Introduction to Set theory:

Set: A Set is collection of well defined objects (or) Elements it is usually denoted by upper Case letters the elements are represented by lower case letters.

Ex: $A = \{a, b, c, d, e\}$ $A = \{1, 2, 3, 4\}$

Finite Set: A Set consisting of a finite number of elements that set is Said to be finite Set. Ex: let $x = \{x_1, x_2, x_3\}$

Infinite Set: A Set consisting of a infinite numbers of elements that Set is Said to be infinite Set.

Ex: let $x = \{x : x \in Z\}$
$x = \{x : x \in N\}$

Empty Set (or) Null Set: A Set is Said to be empty (or) Null Set, it has no elements. It is denoted by ϕ (or) $\{\ \}$.

Disjoint Sets: Let A and B are any two non empty Sets, if A and B are said to be dis-joints. If there is no Common element in the given Sets.

Ex: $A = \{a, b, c, d\}$ $B = \{e, f, g, h\}$

Equal Sets: Two Sets A and B are Said to be equal if both Sets contains common elements.. It is denoted by $A = B$ (or) $A \subseteq B$ & $B \subseteq A$.

Sub Sets: A Set 'A' is a Subset of 'B', if every element of Set A is also an elements of Set B. It is denoted by $A \subseteq B$.

Universal Set: A Set which consist of all elements of given System is called a universal set. It is denoted by μ (or) S (or) U.

Set operations:

Union: Let A and B are any two non empty Sets. The union of two sets is the Set of all elements belonging to either A (or) B.

$A = \{1, 2, 3, 4, 5\}$ $B = \{3, 4, 5, 6, 7\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

Intersection: Let A and B are any two non empty Sets. The intersection of two Sets is Set of all Common elements in A & B.

$A = \{1, 2, 3, 4, 5\}$ $B = \{3, 4, 5, 6, 7\}$

$A \cap B = \{3, 4, 5\}$

Difference: The difference of two Sets A & B is the set of all elements of that do not belonging to B.

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$A = \{1, 2, 3, 4, 5\}$ $B = \{3, 4, 5, 6, 7\}$

$A - B = \{1, 2\}$ $B - A = \{6, 7\}$

Complement: The Complement of a set A is the Set of all elements in μ (or) $\bar{A}$.

$\bar{A} = \mu - \bar{A}$ $(\mu - A)$

(or)

$A \cup \bar{A} = \mu$

$A \cap \bar{A} = \phi$

Laws of Sets:

Let us Consider the three Sets A, B, C are the Subsets of the Universal Set S.

1. Commutative law:

$A \cup B = B \cup A$

$A \cap B = B \cap A$

2. Associative law:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributive law:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. De-Morgan's law:

$(\overline{A \cup B}) = \bar{A} \cap \bar{B}$

$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

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5. Complementary law:

$A \cup \bar{A} = S$

$A \cap \bar{A} = \phi$

6. Identity law:

$A \cap S = A$

$A \cup \phi = A$

Duality principle: A duality principle states that for a given set identity, if we replace all unions by intersections, all intersections by unions, the universal set by null set and null set by the universal set S. Then the identity is preserved. The replaced set is called its duality.

Ex: $A \cap S = A \Rightarrow A \cup \phi = A$

Probability introduced through Sets and Relative frequency:

Probability: The theory of probability is one of the most useful and interesting branches of modern mathematics. It is becoming prominant by its applications in many fields of learning such as insurance, Statistics, biological & physical Sciences, Engineering etc.

If an Experiment is repeated under similar and homogeneous Conditions, generally come across two types of situations.

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1. The net result, what is generally known that outcome is unique (or) certain.
2. The net result is not unique but may be one of the Several possible outcomes.

The Situations covered by
1. Deterministic (or) Predictable
2. Probabilistic (or) unpredictable

Deterministic means there are Some Situations don't predict them selves to the deterministic approach.

Random Experiment: If an experiment is conducted any no. of times, under essential identical Conditions, there is a Set of all possible out comes associated with it.

If the result is not certain and is any one of the Several possible outcomes, the experiment is called a random experiment or random trail.

The outcomes are known as elementary events and a Set of out comes is an event. Thus an elementary event is also an event.

Equally likely events: Events are Said to be equally likely when there is no reason to except any one of them rather than any one of

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the Others.
Ex: When a Card is drawn from a pack, any card may be Obtain. In this trail all the 52 elementary events are equally likely.
Exhaustive events: All possible events in any trial are known as exhaustive events.
Ex: 1) In tossing a coin there are two exhaustive elementary events, head and tail.
2) In drawing three balls out of 9 balls in a
box, there are $9C_3$ exhaustive events.
Simple Event: An event in a trial that cannot be further Split is called a Simple event (or) elementary event.
Sample Space: The Set of all possible Simple events in a trial is called a Sample Space.
Note: Each Element of a Sample Space is called a Sample point.
Any Subset of a Sample Space is an event it is generally denoted by 'E', And the Sample Space is denoted by 'S'.
Mutually Exclusive Events:
Let E_1, E_2 are the two events of a Sample Space 'S' are Said to be mutually exclusive events. if they have no common Sample points.

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Ex: E_1, E_2 CS

$E_1 = \{HH\}$

$E_2 = \{HT, TH\}$

$E_1 \cap E_2 = \phi$

Discrete and Continuous Events:

A discrete Sample Space has discrete events. when
a Coin is tossed the event 'head' is a discrete
and finite event. Events taken from Continuous
Sample Spaces or Continuous Selecting a number
randomly in a given range is a continuous
event. Here there are infinite no. of events.

Probability introduced through relative frequency:

A random experiment is repeated n times. If
the event A occurs n(A) times, then the
Probability of the event A is defined as the
relative frequency of event A. when the no.
of trials n tends to infinity.

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$

where $\frac{n(A)}{n}$ is Called the relative frequency
of the event A.

Properties of Probability

1. $0 \leq P(A) \leq 1$

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2. $P(\bar{A}) = 1 - P(A)$

3. $P(A) \leq P(B)$, if $A \subset B$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

5. $P(\bar{A} \cup B) = P(B) - P(A \cap B)$

6. If $S = A_1 + A_2 + A_3 + \cdots + A_N$ are mutually exclusive events in N. $\sum_{i=1}^{N} P(A_i) = 1$.

7. For null set the probability of ϕ $P(\{\phi\}) = 0$ and for universal set $P(S) = 1$.

8. For equally likely of n events in S, the probability of each event is $P_i = \frac{1}{n}$, $i = 1, 2, 3 \cdots n$

Classical definition of probability:

In a random experiment let their be 'n' mutually exclusive & equally likely elementary events. Let E be an event of the experiment. If m elementary events from an event E [favourable to E] then the probability of E is defined as

$$P(E) = \frac{\text{No. of favourable Events}}{\text{Total no. of possible Events}}$$

$$= \frac{m}{n}$$

Note: If $\bar{E}$ denotes the event of non occurance of E then the no. of elementary events in $\bar{E}$.

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is $(n-m)$ and hence the probability of $\bar{E}$ is

$p(\bar{E}) = \frac{(n-m)}{n}$.

$p(E)$ = Occurance $\quad p(\bar{E})$ = non Occurance

$$p(E) + p(\bar{E}) = \frac{m}{n} + \frac{(n-m)}{n} = \frac{\cancel{m}+n-\cancel{m}}{n} = 1$$

$$p(E) + p(\bar{E}) = 1$$

Important points:

* If $p(E) = 1$, the event E is called certain event.
* If $p(E) = 0$, the event E is called impossible event.
* Suppose E is an event with $p(\bar{E}) < p(E)$

$$1 - p(E) < p(E)$$
$$1 < p(E) + p(E)$$
$$1 < 2p(E) = \frac{1}{2} < p(E)$$

Problems:

1. What is the probability for a leap year to have 52 Mondays & 53 Sundays.

Sol: In a leap year we have 366 days

52 Weeks + 2 days

364 days + 2 days

$\downarrow$

$\begin{bmatrix} 52 \text{ Mon} \\ 52 \text{ Sun} \end{bmatrix}$

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Let E be an event of having 52 Mondays and 53 Sundays in the year

These two days can be any one of the following ways [Sun, Mon] $n = 7$.

[Mon, Tue]

[Tue, Wed]

[Wed, Thur]

[Thur, Fri]

[Fri, Sat]

[Sat, Sun]

Total number of favourable to E [Sat, Sun] i.e; $m = 1$. $P(E) = \frac{m}{n} = \frac{1}{7}$.

2. 5 digit numbers are formed with 0, 1, 2, 3, 4 (repetition is not allowed). Find the probability of getting 2 in 10's place and 0 in the units place always.

Sol: The given digits are 0, 1, 2, 3, 4

The total number of 5 digit numbers are

$n = 4 \times 4 \times 3 \times 2 \times 1$

4×	4×	3×	2×	1

$n = 96$

$\therefore$ 96 ways

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Let E be an event of getting a number having 2 in 10's place and 0 in 1's place.

$m = 3 \times 2 \times 1$

			2	0

$\underbrace{\quad\quad\quad}_{3 \times 2 \times 1}$ $m = 6$

$P(E) = \frac{m}{n} = \frac{6}{96} = \frac{1}{16}$.

3. If a box contains 16 red balls, 12 blue balls and 22 green balls, what is the probability of drawing a ball which is

1. red in colour
2. either red or blue
3. not green

Sol: Total number of balls in a box $n = 50$ $(16+12+22)$

a) Let R be an event of getting only red balls

i.e., $m = 16$

$\therefore P(R) = \frac{m}{n} = \frac{16}{50}$.

b) Let E be an event of getting either red or blue i.e., $m = 16 + 12 = 28$

$\therefore P(E) = \frac{m}{n} = \frac{28}{50}$

c) Let E be an event of not getting green balls

$P(\bar{E}) = \frac{22}{50}$

$P(E) = 1 - P(\bar{E})$
$= 1 - \frac{22}{50} = \frac{28}{50}$.

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4. what is the probability of picking an ace and a king from a deck of cards.

Sol: The total no. of cards $n = 52$

Let E be the event of getting an Ace and a king i.e., m: 4 Ace & 4 Kings

$$m = 4C_1 \times 4C_1 = 16$$

$$P(E) = \frac{4C_1 \times 4C_1}{52C_2}$$

$$= \frac{16}{1326}$$

$$= \frac{8}{663}$$

$$= 0.012.$$

5. A card is drawn at random from a deck of 52 playing cards. Find the probability of drawing.

1. An ace 2. A 6 or a heart in red colour.

3. either 9 or a Spade.

Sol: The total no. of cards $n = 52$

i) An ace

Let E be the event of getting only an ace

i.e., $m = 4$

$$P(E) = \frac{m}{n} = \frac{4}{52}$$

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ii) A 6 (or) a heart in Red colour.

Let E be the event of getting a 6 (or) a heart in red colour i.e.,

$$P(E) = \frac{m}{n} = \frac{4C_1 + 13C_1}{52C_1}$$

iii) either 9 or a Spade

Let E be the event of getting either 9 or a Spade i.e., $m = \frac{4C_1 + 13C_1}{52C_1}$

6. Determine the probability for each of the following events, a non-defective bolt will be found if out of 600 bolts already examine 12 were defective.

Sol: Given that the total no. of balts $n=600$

Let E be an event of getting defective bolts

$m=12$

$P(E) = \frac{m}{n} = \frac{\cancel{12}}{\cancel{600}} = \frac{1}{50}$

The probability of non-defective bolts is denoted by $p(\overline{E})$

We know that $P(E) + P(\overline{E}) = 1$

$$P(\overline{E}) = 1 - P(E)$$
$$= 1 - \frac{1}{50} = \frac{49}{50}.$$

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Addition theorem on probability:

Statement: If S is a Sample Space. E_1, E_2 are any two events in S then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Joint probability:

If a Sample Space Consists of 2 Events A, B which are not mutually exclusive.

Then the probability of these events occuring Simultaneously or jointly is called joint probability.

For 2 Events A and B the common elements from the event $A \cap B$. The probability of this joint event $P(A \cap B)$ is called joint probability.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Problems:

1. A card is drawn from a well shuffled pack of cards. what is the probability that it is either a Spade (or) an ace.

Sol: Let S be a Sample Space of pack of cards

i.e., $n = 52$

Let A be an event of getting a Spade

$$P(A) = \frac{13}{52}$$

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Let B be an event of getting an ace $P(B) = \frac{4}{52}$

$A \cap B$ = The event of getting a Spade and an ace.

$\therefore P(A \cap B) = \frac{1}{52}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$= \frac{16}{52}$

2. Three Students A, B, C are in running race. A and B have the Same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins.

