi_{gcx} = x° - 180°$$

* In the polar graph, the radial line corresponding to ϕ_{gcx} will cut the locus of $G(j\omega)$ ~~at~~ with $K=1$ at point A & magnitude corresponding to that point be G_A.

$$\therefore K = \frac{G_B}{G_A} = \frac{1}{G_A} \quad (\because G_B = 1)$$

① The openloop transfer function of a unity feedback system is given by $G(S) = \frac{1}{S(1+S)(1+2S)}$. Sketch the polar plot & determine the gain margin & phase margin.

Sol Given that $G(S) = \frac{1}{S(1+S)(1+2S)}$

Put $S = j\omega \Rightarrow G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$

Corner frequencies $\omega_{c1} = \frac{1}{1} = 1$ rad/sec.

$\omega_{c2} = \frac{1}{2} = 0.5$ rad/sec

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)} = \frac{1}{\omega\angle 90° \sqrt{1+\omega^2}\angle\tan^{-1}\omega \sqrt{1+4\omega^2}\angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}} \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

Table-1 : Magnitude & phase of $G(j\omega)$ at various frequencies.

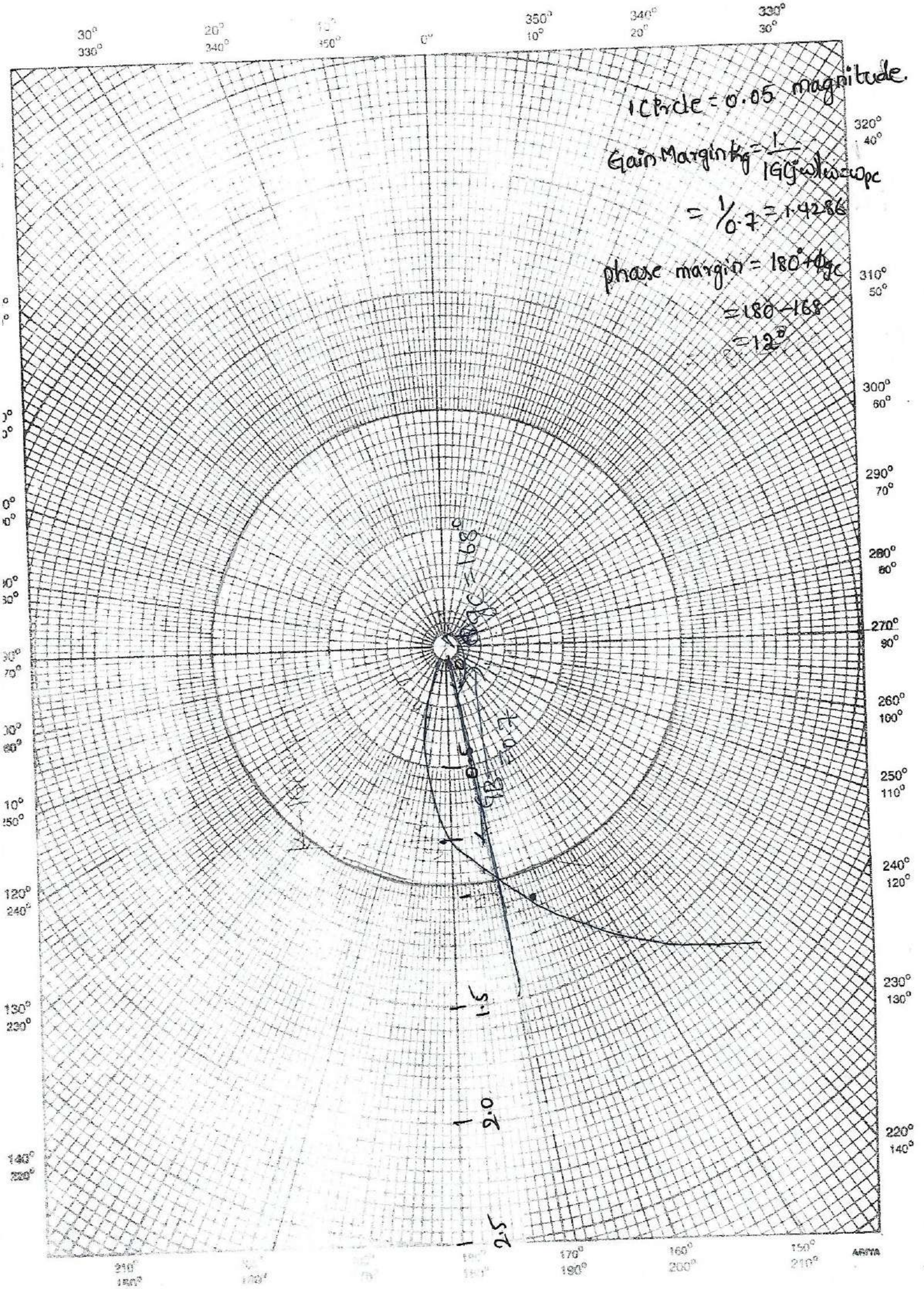
ω rad/sec.	0.1	0.2	0.4	0.5	0.6	0.7	1
$1/\lvert G(j\omega)\rvert$	0.102	0.21	0.55	0.79	1.09	1.47	3.16
$\angle G(j\omega)$.	−107.02	−123.1	-150°	−162	−171	−179.5 $=-180^\circ$	-198°
$\lvert G(j\omega)\rvert$	9.80	4.76	1.81	1.26	0.91	0.68	0.316

Table-2: Real & imaginary part of $G(j\omega)$ at various frequencies.

ω rad/sec.	0.1	0.2	0.4	0.5	0.6	0.7	1
$G_R(j\omega)$	=2.86	−2.59	−1.56	−1.14	−0.89	−0.7	−0.29
$G_I(j\omega)$	−9.37	−3.98	−0.9	−0.37	−0.84	0	0.09

Gain margin $k_g = 1.4286$

phase margin, $\gamma = +12^\circ$.



1 Circle = 0.05 magnitude.

Gain Margin $k_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$

$= 1/0.7 = 1.4286$

phase margin = 180° + ϕgc

= 180 − 168

= 12°

② The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)^2}$. Sketch the polar plot & determine the gain & phase margin.

Sol Given that $G(s) = \frac{1}{s(1+s)^2}$

put $s = j\omega$,

$$\therefore G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

Corner frequency $\omega_{c1} = 1$ rad/sec.

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+\omega^2}\angle\tan^{-1}\omega \sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$

$$= \frac{1}{\omega(\sqrt{1+\omega^2})^2}\angle -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega$$

$$= \frac{1}{\omega(\sqrt{1+\omega^2})^2}\angle -90^\circ - 2\tan^{-1}\omega$$

Gain Margin $k_g = 2$

phase margin $\gamma = 21^\circ$.

$$|G(j\omega)| = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega+\omega^3}$$

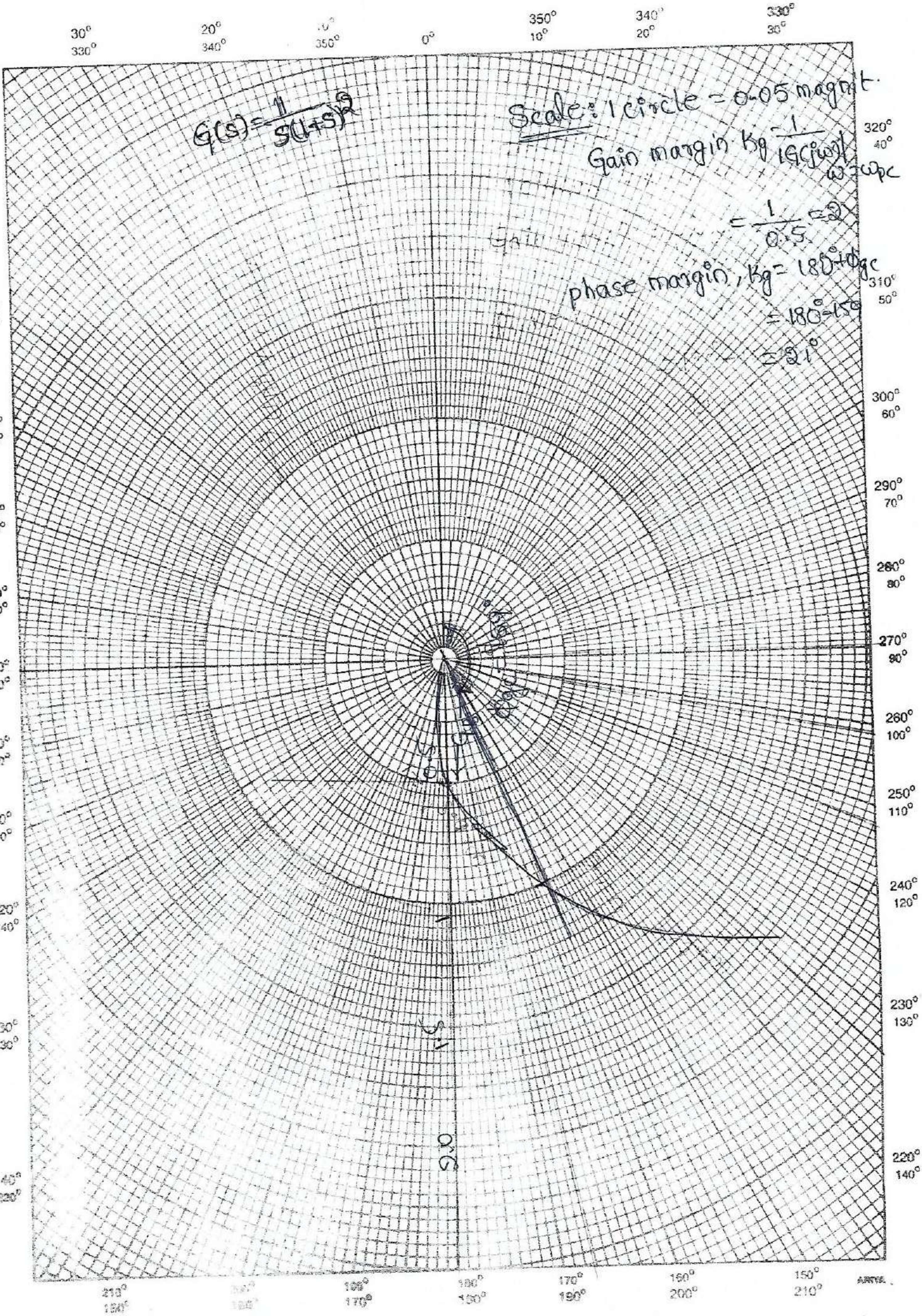
$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}\omega$$

Table 1: Magnitude & phase of $G(j\omega)$

ω rad/sec	0.1	0.2	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
G(jω)	9.9	4.8	2.15	1.6	1.2	1	0.76	0.6	0.5	0.4
∠G(jω)	-101.4	-112.6	-133.6	-143	-151.9	-159	-167.3	-174	-180°	-185

Table 2: Real & Imaginary of $G(j\omega)$

ω rad/sec	0.1	0.2	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$G_R(j\omega)$	-1.95	-1.84	-1.48	-1.28	-1.05	-0.93	-0.74	-0.6	-0.5	-0.4
$G_I(j\omega)$	-9.7	-4.43	-1.55	-0.96	-0.57	-0.36	-0.16	0.06	0	0.03



$G(s) = \frac{1}{s(1+s)^2}$

Scale: 1 circle = 0.05 magnit.

Gain margin $K_g = \frac{1}{|G(j\omega)|}_{\omega=\omega_{pc}}$

$= \frac{1}{0.5} = 2$

phase margin, $K_g = 180^\circ + \phi_{gc}$

$= 180^\circ - 159$

$= 21^\circ$

2. Consider a unity feedback system having an openloop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. sketch the polar plot & determine the value or K so that (i) Gain Margin is 18 db ② phase margin is 60°.

Sol Given that $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$

Let us assume K = 1

K = 1, & s = jω in G(s).

$$\therefore G(j\omega) = \frac{1}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$$

$$\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}.$$

$$\omega_{c2} = \frac{1}{0.05} = 20 \text{ rad/sec}.$$

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+(0.2\omega)^2}\tan^{-1}(0.2\omega)\ \sqrt{1+(0.05\omega)^2}\tan^{-1}(0.05\omega)}$$

$$= \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}} \angle -90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega.$$

⇒ Magnitude & phase or G(jω)

ω rad/sec	0.6	0.8	1	2	3	4	5	6	7	9	10	14
|G(jω)|	1.65	1.23	1.0	0.5	0.3	0.2	0.14	0.1	0.07	0.05	0.04	0.02
∠G(jω)	-98	-101	-104	-117.5	-129.4	-140	-149	-157	-164	-176	-180°	-195

2) Real & Imaginary part of G(jω)

ω radlsec.	0.6	0.8	1	2	3	4	5	6	7	9	10	14
$G_R(j\omega)$	-0.23	-0.23	-0.24	-0.23	-0.19	-0.15	-0.120	-0.092	-0.067	-0.050	-0.04	-0.01
$G_I(j\omega)$	-1.63	-1.21	-0.97	-0.44	-0.23	-0.13	-0.072	-0.039	-0.019	-0.0034	0	0.005

From polar plot, with K = 1.

Gain margin, $K_g = \frac{1}{0.04} = 25$.

Gain margin in db = $20 \log |K_g| = 20 \log (25) = 28$ db

phase margin $\gamma = 76°$

Case (i) with K = 1, $G_B = 0.04$

* The gain margin of 28 db with K=1 has to be reduced to 18db.
The gain margin of 28 db with K=1 has to be increased to a value greater than one.

So K has to be increased to a value greater than one.

G_A at $-180°$ for gain margin is 18db.

$$20 \log \frac{1}{G_A} = 18 \Rightarrow \log \frac{1}{G_A} = 18/20$$

$$G_A = \frac{1}{10^{18/20}} = 0.125.$$

$$K = \frac{G_A}{G_B} = \frac{0.125}{0.04} = 3.125.$$

Case (ii): K=1, gain margin is 76° this is reduced to 60°

gain is increased. Let ϕ_{gc2} is the phase margin of 60°.

$$\therefore 60 = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 60 - 180 = -120°.$$

* polar plot line −120° cut the locus of G(jω) at point C. & cut the unity circle at point D.

G_C = Magnitude of $G(j\omega)$ at point C.

G_D = Magnitude of $G(j\omega)$ at point D.