Sol: Let S be a Sample Space with 3 Students A, B, C they are participated in running race

i.e., $S = A \cup B \cup C$ [A, B, C are independent]

$\therefore P(S) = P(A) + P(B) + P(C)$

WKT., $P(S) = 1$

$P(A) + P(B) + P(C) = 1 \rightarrow (1)$

given that A, B having Same probability and each is twice as likely to win as C.

i.e., $P(A) = P(B) \rightarrow (2)$

$$\left.\begin{array}{l} P(A) = 2P(C) \\ P(B) = 2P(C) \end{array}\right\} \rightarrow (3)$$

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Substitute equation (3) in equation (1)

$$2p(C) + 2p(C) + p(C) = 1$$

$$5p(C) = 1$$

$$p(C) = 1/5$$

from equation (1) $\Rightarrow$ $P(A) + P(A) + \frac{1}{5} = 1$ $\langle \because$ from eqn(2)$\rangle$

$$2P(A) = 1 - \frac{1}{5}$$

$$P(A) = \frac{\cancel{4}}{5} \times \cancel{2}$$

$$P(A) = \frac{2}{5}$$

$$P(B) = \frac{2}{5}$$

$$P(B \cup C) = p(B) + p(C)$$

$$= \frac{2}{5} + \frac{1}{5}$$

$$= \frac{3}{5}.$$

3\. A bag contains 8 red balls and 6 blue balls. Find the two drawing of each 2 balls are made. Find the probability that the first drawing gives 2 red balls and second drawing gives 2 blue balls. If the balls drawn first and replaced before the second draw.

Sol: Let E be the event of the balls i.e., $n = 8+6$

$n = 14C_2$

Let $P(E_1)$ be the event of drawing 2 red balls

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$$P(E_1) = \frac{8C_2}{14C_2}$$

The two balls which are drawn are replaced.
Let $P(E_2)$ be the event of drawing 2 blue balls in Second draw.

$$P(E_2) = \frac{6C_2}{14C_2}.$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = \frac{8C_2}{14C_2} \times \frac{6C_2}{14C_2}$$

$$= \frac{60}{1183}$$

Conditional Events:

If E_1, E_2 are events of a Sample Space 'S' and if E_2 Occurs after the occurance of E_1, then the event of occurance of E_2 after the event E_1 is called Conditional event of E_2 given E_1. It is denoted by $\frac{E_2}{E_1}$.

Ex:

1. 2 coins are tossed. The event of getting two tails given that there is atleast one tail is a Conditional event.
2. A die is thrown 3 times. The event of getting the Some of the numbers thrown is 15. when it is known that the first thrown was a 5 is a Conditional event.

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Conditional probability: If E_1 and E_2 are two events in a Sample Space 'S' and $p(E_1) \neq 0$ then the Probability of E_2 after the event E_1 has Occured is called the Conditional probability of the event E_2 given E_1. And is denoted by $p(\frac{E_2}{E_1})$ and we define $p(\frac{E_2}{E_1}) = \frac{p(E_1 \cap E_2)}{p(E_1)}$

Similarly the probability of E_1 given E_2

$$p\left(\frac{E_1}{E_2}\right) = \frac{p(E_1 \cap E_2)}{p(E_2)}, \quad p(E_2) \neq 0.$$

Independent Events: If $p(E_1) \neq 0$, $p(E_2) \neq 0$ and E_2 is independent of E_1 then E_1 is independent of E_2. In this case we say that E_1, E_2 are mutually independent.

Dependent Event: If the occurance of Event E_2 is affected by the occurance of E_1. Then the events are dependent.

Conditional probability theorem (or) Multiplication theorem on probability

Statement: In a random experiment if E_1, E_2 are two events Such that $p(E_1) \neq 0$, $p(E_2) \neq 0$ then $p(E_1 \cap E_2) = p\left(\frac{E_1}{E_2}\right) \cdot p(E_2)$.

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$$P(E_2 \cap E_1) = P\left(\frac{E_2}{E_1}\right) \cdot P(E_1)$$

Proof: Let S be a Sample Space associated with the random experiment.

Let E_1, E_2 be two events of 'S' Such that

$$P(E_1) \neq 0, \quad P(E_2) \neq 0$$

Since $P(E_1) \neq 0$:

By the definition of conditional probability.

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_2 \cap E_1)}{P(E_1)}$$

$$P(E_2 \cap E_1) = P\left(\frac{E_2}{E_1}\right) \cdot P(E_1)$$

Since $P(E_2) \neq 0$:

By the definition of conditional probability

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cap E_2) = P\left(\frac{E_1}{E_2}\right) \cdot P(E_2)$$

$\therefore$ Hence the theorem is proved.

Problems:

1. 2 Marbles drawn in Succession from a box Contains 10 red, 30 white, 20 blue and 15 orange marbles, with replacement being made after each draw find the probability that i) both are white

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ii) first is red and second is white.

Sol: Given that the total no. of marbles $n = 75$
$(30+20+15+10)$

i) Let E_1 be an event of getting a white marble in the first draw $p(E_1) = \frac{30}{75}$

Let E_2 be an event of getting another white marble in the second draw after the first draw $p\left(\frac{E_2}{E_1}\right) = \frac{30}{75}$

$\therefore$ we know that $p(E_2 \cap E_1) = p\left(\frac{E_2}{E_1}\right) \cdot p(E_1)$

$$= \frac{30}{75} \cdot \frac{30}{75}$$

$$= \frac{4}{25}$$

ii) Let E_1 be an event of getting a red marble in the first draw $p(E_1) = \frac{10}{75}$

Let E_2 be an event of getting a white marble in the second draw after happening to E_1.

$$p\left(\frac{E_2}{E_1}\right) = \frac{30}{75}$$

$$p(E_2 \cap E_1) = p\left(\frac{E_2}{E_1}\right) p(E_1)$$

$$= \frac{30}{75} \cdot \frac{10}{75}$$

$$= \frac{4}{75}.$$

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2. A box contains 5 red & 3 white beads and a box B contains 2 red & 6 white beads. If a bead is drawn from each box, what is the probability that they are both in same colour.

Sol: Let E_1 be an event that the marble (or) bead is drawn from box A & is red.

$\therefore P(E_1) = \frac{1}{2} \times \frac{5}{8} = \frac{5}{16}$

Let E_2 be an event that the bead is drawn from box B and is red.

$P(E_2) = \frac{1}{2} \times \frac{2}{8} = \frac{1}{8}$

The probability that both the beads are red.

i.e., $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

$= \frac{5}{16} \times \frac{1}{8}$

$= \frac{5}{128}$

Let E_3 be an event of getting a white bead from the box A and is white.

$P(E_3) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

Let E_4 be an event of getting a white bead from the box B and is white.

$P(E_4) = \frac{1}{2} \times \frac{6}{8} = \frac{3}{8}$

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The probability that both the beads are white.

$P(E_3 \cap E_4) = P(E_3) \cdot P(E_4)$

$= \frac{3}{16} \times \frac{3}{8}$

$= \frac{9}{128}$

$\therefore$ Required of getting both Same colour

$= P(E_1 \cap E_2) + P(E_3 \cap E_4)$

$= \frac{5}{128} + \frac{9}{128}$

$= \frac{7}{64}$.

Axioms of probability

Let S be a finite Sample Space A real valued function 'p' from the power set of 'S' into 'R' is called a probability function on 'S', if the following 3 axioms are satisfied.

1. Axiom of positivity :- $P(E) \geq 0$, for every subset 'E' of 'S'.

2. Axiom of Certainity :- $P(S) = 1$

3. Axiom of union: If E_1, E_2 are disjoint Subsets of 'S' then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

*The image $P(E)$ of E is called the probability of Event E.

Total probability Theorem:

Let $A_1, A_2, A_3 \ldots A_n$ be 'n' events Such that $S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ and $A_i \cap A_j = \phi$ for $i \neq j$, then for any event B.

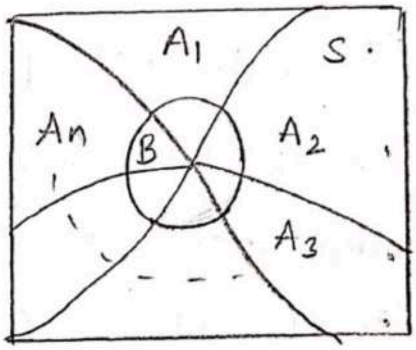
$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)$$

Proof: Given that $A_1, A_2, A_3 \ldots A_n$ be 'n' number of events in 'S'

And $S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

And also we have $A_i \cap A_j = \phi$ for $i \neq j$

Let $B = B \cap A_i$



$$P(B) = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right)$$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$P(B) = \sum_{i=1}^{n} P\left(\frac{B}{A_i}\right) \cdot P(A_i)$$

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P\left(\frac{B}{A_i}\right)$$

$\therefore$ Hence the theorem is proved.

Baye's Theorem:

If $E_1, E_2 \ldots E_n$ are 'n' mutually exclusive and exhaustive events Such that $P(E_i) > 0$ $(i = 1, 2, 3 \ldots n)$. In a Sample Space S & 'A' is any other event in 's' Intersecting with every E_i Such that $P(A)>0$.

(or)

If E_i is any of the events of $E_1, E_2 ---- E_n$ where $P(E_1), P(E_2), ------ P(E_n)$ and $P\left(\frac{A}{E_1}\right), P\left(\frac{A}{E_2}\right)$ ---- $P\left(\frac{A}{E_n}\right)$ are Known then

$$P\left(\frac{E_K}{A}\right) = \frac{P(E_K) \cdot P(A/E_K)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + ---- + P(E_n)P(A/E_n)}.$$

Proof: If $E_1, E_2 ----E_n$ are 'n' no. of events of 'S' Such that $P(E_i)>0$ and $E_i \cap E_j = \phi$ for $i \neq j$, where $i,j = 1,2,3 ----n$.

And also we have $E_1, E_2, E_3 --- E_n$ are exhaustive events of 'S' and 'A' is any other event of 'S' where $P(A)>0$.

$S = E_1 \cup E_2 \cup E_3 \cup ---- \cup E_n$ and $A = A \cap S$

$A = A \cap (E_1 \cup E_2 \cup E_3 \cup ---- \cup E_n)$

$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup ---- \cup (A \cap E_n)$

$\therefore (A \cap E_1), (A \cap E_2), (A \cap E_3) ---$ are exhaustive events.

Let $P\left(\frac{E_K}{A}\right) = \dfrac{P(E_K \cap A)}{P(A)}$

$$= \frac{P(E_K \cap A)}{P[(A \cap E_1) \cup (A \cap E_2) \cup ---- \cup (A \cap E_n)]}$$

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$$= \frac{P(E_K \cap A)}{P(A\cap E_1) + P(A\cap E_2) + \cdots + P(A\cap E_n)}$$

$$P\left(\frac{E_K}{A}\right) = \frac{P(A|E_K)\cdot P(E_K)}{P(A|E_1)\cdot P(E_1) + P(A|E_2)\cdot P(E_2) + \cdots + P(A|E_n)\cdot P(E_n)}$$

$\therefore$ Hence the theorem is proved.

Problems:

1. A bag contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that the red ball is drawn from bag B.

Sol: Let A & B denote the events of selecting bag A and bag B respectively.

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$

Let R denotes the event of drawing a red ball.

Now to select bag A,

the probability to draw a red ball from bag A.

$P(R/A) = \frac{3}{5}$

Similarly, from bag B $P(R|B) = 5/9$

A ball is selected from one of the bag at

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random, it is found to be red.

Then the probability of a red ball from bag B

$P(R/A) = \frac{3}{5}$, $P(R/B) = \frac{5}{9}$

$$P(B/R) = \frac{P(R/B) \cdot P(B)}{P(R/A) \cdot P(A) + P(R/B) \cdot P(B)}$$

$$= \frac{\frac{5}{9} \cdot \frac{1}{2}}{\frac{3}{5} \cdot \frac{1}{2} + \frac{5}{9} \cdot \frac{1}{2}}$$

$$= \frac{25}{52}.$$

2. In a bolt factory machines A, B, C manufacture 20%, 30%, 50% of the total of their output and 6%, 3%, 2% are the defective. A bolt is drawn at random and found to be defective. Find the probabilities that it is manufactured from i) Machine A ii) Machine B iii) Machine C.

Sol: Given that

$P(A) = 20\% = \frac{20}{100} = \frac{1}{5}$

$P(B) = 30\% = \frac{30}{100} = \frac{3}{10}$

$P(C) = 50\% = \frac{50}{100} = \frac{1}{2}$

Let D denotes that the bolt is defective.

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$$P(D/A) = 6\% = \frac{6}{100} = \frac{3}{50}$$

$$P(D|B) = 3\% = \frac{3}{100}$$

$$P(D|C) = 2\% = \frac{2}{100} = \frac{1}{50}$$

i) $$P(A|D) = \frac{P(D|A)\,P(A)}{P(D|A)\,P(A) + P(D|B)\,P(B) + P(D|C)\,P(C)}$$

$$= \frac{\frac{3}{50}\cdot\frac{1}{5}}{\frac{3}{50}\cdot\frac{1}{5} + \frac{3}{100}\cdot\frac{3}{10} + \frac{1}{50}\cdot\frac{1}{2}}$$

$$= \frac{12}{31}$$

ii) $$P(B|D) = \frac{P(D|B)\,P(B)}{P(D|A)\,P(A) + P(D|B)\,P(B) + P(D|C)\,P(C)}$$

$$= \frac{\frac{3}{100}\cdot\frac{3}{10}}{\frac{3}{50}\cdot\frac{1}{5} + \frac{3}{100}\cdot\frac{3}{10} + \frac{1}{50}\cdot\frac{1}{2}}$$

$$= \frac{9}{31}.$$

iii) $$P(C|D) = \frac{P(D|C)\,P(C)}{P(D|A)\,P(A) + P(D|B)\,P(B) + P(D|C)\,P(C)}$$

$$= \frac{\frac{1}{50}\cdot\frac{1}{2}}{\frac{3}{50}\cdot\frac{1}{5} + \frac{3}{100}\cdot\frac{3}{10} + \frac{1}{50}\cdot\frac{1}{2}}$$

$$= \frac{10}{31}$$

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3. A business man goes to hotels x, y, z 20%, 50%, 30% of the rooms respectively. It is known that 5%, 4%, 8% of the rooms in x, y, z hotels have faulty plumbings. What is the Probability that a business man's room having faulty plumbing is assigned to hotel z.

Sol: Given that

$$P(X) = 20\% = \frac{20}{100} = \frac{1}{5}$$

$$P(Y) = 50\% = \frac{50}{100} = \frac{1}{2}$$

$$P(Z) = 30\% = \frac{30}{100} = \frac{3}{10}$$

Let F denotes that the faulty plumbing

$$P(F|X) = 5\% = \frac{5}{100} = \frac{1}{20}$$

$$P(F|Y) = 4\% = \frac{4}{100} = \frac{1}{25}$$

$$P(F|Z) = 8\% = \frac{8}{100} = \frac{2}{25}$$

i) $$P(X|F) = \frac{P(F|X)\,P(X)}{P(F|X)\,P(X) + P(F|Y)\,P(Y) + P(F|Z)\,P(Z)}$$

$$= \frac{\frac{1}{20}\cdot\frac{1}{5}}{\frac{1}{20}\cdot\frac{1}{5} + \frac{1}{25}\cdot\frac{1}{2} + \frac{2}{25}\cdot\frac{3}{10}} = \frac{5}{27}$$

ii) $$P(Y|F) = \frac{P(F|Y)\,P(Y)}{P(F|X)\,P(X) + P(F|Y)\,P(Y) + P(F|Z)\,P(Z)}$$

$$= \frac{\frac{1}{25}\cdot\frac{1}{2}}{\frac{1}{20}\cdot\frac{1}{5} + \frac{1}{25}\cdot\frac{1}{2} + \frac{2}{25}\cdot\frac{3}{10}} = \frac{10}{27}$$

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iii) $P(Z|F) = \dfrac{P(F|Z)P(Z)}{P(F|X)\,P(X) + P(F|Y)\,P(Y) + P(F|Z)\,P(Z)}$

$$= \frac{\frac{2}{25}\cdot\frac{3}{10}}{\frac{1}{20}\cdot\frac{1}{5}+\frac{1}{25}\cdot\frac{1}{2}+\frac{2}{25}\cdot\frac{3}{10}} = \frac{4}{9}.$$

4. A Shipment of Components consists of three identical boxes. One box contains 2000 Components of which 25% are defective, the Second box has 5000 Components of which 20% are defective and the third box contains 2000 Components of which 600 are defective. A box is selected at random and a Component is removed at random from the box. What is the probability that this Component is defective? What is the Probability that it came from the Second box.

Sol: Let the events B_1, B_2 and B_3 are the Selecting of boxes Assume that the Selection is equally likely.

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3} = 0.33$$

Let the event D be Selecting a defective Component.

Given 25% of the Components are defective in B $\quad P(D|B_1) = \dfrac{25}{100} = 0.25$

20% of the Components are defective in B_2.

$$P(D|B_2) = \frac{20}{100} = 0.2$$

600 Components are defective from 2000 in B_3.

$$P(D|B_3) = \frac{600}{2000} = \frac{3}{10} = 0.3$$

The total probability theorem, the probability of getting a defective Component is

$$P(D) = P(D|B_1)\,P(B_1) + P(D|B_2)\,P(B_2) + P(D|B_3)\,P(B_3)$$

$$= (0.25 + 0.2 + 0.3)\frac{1}{3}$$

$$= \frac{0.75}{3} = 0.25$$

$$P(D) = 0.25$$

The probability that the defective Component Comes from the Second box is

$$P(B_2|D) = \frac{P(D|B_2)\ P(B_2)}{P(D)}$$

$$= \frac{0.2 \times \frac{1}{3}}{0.25} = \frac{20}{75} = 0.267.$$

5. A binary Symmetrical channel is used for Commu-nication between a transmitter and a receivers. A transmitter transmits 2 possible inputs 0 and 1 bits. The probability of transmitting a 0 bit is 0.45 and the probability of transmitting a 1 is 0.55.

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At the receiver end, there are 4 possibilities.

1. The probability of transmitting a 0 bit and receiving a 0 bit is 0.8.
2. The probability of transmitting a 0 bit and receiving a 1 bit is 0.2.
3. The probability of transmitting a 1 bit and receiving a 0 bit is 0.2.
4. The probability of transmitting a 1 bit and receiving a 1 bit is 0.8.

Find the probability of possible bits at receiver end.

Sol: Let us Consider B_1 = transmitting 0 bit

B_2 = transmitting a 1 bit

A_1 = receiving a 0 bit

A_2 = receiving a 1 bit

And also given the probability of transmitting a 0 bit is 0.45. $P(B_1) = 0.45$

And the probability transmitting a 1 bit is 0.55

$P(B_2) = 0.55$

And also given the Conditional probabilities are

$P(A_1|B_1) = 0.8$

$P(A_2|B_1) = 0.2$

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$P(A_1|B_2) = 0.2$

$P(A_2|B_2) = 0.8$

The probability of received bits, by using total Probability theorem.

Since A_1, A_2 are mutually exclusive then

The probability of receiving a 0 bit

$$P(A_1) = P(A_1|B_1)\, P(B_1) + P(A_1|B_2)\, P(B_2)$$

$$= 0.8 \times 0.45 + 0.2 \times 0.55$$

$$= 0.47$$

The probability of receiving a 1 bit

$$P(A_2) = P\left(\frac{A_2}{B_1}\right) \cdot P(B_1) + P\left(\frac{A_2}{B_2}\right) \cdot P(B_2)$$

$$= 0.2 \times 0.45 + 0.8 \times 0.55$$

$$= 0.53$$

The Conditional probabilities using baye's theorem.

i) The probability of transmitting a 0 bit when 0 bit is already received.

$$P\left(\frac{B_1}{A_1}\right) = \frac{P\left(\frac{A_1}{B_1}\right) P(B_1)}{P\left(\frac{A_1}{B_1}\right) P(B_1) + P\left(\frac{A_1}{B_2}\right) P(B_2)}$$

$$= \frac{0.8 \times 0.45}{0.8 \times 0.45 + 0.2 \times 0.55} = 0.7660$$

ii) Probability of transmitting a 0 bit when 1 bit is already received.

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$$P\left(\frac{B_1}{A_2}\right) = \frac{P(A_2|B_1)\,P(B_1)}{P\left(\frac{A_2}{B_1}\right)P(B_1) + P\left(\frac{A_2}{B_2}\right)P(B_2)}$$

$$= \frac{0.2 \times 0.45}{0.2 \times 0.45 + 0.8 \times 0.55}$$

$$= 0.1698$$

iii) The probability of transmitting a 1 bit when 0 bit is already received.

$$P\left(\frac{B_2}{A_1}\right) = \frac{P\left(\frac{A_1}{B_2}\right)P(B_2)}{P\left(\frac{A_1}{B_1}\right)P(B_1) + P\left(\frac{A_1}{B_2}\right)P(B_2)}$$

$$= \frac{0.2 \times 0.55}{0.8 \times 0.45 + 0.2 \times 0.55}$$

$$= 0.234$$

iv) The probability of transmitting a 1 bit when 1 bit is already received.

$$P\left(\frac{B_2}{A_2}\right) = \frac{P\left(\frac{A_2}{B_2}\right)P(B_2)}{P\left(\frac{A_2}{B_1}\right)P(B_1) + P\left(\frac{A_2}{B_2}\right)P(B_2)}$$

$$= \frac{0.8 \times 0.55}{0.2 \times 0.45 + 0.8 \times 0.55}$$

$$= 0.8302$$

6. A letter is taken at random out of ASSISTANT and out of STATISTICS, what is the chance that they are the Same letters.

Sol: Given that the two words are ASSISTANT and STATISTICS.

Let the event 1 is letter from the word ASSISTANT.

Let the event 2 is letter from the word STATISTICS.

The probabilities of letters from event 1

$P(A_1) = \frac{2}{9}$

$P(S_1) = \frac{\cancel{3}}{\cancel{9}} = \frac{1}{3}$

$P(I_1) = \frac{1}{9}$

$P(T_1) = \frac{2}{9}$

$P(N_1) = \frac{1}{9}$

Similarly the probabilities of letters from event 2

$P(A_2) = \frac{1}{10}$

$P(S_2) = \frac{3}{10}$

$P(I_2) = \frac{\cancel{2}}{\cancel{10}} = \frac{1}{5}$

$P(T_2) = \frac{3}{10}$

$P(C_1) = \frac{1}{10}$

The probability of taking same letters

$P = P(A_1 \cap A_2) + P(S_1 \cap S_2) + P(I_1 \cap I_2) + P(T_1 \cap T_2)$

$= P(A_1)\,P(A_2) + P(S_1)\,P(S_2) + P(I_1)\,P(I_2) + P(T_1)\,P(T_2)$

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$= \frac{2}{9} \times \frac{1}{10} + \frac{1}{3} \times \frac{3}{10} + \frac{1}{9} \times \frac{1}{5} + \frac{2}{9} \times \frac{3}{10}$

$= \frac{19}{90}$.

7. A letter is known as to have come from either TATANAGAR or CALCUTTA. On the envelope, just two consecutive letters 'TA' are visible. Find the probability that the letter has come from CALCUTTA.

Sol: Let the event A be letter from TATANAGAR

Let the event B be letter from CALCUTTA.

Assume that both events are equally likely

$P(A) = P(B) = \frac{1}{2}$

Let the event C be the two consecutive letters TA.

The probability of the letters TA being selected from TATANAGAR is $P(C|A) = \frac{2}{7}$

The probability of the letters TA being selected from CALCUTTA is $P(C|B) = \frac{1}{7}$

The probability of the letter coming from CALCUTTA when the event C selected is

$$P(B|C) = \frac{P(C|B)\,P(B)}{P(C)} = \frac{P(C|B)\,P(B)}{P(C|A)\,P(A) + P(C|B)\,P(B)}$$

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$$P(B/C) = \frac{\frac{1}{7}\cdot\frac{1}{2}}{\frac{2}{7}\cdot\frac{1}{2}+\frac{1}{7}\cdot\frac{1}{2}}$$

$$= \frac{\frac{1}{7}}{\frac{2}{7}+\frac{1}{7}}$$

$$= \frac{1}{7} \times \frac{7}{3} = \frac{1}{3} = 0.33.$$

8. In a factory there are four machines. The machines produce 10%, 20%, 30%, 40% of an item respectively. The defective items produced by each machine are 5%, 4%, 3% and 2% respectively. Now an item is Selected which is to be defective, what is the probability of it being from the Second machine.

Sol: Let B_1, B_2, B_3, B_4 are the events of producing an item by the four machines respectively.

$P(B_1) = 10\% = 0.1$

$P(B_2) = 20\% = 0.2$

$P(B_3) = 30\% = 0.3$

$P(B_4) = 40\% = 0.4$

Let D event be producing a defective item

The probability of producing defective items by the machines are

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$P(D|B_1) = 5\% = 0.05$

$P(D|B_2) = 4\% = 0.04$

$P(D|B_3) = 3\% = 0.03$

$P(D|B_4) = 2\% = 0.02$

The probability of Selecting a defective item is

$$P(D) = P(D|B_1)\,P(B_1) + P(D|B_2)\,P(B_2) + P(D|B_3)\,P(B_3) + P(D|B_4)\,P(B_4)$$

$$= 0.1 \times 0.05 + 0.2 \times 0.04 + 0.3 \times 0.03 + 0.4 \times 0.02$$

$$P(D) = \frac{3}{100} = 0.03.$$

The probability of selecting a defective item from the second machine is, $P(B_2/D)$

$$\therefore P(B_2/D) = \frac{P(D/B_2) \cdot P(B_2)}{P(D)}$$

$$= \frac{0.2 \times 0.04}{0.03} = 0.267.$$

$$\therefore \boxed{P(B_2/D) = 0.267}$$

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Random Variables:-

A Real Variable "x" whose value is determined by the Outcome of a Random Experiment is called @ random Variable. A Random Variable x can also be Regraded as a real value function defined on the sample spaces of a Random Experiment. Such that for Each point x of the sample space.