$G_C = 0.425$, $G_D = 1$

$$K = \frac{G_D}{G_C} = \frac{1}{0.425} = 2.353.$$

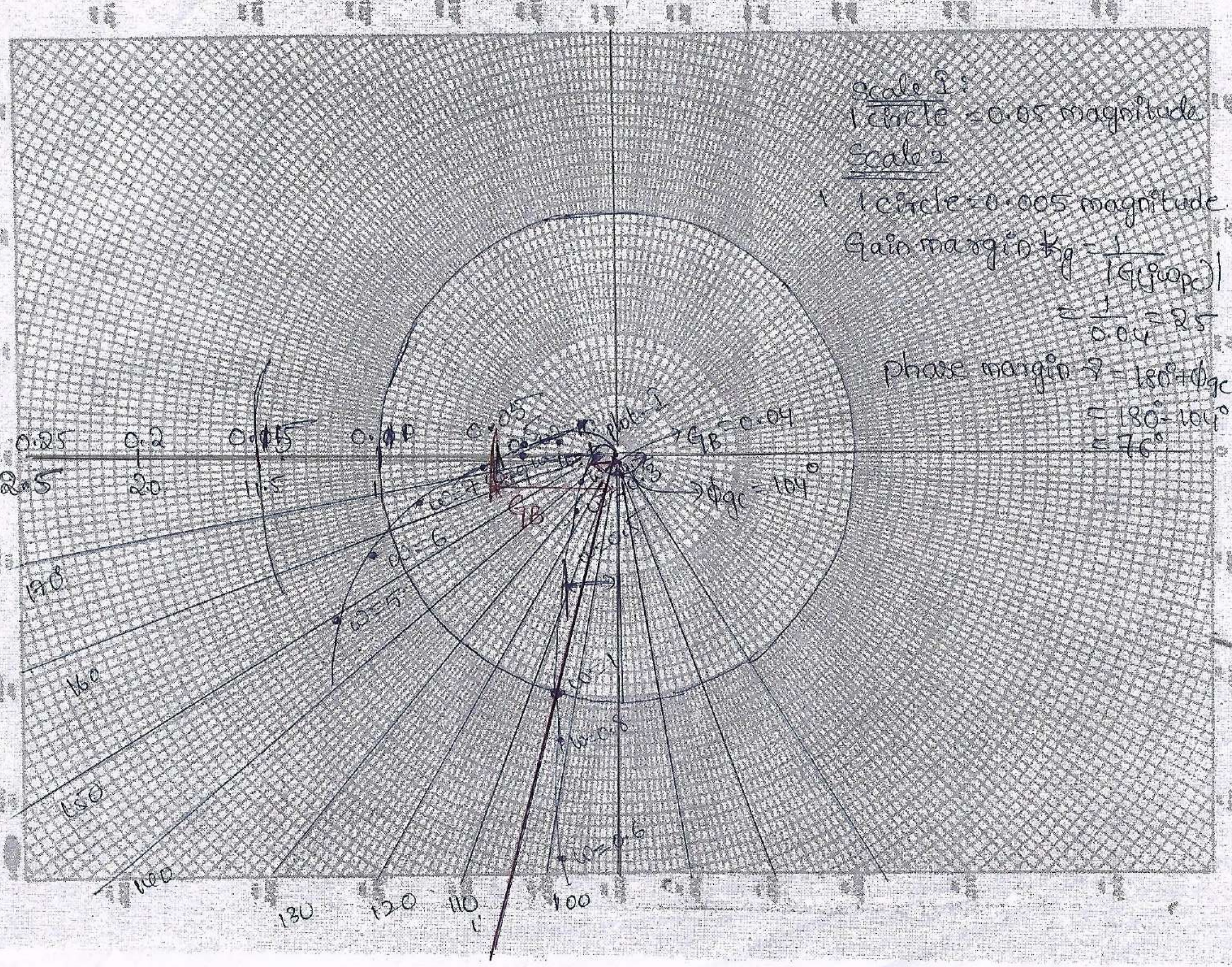
Result: (i) $K = 1$, $K_g = 25$

K_g in db = 28 db

(ii) when $K = 1$, phase margin, $\gamma = 76°$

3) gain margin of 18 db, $K = 3.125$

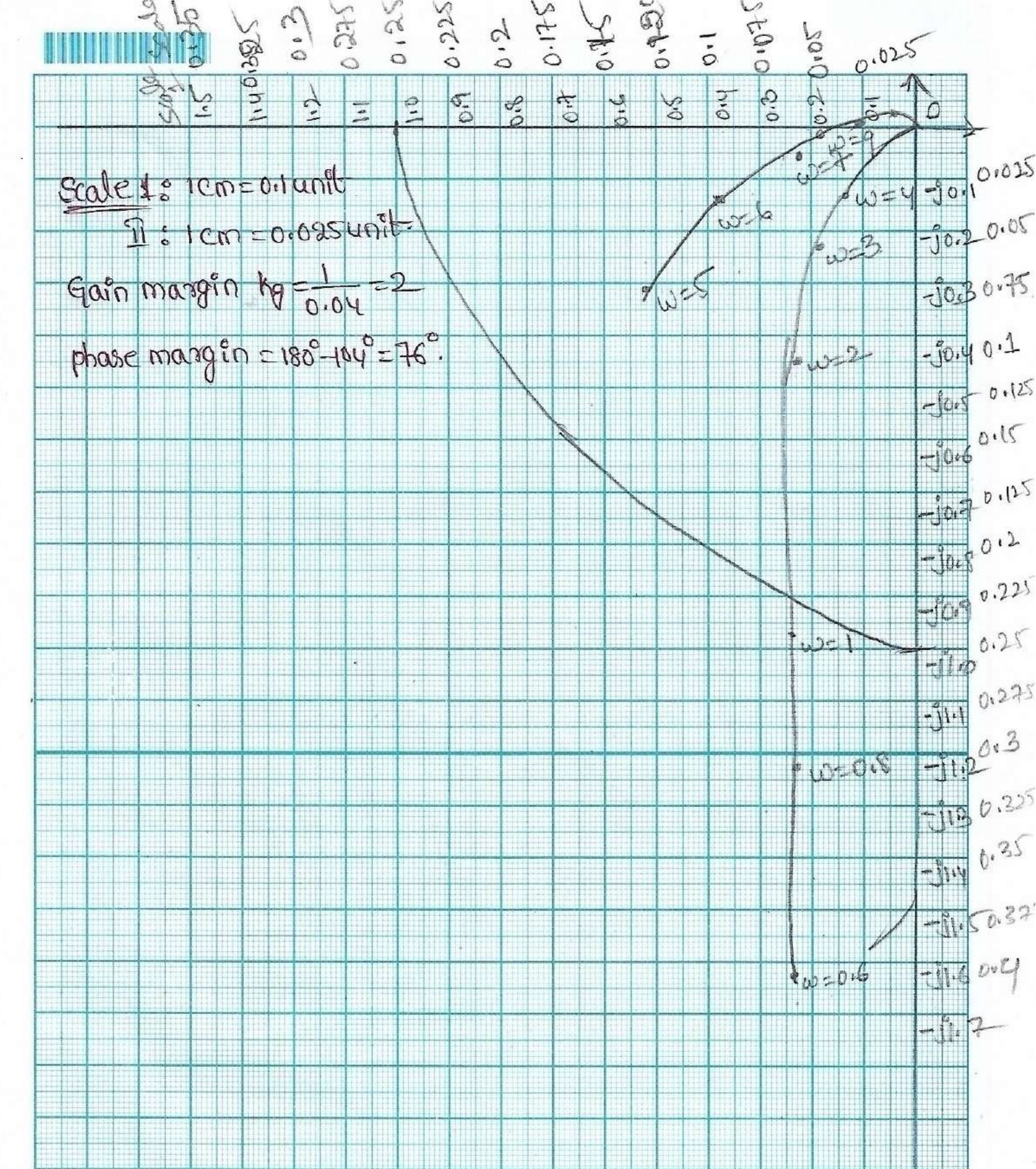
4) For phase margin of 60°, $K = 2.353$.



Scale 1:
1 circle = 0.05 magnitude
Scale 2
1 circle = 0.005 magnitude

Gain margin $k_g = \frac{1}{|G(j\omega_{pc})|}$
$= \frac{1}{0.04} = 25$

Phase margin $\gamma = 180^\circ + \phi_{gc}$
$= 180^\circ - 104^\circ$
$= 76^\circ$



Scale I: 1cm = 0.1 unit

II: 1cm = 0.025 unit

Gain margin $k_g = \frac{1}{0.04} = 2$

phase margin = 180° - 104° = 76°.

HW

① The openloop transfer function of a unity feedback system is given by $G(s) = 1/s^2(1+s)(1+2s)$. sketch the polar plot and determine the gain margin and phase margin.

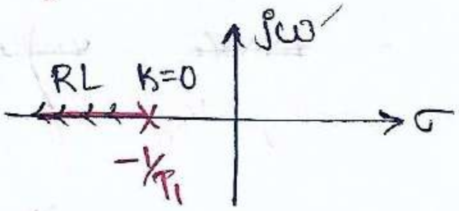
② The openloop transfer function of a unity feedback system is given by $G(s) = \frac{(1+0.2s)(1+0.25s)}{s^3(1+0.005s)(1+0.001s)}$. sketch the polar plot and determine the phase margin.

③ Consider a unity feedback system having an openloop transfer function, $G(s) = \frac{K}{s(1+0.5s)(1+4s)}$. sketch the polar plot and determine the value of K so that (i) Gain margin is 20 db and ② phase margin is 30°.

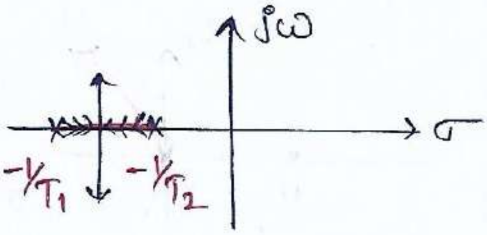
Root locus plots for Typical Transfer functions :-

G(s) Root locus.

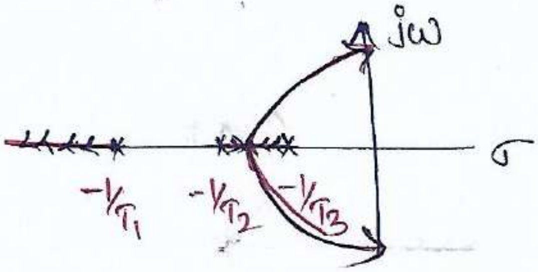
1) $\frac{K}{1+ST_1}$



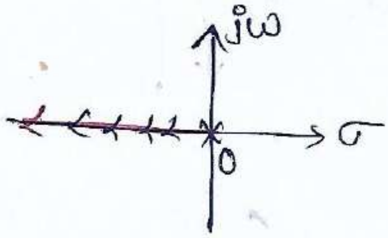
② $\frac{K}{(1+ST_1)(1+ST_2)}$



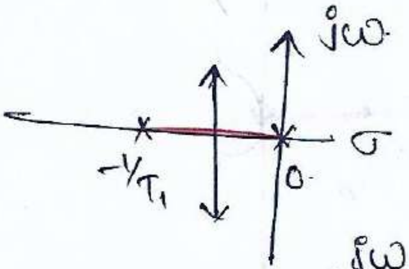
3) $\frac{K}{(1+ST_1)(1+ST_2)(1+ST_3)}$



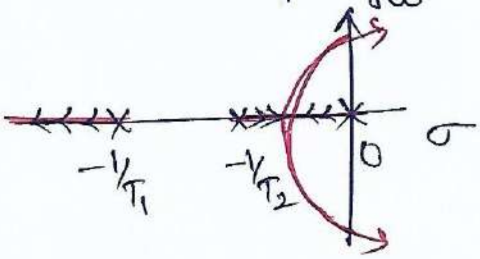
④ $\frac{K}{S}$



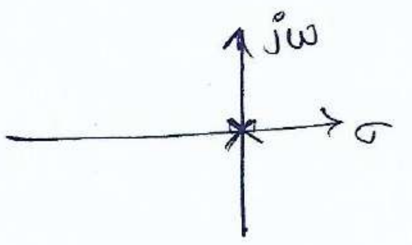
⑤ $\frac{K}{S(1+ST_1)}$



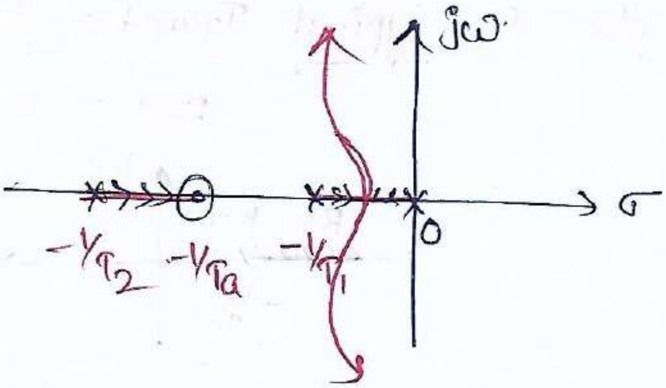
⑥ $\frac{K}{S(1+ST_1)(1+ST_2)}$



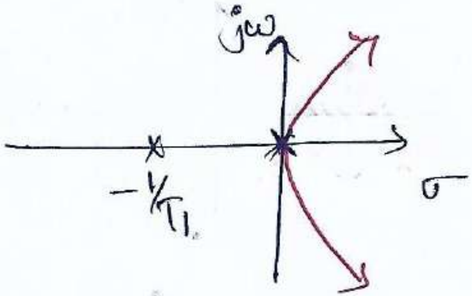
⑦ $\frac{K}{S^2}$



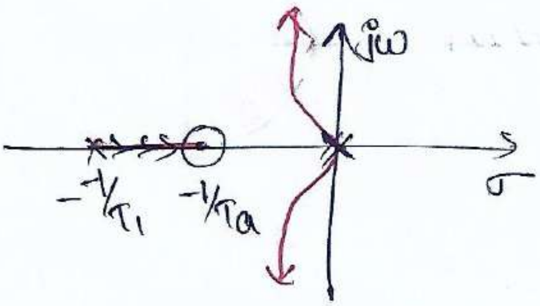
⑧ $\frac{K(1+ST_a)}{S(1+ST_1)(1+ST_2)}$



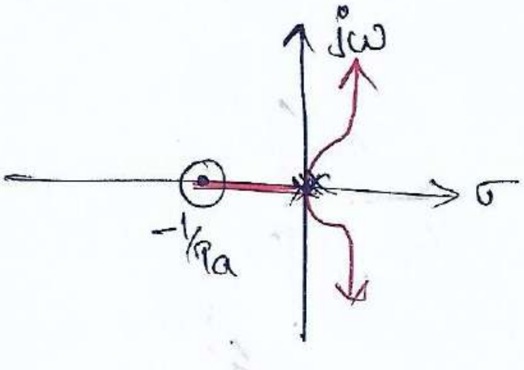
⑨ $\frac{K}{S^2(1+ST_1)}$



⑩ $\frac{K(1+ST_a)}{S^2(1+ST_1)}$



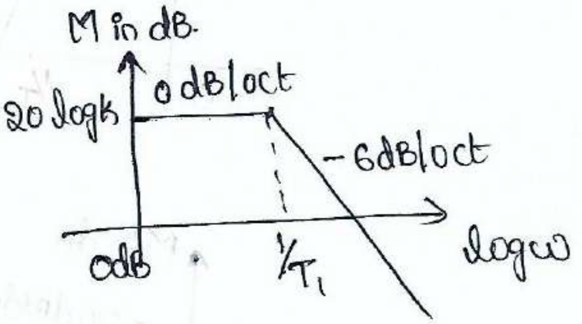
⑪ $\frac{K(1+ST_a)}{S^3}$



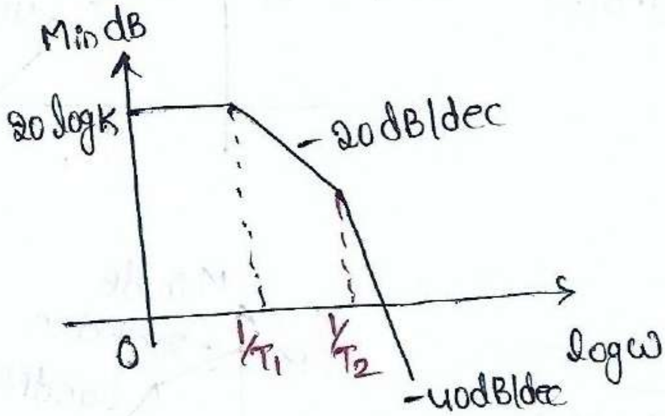
Bodeplots for Typical Transfer functions:-