F(x) is the probability of Occurance of the Event Represented by "x".

Eg:- The sample space Corresponding to tossing of two coins.

$\therefore$ The Outcomes S = {HH, TH, HT, TT}

After the performance of the Experiment we count the number of tails @nd denoted it by.

The first, Outcome "HH" has on tail

i.e $\boxed{x=0}$

The second outcome "HT" has one Tail

$\therefore \boxed{x=1}$

The third outcome "TH" has one Tail

$\therefore \boxed{x=1}$

The fourth outcome "TT" has two Tails

$\therefore \boxed{x=2}$

$\therefore$ x takes the values 0, 1, 2

$x = 0, 1, 2.$

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Types Of Random Variable :-

Random Variables are classified into Continoues discrete and Mixed random Variables.

i) Discrete Random Variables :-

A Random Varible x which can take only a finite number of discrete value in an interval of domain is called a discrete Random Variable.

Tossing @ coin, throwing a dice. The No.of defectivies in a sample of Electric Bulbs, the no.of Mistakes in Each page of the Book. The no.of telephone calls Received by the telephone Operator are the Example of a discrete Random Variables.

Continoues Random Variables :-

A Random Variable x which can taken the values Continously which takes all possible values in @ given intervel is called @ continoues Random Variable.

Ex :- The height, age and weight individuals are the Examples of continoues Random Variable and also temperature and time are the Continoues random Variable.

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Mixed Random Variable :-

The values of a Mixed Random Variables are Both continoues and discrete in a given sample space.

The Random Varible maps same points as continoues and some points as discrete values.

Ex:- In a Combined Experiment, the wheel of chance and tossing 2 coins, the Mixed Random Variable maps continoues points in the Range $0 \le s < 12$ and discrete points $S_2 = \{0, 1, 2\}$

Probability distrubution function :-

let "x" be a Random Variable then the probability distrubution function associated with x is defined as the probability that the outcomes of the Experiment will be one of the outcomes for which $x(s) \le x,\ x \in R$.

The Function $F(x)$ defined by $F_X(x) = p(X \le x)$

$= p\{s : x(s) \le x\}$, $-\infty \le x \le \infty$ is called the distrubution function of x.

Properties of distrubution function :-

1):- $p(a < x \le b) = F(b) - F(a)$

2):- $p(a \le x \le b) = p(x = a) + F(b) - F(a)]$

3):- $p(a < x < b) = [F(b) - F(a)] - p[x = b]$

$a < b$

4):- $p(a \le x \le b) = [F(b) - F(a)] - p[x = b] + p(x = a)$

5):- If $p(x = a) = p(x = b) = 0$ then the probability $p(a \le b)$

$p(a \le x < b) = p(a < x < b) = p(a \le x \le b) = f(b) - f(a)$

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All distrubution functions are monotonically increasing and lie between 0 and 1.

If F is the Distrubution function Of the Random Variable x.

Then

i):- $0 \le F(x) \le 1$

ii):- $F(x) < F(y)$ when $x < y$

iii):- $F(-\infty) = \lim_{x \to \infty} F(x) = 0$

iv):- $F(\infty) = \lim_{x \to \infty} F(x) = 1$

Discrete Probability distrubution probability Mass function:-

Probability distrubution of a Random Variable is the set of it's possible values together with their Respective. Probabilites. Suppose x is a discrete Random variable with possible outcomes.

x_1, x_2, x_3 - - - - The probability of Each outcome. x_i is p_i $p[x=x_i] = p(x_i)$ for $i = 1, 2, 3 \ldots$

It satisfies the two Condition

i) $p(x_i) > 0$, there $i = 1, 2, 3 \ldots$ (or) $0 < p_i \le 1$

ii) $\Sigma p(x_i) = 1$, $i = 1, 2, 3 \ldots$ Then function $p(x)$ is called discrete probability distrubution Of a random Variable x is given by Means of following table.

x_i	x_1	x_2	x_3	- - -	x_n
$p(x_i)$	p_1	p_2	p_3	- - -	p_n

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Cumulative distrubution function of a discrete Random Varible:-

Suppose that "x" is a discrete Random Variable. Then the discrete distrubution function (or) Cumulative distrubution function F(x) is defined by

$$F(x) = p(x \leq x) = \sum_{i=1}^{n} p(xi)$$

(or)

$$F(x) = p(x \leq x) = \sum_{t=x} f(t) \quad -\infty < x < \infty$$

Here f(t) is the value of t probability distrbution on of of x at t.

Expectation of a discrete Variable:-

Suppose a Random Variable x assumes the values $x_1, x_2, x_3 - - - x_n$ with their Respective probabilites $p_1, p_2, p_3 - - - p_n$. Then the Mathematical Expectation (or) Mean (or) Expected value of x denoted by F(x) is defined as the sum of products of different values of x and the corresponding probabilites.

$$\therefore E[x] = x_1p_1 + x_2p_2 + - - - x_np_n$$

$$E[x] = \sum_{i=1}^{n} x_ip_i \text{ (or) } E[x] = \sum_{i=1}^{n} x_ip(x_i)$$

Note:- Expected value of "x" is @ population Mean if population Mean "μ" then E[x] = μ

properties:-

1):- $\sum_{i=1}^{n} p(x_i) = 1$

2):- $E[x+k] = E[x]+k$

3):- $E[kx] = k\,E[x]$

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4):- $E[x+y] = E[x] + E[y]$

5):- $E[xy] = E[x] \cdot E[y]$

Note :- $\frac{1}{E[x]}$, $E\left[\frac{1}{x}\right]$ are not same.

Mean :- The Mean value μ of the discrete distribution function is given by $\mu = \frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} p_i}$

$$\mu = \frac{\sum_{i=1}^{n} p_i x_i}{\sum p_i (1)} \Rightarrow \mu = \sum_{i=1}^{n} p_i x_i$$

we know that $\sum_{i=1}^{n} p_i = 1$

Variance :- Variance charaterstics the Variability in the Distrubution since two distrubutions with same name can still have different dispersion of data about their Means. Varience of the probability distrubution of a Random Variable x is

$$E[x-\mu]^2 \quad Var(x) \text{ (or) } \sigma^2 \; E[x-\mu]^2$$
$$= E[x-E(x)]^2$$

(or)

$$Var(x) = E[x^2] - \mu^2$$

Standard deviation :-

It is the positive Square root of the Variation SD (or) $\sigma = \sqrt{E(x^2) - \mu^2}$.

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1) Two dice are thrown. let x assign to Each point (a,b) in "s" the Maximum of its numbers x(a,b) = max (a,b). Find the probability distrubution. x is @ random variable with the $x(s) = \{1,2,3,4,5,6\}$. Also find the Mean & Varience of the distrubution.

x	1	2	3	4	5	6	
$p(x)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$.	

Mean (μ) = $\sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{6} x_i p_i$

$= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6$

$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(5/36\right) + 4\left(7/36\right) + 5\left(9/36\right) + 6\left(1/36\right)$

$\boxed{\mu = 4.47}$

Variance (σ^2) :- $E(x^2) - \mu^2$

$= \sum_{i=1}^{6} x_i^2 p_i - \mu^2$

$= x_1^2 p_1 + x_2^2 \cdot p_2 + x_3^3 \cdot p_3 + x_4^4 \cdot p_4 + x_5^5 \cdot p_5 + x_6^6 p_6.$

$= 1^2\left(\frac{1}{36}\right) + 2^2\left(\frac{3}{36}\right) + 3^2\left(5/36\right) + 4^2\left(\frac{7}{36}\right) + 5^2\left(\frac{9}{36}\right) + 6^2\left(\frac{11}{36}\right) - (4.47)^2$

$= \boxed{1.99}$

Standard deviation $\sigma = \sqrt{E(x)^2 - \mu^2}$

$= \sqrt{1.99}$

$= 1.41$

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2) A Random Variable x has the following probability dis-
-tribution. Determine k.

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i):- Determine k

ii):- Evaluate $p(x<6)$, $p(x \geq 6)$, $p(0<x<5)$, $p(0 \leq x \leq 4)$

iii):- if $p(x \leq k) > 1/2$ Find the minimum value of k

iv):- Determine the distrubution function of x

v):- Mean

vi):- Variance

i):- We know that $\Sigma p(xi) = 1$

$$0+k+2k+3k+k^2+2k^2+7k^2+k$$

$$10k^2+9k=1$$

$$10k^2+9k-1=0$$

$$k=\frac{1}{10}, -1$$

$$\boxed{k=0.1}$$

ii):- $p(x<6) = p(x=0)+p(x=1)+p(x=2)+p(x=3)+p(x=4)+p(x=5)$

$$= 0+k+2k+2k+3k+k^2$$

$$= k^2+8k$$

$$= (0.1)^2+8(0.1)$$

$$= \boxed{0.81}$$

$p(x \geq 6) = p(x=6)+p(x=7)$

$$= 2k^2+7k^2+k$$

$$= 9k^2+k$$

$$= 9(0.1)^2+(0.1)$$

$$= 0.19$$

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$p(0 < x < 5) = p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$= 0 + k + 2k + 2k + 3k$

$= 8k$

$= 8(0.1)$

$= \boxed{0.8}$

$p(0 \leq x \leq 4) = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$= 0 + k + 2k + 2k + 3k$

$= 8k$

$= 0.8$

iii):- $p(x \leq k) > 1/2$

Let $p(x \leq 1)$

$p(x \leq 1) = p(x=0) + p(x=1)$

$= 0 + k$

$= 0.1 < 1/2$

$p(x \leq 2) = p(x=0) + p(x=1) + p(x=2) \cancel{+ p(x=3)}$

$= 0 + 2k + \cancel{2k} + k$

$= 0.3 < 1/2$

$p(x \leq 3) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$= 0 + 2k + 2k + k$

$= 0.5$

$= 1/2$

$p(x \leq 4) = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$= 0 + k + 2k + 2k + 3k$

$= 0.8 > 1/2$

if $p(x \leq k) > \frac{1}{2}$, the minimum value for k is 4.

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iv):- The distrubution function for $f(x)$.

x	0	1	2	3	4	5	6	7
$p(x)$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7

v) Mean (μ) $\sum_{i=1}^{n} x_i p_i$

$= \sum_{i=1}^{7} x_i p_i$

$= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6 + x_7p_7$

$= 0\times0 + 1\times0.1 + 2\times0.2 + 3\times0.2 + 4\times0.3 + 5\times0.01 + 6\times0.02 + 7\times0.17$

$= \boxed{3.66.}$

vi):- Variance $(\sigma^2) = E(x^2) - \mu^2$

$= \sum_{i=1}^{7} x_i^2 p_i - \mu^2$

$= x_1^2 p_1 + x_2^2 p_2 + x_3^3 p_3 + x_4^4 p_4 + x_5^5 p_5 + x_6^6 p_6 + x_7^7 p_7 - \mu^2$

$= 0\times0 + 1^2\times0.1 + 2^3\times0.2 + 3^4\times0.2 + 4^5\times0.3$
$+ 5^6\times0.01 + 6^6\times0.02 + 7^7\times0.17 - (3.66)^2$

$\boxed{\sigma^2 = 3.4044}$

③:- From @ lot of 10 items containing 3 defectives @ sample of 4 items is drawn at Random let the Random Variable. denote the no.of defective item in the sample. Find the probability distribution of x. when the sample is drawn without replacement.

Given,

Number of items $= 10$

Number of defectives $= 3$

A sample of four items is drawn at a random $= {}^{10}C_4$

Let "x" be the Random Variable denote the number when we assumed that $x = 1, 2, 3$.

When $x = 1$

$$p(x_i) = \frac{{}^7c_3 \times {}^3c_1}{{}^{10}c_4} = 0.5$$

when $x = 2$.

$$p(x_i) = \frac{{}^7c_2 \times {}^3c_2}{{}^{10}c_4} = 0.3$$

When $x = 3$.

$$p(x_i) = \frac{{}^7c_1 \times {}^3c_3}{{}^{10}c_4} = 0.03.$$

The probability distribution of x

X	1	2	3
$p(x_i)$	0.5	0.3	0.03.

probability density function :- The probability density function of the Random Variable "x" is defined as the values of probabilites at a given value of x. It is derivative of the distribution function $F_X(x)$ & is defined as $f_X(x)$.

$$\boxed{f_X(x) = \frac{d}{dx}\left[F_X(x)\right] \quad -\infty \leq x \leq \infty}$$

Expression for Density function for discrete random variables:-

If the values of X are $\{x_i\}$ the distribution function (M) can be written Mathematically $(F_X(x))$

$$F_X(x) = \sum_{i=1}^{N} p(x_i)\, u(x - x_i), \quad u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The Expression of density function for a discrete random variable $f_X(x) = \sum_{i=1}^{N} p(x_i)\, \delta(x - x_i)$

(since the derivate of a unit step function $u(t)$ is the unit impulse function $\delta(t)$ that is

$$\frac{d}{dt}(u(t)) = \delta(t)$$

where $\delta(t) = \begin{cases} 1, & t=0 \\ 0, & \text{otherwise.} \end{cases}$

Q: The Random Variable "x" has the discrete variable in the set $\{-1, -0.5, 0.7, 1.5, 3\}$. The corresponding probabilities are assumed to be set of $\{0.1, 0.2, 0.1, 0.4, 0.2\}$. plot the distrubution function & State wheather it is a discrete or continoues function.