$G(s)H(s)$ Bodeplot

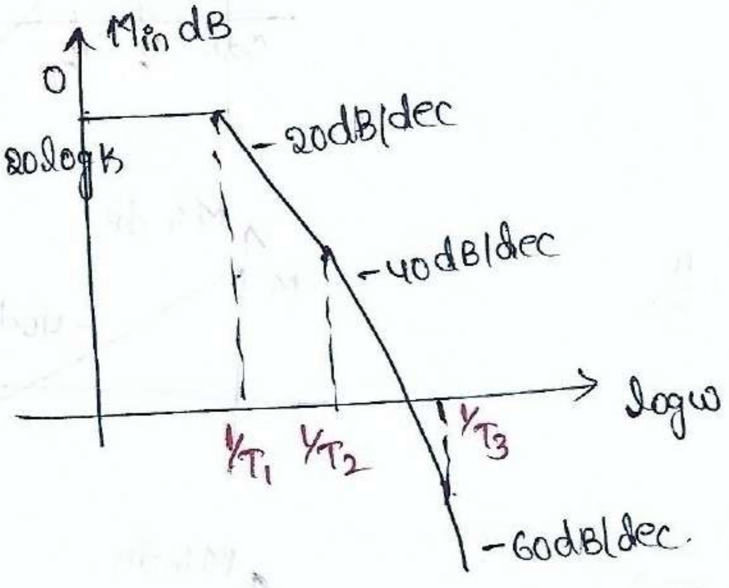
1) $\frac{K}{(1+sT_1)}$



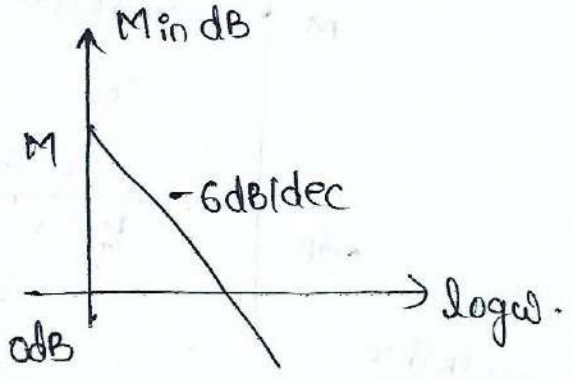
2) $\frac{K}{(1+sT_1)(1+sT_2)}$



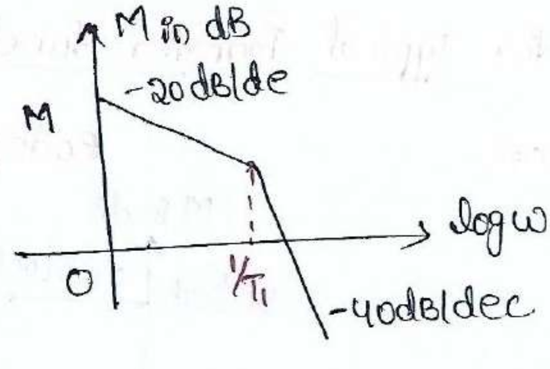
3) $\frac{K}{(1+sT_1)(1+sT_2)(1+sT_3)}$



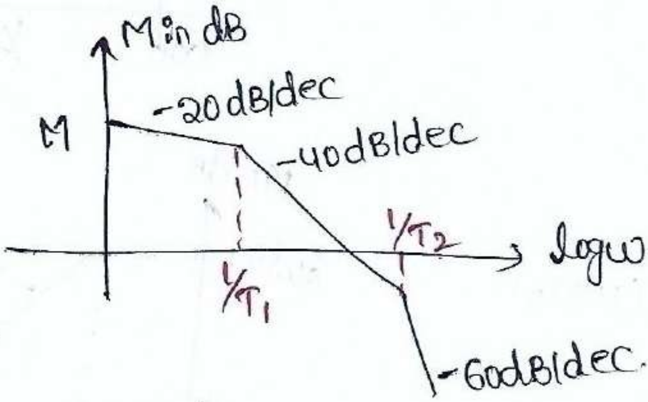
4) $\frac{K}{s}$



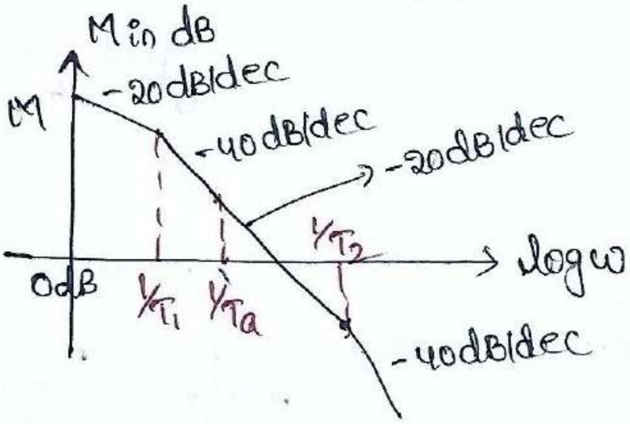
5) $\frac{K}{S(1+ST_1)}$



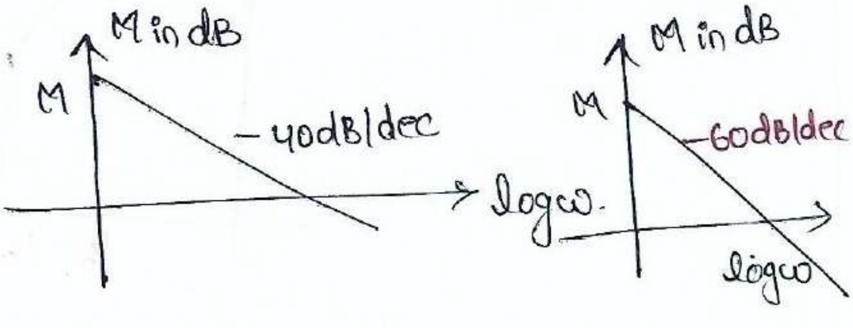
⑥ $\frac{K}{S(1+ST_1)(1+ST_2)}$



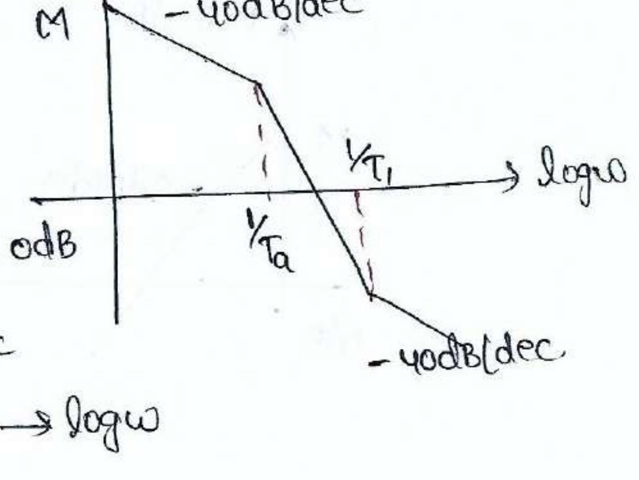
⑦ $\frac{K(1+ST_a)}{S(1+ST_1)(1+ST_2)}$



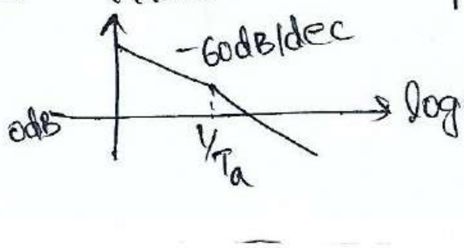
⑧ $\frac{K}{S^2}$, $\frac{K}{S^3}$



⑨ $\frac{K(1+ST_a)}{S^2(1+ST_1)}$



⑩ $\frac{K(1+ST_a)}{S^3}$



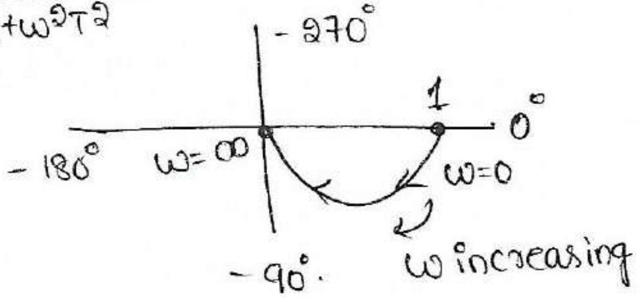
Typical sketches of polar plot :-

Type 0, Order 1: $G(s) = \frac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}}\angle -\tan^{-1}\omega T$$

$\omega \to 0 \Rightarrow G(j\omega) \to 1\angle 0^\circ$

$\omega \to \infty \Rightarrow G(j\omega) \to 0\angle -90^\circ$



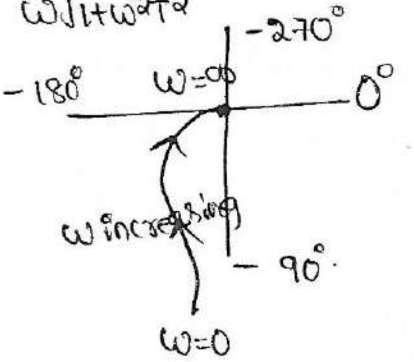
Type 1, Order 2!

$G(s) = \frac{1}{s(1+sT)}$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90^\circ \sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T} = \frac{1}{\omega\sqrt{1+\omega^2T^2}}\angle -90^\circ - \tan^{-1}\omega T$$

$\omega = 0 \Rightarrow G(j\omega) = \infty\angle -90^\circ$

$\omega = \infty \Rightarrow G(j\omega) = 0\angle -180^\circ$



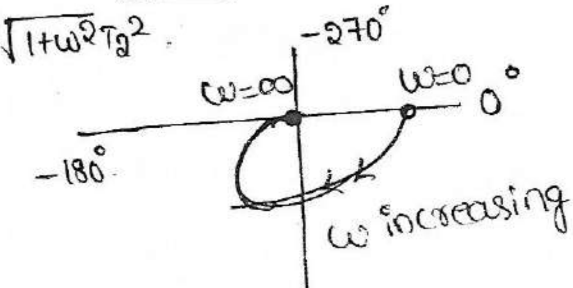
Type 0, Order 2 :-

$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}\angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$\omega = 0 \Rightarrow G(j\omega) =$ [illegible] $\angle 0^\circ$

$\omega = \infty \Rightarrow G(j\omega) = 0\angle -180^\circ$

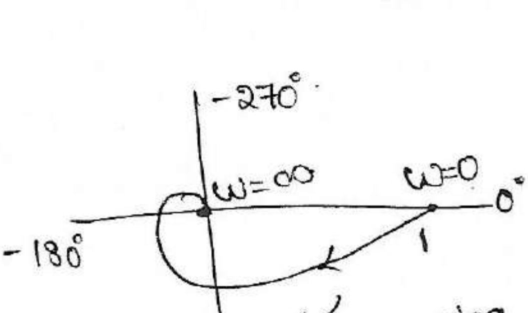


Type 0, Order 3 :-

$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

$\omega = 0 \Rightarrow G(j\omega) = 1\angle 0^\circ$

$\omega = \infty \Rightarrow G(j\omega) = 0\angle -270^\circ$

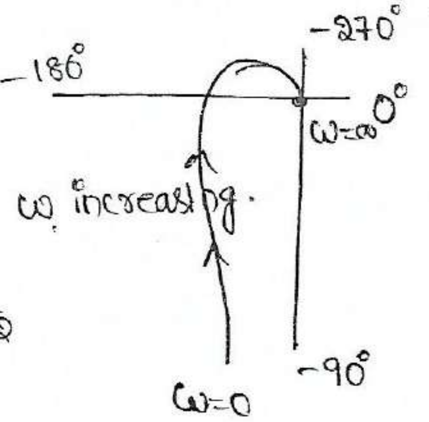


Type 1 Order 3:

$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}.$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$



$\omega \to 0 \Rightarrow G(j\omega) = \infty \angle -90^\circ$

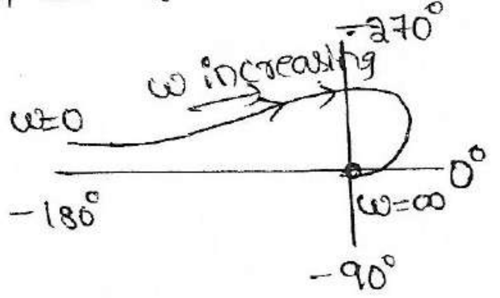
$\omega = \infty \Rightarrow G(j\omega) = 0 \angle -270^\circ.$

Type 2 order 4:

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2.$$



$\omega = 0 \Rightarrow G(j\omega) = \infty \angle -180^\circ$

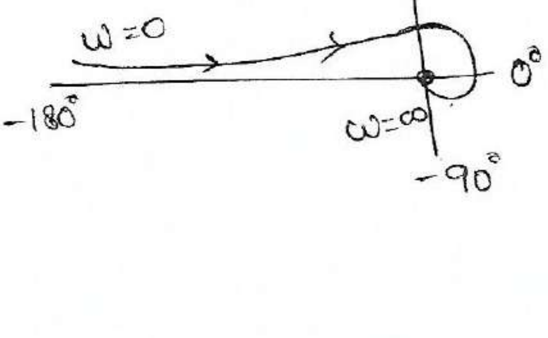
$\omega = \infty \Rightarrow G(j\omega) = 0 \angle -360^\circ$

Type 2 Order 5:-

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3.$$



$\omega = 0 \to G(j\omega) = \infty \angle -180^\circ$

$\omega = \infty \to G(j\omega) = 0 \angle -450^\circ = 0 \angle -90^\circ.$

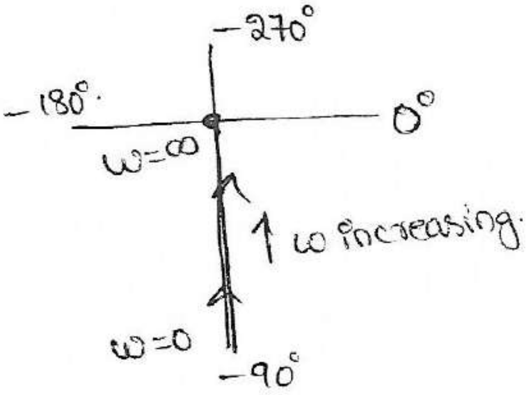
Type 1 Order 1:-

$G(s) = \frac{1}{s}$

$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega\angle 90^\circ} = \frac{1}{\omega}\angle -90^\circ.$

$\omega=0 \rightarrow G(j\omega) = \infty\angle -90^\circ$

$\omega=\infty \Rightarrow G(j\omega) = 0\angle -90^\circ.$

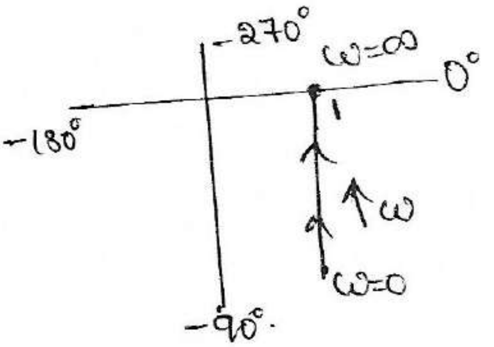


$G(s) = \frac{1+sT}{sT}$

$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T}+1 = 1+\frac{1}{\omega T}\angle -90^\circ$

$\omega=0 \rightarrow G(j\omega) = 1+\infty\angle -90^\circ$

$\omega=\infty \Rightarrow G(j\omega) = 1+0\angle -90^\circ.$

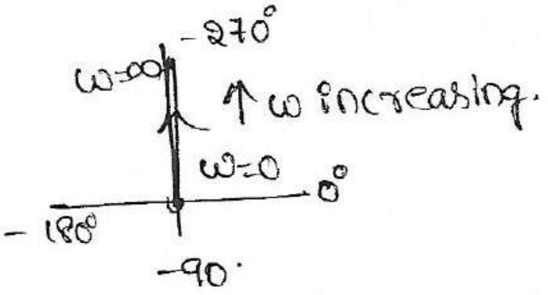


$G(s) = s$

$G(j\omega) = j\omega = \omega\angle 90^\circ$

$\omega=0 \rightarrow G(j\omega) = 0\angle 90^\circ$

$\omega=\infty \Rightarrow G(j\omega) = \infty\angle 90^\circ$

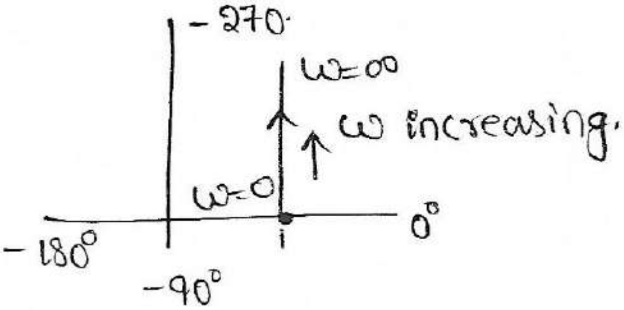


$G(s) = 1+sT$

$G(j\omega) = 1+j\omega T = 1+\omega T\angle 90^\circ$

$\omega=0 \Rightarrow G(j\omega) = 1+0\angle 90^\circ$

$\omega=\infty \Rightarrow G(j\omega) = 1+\infty\angle 90^\circ.$



Compensators :-

The first step in design is the adjustment of gain to meet the desired specifications. In practical systems, adjustment of gain alone will not be sufficient to meet the given specifications.

* In many cases, increasing the gain may result in poor stability or instability.

* In such cases it is necessary to introduce additional devices or components in the system to alter the behaviour & to meet the desired specifications.

* Such a redesign or addition of a suitable device is called compensation.

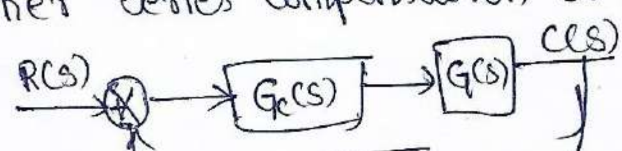
* A device inserted into the system for the purpose of satisfying the specifications is called compensator.

* The compensators basically introduce pole and/or zero in open loop transfer function to modify the performance of the system.

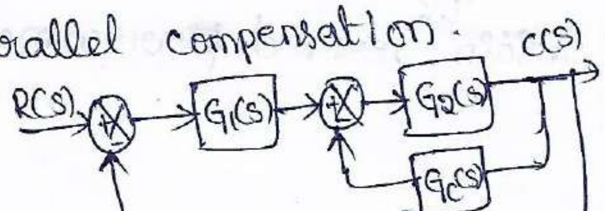
The design problem may be stated as follows:- When a set of specifications are given for a system, then a suitable compensator should be designed so that the overall system will meet the given specification.

⇒ The compensation schemes used for feedback control system is either series compensation or parallel compensation.

ex:



Series Compensation



parallel/Feedback

The choice between the Series compensation & parallel compensation depends on

① Nature of Signals in the system
② power levels at various points
③ Components available
④ Designers experience
⑤ Economic Considerations.

* The compensator may be electrical, mechanical, hydraulic, Pneumatic or other type of device or network.

* Usually, an electric network or electronic device serves as compensator in many control systems.

* The different types of electrical or electronic compensators used are

① Lag Compensator.
② Lead Compensator
③ Lag-Lead compensator.

* In ~~compensator~~ Control system, compensation is required in the following situations

① When the system is absolutely unstable, then compensation is required to stabilize the system and also meet the desired performance.

* When the system is stable, compensation is provided to obtain the desired performance.

Lag Compensator :-

A compensator having the characteristics of a lag network is called a lag compensator.

* If a sinusoidal signal is applied to a lag network, then in steady state the output will have a phase lag with respect to input.

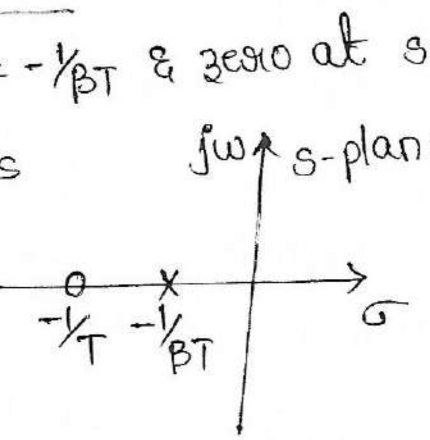
* Lag compensation results in a large improvement in steady state performance but results in slower response due to reduced bandwidth.

* lag compensator is a low pass filter & high frequency noise signals are attenuated.

S-plane representation of lag compensator :-

* The lag compensator has a pole at $s = -1/\beta T$ & zero at $s = -1/T$

* The pole-zero plot of lag compensator is



* Here $\beta > 1$, so the zero is located left of the pole on the negative real axis.

Transfer function of lag compensator,

$$G_c(s) = \frac{s+z_c}{s+p_c} = \frac{s+1/T}{s+1/\beta T} \quad \text{where } T>0,\ \beta>1$$

zero of lag compensator, $z_c = 1/T \longrightarrow ①$

pole of lag compensator, $p_c = 1/\beta T \longrightarrow ②$

where from ① $T = 1/z_c$

from ② $\beta = \frac{1}{T p_c}$

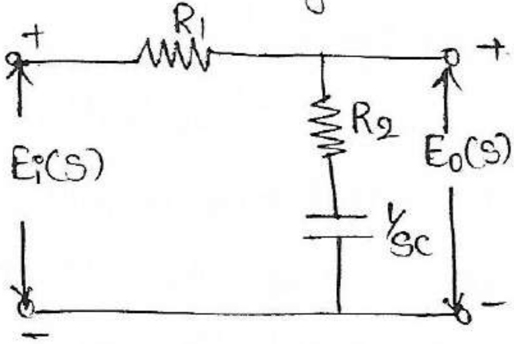
$$\boxed{\beta = \frac{z_c}{p_c}} \quad [\because T = 1/z_c]$$

Realisation of Lag Compensator Using Electrical Network:-

The lag compensator can be realised by the R-C network

Let, $E_i(S)$ = Input voltage

$E_0(S)$ = Output voltage



Electrical lag compensator

* The input voltage is applied to the series combination of R_1, R_2 & C.

* The output voltage is obtained across the series combination of R_2 & C.

By voltage division rule,

$$E_0(S) = E_i(S)\frac{(R_2+1/SC)}{(R_1+R_2+1/SC)} = E_i(S)\frac{(SCR_2+1)/\cancel{SC}}{\frac{(R_1+R_2)SC+1}{\cancel{SC}}} = E_i(S)\frac{1+SCR_2}{(R_1+R_2)SC+1}$$

The transfer function of the electrical network is the ratio of output voltage to input voltage.

$$\frac{E_0(S)}{E_i(S)} = \frac{CR_2\left(S+\frac{1}{CR_2}\right)}{C(R_1+R_2)\left[S+\frac{1}{C(R_1+R_2)}\right]}$$

$$= \frac{\left(S+\frac{1}{R_2C}\right)}{\left(\frac{R_1+R_2}{R_2}\right)\left[S+\frac{1}{((R_1+R_2)/R_2)R_2C}\right]} \longrightarrow ①$$

Transfer function of lag compensator is,

$$G_c(S) = \frac{(S+1/T)}{\left(S+\frac{1}{\beta T}\right)} \longrightarrow ②$$

On comparing (1) & (2) $\boxed{\frac{E_0(S)}{E_i(S)} = \frac{1}{\beta}\frac{(S+1/T)}{\left(S+\frac{1}{\beta T}\right)}}$ where

$T = R_2C$

$\beta = (R_1+R_2)/R_2$

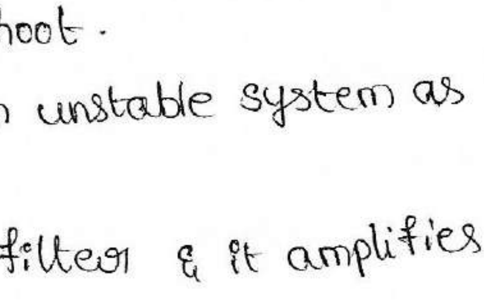
Lead Compensator:-

A compensator having the characteristics of a lead network is called a lead compensator.

- If a sinusoidal signal is applied to the lead network, then in steady state the output will have a phase lead with respect to the input.
- The lead compensation increases the bandwidth, which improves speed of response & reduces the overshoot.
- lead compensation provides to make an unstable system as a stable system.
- A lead compensator is a high pass filter & it amplifies high frequency noise signals.

S-plane representation of lead compensator:-

- The lead compensator has zero at $s = -1/T$ & pole at $s = -1/\alpha T$
- The pole-zero plot of lead compensator is
- $\alpha < 1$, so the zero is closer to the origin than the pole.



pole-zero plot of lead compensator

$$G_c(s) = \frac{s+z_c}{s+p_c} = \frac{(s+1/T)}{(s+1/\alpha T)}$$

where $T > 0$, $\alpha < 1$

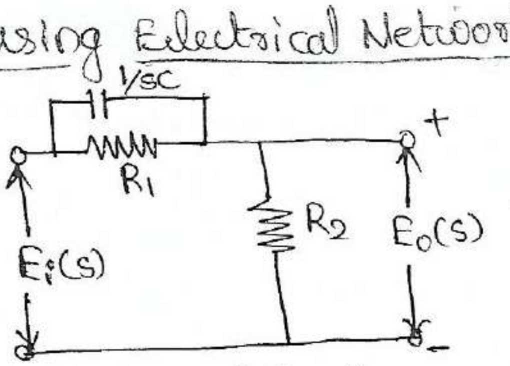
zero of lead compensator $z_c = 1/T \rightarrow$ ①

pole of lead compensator $p_c = 1/\alpha T \rightarrow$ ②

from ① $T = 1/z_c$

from ② $\alpha = \frac{1}{p_c T} = \frac{z_c}{p_c}$

Realisation of lead compensator using Electrical Network:-



Electrical lead compensator

* The lead compensator can be realised by the RC network.

let $E_i(S)$ = Input voltage,

$E_o(S)$ = Output voltage.

The input voltage is applied to the series combination of $(R_1 \| C)$ & R_2.

The output voltage is obtained across R_2.

By voltage division rule

Output voltage, $E_0(S) = E_i(S)\dfrac{R_2}{R_2+\dfrac{(R_1\times 1/SC)}{(R_1+1/SC)}}$

$$E_0(S) = E_i(S)\frac{R_2}{R_2+\frac{R_1}{R_1CS+1}} = E_i(S)\frac{R_2}{\frac{R_2(R_1CS+1)+R_1}{R_1CS+1}}$$

Transfer function of electrical network

$$\frac{E_0(S)}{E_i(S)} = \frac{R_2(R_1CS+1)}{(R_1R_2CS+R_2+R_1)} = \frac{R_1\cancel{C}R_2\left(S+\frac{1}{R_1C}\right)}{\cancel{R_1}CR_2\left(S+\frac{(R_1+R_2)}{R_1CR_2}\right)}$$

$$\boxed{\frac{E_0(S)}{E_i(S)} = \frac{\left(S+\frac{1}{R_1C}\right)}{S+\left(\frac{1}{R_2/(R_1+R_2)}\right)\frac{1}{R_1C}}} \longrightarrow ①$$

General form of lead compensator transfer function is

$$G_c(S) = \frac{(S+1/T)}{\left(S+\frac{1}{\alpha T}\right)} \longrightarrow ②$$

Comparing (1) & (2) $\dfrac{E_0(S)}{E_i(S)} = \dfrac{S+1/T}{S+1/\alpha T}$ where $T = R_1C$

$\alpha = \dfrac{R_2}{R_1+R_2}$.

∴ The transfer function of the RC network is similar to the general form of transfer function of lead compensator.

Lag-Lead Compensator:-

A compensator having the characteristics of lag-lead network is called lag-lead compensator.

* In a lag-lead network when sinusoidal signal is applied, both phase lag & phase lead occurs in the output, but in different frequency regions. phase lag occurs in the low frequency region & phase lead occurs in the high frequency regions.

* A lead compensator increases bandwidth & decreases the max. overshoot in the step response.

* lag compensation increases the low frequency gain & improves the steady state accuracy of the system.

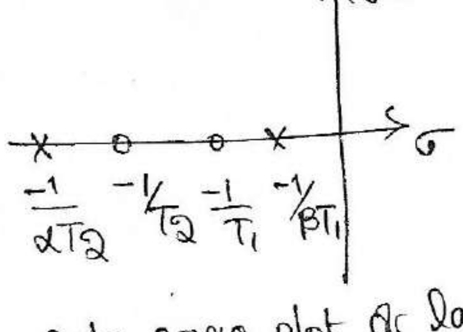
* A lag-lead compensation combines the advantage of lag & lead compensations.

* Lag-lead compensator posses two poles & two zeros.

S-plane Representation of Lag-lead compensator:-

* The lag section has one real pole & one real zero with pole to the right of zero.

* The lead section has one real pole & one real zero but the zero is to the right of the pole.



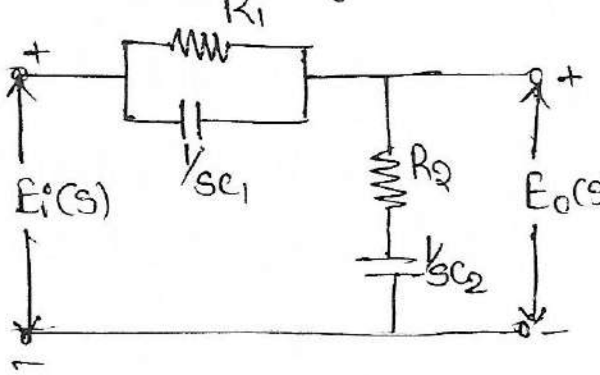
Pole-zero plot of lag-lead compensator.

Transfer function of lag-lead compensator

$$G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{\underbrace{(s+1/\beta T_1)}_{lag}\underbrace{(s+1/\alpha T_2)}_{lead}}. \quad \beta>1,\ 0<\alpha<1$$

Realisation of Lag-Lead compensator using Electrical network:-

The lag-lead compensator can be realised by the R-C network.



Let $E_i(s)$ = Input voltage.

$E_o(s)$ = Output voltage.

* Input voltage is applied to the series combination of $R_1 || C_1$, R_2 & C_2.

* Output voltage is obtained across the series combination of R_2 & C_2.

By voltage division rule

$$E_o(s) = E_i(s) \frac{R_2 + 1/sC_2}{\left(R_1 || \frac{1}{sC_1}\right) + R_2 + \frac{1}{sC_2}}$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{\frac{sR_2C_2+1}{sC_2}}{\frac{R_1 \frac{1}{sC_1}}{R_1 + 1/sC_1} + \frac{sR_2C_2+1}{sC_2}} = \frac{\frac{sR_2C_2+1}{sC_2}}{\frac{R_1}{sR_1C_1+1} + \frac{sR_2C_2+1}{sC_2}}$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{\frac{sR_2C_2+1}{sC_2}}{\frac{sR_1C_2 + (sR_1C_1+1)(sR_2C_2+1)}{(sR_1C_1+1)sC_2}}$$

$$= \frac{(sR_1C_1+1)(sR_2C_2+1)}{sR_1C_2 + (sR_1C_1+1)(sR_2C_2+1)}$$

$$= \frac{R_1C_1R_2C_2\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{sR_1C_2 + R_1C_1R_2C_2\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}$$

On dividing the numerator & denominator by $R_1C_1R_2C_2$ we get

$$\frac{E_0(s)}{E_i(s)} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{\frac{s}{R_2C_1}+\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}$$

$$\boxed{\frac{E_0(s)}{E_i(s)} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}}} \rightarrow ①$$

Transfer function of lag-lead compensator is given by

$$G_c(s) = \frac{(s+1/T_1)(s+1/T_2)}{(s+1/\beta T_1)(s+1/\alpha T_2)} = \frac{(s+1/T_1)(s+1/T_2)}{s^2+s\left[\left(\frac{1}{\beta T_1}\right)+\left(\frac{1}{\alpha T_2}\right)\right]+\frac{1}{\alpha\beta T_1T_2}} \leftrightarrow ②$$

comparing (1) & (2)

$$T_1 = R_1C_1 \longrightarrow ③$$

$$T_2 = R_2C_2. \longrightarrow ④$$

$$R_1R_2C_1C_2 = \alpha\beta T_1T_2 \longrightarrow ⑤$$

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1} = \frac{1}{\beta T_1}+\frac{1}{\alpha T_2} \longleftrightarrow ⑥$$

$$\alpha\beta = \frac{R_1R_2C_1C_2}{T_1T_2.} \longrightarrow ⑦$$

from ③, ④, ⑥ => we can say $\boxed{\alpha\beta = 1}$ $\leftrightarrow$ ⑧

eq⑧ implies that a single lag-lead network does not allow an independent choice of α and β. Hence in the T.F of electrical lag-lead compensator replace α by $1/\beta$.

$$\therefore \frac{E_o(s)}{E_i(s)} = G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \longrightarrow ⑨$$

where $\beta > 1$, $T_1 = R_1C_1$, $T_2 = R_2C_2$.

$$\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} = \frac{1}{\beta T_1} + \frac{\beta}{T_2}.$$

Comparision between Lead & Lag Compensator:-

$$G_c(s) = \frac{1+aTs}{1+Ts}$$

$$\angle G_c(s) = \phi = \tan^{-1}\omega aT - \tan^{-1}\omega T.$$

ϕ is positive if $a>1$ → lead

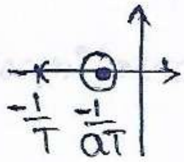
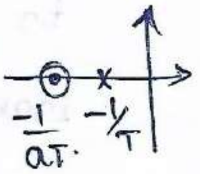
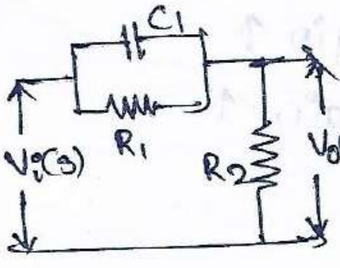
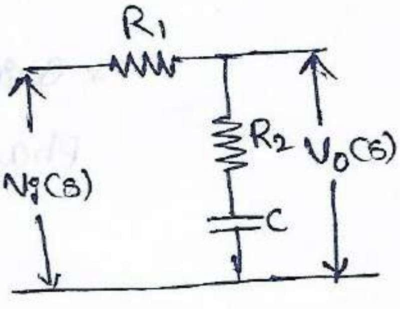
ϕ is negative if $a<1$ → lag.

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_m} = 0.$$

ω_m = Maximum phase lead/lag frequency.