Sol: Given that probability distrubution table.

x	-1	-0.5	0.7	1.5	3
$P(x_i)$	0.1	0.2	0.1	0.4	0.2

Distrubution function $F_X(x) = P(X \le x)$

$$F_X(-1) = P(X \le -1) = P(X = -1) = 0.1$$

$$F_X(-0.5) = P(X \le -0.5) = P(X=-1) + P(X=-0.5) = 0.1 + 0.2 = 0.3$$

$$F_X(0.7) = P[X \le 0.7] = P(X=-1) + P(X=-0.5) + P(X=0.7)$$

$$= 0.1 + 0.2 + 0.1$$

$$= 0.4$$

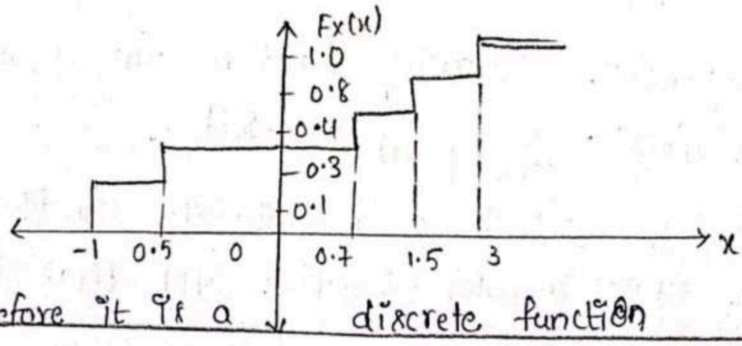
$$F_X(1.5) = P[X \le 1.5] = P[X=-1] + P(X=-0.5] + P(X=0.7) + P(X=1.5] = 0.4 + 0.4 = 0.8$$

$$F_X(3) = P(X \le 3)$$

$$= 0.8 + 0.2 = 1.0$$

The probability distribution function $F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x-i)$

$$= 0.1\, u(x+1) + 0.2\, u(x+0.5) + 0.1\, u(x-0.7) + 0.4\, u(x-15) + 0.2 u(x-3)$$



Therefore it is a discrete function

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(3) Consider the Experiment of tossing four fair coins the Random Variable x is associated the no. of tails showing Compute & Sketch the cumulative distrubution function of x & probability density function.

Consider the Experiment of tossing 4 fair coins

Let "x" be a discrete Random Variable that shows the no. of tails.

x	0	1	2	3	4
p(xi)	1/16	4/16	6/16	4/16	1/16

distrubution function $F_X(x) = p(X \leq x)$

$F_X(0) = p(X \leq 0) = 1/16$

$F_X(1) = p(X \leq 1)\ p(X=0) + p(X=1)$

$= 1/16 + 4/16 = 5/16$

$F_X(2) = p(X \leq 2) = p(X=0) + p(X=1) + p(X=2)$

$= 1/16 + 4/16 + 6/16 = 11/16$

$F_X(3) = p(X \leq 3) = p(X=0) + p(X=1) + p(X=2) + p(X=3)$

$= \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$

$F_X(4) = p(X \leq 4) = p(X=0) + p(X=1) + p(X=2) + p(X=3) + p(X=4)$

$= \frac{16}{16} = 1$

$$F_X(x) = \sum_{i=1}^{N} p(x_i) \cdot u(x - x_i)$$

$$F_X(x) = 1/6 \cdot u(x) + \frac{4}{16} \cdot u(x-1) + \frac{6}{16} u(x-2) + \frac{4}{16} u(x-3) + 1 \cdot u(x-4)$$

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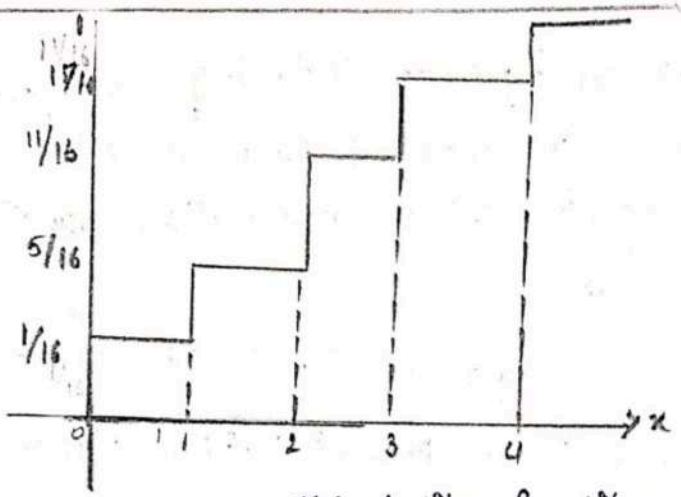
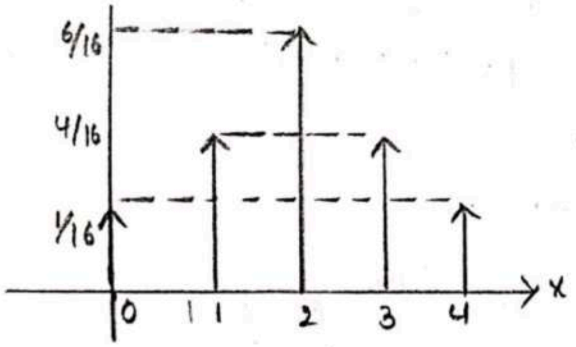
$F_X(0) = 1/16$
$F_X(1) = 5/16$
$F_X(2) = 11/16$
$F_X(3) = 15/16$
$F_X(4) = 1$



Probability distrubution function.

Probability density function:-



(6) A Random Variable "x" has probabilities show in the table.

x	-3	-2	-1	0	1	2
$p(x)$	0.2	0.5k	k	0.1	0.3k	k

Find the value of k, i) find $f_X(x)$, $F_X(x)$ and draw the plots.

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i) $p(x_i) = 0.2, 0.5k, k, 0.1, 0.3k, k$

$$0.2 + 0.5k + k + 0.1 + 0.3 + k = 1$$

$$0.3 + 2.8k = 1$$

$$2.8k = 1 - 0.3$$

$$2.8k = 0.7$$

$$k = \frac{0.7}{2.8} \qquad \boxed{k = 0.25}$$

ii) $F_X(x) = F_X(-3) = p(x \leq -3) = p(x = -3) = \boxed{0.2}$

$F_X(-2) = p(x \leq -2) = p(x=-3) + p(x=-2) = 0.2 + 0.5(0.25) = \boxed{0.325}$

$F_X(-1) = p(x \leq -1) = p(x=-3) + p(x=-2) + p(x=-1) = 0.2 + 0.125 + 0.25$

$= \boxed{0.578}$

$F_X(0) = p(x \leq 0) = p(x=-3) + p(x=-2) + p(x=-1) + p(x=0) = 0.575 + 0.3k$

$= 0.575 + 0.3(0.25)$

$= \boxed{0.675}$

$F_X(1) = p(x \leq 1) = p(x=-3) + p(x=-2) + p(x=-1) + p(x=0) + p(x=1)$

$= 0.2 + 0.125 + 0.25 + 0.1 + 0.075 = \boxed{0.75}$

$F_X(2) = p(x \leq 2) = p(x=-3) + p(x=-2) + p(x=-1) + p(x=0) + p(x=1) + p(x=2)$

$= 0.75 + 0.25$

$= 1$

The probability distrubution function can also Expressed as $F_X(x) = \sum_{i=1}^{N} p(x_i)\, u(x - x_i)$

$$= 0.2\, u(x+3) + 0.5\, u(x+2) + 0.25\, u(x+1) + 0.1\, u(x-0) + 0.075\, u(x-1) + 0.25\, u(x-2).$$

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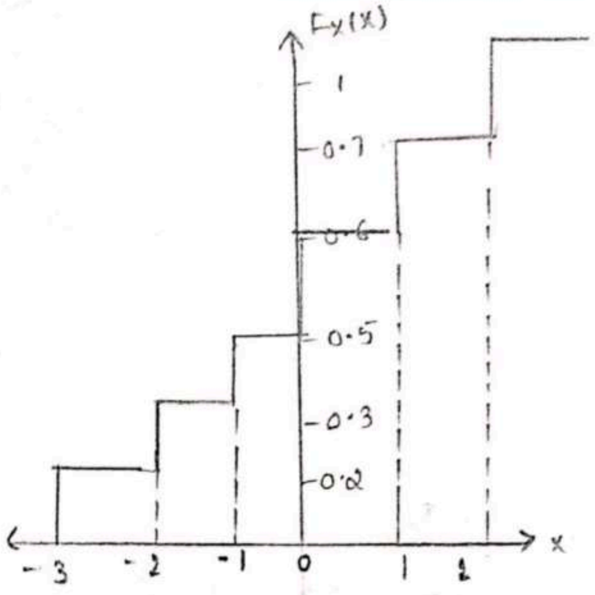
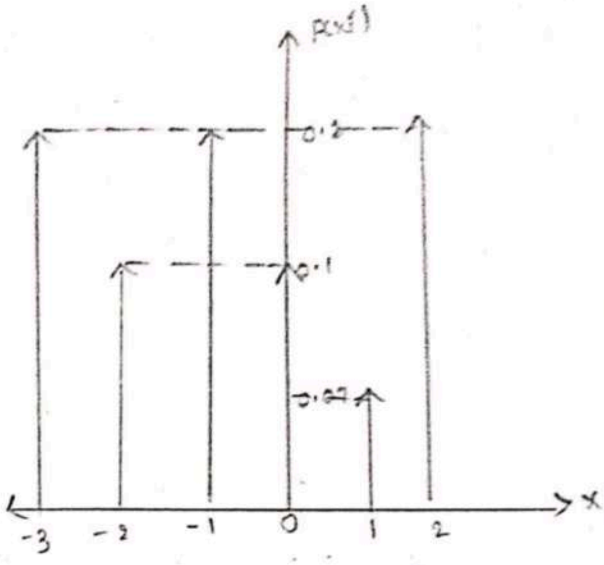


Probability distrubution function



probability density function

Continuous probability distrubution:-

let x be a random Variable. Takes Every value in an interval it gives rise to continoues distrubution of x. The distrubution defined by the Variates Light, Temp Height weight are Continoues distrubutions.

Probability density functions:-

let $f(x)$ be any Continoues function of x. So that $f(x)\,dx$ represents the the probability that the Variable x falls in the infinite decimal interval

$$\left[x-\frac{dx}{2},\ x+\frac{dx}{2}\right]$$

Symbolically it can be Expressed as $p\left(x-\frac{dx}{2}\le x\le x+\frac{dx}{2}\right)$

$$= f(x)\cdot dx.$$

Then $f(x)$ is called probability density function.

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The probability for a varient value to lie in the interval dx f(x) dx. So that probability for a varient value in the finite interval (a,b) is $\int_a^b f(x) \cdot dx$. which Represents the area between the curve $y = f(x)$, the x-axis & the ordinates $x = a$ & $x = b$.