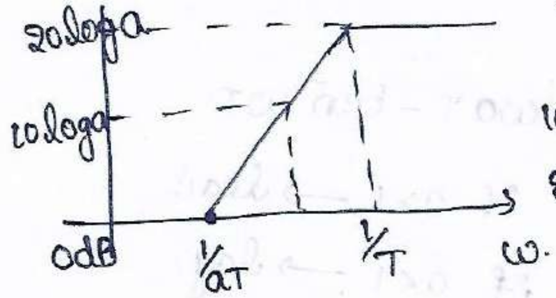
$$\omega_m = \frac{1}{T\sqrt{a}}.$$

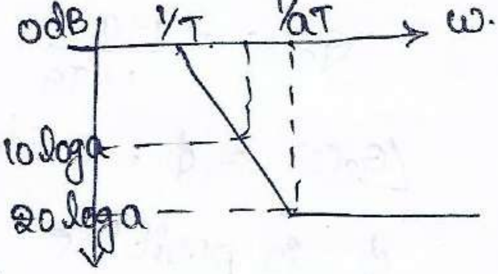
$$\phi_m = \sin^{-1}\left[\frac{a-1}{a+1}\right]$$

Characteristics	Lead	Lag.
1) $G_c(s) = \frac{1+aTs}{1+Ts}$	$a>1$	$0<a<1$
2) $\omega_m = \frac{1}{T\sqrt{a}}$, $\phi_m = \sin^{-1}\left(\frac{a-1}{a+1}\right)$	$0<\phi_m<90^\circ$	$-90^\circ<\phi_m<0^\circ$
3) Pole-zero plot		
4) Filter	Highpass	Lowpass
5) Similar to	PD	PI
6) Electrical n/w $TF = G_c(s) = \frac{V_o(s)}{V_i(s)}$	 $T = \frac{R_1R_2}{R_1+R_2}C$, $a = \frac{R_1+R_2}{R_2}$	 $T = (R_1+R_2)$

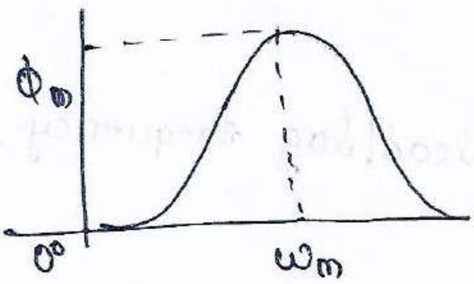
Characteristics Lead Lag

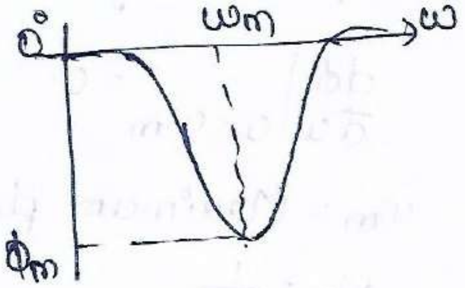
7) Magnitude plot



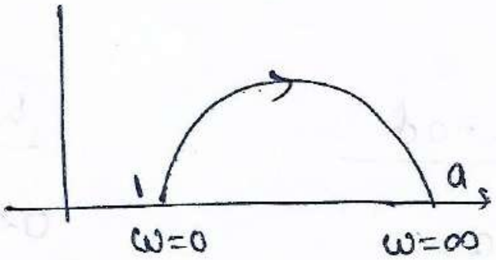


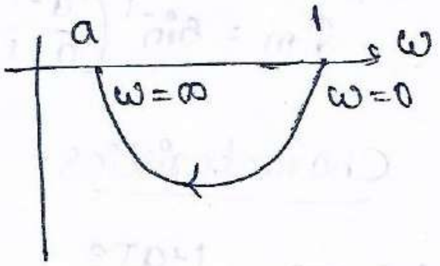
8) phase plot





9) polar plot





10) Effects

	Lead	Lag
10) Effects	Bandwidth ↑ t_r ↓ , speed ↑ transient response improved	* steady state response is improved. * ω_{gc} ↓, BW ↓, t_r ↑, Speed ↓ * Transient response becomes slower.
	* M_p ↓ , M_r ↓	* M_r ↓, M_p ↓
	* Gain margin ↑ Phase margin ↑	* ξ ↑, phase margin ↑ gain margin ↑, signal to Noise ratio ↑.

① Write the transfer functions of lag, lead, lag-lead compensator? Draw its pole-zero plots?

Sol: **Lag**

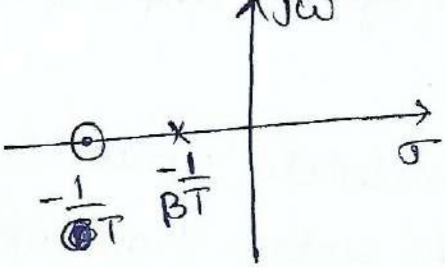
Transfer function of lag compensator $g_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$

* The lag compensator has a pole at $s = -\frac{1}{\beta T}$.

a zero at $s = -\frac{1}{T}$.

Since $\beta > 1$ & $T > 0$, the pole of lag compensator is nearer to origin.

* The pole-zero plot of lag compensator is



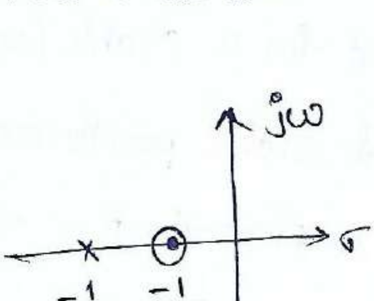
Lead compensator:

Transfer function of lead compensator $g_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$

The lead compensator has a pole at $s = -\frac{1}{\alpha T}$

zero at $s = -\frac{1}{T}$.

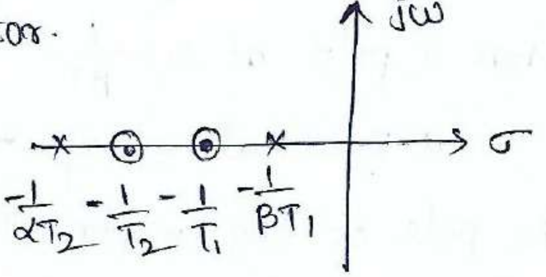
Since $\alpha < 1$ & $T > 0$, the zero of lead compensator is nearer to origin.



lag-lead compensator:-

Transfer function of lag-lead compensator $G_c(s) = \underbrace{\frac{(s+1/T_1)}{(s+\frac{1}{\beta T_1})}}_{lag} \underbrace{\frac{(s+1/T_2)}{(s+\frac{1}{\alpha T_2})}}_{lead}$

Pole-zero plot of lag-lead compensator.



10 Q Why compensation is necessary in feedback control system?

In feedback control system compensation is required in the following situations.

1. When the system is absolutely unstable, then compensation is required to stabilize the system & also meet the desired Performance.

2) When the system is stable, compensation is ~~required~~ provided to obtain the desired performance.

Q When lag/lead/lag-lead compensation is employed?

* Lag compensation is employed for a stable system for improvement in steady state performance.

* Lead compensation is employed for a stable/unstable system for improvement in transient state performance.

* Lag-Lead compensation is employed for stable/unstable system for improvement in both steady-state & transient state performance.

① The damping ratio and natural frequency of oscillation of second order system is 0.5 and 8 rad/sec respectively. Calculate the resonant peak and resonant frequency?

Sol Resonant peak, $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2(0.5)\sqrt{1-(0.5)^2}} = 1.154$

Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\xi^2} = 8\sqrt{1-2(0.5)^2} = 5.657$ rad/sec.

② The resonant peak of a proto type 2nd order system is 1.042 the damping ratio of system is ?

Sol Resonant peak, $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

$$1.042 = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$2\xi\sqrt{1-\xi^2} = \frac{1}{1.042}$$

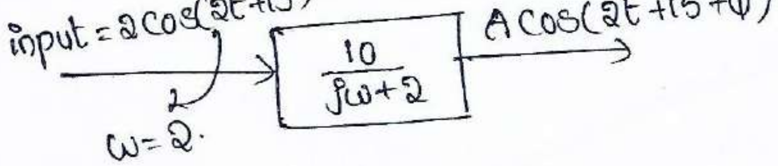
$$4\xi^2(1-\xi^2) = 0.921$$

$$\boxed{\xi = 0.632}$$

③ $TF = \frac{10}{s+2}$ the steady state o/p of a system to the I/p of $2\cos(2t+15^\circ)$.

Sol $TF = \frac{10}{s+2}$, input $= 2\cos(2t+15^\circ)$

$s = j\omega \rightarrow$ for sinusoidal, $TF = \frac{10}{j\omega+2}$



$$\frac{10}{j\omega+2} = \frac{10}{j2+2}$$

Amplitude $= \left|\frac{10}{j\omega+2}\right| = \left|\frac{10}{j2+2}\right| = \frac{10}{\sqrt{2^2+2^2}} = \frac{10}{2\sqrt{2}}$

$\phi = \angle\frac{10}{j2+2} = -\tan^{-1}\frac{2}{2} = -45^\circ.$

$\therefore$ steady state output $= \frac{10}{2\sqrt{2}} 2\cos(2t+15^\circ-45^\circ)$

$= 5\sqrt{2}\cos(2t-30^\circ).$

UNIT-V

STATE SPACE ANALYSIS

Introduction:-

The state variable approach is a powerful tool/technique for the analysis & design or control systems.

* The analysis and design or the following can be carried by using state space method.

1) Linear system
2) Non-linear system.
3) Time invariant system.
4) Time variant system
5) Multiple input & multiple output system.

* The state space analysis is a modern approach & also easier for analysis using digital computers.

* The conventional methods or analysis employs the transfer function or the system.

The drawbacks in the transfer function model &analysis are,

1. Transfer function is defined under zero initial conditions.
2. Transfer function is applicable to linear time invariant systems.
3. Transfer function analysis is restricted to single input & single output systems.
4. Does not provide information regarding the internal state or the system.

* The state variable analysis can be applied for any type of systems.

* The analysis can be carried with initial conditions & can be carried on multiple input & multiple output system.

Significance of state space analysis:-

It is a time domain analysis which uses state space model of representation.

Advantages of state space analysis:-

* It is applicable to any type of the system such as linear, non-linear, time variant, time invariant etc.
* It gives the complete information about the state of the system as it also consider the initial conditions.
* It is more accurate than transfer function.
* It is equally applicable to SISO & MIMO (Single input Single output & Multi input Multi output).
* It is quite simple when compare to other methods as it uses matrix algebra.
* It is possible to optimise the systems useful for optimal design.
* It gives the information about the controllability & observability.

Disadvantages:-

* Techniques are complex.
* Many computations are required.

Basic Concepts:-

State:- The state of a dynamic system is defined as the minimal set of variables (known as state variables), such that the knowledge of these variables at $t=t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

State variables:-

A set of variables which describes the system at any time instant are called state variables.

(or)

The variables which are involved in determining the state of a dynamic system is called State variables. i.e minimal set of variables required at $t=t_0$, along with the knowledge of inputs are sufficient to determine the dynamic behaviour at $t \geq t_0$.

State vector:- It is the vector sum of all state variables. It describes the behaviour of a given system.

* A state vector is unique $(x(t))$

State space:-

The 'n' dimensional space whose coordinates consist of $x_1, x_2 .. x_n$ axes is called as state space. and x_1, x_2, x_3 are state variables.

Input space:- The set of all possible values which the i/p vector $(u(t))$ can have(or assume) at time 't' forms.

Output space:- set of all possible values which the o/p vector $y(t)$.

State model :(or) state diagram:-

The pictorial representation of the state model of a system is called state diagram. The state diagram of a system can be either in block diagram or in signal flow graph.

State space formulation:-

In the state space formulation, a system consist of 'm' inputs, p' outputs, and n-state variables.

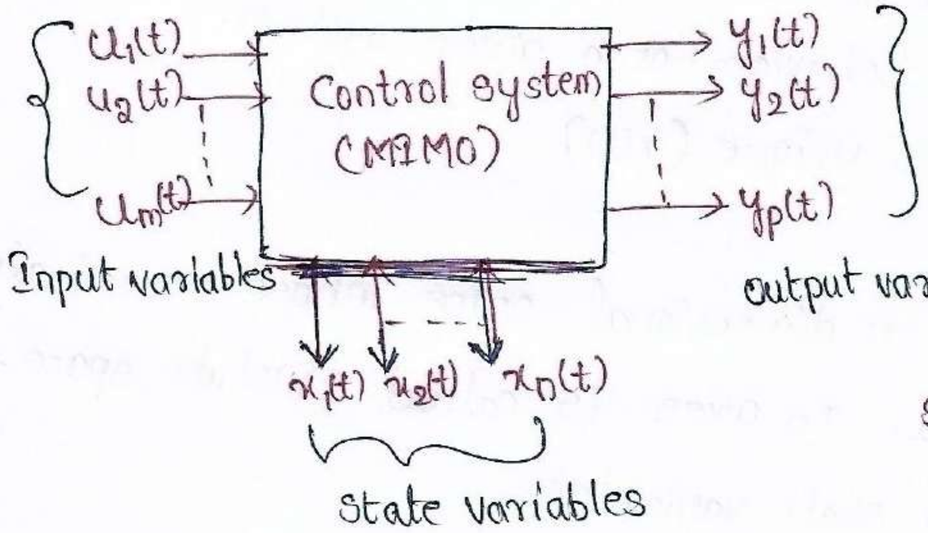
Let us consider

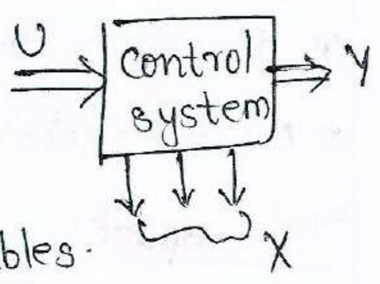
$x_1(t), x_2(t), x_3(t) \ldots x_n(t)$ = state variables.

$u_1(t), u_2(t), u_3(t) \ldots u_m(t)$ = Input variables.

$y_1(t), y_2(t), y_3(t) \ldots y_p(t)$ = Output variables.

The state-space representation of block diagram is shown below (or state model of a system).





State space Representation of a system.

The different variables can be represented by the vectors (column matrix) as shown below.

Input vector

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m\times 1}$$

Output vector

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1}$$

State variable vector

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1}.$$

The nth order time invariant differential equations are written as

$$x_1^{\circ} = (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n) + (b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m).$$

$$x_2^{\circ} = (a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n) + (b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m)$$

$$\vdots$$

$$x_n^{\circ} = (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n) + (b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m)$$

where $x_1^{\circ} = \frac{dx_1}{dt} = f_1(x_1, x_2 \dots x_n ; u_1, u_2 \dots u_m)$

$$x^{\circ}(t) = f(X(t), U(t))$$

$$\begin{bmatrix} x_1^{\circ}(t) \\ x_2^{\circ}(t) \\ \vdots \\ x_n^{\circ}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

The above matrix can be written as

$$\boxed{X^{\circ}(t) = AX(t) + BU(t).}$$

⇒ This equation is known as state equation.

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$ ⇒ system matrix (or) Evaluation matrix.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}_{n \times m} \Rightarrow$$ Input matrix (or) Control matrix.

$\dot{x} = (n \times 1)$ column matrix.

$x = (n \times 1)$ column matrix — state vector

$u = (m \times 1)$ column matrix — Input vector.

$A = (n \times n)$ = system matrix (or) Evaluation matrix

$B = (n \times m)$ = Input matrix (or) Control matrix.

Similarly the output variables are represented by

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

$$Y(t) = f[x(t), U(t)].$$

The above equations can be written in the matrix form is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

The above matrix can be written as,

$$\boxed{Y(t) = CX(t) + DU(t)} \rightarrow$$ This equation is called Output equation.

Where $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$ ⇒ output matrix (or) observation matrix.

$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \vdots \\ d_{pm} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$ ⇒ Direct transistion matrix.

$y(t)$ = output vector = $(p \times 1)$

$u(t)$ = Input vector = $(m \times 1)$

$x(t)$ = state vector = $(n \times 1)$

C = Output matrix (or) observation matrix = $(p \times n)$

D = Direct transistion matrix = $(p \times m)$.

By using the system, the dynamic equation can be written as

$$\boxed{\begin{array}{l} x^{\circ}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{array}}$$

→ ① - state equation

→ ② - Output equation

Where $x^{\circ}(t) = \frac{dx(t)}{dt}$.

$$\begin{bmatrix} \overset{\circ}{x}_1 \\ \overset{\circ}{x}_2 \\ \vdots \\ \overset{\circ}{x}_n \end{bmatrix}_{n\times 1} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} + B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{m\times 1}$$

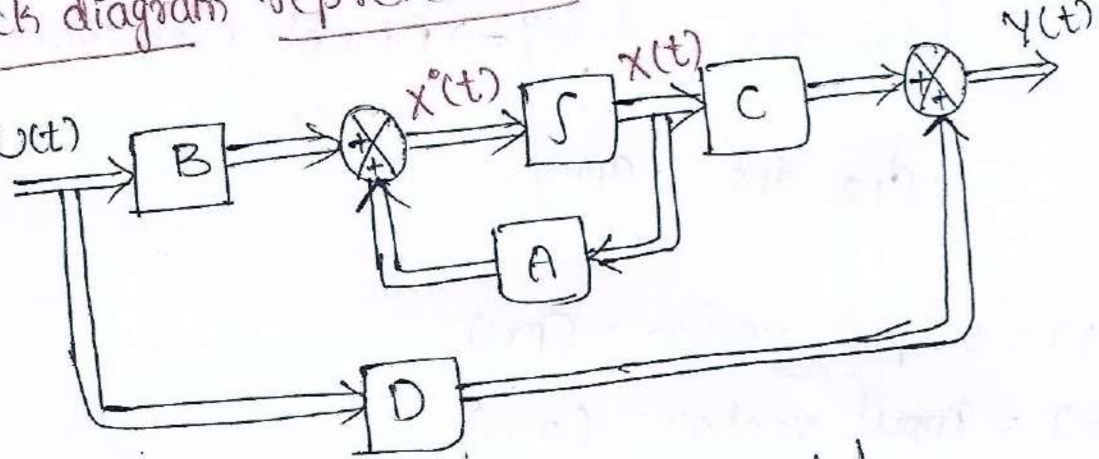
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}_{p\times 1} = C \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} + D \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}_{m\times 1}.$$

* The state model of linear time invariant system is given by

$\overset{\circ}{X}(t) = AX(t) + BU(t)$ → state equation

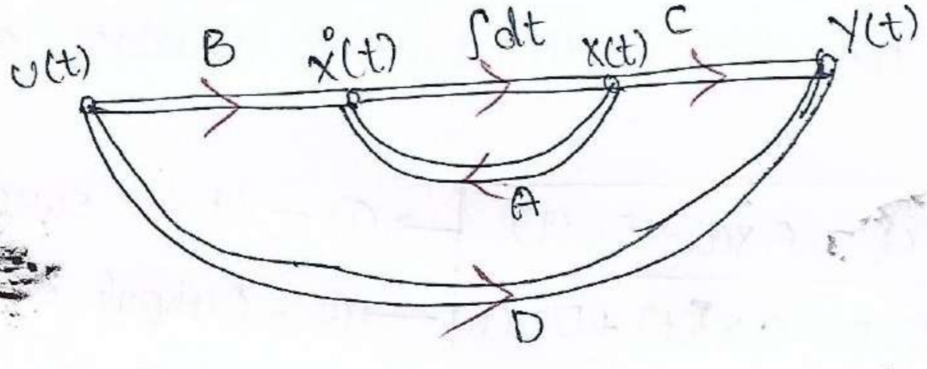
$Y(t) = CX(t) + DU(t)$ → Output equation.

Block diagram representation of state model :-



Block diagram of state model.

Signal flow graph representation of state model :-



Signal flow graph of state model.

Limitations of Transfer function :-

1. Applicable to only linear time-invariant systems only.
2. Only for single input & single output (SISO) systems.
3. Initial conditions must be zero.
4. Give analysis only for specific inputs like step, ramp etc.

Difference between Transfer function & State variable Approach :-

Transfer function	State variable
1. It is called conventional or classical approach.	1. It is the modern approach.
2. Depends upon I/P & O/P relation Transfer function.	2. Based on the description of system equations interms of 'n' first order differential equations.
3. Applicable to Linear-Time invariant single input single output (SISO)	③ Also applicable to Non-linear, time variant, multi input & multi output (MIMO).
④ Initial conditions are neglected	④ Initial conditions are considered.
⑤ It is frequency domain approach	⑤ It is time-domain approach.
⑥ Only i/p, o/p & error signals are considered. The i/p, o/p variables must be measurable.	⑥ The state variables need not represent physical variables They need not to be measurable.
⑦ Requires laplace transform for continuous & z-transform for discrete control system.	⑦ Formulates continuous & discrete data in same way.
⑧ Transfer function is unique.	⑧ State-model is non-unique.

Solution of State equation (or) dynamic equation :-

(i) Homogeneous solution of state equation (without input, without excitation) :-

Consider nth order of state equation of a linear time-invariant sys is

$$\dot{x}(t) = Ax(t) + BU(t) \rightarrow ①$$

Since, input is zero $u(t)=0$. [for a homogeneous (unforced sys)]

$$\dot{x}(t) = Ax(t).$$

Apply Laplace transform on bothsides.

$$sX(s) \overset{-x(0)}{=} AX(s).$$

Consider a initial condition at p=0.

$$sx(s) - x(0) = Ax(s).$$

$$sI\,x(s) - x(0) = Ax(s)$$

$$x(s)\left[sI - A\right] = x(0).$$

$$x(s) = \frac{1}{(sI-A)} x(0).$$

$$\boxed{x(s) = \left(sI-A\right)^{-1} x(0)} \rightarrow ②$$

Apply Laplace inverse to eq②.

$$\boxed{x(t) = L^{-1}\left[sI-A\right]^{-1} x(0)} \rightarrow ③$$

$$\boxed{x(t) = e^{At} x(0)}$$

Where $e^{At} = L^{-1}\left[(sI-A)^{-1}\right] = \phi(t).$

e^{At} = state transmission matrix (STM).—

$\Rightarrow [sI-A]^{-1}$ is called ~~rediuent~~ resolvant matrix denoted by $\phi(s)$.

Solution of Non-Homogeneous state equations:-

(Solution of state equations with input or excitation.)

The state equation of nth order system is given by

$$\dot{x}(t) = Ax(t) + BU(t).$$

The state equation can be rearranged as shown below

$$\dot{x}(t) - Ax(t) = BU(t)$$

Pre multiply both sides of above equation by e^{-At}.

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At} BU(t).$$

$$e^{-At}\dot{x}(t) + e^{-At}(-A)\,x(t) = e^{-At} BU(t). \longrightarrow (1)$$

Consider the differential eqⁿ of $e^{-At}x(t)$.

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}\dot{x}(t) + e^{-At}(-A)\dot{x}(t) \rightarrow ②$$

on comparing eq ① & ② we can write

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At} B\,U(t).$$

$$d\left(e^{-At}x(t)\right) = e^{-At} B\,U(t)\,dt. \longrightarrow ③$$

On integrating the eqⁿ ③ b/w limits '0' & 't'.

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau}.BU(\tau)\,d\tau \longrightarrow ④$$

where $x(0)$ = Initial condition vector.

τ = Dummy variable substituted for 't'.

Pre multiply both sides of eqⁿ(4) by e^{At}.

$$e^{At}.e^{-At}x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}.B\,U(\tau)\,d\tau.$$

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau} B.U(\tau)\,d\tau.$$

$$\boxed{\therefore X(t) = \underbrace{e^{At} x(0)}_{\text{Homogeneous solution}} + \underbrace{\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau}_{\text{Forced Solution}}}$$

State transistion Matrix:-

$$\phi(t) = e^{At} = L^{-1}\{(SI-A)^{-1}\} = L^{-1}[\phi(s)].$$

where $\boxed{\phi(s) = [SI-A]^{-1}}$ is called Resolvent matrix.

As e^{At} represents a power series of the matrix.

$$\phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

Properties of state transistion matrix:-

1. $\phi(0) = 1$

Proof: $\phi(t) = e^{At}$

put $t=0$

where $\phi(0) = e^{A(0)}$.

$$\boxed{\phi(0) = 1.}$$

2) $\phi^{-1}(t) = \phi(-t)$.

Proof: $\phi(t) = e^{At}$.

post multiply both sides by e^{-At}.

$$\phi(t)\, e^{-At} = e^{At} \cdot e^{-At}$$

(or) $\phi(t)\, \phi(-t) = I$.

Pre multiply both sides by $\phi^{-1}(t)$. $\qquad e^{-At} = \phi(-t)$.

$$\phi^{-1}(t)\,\phi(t)\,e^{-At} = I\,\phi^{-1}(t) \Rightarrow I\,\phi(-t) = \phi^{-1}(-t)$$

$$\boxed{\phi^{-1}(-t) = \phi(-t)}$$

③ $\phi(t_1+t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$.

Proof: $\phi(t_1+t_2) = e^{A(t_1+t_2)}$

$= e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\,\phi(t_2)$.

$\phi(t_1+t_2) = e^{A(t_1+t_2)}$

$= e^{At_2} \cdot e^{At_1}$

$= \phi(t_2)\phi(t_1)$

$$\boxed{\phi(t_1+t_2) = \phi(t_1)\,\phi(t_2) = \phi(t_2)\,\phi(t_1)}$$

④ $\phi(t)^n = \phi(nt)$; n = Integer.

Proof: $[\phi(t)]^n = [e^{At}]^n$.

$= e^{Ant}$.

$$\boxed{[\phi(t)]^n = \phi(nt)}$$

⑤ $\dot\phi(t) = A\phi(t)$.

Proof: $\phi(t) = e^{At}$.

$\dot\phi(t) = \frac{d}{dt}(e^{At})$

$\dot\phi(t) = Ae^{At}$.

$$\boxed{\dot\phi(t) = A\phi(t)}$$

⑥ $\phi(t_2-t_1)\,\phi(t_1-t_0) = \phi(t_2-t_0) = \phi(t_1-t_0)\,\phi(t_2-t_1)$.

Proof: $\phi(t_2-t_1)\,\phi(t_1-t_0) = e^{A(t_2-t_1)}\,e^{A(t_1-t_0)}$

$= e^{A(t_2-t_0)} = \phi(t_2-t_0)$

$\phi(t_1-t_0)\,\phi(t_2-t_1) = e^{A(t_1-t_0)}\,e^{A(t_2-t_1)}$

$= e^{A(t_2-t_0)} = \phi(t_2-t_0)$.

Transfer function of state model :-

Consider a state model

$$\dot{x}(t) = AX(t) + BU(t) \rightarrow \text{state eq}^n$$

$$Y(t) = CX(t) + DU(t) \rightarrow \text{output eq}^n.$$

Taking Laplace transform of above eq^{n}s.

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$Y(s) \quad = CX(s) + DU(s).$$

consider the eq^n.

$$(sI - A] X(s) = BU(s) + X(0)$$

Assume zero initial conditions i.e $X(0) = 0$, we get

$$(sI - A] X(s) = BU(s)$$

$$X(s) = (sI - A]^{-1} BU(s).$$

Put $X(s)$ from eq^n above. $Y(s)$

$$Y(s) = CX(s) + DU(s).$$

$$= C[(sI - A)^{-1} B\, U(s)] + DU(s).$$

$$Y(s) = U(s)[C(sI - A)^{-1} B + D]$$

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D.}$$

The above equation is called as transfer function of state model.

** State space Representation (SSR):-

$$\frac{dx(t)}{dt} = (A)_{n\times n} X(t) + [B]_{n\times p} U(t) \longrightarrow ① \text{ state equation.}$$

$$Y(t) = [C]_{q\times n} X(t) + [D]_{q\times p} U(t) \longrightarrow \text{Output equation.}$$

* Eigen values of matrix is the roots of the characteristic equation.

Where $X(t) = [x_1 \; x_2 \; x_3 \; \cdots \; x_n(t)]^T \rightarrow$ column vector.

$Y(t)$ = Output.

$U(t)$ = Input.

$[A]$ = system matrix.

* Eigen values of matrix A = roots of the characteristic eqⁿ.

Transfer function from the state space representation (SSR):-

$$TF = \frac{Y(s)}{U(s)} = C[SI-A]^{-1}B + D.$$

$|SI-A| = 0 \longrightarrow$ characteristic equation.

Note: state space representation of a system is not unique.

Some of the SSR's are.

(i) Controllable canonical form (CCF)

2) Observable canonical form (OCF)

3) Normal form $\longrightarrow$ Diagonal Canonical Form (DCF)
(Distinct Eigen Values)
$\rightarrow$ Jordon Canonical Form (JCF)
(Multiple / Repeated eigen values).

Transfer Function -SSR - OCF / CCF : $TF = \frac{Y(s)}{U(s)} = \frac{b[c_3 s^3 + c_2 s^2 + c_1 s + c_0]}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$

Controllable canonical Form (SSR-CCF)

$$\underset{X^{\circ}}{\begin{bmatrix} x_1^{\circ} \\ x_2^{\circ} \\ x_3^{\circ} \\ x_4^{\circ} \end{bmatrix}} = \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} + \underset{B}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix}} \underset{U}{U(t)}$$

$$y(t) = \underset{C}{\begin{bmatrix} C_0 & C_1 & C_2 & C_3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

$$X^{\circ} = AX + BU$$
$$Y = CX$$

Observable Canonical form (SSR-OCF)

$$X^{\circ}(t) = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} U(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & b \end{bmatrix} x(t).$$

Diagonal canonical form (SSR-DCF):

$$TF = \frac{Y(s)}{U(s)} = \frac{(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)(s+P_3)}.$$

$$= \frac{b_1}{s+P_1} + \frac{b_2}{s+P_2} + \frac{b_3}{s+P_3}.$$

$$X^{\circ}(t) = \begin{bmatrix} -P_1 & 0 & 0 \\ 0 & -P_2 & 0 \\ 0 & 0 & -P_3 \end{bmatrix} X(t) + \begin{bmatrix} K_1 \\ l_1 \\ m_1 \end{bmatrix} U(t).$$

$$y(t) = \begin{bmatrix} K_2 & l_2 & m_2 \end{bmatrix} X(t).$$

Where $b_1 = K_1 K_2$, $b_2 = l_1 l_2$, $b_3 = m_1 m_2$.

Jordon Canonical form:- Matrix (A).

If roots | eigen values are repeated | same values.

ex: -2, -2, -3.

$$A = \left[\begin{array}{c:cc} -3 & 0 & 0 \\ \hdashline 0 & -2 & 1 \\ 0 & 0 & -2 \end{array}\right]$$

→ Jordon block. (JB)

eg: If the roots are -2, -2, -2

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

① Construct the state model using the phase variable if the system is described by the following differential eqns.

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t).$$

sol Let $y(t) = x_1(t) \longrightarrow ①$

$y'(t) = x_2(t) = \dot{x}_1(t) \longrightarrow ②$

$y''(t) = x_3(t) = \dot{x}_2(t) = \ddot{x}_1(t) \longrightarrow ③$

$\dddot{y}(t) = x_4(t) = \dot{x}_3(t) = \ddot{x}_2(t) = \dddot{x}_1(t) \longrightarrow ④$

$x_3'(t) + 4x_3(t) + 7x_2(t) + 2x_1(t) = 5u(t)$ —

$\dot{x}_3(t) = 5u(t) - 4x_3(t) - 7x_2(t) - 2x_1(t) \longrightarrow ⑤$

$\dot{x}_2(t) = x_3(t) \longrightarrow ⑥$

$\dot{x}_1(t) = x_2(t) \longrightarrow ⑦$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t).$$

② Find the state transistion matrix (STM) for $A = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}$

Sol) State transistion matrix $= e^{At} = L^{-1}[sI-A]^{-1}$.

$$(sI-A) = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} s & 1 \\ -2 & s+3 \end{pmatrix}$$

$$(SI-A)^{-1} = \begin{pmatrix} S+3 & -1 \\ 2 & S \end{pmatrix} \frac{1}{S(S+3)-(-2)}$$

$$= \frac{1}{S(S+3)+2} \begin{bmatrix} S+3 & -1 \\ 2 & S \end{bmatrix}$$

$$= \frac{1}{S^2+3S+2} \begin{bmatrix} S+3 & -1 \\ 2 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(S+3)}{(S+1)(S+2)} & \frac{-1}{(S+1)(S+2)} \\ \frac{2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

1) $\frac{S+3}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{(S+2)}$

$(S+3) = A(S+2) + B(S+1).$

put $S=-1 \Rightarrow (-1+3) = A(-1+2)+B(0) \Rightarrow \boxed{A=2}$.

put $S=-2 \Rightarrow (-2+3) = A(-2+2)+B(-2+1) \Rightarrow \boxed{B=-1}$

2) $\frac{S+3}{(S+2)(S+1)} = \frac{2}{(S+1)} + \frac{(-1)}{(S+2)}$

2) $\frac{-1}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{(S+2)}$

$-1 = A(S+2) + B(S+1)$

put $S=-2 \Rightarrow -1 = A(0)+B(-2+1) \Rightarrow \boxed{B=1}$

put $S=-1 \Rightarrow -1 = A(-1+2)+B(0) \Rightarrow \boxed{A=-1}$

$\frac{-1}{(S+1)(S+2)} = \frac{-1}{(S+1)} + \frac{1}{(S+2)}$

3) $\frac{2}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{(S+2)}$

$2 = A(S+2) + B(S+1)$

put $S=-1 \Rightarrow 2 = A(-1+2)+B(0) \Rightarrow \boxed{A=2}$

put $S=-2 \Rightarrow 2 = A(0)+B(-2+1)$

$\boxed{B=-2}$

$$\frac{2}{(S+1)(S+2)} = \frac{2}{(S+1)} + \frac{(-2)}{(S+2)}.$$

4) $$\frac{S}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$S = A(S+2) + B(S+1)$$

put $S=-1 \Rightarrow -1 = A(-1+2)+B(0) \Rightarrow \boxed{A=-1}$

put $S=-2 \Rightarrow -2 = A(0)+B(-2+1) \Rightarrow \boxed{B=2}$

$$\frac{S}{(S+1)(S+2)} = \frac{-1}{S+1} + \frac{2}{S+2}$$

$$(SI-A)^{-1} = \begin{pmatrix} \frac{-1}{S+2} + \frac{2}{S+1} & \frac{-1}{S+1} + \frac{1}{S+2} \\ \frac{-2}{S+2} + \frac{2}{S+1} & 2/S+2 - \frac{1}{S+1} \end{pmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} -1e^{-2t} + 2e^{-t} & -1e^{-t} + 1e^{-2t} \\ -2e^{-2t} + 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} = e^{At}.$$

② A Linear time Invariant system is characterised by homogeneous equation Assuming the initial state vector as $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ compute the solution of homogeneous eq^n

Sol $x(t) = e^{At} x(0).$

$$(SI-A) = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s-1)(s-1)} \begin{pmatrix} s-1 & 0 \\ +1 & s-1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(s-1)}{(s-1)^2} & 0 \\ \frac{1}{(s-1)^2} & \frac{(s-1)}{(s-1)^2} \end{pmatrix}$$

$$\frac{1}{(s-1)^2} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} \Rightarrow 1 = A(s-1)+B.$$

put $s=1 \Rightarrow \boxed{B=1}$

put $s=0, B=1 \Rightarrow 1 = A(-1)+1 \Rightarrow \boxed{A=0}$

$$\frac{1}{(s-1)^2} = \frac{0}{(s-1)} + \frac{1}{(s-1)^2} = \frac{1}{(s-1)^2}$$

$$L^{-1}(sI-A)^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}$$

$$x(t) = e^{At}x(0)$$

$$= \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^t + 0 \\ te^t + 0 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}$$

③ Given $\begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \theta(t)$. Find the unit step response when $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the Non-homogeneous solution.