∴ The total probability is unity $\int_a^b f(x) \cdot dx = 1$

Properties:-

* $f(x) \geq 0$ for all $x \in R$
* $\int_{\infty}^{-\infty} f(x) \cdot dx = 1$

Cummulative Distrubution function of a ~~Continouus~~ Random Variable:-

The Cummulative distrubution function of a Continouies Random Variable of f(x) is denoted by F(x)

$$\boxed{F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x \cdot dx}$$

Properties:-

i) $0 \leq F(x) \leq 1$, $-\infty \leq x \leq \infty$

ii) $F(x) = F(x)$

iii) $F(-\infty) = 0$

$F(\infty) = 1$

The probability of $p(a \leq x \leq b) = \int_a^b f(x) \cdot dx = F(b) - F(a)$

Mean:- Mean of a distrubuction is given by $\mu = E[x] = \int_{-\infty}^{\infty} x f(x)\, dx$

In the interval (a,b)

$$\boxed{E(x) = \int_a^b x f(x) \cdot dx}$$

Varience :-

The varience of the distrubution is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \cdot dx$$

(or)

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = -\mu^2.$$

1):- The probability Density of a Continuous Random Variable is given by $f(x) = c \cdot e^{-|x|}$, $-\infty < x < \infty$. Show that $c = 1/2$ and find the Mean & varience of the distrubution and also find the probability that variets like b/w 0 & 4.

sol):- Given that $f(x) = c \cdot e^{-|x|}$

we know that $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$= \int_{-\infty}^{\infty} c \cdot e^{-|x|} \cdot dx = 1$$

$$= 2c \int_{-\infty}^{\infty} e^{-x} \cdot dx = 1$$

$$= -2c \left[e^{-x}\right]_0^{\infty} = 1$$

$$= -2c \left[e^{-\infty} - e^{0}\right] = 1$$

$$= -2c [0-1] = 1$$

$$2c = 1 \Rightarrow c = 1/2 = f(x) = 1/2 \cdot e^{-|x|}$$

$$= \boxed{c = 1/2}$$

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$$\mu = \int_{-\infty}^{\infty} x \cdot 1/2 \, e^{-|x|} \cdot dx$$

$$= 1/2 \; x \int_{-\infty}^{\infty} (x \cdot e^{-|x|}) \cdot dx.$$

$$= \boxed{\mu = 0}$$

$$\text{varience} (\sigma^2) = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot c \cdot e^{-|x|} \cdot dx - 0$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|} \cdot dx$$

$$= 1/2 \cdot 2 \int x^2 e^{-x} \cdot dx.$$

$u = x^2$	$v = e^{-x} \cdot dx$
$u' = 2x$	$v' = -e^{-x} \cdot dx$
$u'' = 2$	$v'' = e^{-x} \cdot dx$
$u''' = 0$	$v''' = -e^{-x} dx.$

$$\sigma^2 = \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$\sigma^2 = -[0 - (0 + 0 + 2)]$$

$$\boxed{\sigma^2 = 2}$$

iii) $P(0 \le x \le 4) = \int_0^4 f(x) \cdot dx$

$$= \int_0^4 c \cdot e^{-|x|} \cdot dx$$

$$= 1/2 \int_0^4 e^{-x} \cdot dx$$

$$= 1/2 \left[-e^{-x}\right]_0^4$$

$$= -1/2 \left[e^{-4} - e^{0}\right]$$

$$= -1/2 \left[e^{-4} - 1\right]$$

$$= 1/2 \left(1 - e^{-4}\right)$$

$$= 0.5$$

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2) Consider that PDF of a Random Varible "x" $f_x(x)$ $= \begin{cases} \frac{1}{k} & , -2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$ and other Variable $Y = 2x$. Then find i) :- k

ii) $E[x]$ iii) $E[Y]$ iv) $E(X)\cdot(Y)$

Given that $f_X(x) = \begin{cases} \frac{1}{k}, & -2 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$

we know that $\int_{-\infty}^{\infty} f(x)\cdot dx = 1$

$$= \int_{-\infty}^{-2} f(x)\cdot dx + \int_{-2}^{3} f(x)\cdot dx + \int_{3}^{\infty} f(x)\cdot dx = 1$$

$$= 0 + \int_{-2}^{3} f(x)\cdot dx + 0 = 1$$

$$= 1/k \int_{-2}^{3} 1\cdot dx = 1$$

$$= 1/k\,[x]_{-2}^{3} = 1 \Rightarrow \frac{1}{k}[3-[-2]] = 1 \Rightarrow \boxed{x = 5}$$

ii) $E[x] = \text{Mean} = \int_{-\infty}^{\infty} x\, f(x)\cdot dx$

$$= \int_{-2}^{3} x\left(\tfrac{1}{5}\right) dx = 1/5\left[\frac{x^2}{2}\right]_{-2}^{3}$$

$$= \frac{1}{10}[9-4] = 1/2$$

iii) :- $E[Y] = E[2x]$

$$= 2\cdot E[x]$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}} = E[x] = 1$$

iv) :- $E[x]\cdot[Y]$

$$= E[x]\cdot E[Y]$$

$$= 1/2 \cdot 1$$

$$E[xY] = 1/2$$

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(58)

(3) If the probability density function of a random variable is given by $f_X(x) = \begin{cases} c \cdot e^{-x/4}, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$

find the value that 'c' must have and evaluate $F_X(0.5)$.

Sol:- Given pdf is $f_X(x) = \begin{cases} c \cdot e^{-x/4}, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$

If $f_X(x)$ is a valid density function, then.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^1 c\, e^{-x/4}\,dx = 1$$

$$\Rightarrow c \cdot \left[e^{-x/4} \cdot (-4) \right]_0^1 = 1$$

$$\Rightarrow -4c \left[e^{-1/4} - e^0 \right] = 1$$

$$\Rightarrow c = \frac{1}{4(e^{-1/4} - 1)} = 1.13.$$

$$\Rightarrow \boxed{c = 1.13}$$

Now to find $F_X(0.5) = \int_0^{0.5} 1.13\, e^{-x/4}\,dx$

$$= 1.13 \cdot \left[e^{-x/4} \cdot (-4) \right]_0^{0.5}$$

$$= -4(1.13) \left[e^{-0.5/4} - e^0 \right]$$

$$= -4.52 \left[e^{-0.5/4} - 1 \right]$$

$$\boxed{F_X(0.5) = 0.5311}$$

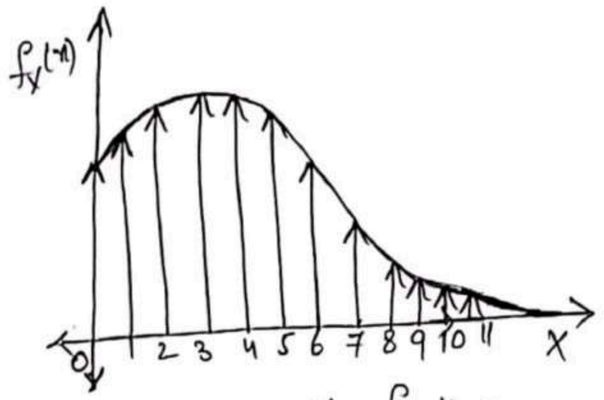
Binomial probability Density Function:-

Consider an experiment having only two possible outcomes per trial, such as, "success" (or) "failure", "tails" (or) "heads", "yes" (or) "no". etc.,

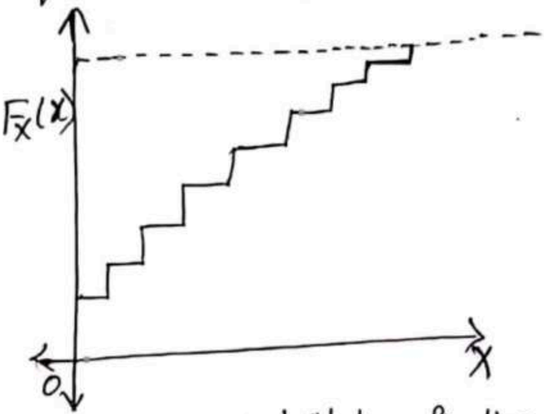
If the experiment is repeated for 'n' trials, then the binomial probability density function of a discrete random variable 'x' is defined as $f_X(x) = \sum_{k=0}^{n} nC_k \cdot p^k \cdot q^{n-k} \cdot \delta(x-k)$

∴ The Binomial Distribution function is

$$F_X(x) = \sum_{k=0}^{n} nC_k \; p^k \; q^{n-k} \cdot u(x-k).$$



Binomial Density function



Binomial Distribution function.

Conditions of Binomial Distribution:-

① Trials are repeated under identical conditions for a fixed number of times, say 'n' times. ['n' should be finite and small].

② There are two possible outcomes only, Eg:- success (or) failure for each trial.

③ The probability of success in each trial remains constant and does not change from trial to trial.

④ The trials are independent ie, the probability of an event in any trial is not affected by the results of any other trial.

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Applications of a Binomial Distribution:—

1. The number of defective bolts in a box containing 'n' bolts.
2. The number of machines lying idle in a factory having 'n' machines.
3. The number of post-graduates in a group of 'n' men.
4. The number of coins are tossed.
5. The number of oil wells yielding natural gas in a group of 'n' wells test drilled.

Mean of the Binomial Distribution:—

The mean of the Binomial probability distribution is given by,

$$\text{Mean } (\mu) = E[X] = \sum_{i=0}^{n} x_i \, p(x_i) = np.$$

where $p(x_i) = P(X=x) = nC_x \, p^x \cdot q^{n-x}$, where $p+q=1$
$\Rightarrow q = (1-p)$

$$\therefore \boxed{E[X] = \mu = np}$$

Variance of the Binomial Distribution:—

The variance of the Binomial distribution is given by

$$Var(x) = \sigma_x^2 = E[X^2] - (E[X])^2$$

(or)

$$\sigma_x^2 = E[(x-\bar{x})^2]$$

(or)

$$\sigma_x^2 = \sum_{i=0}^{n} x_i^2 \cdot p(x_i) - \mu^2$$

$$\therefore \boxed{Var(x) = \sigma_x^2 = npq}$$

Standard Deviation of the Binomial distribution:—

Standard Deviation is the square root of the Variance.

i.e, $S.D = \sqrt{Var(x)} = \sqrt{npq}$.

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Problems:-

① A fair coin is tossed six times. Find the probability of getting four heads.

Sol:- probability of getting head $(p) = 1/2$

probability of getting not head $(q) = 1/2$

$\therefore n = 6, \quad x = 4.$

$\therefore$ we know that $P(X = x) = nC_x \cdot p^x \cdot q^{n-x}$

$$= 6C_4 \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{6-4}$$

$$= 6C_4 (1/2)^4 \left(\frac{1}{2}\right)^2 = \frac{15}{64} = 0.2344.$$

② Ten coins are thrown simultaneously. Find the probability of getting at least (i) seven heads and (ii) six heads

Sol:- probability of getting head $(p) = 1/2$

probability of getting not getting head $(q) = 1/2$

ie, $n = 10$,

(i) probability of getting atleast seven heads:-

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= 10C_7 (1/2)^7 (1/2)^3 + 10C_8 (1/2)^8 \cdot (1/2)^2 + 10C_9 (1/2)^9 \cdot (1/2)^1 + 10C_{10} (1/2)^{10} (1/2)^0$$

$$= 10C_7 (1/2)^{10} + 10C_8 (1/2)^{10} + 10C_9 (1/2)^{10} + 10C_{10} (1/2)^{10}.$$

$$= \left(\frac{1}{2}\right)^{10} \left[10C_7 + 10C_8 + 10C_9 + 10C_{10}\right]$$

$$= \frac{1}{2^{10}} [120 + 45 + 10 + 1] = \frac{176}{1024} = 0.1719.$$

(ii) probability of getting atleast six heads:-

$$P(X \geq 6) = P(X=6) + P(X \geq 7)$$

$$= 10C_6 (1/2)^6 \cdot (1/2)^4 + \frac{1}{2^{10}} (176)$$

$$= \frac{1}{2^{10}} [210 + 176] = \frac{386}{1024} = 0.3769$$

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③ The mean and Variance of binomial distribution are 4 and 4/3 respectively. Find the probability of (i) $P(X \geq 1)$ (ii) $P(X \geq 2)$

Sol:- Mean of binomial distribution $= np = 4$ — ①

Variance of binomial distribution $= npq = 4/3$ — ②

$$\frac{eq ②}{eq ①} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{(4/3)}{4} = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3} \Rightarrow \boxed{q = 1/3}$$

$$\therefore p+q=1 \Rightarrow p = 1-q = 1-\frac{1}{3} = 2/3 \Rightarrow \boxed{p = 2/3}$$

$$\therefore np=4 \Rightarrow n = 4/p = 4/(2/3) = 4 \times \frac{3}{2} = 6 \Rightarrow \boxed{n=6}$$

By the Binomial distribution $P(X=x) = nC_x \cdot p^x \cdot q^{n-x}$

$$P(X=6) = 6C_x (2/3)^x \cdot (1/3)^{6-x}$$

(i) $P(X \geq 1) = 1 - P(X < 1)$
$$= 1 - [P(X=0)] = 1 - P(X=0)$$
$$= 1 - 6C_0 (2/3)^0 (1/3)^6 = 0.99.$$

(ii) $P(X \geq 2) = 1 - P(x < 2)$
$$= 1 - [P(x=0) + P(X=1)]$$
$$= 1 - \left[6C_0 \cdot (2/3)^0 (1/3)^6 + 6C_1 (2/3)^1 (1/3)^5\right] = 0.98.$$

④ Determine the binomial distribution for which the mean is 4 and Variance - 3.

Sol:- Mean of the B.D = 4 ie, $np = 4$ — ①

and Variance of the B.D = 3 ie, $npq = 3$ — ②

$$\frac{②}{①} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{3}{4} \Rightarrow \boxed{q = 3/4}.$$

$$\therefore p = 1-q = 1-3/4 = 1/4 \Rightarrow \boxed{p = 1/4}$$

$n = 4/p$

$\boxed{n = 16}$

Hence the Binomial Distribution is

$$P(X=x) = P(x) = \begin{cases} 16C_4 (1/4)^x (3/4)^{16-x}, & x = 0,1,2,\dots n \\ 0, & \text{otherwise.} \end{cases}$$

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Poisson Distribution :—

A random variable X is said to follow a poisson distribution if it assumes only non-negative values and its probability density function is given by

$$P(X=x) = p(x,\lambda) = \begin{cases} \dfrac{e^{-\lambda}\cdot\lambda^{x}}{x!}, & x=0,1,2,3,\cdots \\ 0, & \text{otherwise.} \end{cases}$$

where $\lambda > 0$ is called the parameter of the distribution.