Sol

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau.$$

$$= e^{At}x(0) + L^{-1}\{\phi(s)\,B\,U(s)\} \longrightarrow ①$$

$$x(t) = x_1(t) + x_2(t).$$

$$x_1(t) = e^{At}x(0)$$

$$x_2(t) = L^{-1}[\phi(s)BU(s)]$$

$$x_1(t) = L^{-1}(sI-A)^{-1}x(0).$$

$$(sI-A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}$$

$$(sI-A)^{-1} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} = \frac{1}{s(s+3)+2}\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} = \frac{1}{s^2+3s+2}\begin{pmatrix} s+3 & 1 \\ 2 & s \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{pmatrix}$$

⇒ Apply Partial fractions

$$= \begin{pmatrix} \frac{-1}{s+2} + \frac{2}{s+1} & \frac{-1}{s+2} + \frac{1}{s+1} \\ \frac{2}{s+2} - \frac{2}{s+1} & \frac{2}{s+2} - \frac{1}{s+1} \end{pmatrix}$$

$$L^{-1}(sI-A)^{-1} = \begin{pmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{pmatrix}$$

$$e^{At}x(0) = L^{-1}[sI-A]^{-1}x(0) = \begin{pmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-2t}+2e^{-t}-e^{-2t}+e^{-t} \\ 2e^{-t}-2e^{-t}+2e^{-t}-e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-2t}+3e^{-t} \\ 4e^{-2t}-3e^{-t} \end{pmatrix}$$

$$x_2(t) = L^{-1}\left[\phi(s)\, B \cdot U(s)\right]$$

$$\phi(s) = (sI - A)^{-1}$$

$$= \begin{bmatrix} \frac{(s+3)}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}$$

$$\phi(s)\, B\, U(s) = \begin{bmatrix} \frac{(s+3)}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(s+3)}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s}{(s+2)(s+1)} \\ \frac{s}{(s+2)(s+1)} \end{bmatrix} \left[1/s\right] \qquad \left[\begin{matrix} \because u(t) = 1 \\ U(s) = 1/s \end{matrix}\right]$$

$$= \begin{bmatrix} \frac{1}{(s+2)(s+1)}\left(\frac{1}{s}\right) \\ \frac{1}{(s+2)(s+1)(s)} \end{bmatrix} (\text{P.F})$$

$$\phi(s)\, B\, U(s) = \begin{bmatrix} \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$x_2(t) = L^{-1}\left[\phi(s)\, B\, U(s)\right]$$

$$= \begin{bmatrix} 0.5 - 1e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$x(t) = x_1(t) + x_2(t) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 + 2e^{-t} - 1.5e^{-2t} \\ 3e^{-2t} - 2e^{-t} \end{bmatrix}$$

③ closed loop T.F $\frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3+8s^2+192s+640}$. obtain the state variable model using signal flow graph.

Sol

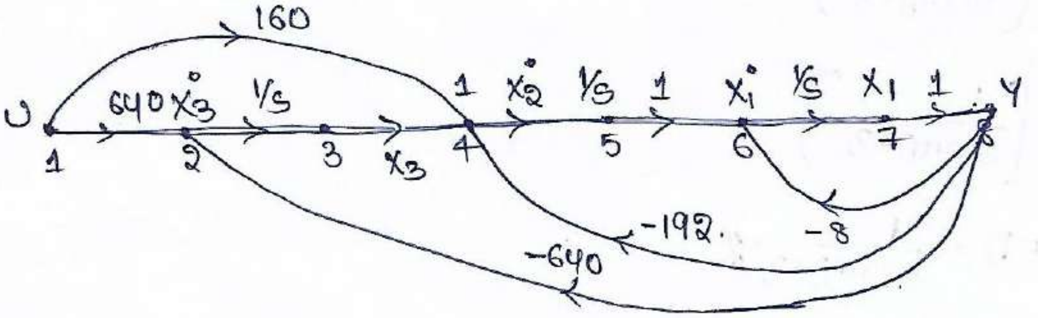
$$\frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3+8s^2+192s+640}$$

$$= \frac{160(s+4)/s^3}{\frac{s^3+8s^2+192s+640}{s^3}}$$

$$= \frac{160/\left(s^2+\frac{640}{s^3}\right)\text{(N-paths)}}{1-\left(-8/s-\frac{192}{s^2}-\frac{640}{s^3}\right)\text{(Loops)}}$$

No.of nodes equal to highest degree x2 + input node + output node

$3x_2 = 6+1+1 = 8.$



Let x_1, x_2, x_3 be 3 state variables from the above signal flowgraphs weget Let $\boxed{Y=x_1}$

$$\dot{x}_1 = x_2 - 8\cancel{y} = x_2 - 8x_1$$

$$\dot{x}_2 = x_3 - 192x_1 + 160U.$$

$$\dot{x}_3 = -640x_1 + 640U.$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-8 & 1 & 0\\ -192 & 0 & 1\\ -640 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 160\\ 640\end{bmatrix}U.$$

$$Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

④ A single input & single output system has the matrix equation $\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [1 \ 0]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ find the T.F.

Sol

$$T.F = C(SI-A)^{-1}B + D$$

$$|SI-A| = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 3 & s+4 \end{pmatrix}$$

$$(SI-A)^{-1} = \frac{1}{S(S+4)+3}\begin{pmatrix} s+4 & +1 \\ -3 & s \end{pmatrix}$$

$$(SI-A)^{-1}B = \frac{1}{S(S+4)+3}\begin{bmatrix} s+4 & +1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(SI-A)^{-1}B \cdot C = \begin{pmatrix} \frac{1}{S(S+4)+3} \\ \frac{s}{S(S+4)+3} \end{pmatrix}[1 \ 0]$$

$$= \left[\frac{1}{S(S+4)+3}\right]$$

$$C(SI-A)^{-1}B + D = \frac{1}{S(S+4)+3} \,//$$

⑤ Transfer function of (a system) a certain 2nd order system are ~~$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -4 & -a \end{pmatrix}x$~~ $\frac{1}{(s^2+5s+4)}$. The state space representation is

$\dot{x}(t) = Ax(t) + BU(t)$

$y(t) = Cx(t)$. Find the eigen values of a matrix 'A'.

Sol Given that $TF = \frac{Y(s)}{U(s)} = \frac{(s+z_1)(s+z_2)}{(s+P_1)(s+P_2)} = \frac{b_1}{s+P_1} + \frac{b_2}{s+P_2}$.

$$Tf = \frac{1}{s^2+5s+4} = \frac{1}{(s+1)(s+4)} = \frac{A}{(s+1)} + \frac{B}{s+4}.$$

$1 = A(s+4) + B(s+1)$

$s=-1 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

$s=-4 \Rightarrow B = -\frac{1}{3}$.

Roots of characteristic equation is called as eigen values of a matrix. $s = -1, -4$.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

① A state variable system $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$ with the initial condition $x(0) = [-1 \;\; 3]^T$ and unit step input $u(t)$.

Find state transistion matrix & state transistion equation.

Sol: state transistion matrix $\phi(t) = L^{-1}(SI-A)^{-1}$

$$(SI-A) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 0 & S+3 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{S(S+3)-0} \cdot \begin{bmatrix} S+3 & 1 \\ 0 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cancel{S+3}}{S(\cancel{S+3})} & \frac{1}{S(S+3)} \\ \frac{0}{S(S+3)} & \frac{\cancel{S}}{\cancel{S}(S+3)} \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{1}{S} & \frac{1}{S(S+3)} \\ 0 & \frac{1}{S+3} \end{bmatrix}$$

$$\frac{1}{S(S+3)} = \frac{A}{S} + \frac{B}{S+3}$$

$1 = A(S+3) + BS$

$S=0 \Rightarrow A = 1/3$

$S=-3 \Rightarrow B = -1/3$

$$= \begin{bmatrix} \frac{1}{S} & \frac{1}{3S} - \frac{1}{3(S+3)} \\ 0 & \frac{1}{S+3} \end{bmatrix}$$

$$\phi(t) = L^{-1}(SI-A)^{-1} = L^{-1}\begin{bmatrix} \frac{1}{S} & \frac{1}{3S} - \frac{1}{3(S+3)} \\ 0 & \frac{1}{S+3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{bmatrix}$$

state transistion equation $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\, U(\tau) d\tau$.

$U(\tau)$ = unit step input.

$$x(t) = \begin{bmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & \frac{1}{3}(1-e \\ 0 & e^{-3(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1)\,d\tau$$

$$= \begin{bmatrix} -1+\frac{\cancel{3}}{\cancel{3}}(1-e^{-3t}) \\ 3e^{-3t} \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau.$$

$$= \begin{bmatrix} -e^{-3t} \\ 3e^{-3t} \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau.$$

$$= \begin{bmatrix} -e^{-3t} \\ 3e^{-3t} \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} t-e^{-3t} \\ 3e^{-3t} \end{bmatrix}$$

② The system dynamics is represented by the state space equations. The system dynamics is represented by the transfer function.

$\dot{x} = \begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix}x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$ & $y = [1 \ 1]\,x$. find the transfer function.

$$2 = \frac{A}{S-2}+\frac{B}{S-4}$$

$$2 = A(S-4)+B(S-2)$$

put $s=2 \Rightarrow$

Sol

$$TF = \frac{Y(S)}{U(S)} = C(SI-A)^{-1}B+D.$$

$$(SI-A) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} S-4 & 0 \\ 2 & S-2 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{(S-2)(S-4)}\begin{bmatrix} S-2 & 0 \\ -2 & S-4 \end{bmatrix} = \begin{bmatrix} \frac{S-2}{(S-2)(S-4)} & 0 \\ \frac{-2}{(S-2)(S-4)} & \frac{(S-4)}{(S-2)(S-4)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(S-4)} & 0 \\ \frac{-2}{(S-2)(S-4)} & \frac{1}{S-2} \end{bmatrix}$$

$$\frac{2}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4} = \frac{-1}{s-2} + \frac{1}{s-4}$$

$$2 = A(s-4) + B(s-2)$$

put $s=2 \Rightarrow -2 = A(-2) \Rightarrow \boxed{A=+1}$

$s=4 \Rightarrow -2 = B(2) \Rightarrow \boxed{B=1}$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-4} & 0 & 0 \\ \frac{-1}{s-2}+\frac{1}{s-4} & & \frac{1}{s-2} \end{bmatrix}$$

$$TF = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D.$$

$$= [1 \quad 1]\begin{bmatrix} \frac{1}{s-4} & 0 \\ \frac{-1}{s-2}+\frac{1}{s-4} & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + [0)$$

$$= [1 \quad 1]\begin{bmatrix} \frac{1}{s-4} \\ \frac{-1}{s-2}+\frac{-1}{s-4} \end{bmatrix}$$

$$= \left(\frac{1}{s-4} + \frac{1}{s-2} - \frac{1}{s-4}\right]$$

$$\boxed{TF = \frac{1}{s-2}}$$

① A state variable formation of a system is given by the equations.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u \quad ; \quad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

find the transfer function.

Sol Given that. $A = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, $B = \begin{bmatrix}1\\ 1\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\end{bmatrix}$

Transfer function $\frac{Y(S)}{X(S)} = T(S) = C(SI-A)^{-1}B+D$

$$SI-A = \begin{pmatrix}S & 0\\ 0 & S\end{pmatrix} - \begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix} = \begin{pmatrix}(S+1) & 0\\ 0 & S-3\end{pmatrix}$$

$$(SI-A)^{-1} = \frac{1}{(S+1)(S-3)}\begin{pmatrix}S-3 & 0\\ 0 & S+1\end{pmatrix}$$

$$\frac{Y(S)}{X(S)} = \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}S-3 & 0\\ 0 & S+1\end{bmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}}{(S+1)(S-3)}$$

$$= \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{pmatrix}S-3\\ S+1\end{pmatrix}}{(S+1)(S-3)}$$

$$= \frac{S-3}{(S+1)(S-3)}$$

$$\boxed{\frac{Y(s)}{X(S)} = \frac{1}{S+1}}$$

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⑨ A state variable formulation of a system is given by the equations

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}u \;;\; x_1(0) = x_2(0) = 0$$

$$y = [1\ \ 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

find the response y(t) to a unit-step input.

Sol Given that $A = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, $B = \begin{bmatrix}1\\ 1\end{bmatrix}$, $C = [1\ \ 0]$

$$\frac{Y(S)}{X(S)} = T(S) = C[SI-A]^{-1}B + D$$

$$(SI-A) = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}S+1 & 0\\ 0 & S-3\end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{(S+1)(S-3)}\begin{bmatrix}S-3 & 0\\ 0 & S+1\end{bmatrix}$$

$$\frac{Y(S)}{X(S)} = \frac{[1\ \ 0]\begin{bmatrix}S-3 & 0\\ 0 & S+1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}}{(S+1)(S-3)} = \frac{[1\ \ 0]\begin{bmatrix}S-3\\ S+1\end{bmatrix}}{(S-3)(S+1)} = \frac{\cancel{S-3}}{\cancel{(S-3)}(S+1)}$$

$$\frac{Y(S)}{X(S)} = \frac{1}{S+1}$$

Given $X(S) = \frac{1}{S}$

$$Y(S) = X(S)\left(\frac{1}{S+1}\right) = \frac{1}{S}\left(\frac{1}{S+1}\right) = \frac{A}{S} + \frac{B}{S+1}$$

$$\frac{1}{S(S+1)} = \frac{A}{S} + \frac{B}{S+1}$$

$1 = A(S+1) + BS$

$S = 0 \Rightarrow 1 = A$

$S = -1 \Rightarrow 1 = -B \Rightarrow B = -1$

$$Y(S) = \frac{1}{S} - \frac{1}{S+1}$$

Apply inverse laplace transform

$$L^{-1}(Y(S)) = L^{-1}\left[\frac{1}{S} - \frac{1}{S+1}\right] = \boxed{Y(t) = 1 - e^{-t}}$$

5) Find the eigen values of a matrix, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{pmatrix}$

Sol

$|SI - A| = 0 \rightarrow$ roots are called as eigen values of matrix A.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{pmatrix}$$

$$SI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{bmatrix}$$

$$|SI - A| = 0 \Rightarrow \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{vmatrix} = 0$$

$$s\left[(s(s+4)+3)\right] + 1(0) + 0(0) = 0$$

$$s\left[s^2 + 4s + 3\right] = 0$$

$$s = 0, \quad s^2 + 4s + 3 = 0$$

$$(s+3)(s+1) = 0.$$

Roots are $s = -1, -3, 0$.

④ Given the system $\dot{x} = \begin{bmatrix} 0 & 0 & -20 \\ 1 & 0 & -24 \\ 0 & 1 & -9 \end{bmatrix} x + \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} u \; ; \; y = (0 \; 0 \; 1) x$ For a mod

Find the characteristic equation?

Sol characteristic equation $|sI-A| = 0$.

$$|sI-A| = \begin{vmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 0 & -20 \\ 1 & 0 & -24 \\ 0 & 1 & -9 \end{vmatrix} = 0$$

$$\begin{vmatrix} s & 0 & 20 \\ -1 & s & 24 \\ 0 & -1 & s+9 \end{vmatrix} = 0.$$

$$s(s(s+9)+24) - 0((s+9)-0) + 20(1-0) = 0$$

$$s(s^2+9s+24) + 20 = 0.$$

$$s^3+9s^2+24s+20 = 0.$$

Characteristic equation is $s^3+9s^2+24s+20 = 0$.

IInd method

By SSR-OCF

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} u(t) \quad ; \; y(t) = (0 \; 0 \; b) x(t)$$

$$T.F = \frac{Y(s)}{U(s)} = \frac{b(C_2 s^2 + C_1 s + C_0)}{s^3 + a_2 s^2 + a_1 s + a_0}$$

T.F = From above eqn $a_0 = 20, a_1 = 24, a_2 = 9, b = 1, C_0 = 3, C_1 = 1, C_2 = 0$.

$$T.F = \frac{1(s+3)}{s^3+9s^2+24s+20}$$

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For a dynamic system, various matrices for a state space model are given by $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \; 0]$. Find the system poles?

Sol system poles are determined by the system matrix, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

poles are given by $|sI - A| = 0$

$$\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = 0 \Rightarrow s(s+3)+2=0$$

$$s^2+3s+2=0$$

$$(s+1)(s+2)=0$$

poles $p_1=-1$, $p_2=-2$.

⑥ A state space representation for all the transfer function $\frac{Y(s)}{U(s)} = \frac{s+6}{s^2+5s+6}$ is $\dot{x} = Ax+BU$, $Y=CX$ where $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & find value of $C=$?

Sol Given that $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$(sI-A)^{-1} = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}^{-1} = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \frac{1}{s(s+5)+6}.$$

$$(sI-A)^{-1}B = \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} \begin{pmatrix} 1 \\ s \end{pmatrix}$$

$$T.F = C(sI-A)^{-1}B = [C_1 \; C_2] \begin{pmatrix} 1 \\ s \end{pmatrix} \frac{1}{(s^2+5s+6)} = \frac{C_1+C_2 s}{s^2+5s+6} \rightarrow ①$$

$$T.F = \frac{s+6}{s^2+5s+6} = \frac{C_1+C_2 s}{s^2+5s+6}$$

$C_1 = 6, C_2 = 1$

(or) SSR - CCF

$$T.F = \frac{b(C_3 s^3+C_2 s^2+C_1 s+C_0)}{s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

$$\overset{\circ}{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u \quad ; y = [c_0 \; c_1 \; c_2]\, x$$

Given $T.F = \frac{s+6}{s^2+5s+6}$

$a_0 = 6, a_1 = 5, a_2 = 1, b = 1, C_0 = 6, C_1 = 1$

So $Y = [6 \;\; 1]$

⑦ A linear time-invariant single input ~~input~~ single output system has a state space model given by

$$\frac{dx}{dt} = Fx+Gu \; ; y = Hz \quad \text{where } F = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} ; G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = [1 \;\; 0]$$

Here 'x' is the state vector, u is the input & y is the output.

find the damping ratio of the system?

Sol poles of the system obtained from the system matrix 'F'.

$$|sI - F| = 0 \Rightarrow \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} s & -1 \\ s+4 & s+2 \end{vmatrix} = 0$$

$$s(s+2)+4 = 0 \Rightarrow s^2+2s+4.$$

$$\omega_n^2 = 4, \; \omega_n = 2. \quad 2\xi\omega_n = 2 \Rightarrow \xi = 2/2\omega_n = \frac{\cancel{2}}{2\times\cancel{2}} = 0.5.$$

Find the transfer function of the system described by the state-space equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \ ; \ y = [1 \ \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol

$$T.F = \frac{Y(s)}{X(s)} = C(sI-A)^{-1}B + D \quad ; \ A = \begin{bmatrix} -4 & -1 \\ -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [1 \ \ 0]$$

$$(sI-A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{bmatrix} -4 & -1 \\ -3 & -1 \end{bmatrix} = \begin{pmatrix} s+4 & 1 \\ 3 & s+1 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{\cancel{s+1} \ 1}{(s+1)(s+4)-3} \begin{bmatrix} s+1 & -1 \\ -3 & s+4 \end{bmatrix} = \frac{1}{s^2+5s+1} \begin{pmatrix} s+1 & -1 \\ -3 & s+4 \end{pmatrix}$$

$$C(sI-A)^{-1}B = (1 \ \ 0) \frac{1}{s^2+5s+1} \begin{pmatrix} s+1 & -1 \\ -3 & s+4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{s^2+5s+1} [1 \ \ 0] \begin{pmatrix} s+1-1 \\ -3+ \\ s+4 \end{pmatrix}$$

$$T.F = \frac{[1 \ \ 0]}{s^2+5s+1} \begin{pmatrix} s \\ s+1 \end{pmatrix} = \frac{s}{s^2+5s+1}$$

⑨ The SSR & TF of a system is $\frac{dx(t)}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} U(t)$

$Y(t) = [C_1 \ C_2] x(t)$, $TF = [C_1 \ C_2] x(t)$

$TF = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$ the elements C_1 & C_2 respectively.

Sol

$$TF = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$1 = A(s+2) + B(s+1)$

$s = -1 \Rightarrow \boxed{1 = A}$

$s = -2 \Rightarrow B = -1$

$$TF = \frac{1}{s+1} - \frac{1}{s+2}$$

$Y = X_1 + X_2$; $X_1 = \frac{1}{s+1}$, $X_2 = \frac{-1}{s+2}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$Y = (1 \ \ 1] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\therefore C_1 = 1, C_2 = 1$.

IInd method

SSR - DCF

$$TF = \frac{(S+Z_1)(S+Z_2)}{(S+P_1)(S+P_2)(S+P_3)} = \frac{b_1}{S+P_1} + \frac{b_2}{S+P_2} + \frac{b_3}{S+P_3}.$$

$$\overset{\circ}{X}(t) = \begin{bmatrix} -P_1 & 0 & 0 \\ 0 & -P_2 & 0 \\ 0 & 0 & -P_3 \end{bmatrix} X(t) + \begin{pmatrix} K_1 \\ l_1 \\ m_1 \end{pmatrix} U(t) \quad ; \; Y(t) = \begin{bmatrix} K_2 & l_2 & m_2 \end{bmatrix} X(t)$$

where $b_1 = K_1K_2$, $b_2 = l_1l_2$, $b_3 = m_1m_2$.

$C_1 = [1 \quad 1]$.

(10) A first order matrix differential equation of a system is given as $\begin{bmatrix} \overset{\circ}{x}_1 \\ \overset{\circ}{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$, $y = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(a) Find the transfer function of the system (b) Find the solution of the states & the output when the input is a unit step & the initial condition of the states is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Sol) $T.F = G(s) = C(SI-A)^{-1}B + D.$

$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $C = [1 \quad 1]$

$$SI - A = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} S & -1 \\ 2 & S+3 \end{pmatrix}$$

$$(SI-A)^{-1} = \frac{1}{S(S+3)+2} \begin{pmatrix} S+3 & 1 \\ -2 & S \end{pmatrix} = \frac{1}{S^2+3S+2} \begin{pmatrix} S+3 & 1 \\ -2 & S \end{pmatrix}.$$

$$C(SI-A)^{-1}B = [1 \quad 1] \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{S^2+3S+2} = \frac{[1 \quad 1]}{S^2+3S+2} \begin{bmatrix} 2 \\ 2S \end{bmatrix}$$

$$T.F = \frac{2+2S}{(S+1)(S+3)} = \frac{2\cancel{(S+1)}}{\cancel{(S+1)}(S+3)} = \frac{2}{S+3}.$$

(b)ii)

SSR - CCF

$$T.f = \frac{b(c_2 s^2 + c_1 s + c_0)}{s^3 + a_2 s^2 + a_1 s + a_0} \quad \dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} U \; ; y = [c_0 \; c_1 \; c_2] x$$

$a_0 = 2, a_1 = 3, c_0 = 1, c_1 = 1, b = 2.$

$$T.f = \frac{2(s+1)}{s^2 + 3s + 2} = \frac{2\cancel{(s+1)}}{\cancel{(s+1)}(s+2)} = \frac{2}{s+2}$$

Given that $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u(t)$ = unit step - $1/s$

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = (sI - A)^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + (sI - A)^{-1} B \, U(s).$$

$$= \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{bmatrix} + \begin{bmatrix} \frac{2}{s(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} \end{bmatrix}$$

$Y(s) = X_1(s) + X_2(s).$

$y(t) = x_1(t) + x_2(t).$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \Rightarrow \frac{s}{s+1(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}$$

$$X_1(s) = \frac{1}{s+1} - \cancel{\frac{1}{s+2}} + \frac{1}{s} - \frac{2}{s+1} + \cancel{\frac{1}{s+2}} = \frac{1}{s} - \frac{1}{s+1}$$

$$X_2(s) = -\frac{1}{s+1} + \frac{2}{s+2} + \frac{2}{s+1} - \frac{2}{s+2} = \frac{1}{s+1}.$$

$x_1(t) = 1\,u(t) - e^{-t}u(t) \; ; \; x_2(t) = e^{-t}u(t)$

$y(t) = x_1(t) + x_2(t) = u(t) - e^{-t}u(t) + e^{-t}u(t) = u(t).$

** Concepts of Controllability and Observability:-

Controllability:-

The controllability verifies the usefulness of a state variable. In the controllability test we can find whether the state variable can be controlled to achieve the desired output.

* The choice of a state variables is arbitrary while forming the state model. After determining the state model, the controllability of the state variable is verified.

* If the state variable is not controllable then we have to ~~(find)~~ go for another choice of state variable.

* All the state variables are controllable then system is controllable.

* If pole-zero cancellation occur in the system it is not controllable & observable.

* If pole-zero cancellation is not occur in the system then the system is pure controllable & observable.

** Definition of Controllability:-

"A system is said to be completely state Controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_d)$ in the specified finite time by a Control vector $U(t)$".

* System is controllable, if it is possible to transform the initial state $x(t_0)$ to any desired state $x(t)$ by applying the input.

The controllability of a state model can be tested by kalman's test or Gilbert's test.

Gilbert's method of testing Controllability:-

Case (i): When the system matrix has distinct eigenvalues.

In this case the system matrix can be diagonalized and the state model can be converted to canonical form.

Consider the state model of a system.

$$\dot{X} = AX + BU$$
$$Y = CX + DU.$$

The state model can be converted to canonical form by a transformation, $X = MZ$, where 'M' is the modal matrix and 'Z' is the transformed state variable vector.

The transformed state model is given by

$$\dot{Z} = AZ + \tilde{B}U \qquad \text{Where } A = M^{-1}AM.$$
$$Y = \tilde{C}Z + DU. \qquad \tilde{B} = M^{-1}B$$
$$\tilde{C} = CM.$$

***

* In this case the necessary and sufficient condition for complete controllability is that, the matrix $\tilde{B}$ must have no rows with all zeros.

* If any row of the matrix $\tilde{B}$ is the zero then the corresponding state variable is ~~contro~~ uncontrollable.

Case (ii) :-

In this case, the system matrix cannot be diagonalized but can be transformed to Jordon canonical form.

Consider the state model of the system,

$$\overset{\circ}{x} = AX + BU$$
$$Y = CX + DU$$

* The state model can be transformed to Jordon canonical form by transformation, $x = MZ$. where M = Modal matrix
Z = transformed state variable vector.

$$\overset{\circ}{Z} = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

where $J = M^{-1}AM$
$\tilde{B} = M^{-1}B$
$\tilde{C} = CM$.

***

* In this case, the system is completely controllable if the elements of any row of $\tilde{B}$ that correspond to the last row of each jordon block are not all zero & the rows corresponding to other state variables must not have all zeros.

Kalaman's test :-

Consider a system with state equation, $\overset{\circ}{x} = AX + BU$.

$$M = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

where 'n' is the order of the system ('n' is also equal to number of state variables).

* If the system is completely state controllable, the rank of the composite matrix M is 'n'.

* For controllability $|M| \neq 0$, 'M' must be non-singular.

Advantage:- calculations are simpler.

Disadvantage:- We can't find the state variable which is uncontrollable.

Condition for completestate Controllability :- in the S-plane :-

- A necessary and sufficient condition for complete state controllability is that no cancellation of poles & zeros occur in the transfer function of the system.
- If cancellation occurs then the system cannot be controlled in the direction of cancelled mode.

Observability :-

- In observability test we can find whether the state variable is observable or measurable.
- It is useful in solving the problem of reconstructing unmeasurable state variables from measurable ones in the minimum possible length of time.
- In state feedback control the estimation of unmeasurable state variable is essential in order to construct the control signals.

Definition of observability :-

A system is said to be completely observable if every state x(t) can be completely identified by measurments of the output y(t) over a finite time interval.

The observability of a system can be tested by either Gilbert's test or Kalman's test.

Gilberts test :-

Consider a state model of n^{th} order system

$$X^{\circ} = AX + BU, \quad Y = CX + DU.$$

The state model can be transformed to a canonical or Jordon canonical form by a transformation,

$X = MZ$ where M = Modal matrix, Z = Transformed state variable vector.

The transformed state model is,

$$\dot{Z} = AZ + \tilde{B}U \qquad (\text{or}) \qquad \dot{Z} = JZ + \tilde{B}U$$
$$Y = \tilde{C}Z + DU. \qquad\qquad Y = \tilde{C}Z + DU.$$

where $A = M^{-1}AM$: eigen values are distinct., $\tilde{B} = M^{-1}B$.

$J = M^{-1}AM$; eigen values are multiple, $\tilde{C} = CM$.

* The necessary & sufficient condition for complete observability is that none of the columns of the matrix $\tilde{C}$ be zero. if any column's of the $\tilde{C}$ has all zeros then the corresponding state variable is not observable.

Kalman's test:-

Consider a system with state model, $\dot{X} = AX + BU$, $Y = CX + DU$.

$$N = \left[C^T \ \ A^TC^T \ \ (A^T)^2C^T \ \ldots \ (A^T)^{n-1}C^T \right]_{n \times n}.$$

$$N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n \times n}.$$

For observability, $|N| \neq 0$, 'N' must be non-singular.

* The system is completely observable, if the rank of 'N' is 'n'.

Disadvantage:- The nonobservable state variables cannot be determined.

Condition for complete observability:- in the s-plane:

* The necessary & sufficient condition for complete observability is that no cancellation of poles & zeros in the transfer function.
* If occurs cancelled mode cannot be observed in the output.

Effect of pole-zero cancellation in transfer function:-

* The concepts of controllability & observability are closely related to the properties of the transfer function.

Consider nth order system with distinct eigen values.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s_1 + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^1 + a_n} \quad m<n.$$

$$= \frac{K(s+\beta_1)(s+\beta_2)\cdots(s+\beta_m)}{(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)}$$

Apply partial fractions.

$$\frac{Y(s)}{U(s)} = \frac{c_1}{s_1+\lambda_1} + \frac{c_2}{s_2+\lambda_2} + \cdots + \frac{c_n}{s+\lambda_n}.$$

* If pole-zero cancellation occurs in the T/F then the system will be either not state controllable or unobservable. depending on how the state variables are defined.

* If pole-zero cancellation does not occur then the system always completely observable & controllable in the state model.

Observability test:-

* column of a 'C' matrix that correspond to the distinct eigenvalues should not have all zero elements.

* Column of 'c' matrix that corresponds to the first row of a Jordon block should not have all zero elements.

Note:- For complete controllability & observability there should not be any pole zero cancellation in the transfer function.

Transfer function (TF) = $C[SI-A]^{-1}B$.

Duality property:-

* If the pair (A,B) is controllable then the pair (A^T, B^T) is Observable.

* If the pair (A,C) is Observable then the pair (A^T, C^T) is controllable.

① A state variable model of a system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u \; ; \quad y = [1 \;\; 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the controllability & observability.

sol W.K.T $\dot{X} = AX + BU$

$Y = CX + DU$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [1 \;\; 0].$$

Test for controllability:

$$M = [B \;\; AB \;\; A^2B \; \dots \; A^{n-1}B] \text{ for } M \neq 0.$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; AB = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M = \left|B \;\; AB\right| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0-1 \neq 0.$$

$|M| \neq 0 \rightarrow$ Given system is Controllable.

Test for Observability:-

$$N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n\times n} \quad \& \; |N| \neq 0.$$

$$C = [1 \;\; 0], \quad CA = [1 \;\; 0]\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = [1 \;\; \text{[illegible]}]$$

$$N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; |N| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1-0 \neq 0$$

$|N| \neq 0$.

The given system is observable.

② Consider the system $A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = (0 \;\; 1)$. The transferfunction of the system has pole-zero cancellation. Find the controllability & observability.

sol W.K.T $\dot{X} = AX + BU$ Given that $A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = [0 \;\; 1]$

$Y = CX + DU$

Test for controllability:-

$$[M] = \left[B \;\; AB \;\; A^2B \cdots A^{n-1}B\right]_{n\times n}.$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{[illegible]} \qquad AB = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} -2 \\ 1-3 \end{pmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$M = \left|B \;\; AB\right| = \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} = -2+2 = 0.$$

$|M| = 0 \rightarrow$ not controllable.

Test for observability:-

$$N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad C = [0\ 1],\ CA = [0\ \ 1]\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

$$CA = [1\ \ -3]$$

$$N = \begin{vmatrix} C \\ CA \end{vmatrix} \cancel{CA} = \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} = |-1| \neq 0$$

∴ system is observable.

Method 2:-

* state space representation is in Observable Canonical Form(OCF) hence the system is observable, but there is a pole zero cancellation hence not controllable.