It is a distribution suitable for rare events for which the probability of occurrence 'p' is very small and the number of trials 'n' is very large, where np is finite.

* The probability distribution function can be expressed as the form $F_X(x) = P(X=x) = e^{-\lambda}\sum_{K=0}^{\infty}\dfrac{\lambda^{K}}{K!}\cdot u(x-K)$. ——— ①

* The probability density function for a discrete random variable X is given by $f_X(x) = \dfrac{d}{dx}\left[F_X(x)\right]$

$$\Rightarrow f_X(x) = e^{-\lambda}\sum_{K=0}^{\infty}\frac{\lambda^{K}}{K!}\,\delta(x-K) \text{ ——— ②}$$

Conditions of poisson distribution :—

① The variable is a discrete variable.

② The occurrences are rare.

③ The number of trials (n) is large.

④ The probability of success (p) is very small (very close to zero)

⑤ $np = \lambda$ is finite.

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Applications of the poisson distribution:—

1. The number of telephone calls made during a period of time.
2. The number of defective elements/items in a given sample.
3. The number of electrons emitted from a cathode in a given time interval.
4. The number of items waiting in a queue etc..
5. The number of persons born blind per year in a large city.

Mean of the poisson distribution:—

$$\text{Mean } (\mu) = E[x] = \sum_{x=0}^{\infty} x \, p(x) = \sum_{x=0}^{\infty} x \cdot \left[\frac{e^{-\lambda} \cdot \lambda^x}{x!}\right]$$

$$= \sum_{x=0}^{\infty} \frac{x \, e^{-\lambda} \cdot \lambda^x}{x(x-1)!}$$

$$= e^{-\lambda} \cdot \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!} \quad \langle \because \text{ let } y = x-1, \; x = y+1 \rangle$$

$$= e^{-\lambda} \cdot \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \cdot \lambda$$

$$= \lambda \cdot e^{-\lambda} \left[e^{+\lambda} \right] = \lambda.$$

$$\Rightarrow \text{Mean } (\mu) = \lambda$$

ie, $\boxed{\text{Mean} = \lambda = np}$

Variance of the poisson distribution:—

$$\text{Variance } (\sigma^2) \text{ (or) } V(x) = \sum_{x=0}^{\infty} x^2 \cdot p(x) - \mu^2$$

$$\text{ie, } V(x) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \lambda^x}{x!} - \lambda^2$$

$$\Rightarrow \boxed{V(x) = \lambda}$$

Note:—

The Mean and variance of the poisson distribution is same.

ie, $\boxed{\mu = V(x) = \lambda = np}$

problems :-

① A hospital switch board receives an average of 4 emergency calls in a 10 minute interval. what is the probability that

(i) there are atmost 2 emergency calls in a 10 minute interval.

(ii) there are exactly 3 emergency calls in a 10 minute interval.

Sol:- Mean $(\lambda) = 4$ calls/10 min

By the poisson distribution, $p(X=x) = p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

(i) $p(\text{atmost } 2 \text{ calls}) = p(X \leq 2) = p(X=0) + p(X=1) + p(X=2)$

$$= \frac{e^{-4} \cdot \lambda^0}{0!} + \frac{e^{-4} \cdot \lambda^1}{1!} + \frac{e^{-4} \cdot \lambda^2}{2!}$$

$$= e^{-4}[1+4+8] = 0.2381$$

(ii) $p(\text{exactly } 3 \text{ calls}) = p(X=3) = \frac{e^{-4} \cdot 4^3}{3!} = 0.1954$.

② If a bank received on the average 6 bad cheques. per day find the probability that it will receive 4 bad cheques on any given day.

Sol:- Mean $(\lambda) = 6$,

we have $p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$p(X=4) = \frac{e^{-6} \cdot 6^4}{4!} = 0.1339.$$

③ A manufacturer of cotter pins knows that 5% of his product is defective. pins are sold in boxes of 100. He guarantees that not more than 10 pins will be defective. what is the approximate probability that a box will fail to meet the guaranteed quality?.

Sol:- No: of defectives $= 5\% = 0.05 = p$.

Total no: of pins in a box $= 100 = n$

Mean $(\lambda) = np = 100 \times 0.05 = 5$

$p(\text{a box will fail to meet the guarantee}) = p(X>10) = 1 - p(X \leq 10)$

$$= 0.0137.$$

④ A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. plot the probability distribution function and probability density function.

Sol:- The probability of defective $(p) = \frac{4}{10} = \frac{2}{5}$.

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No: of items are choosen, $n=3$.

$$\text{Mean}(\lambda) = np = 3 \times \frac{2}{5} = \frac{6}{5} = 1.2 \quad , \quad p(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$\therefore \; p(X=0) = \frac{e^{-1.2} \cdot (1.2)^0}{0!} = 0.301$$

$$p(X=1) = \frac{e^{-1.2} \cdot (1.2)^1}{1!} = 0.361$$

$$p(X=2) = \frac{e^{-1.2} \cdot (1.2)^2}{2!} = 0.216$$

$$p(X=3) = \frac{e^{-1.2} \cdot (1.2)^3}{3!} = 0.0867$$

The probability density function,

X	0	1	2	3
$p(x)$ (or) $f(x)$	0.301	0.361	0.216	0.0867

The probability distribution function:—

$$F_X(x) = p(X \le x)$$

$$F_X(0) = p(X \le 0) = 0.301$$

$$F_X(1) = p(X \le 1) = p(X=0) + p(X=1) = 0.301 + 0.361 = 0.662$$

$$F_X(2) = p(X \le 2) = 0.301 + 0.361 + 0.216 = 0.878 .$$

$$F_X(3) = p(X \le 3) = 0.301 + 0.361 + 0.216 + 0.0867 = 0.964$$

$\Rightarrow$ The probability distribution function can be expressed as

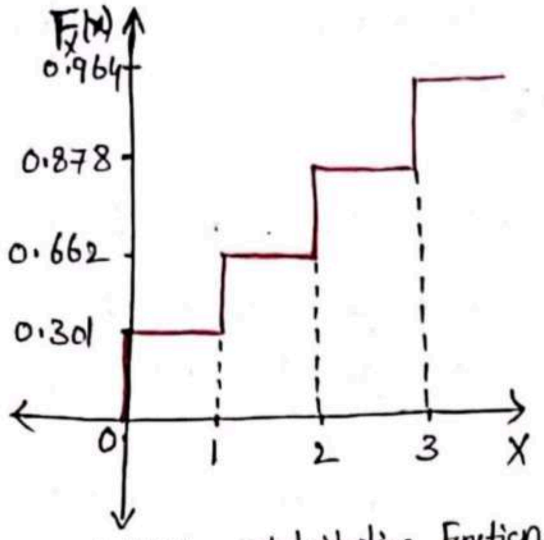
$$F_X(x) = \sum p(x_i) \cdot u(x - x_i) \quad (\text{or}) \quad \sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{k}}{k!} u(x-k) .$$

$$F_X(x) = 0.301 \cdot u(x-0) + 0.361 \cdot u(x-1) + 0.216\, u(x-2) + 0.0867 \cdot u(x-3)$$

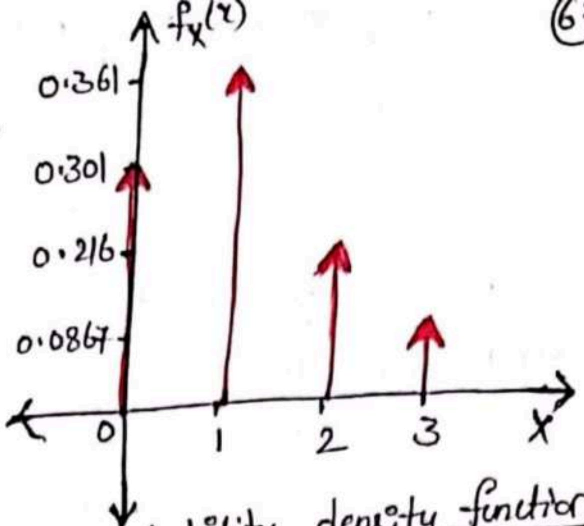
$\Rightarrow$ The probability density function can be expressed as

$$f_X(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{k}}{k!} \cdot \delta(x-k) \quad (\text{or}) \quad \sum p(x_i) \cdot \delta(x - x_i)$$

$$f_X(x) = 0.301\, \delta(x-0) + 0.361\, \delta(x-1) + 0.216\, \delta(x-2) + 0.0867\, \delta(x-3)$$



probability Distribution Function.



probability density function.

(5) A certain large city experiences, on an average, three murders per week. Their occurrence follows a poisson distribution.

a) what is the probability that there are 5 (or) more murders in a given week?.

b) on an average, how many weeks in a year can this city expect to have no murders?.

c) How many weeks per year (average) can the city expect the number of murders per week to equal (or) exceed the average number per week?.

Sol:— Average no: of murders, $\lambda = 3$. , $p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$,

(a) probability that there will be five (or) more murders in a given week: $P(X \geq 5) = 1 - P(X < 5)$

$$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)]$$

$$= 1 - e^{-3}\left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}\right]$$

$$= 1 - e^{-3}\left[\frac{131}{8}\right] = 0.1847$$

(b) probability of no murders: $P(x=0) = \frac{e^{-3} \cdot 3^0}{0!} = 0.0498$.

Average no: of weeks a year, the city has no murders is $52 \times 0.0498 = 2.5889$ weeks.

(c) The probability that the number of murders per week is equal to (or) greater than the average number of murders per week is $P(x \geq 3) = 1 - P(x < 3)$

$\therefore P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$$= 1 - e^{-3}\left[1 + 3 + \frac{3^2}{2!}\right]$$

$$= 1 - e^{-3}\left(4 + \frac{9}{2}\right) = 1 - e^{-3}\left(\frac{17}{2}\right) = 0.5768.$$

The average number of weeks in a year when the number of murders exceeds the average value is

$52 \times 0.5768 = 30$ weeks.

Gaussian Density Function:- The gaussian density function and distribution function of a random variable 'X' are given by

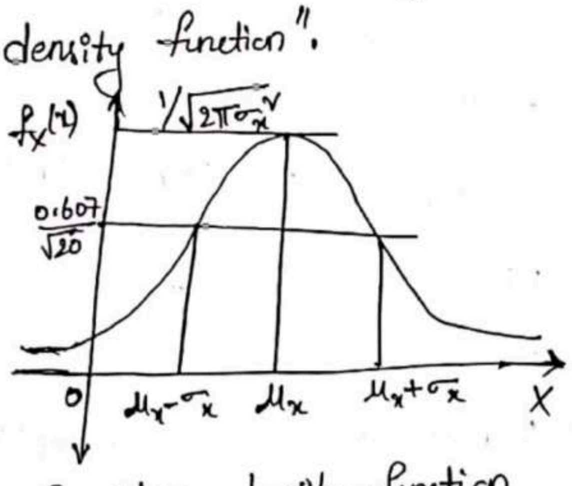
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{(x-\mu_x)^2}{2}\right)} \quad \text{for all } x.$$

and

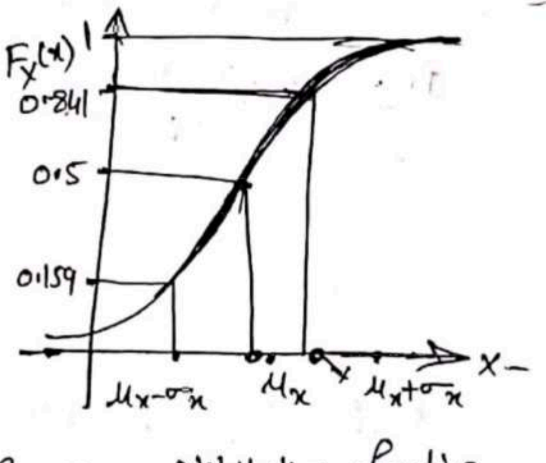
$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\left(\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)} \cdot dx, \quad \text{for all } x.$$

for $\sigma_x > 0$, and $-\infty < \mu_x < \infty$ are constants called standard deviation and mean values of 'X' respectively.

The Gaussian density function is also called the "normal density function".



Gaussian density function



Gaussian Distribution function

* The plot of the density function of Gaussian is bell shaped and symmetrical about its mean value μ_x.
* The total area under the density function is one.

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ie, $$\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{\infty} e^{-\left(\frac{(x-\mu)^2}{2\sigma_x^2}\right)}\,dx = 1$$

Normalised Gaussian distribution function:-

If the Gaussian distribution function has $\mu_x = 0$ and $\sigma_x = 1$ then it is called a normalised Gaussian distribution function. It is denoted as $F(x)$, and given by

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}\,dx \quad , \text{ for } x \geq 0 \quad \text{——} (1)$$

and $F(-x) = 1 - F(x)$. , for $x < 0$

The relation ship between $F_X(x)$ and $F(x)$,

By the Gaussian distribution function, we have

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{x} e^{-\left(\frac{(x-\mu)^2}{2\sigma_x^2}\right)}\,dx \quad \text{——} (2)$$

Let $u = \frac{x-\mu_x}{\sigma_x} \Rightarrow du = \frac{dx}{\sigma_x}$

$\Rightarrow dx = \sigma_x\,du.$

$$(2) \Rightarrow F_X(x) = \frac{1}{\sqrt{2\pi}\cdot\cancel{\sigma_x}}\int_{-\infty}^{u} e^{-u^2/2}\,du\cdot\cancel{\sigma_x}$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-u^2/2}\,du.$$

$$F_X(x) = F(u) \qquad \langle \because \text{from } (1)\rangle$$

$$\boxed{\therefore\ F_X(x) = F\left(\frac{x-\mu_x}{\sigma_x}\right)}$$

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The function $F(x)$ can be evaluated using either the Q-function approximation (or) {the complementary error function}.

* The Q-function approximation is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-x^2/2}\, dx.$$

The approximation of $Q(x)$ given by Borjesson and Sundberg is

$$Q(x) = \left[\frac{1}{0.661x + 0.339\sqrt{x^2+5.51}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad x \geq 0.$$

Then $\boxed{F(x) = 1 - Q(x)}$

problems:—

① Find the probability of event $\{X \leq 5.5\}$ for a Gaussian random variable having $\mu_X = 3$ and $\sigma_X = 2$.

Sol:— Given a gaussian random variable, the probability of event $\{X \leq 5.5\}$ is given by.

$$P\{X \leq x\} = F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\left(\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)} \cdot dx.$$

Given that $\mu_X = 3$ and $\sigma_X = 2$, for normalised function.

$$F_X(x) = F\left(\frac{x-\mu_x}{\sigma_x}\right)$$

$$\therefore P\{X \leq 5.5\} = F_X(5.5) = F\left(\frac{5.5-3}{2}\right) = F(1.25)$$

Now to find $F(1.25)$ by using Q-function,

we know that $F(x) = 1 - Q(x)$

$$F(1.25) = 1 - Q(1.25) \quad \text{———} \; ①$$

where $Q(x) = \left(\frac{1}{0.661 \cdot x + 0.339\sqrt{x^2+5.51}}\right) \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

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$$Q(1.25) \simeq \left[\frac{1}{(0.661)(1.25)+0.339\sqrt{(1.25)^2+5.51}}\right]\left[\frac{e^{-(1.25)^2/2}}{\sqrt{2\pi}}\right]$$

$$Q(1.25) \simeq 0.1056.$$

$\therefore$ ① $\Rightarrow$ $F(1.25) = 1-0.1056 = 0.8944.$

$\therefore P\{X \le 5.5\} = F_X(5.5) = F(1.25) = 0.8944.$

② A Gaussian random variable X with $\mu_x = 4$ and $\sigma_x = 3$ is generated. Find the probability of $X \le 7.75$.

Sol:- Given a Gaussian random variable with $\mu_x = 4$ and $\sigma_x = 3$.

Now to find the probability of $P(X \le 7.75)$

ie, we have $P(X \le 7.75) = F_X(7.75) = F\left(\frac{7.75-4}{3}\right)$

$$= F(1.25)$$

By the Q-function approximation, we have

$$F(x) = 1 - Q(x), \text{ where } Q(x) = \left[\frac{1}{0.66(x)+0.34\sqrt{x^2+5.5}}\right]\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$F(1.25) = 1 - \left[\frac{1}{0.66(1.25)+0.34\sqrt{(1.25)^2+5.5}}\right]\frac{e^{-\frac{(1.25)^2}{2}}}{\sqrt{2\pi}}$$

$$F(1.25) = 1 - 0.1056 = 0.8944.$$

$\therefore$ The Gaussian density function is,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\left(\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi(9)}} \cdot e^{-\frac{(x-4)^2}{18}} \quad \langle \because \mu_x = 4,\ \sigma_x = 3 \rangle$$

$$f_X(x) = 0.133\, e^{-\frac{(x-4)^2}{18}}.$$

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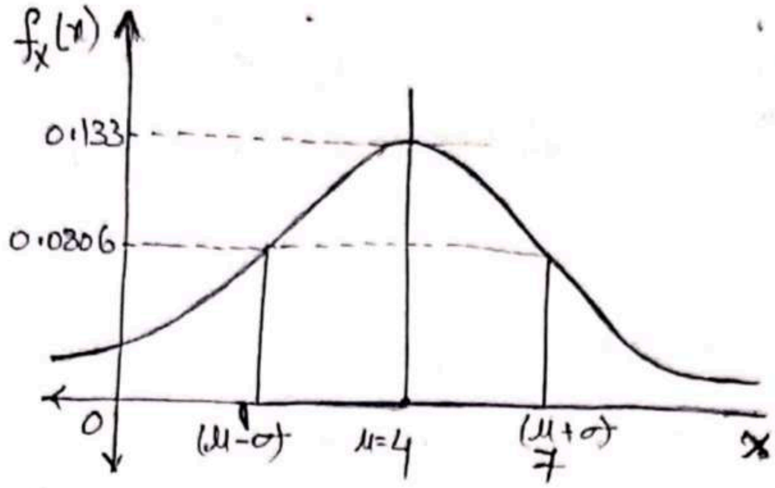

Since $f_X(1) = 0.0806$

$f_X(\mu) = 0.133$

$f_X(7) = 0.0806$

Gaussian density function for $\mu_x = 4$, & $\sigma_x = 3$

(3) Assume that the height of clouds σ_x above the ground at some location is a Gaussian random variable 'X' with mean value 2km and $\sigma_x = 0.25$ km. Find the probability of clouds higher than 2.5 km.

Sol:— Given a Gaussian random variable X.

Mean Value $\mu_x = 2km$ and $\sigma_x = 0.25 km$.

Now to find $P\{x > 2.5 km\} = 1 - P(x \le 2.5)$ —— (1)

$\therefore$ we have $P(x \le 2.5) = F_X(2.5) = F\left(\frac{2.5-2}{0.25}\right) = F(2)$

$\therefore F(2) = 1 - Q(2)$

$$= 1 - \left[\frac{1}{(0.66 \times 2) + 0.34\sqrt{2^2 + 5.51}}\right] \frac{e^{-2^2/2}}{\sqrt{2\pi}}$$

$= 1 - 0.0228$.

(1) $\Rightarrow P(x > 2.5) = 1 - F(2)$

$= 1 - [1 - 0.0228]$

$$\boxed{P(x > 2.5) = 0.0228}$$

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Uniform Density Function :-

The uniform probability density function is defined as

$$f_X(x) = \begin{cases} \frac{1}{b-a} , & a \le x \le b \\ 0 , & \text{else where.} \end{cases}$$

where a, b are real constants, $-\infty < a < \infty$, and $b > a$.

The uniform distribution function is

$$F_X(x) = \int_a^x f_X(x)\, dx = \int_a^x \frac{1}{b-a}\, dx$$

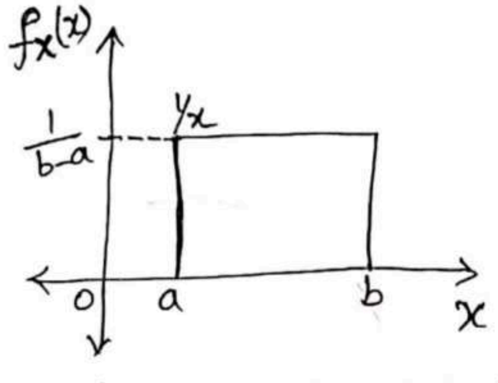
$$= \frac{1}{b-a} \left[x \right]_a^x$$

$$F_X(x) = \frac{x-a}{b-a} . \quad \text{——} \ (1)$$

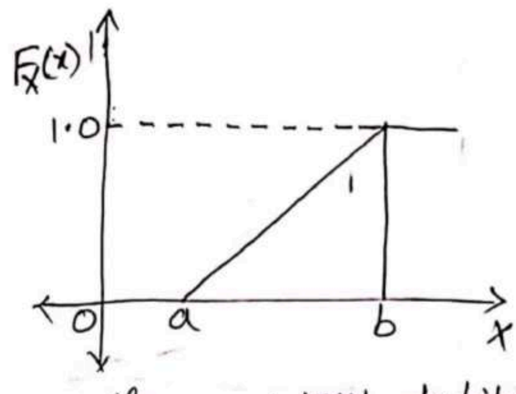
$$\therefore \ F_X(a) = \frac{a-a}{b-a} = 0.$$

$$F_X(b) = \frac{b-a}{b-a} = 1$$

$$\therefore \ F_X(x) = \begin{cases} 0 , & x > a \\ \frac{x-a}{b-a} , & a \le x < b \\ 1 , & x \ge b \end{cases}$$



uniform probability density function.



uniform probability distribution function.

The uniform probability density function has a constant amplitude in the given range. The total area of the function is always one.

Exponential probability Density function:-

The exponential probability density function for a continuous random variable x is defined as
$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b}, & x > a \\ 0, & x < a. \end{cases}$$
where a and b are real constants, $-\infty < a < \infty$ and $b > 0$.

The distribution function is, $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

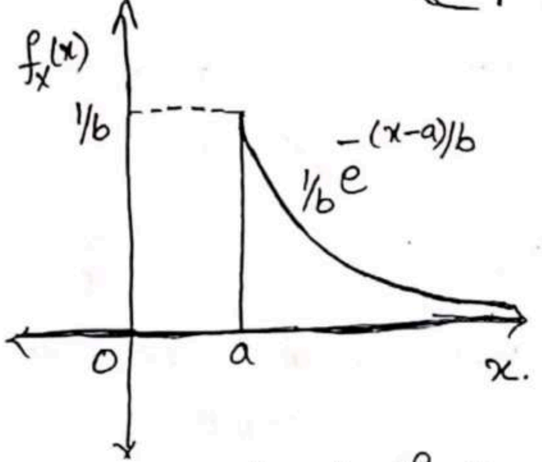
$$F_X(x) = \int_a^x \frac{1}{b} e^{-(x-a)/b} \cdot dx$$

$$F_X(x) = -\frac{1}{\cancel{b}}\left[\cancel{b} \cdot e^{-(x-a)/b}\right]_a^x$$

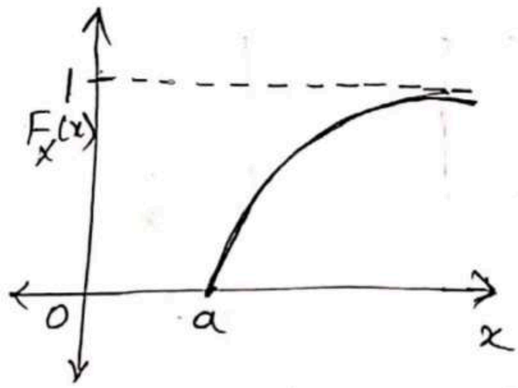
$$F_X(x) = e^0 - e^{-(x-a)/b}$$

$$F_X(x) = 1 - e^{-(x-a)/b}.$$

$$\therefore F_X(x) = \begin{cases} 0, & x \le a \\ 1 - e^{-(x-a)/b}, & x > a \\ 1, & x = \infty \end{cases}$$



Exponential density function.



Exponential distribution function.

The exponential probability density function has a maximum value of $\frac{1}{b}$ at $x = a$. The function starts at $x = a$, and exponentially decreases to zero at $x = \infty$.

The distribution function starts at $x=a$ and exponentially increases to one at $x=\infty$.

Problems:—

① The power reflected from an aircraft received by a radar is described by an exponential distribution. The pdf. is given by $f_X(x) = \frac{1}{10} e^{-x/10}, \; x>0$. The average power is 10W. what is the probability that the received power is greater than the average power?

Sol:– Given that the power reflected is an exponential distribution.

The pdf is given as $f_X(x) = \frac{1}{10} e^{-x/10}, \quad x>0.$

$$F_X(x) = \int_0^x f_X(x)\, dx = \frac{1}{10}\int_0^x e^{-x/10}\, dx$$

$$= \frac{1}{10}\left[-10\, e^{-x/10}\right]_0^x$$

$$= -\left[e^{-x/10} - e^0\right]$$

$$F_X(x) = 1 - e^{-x/10}, \quad x>0$$

The required probability is $P(x>10)$

$$\therefore P(x>10) = 1 - P(x \le 10)$$

$$= 1 - F_X(10)$$

$$= 1 - \left(1 - e^{-10/10}\right)$$

$$= \cancel{1} - \cancel{1} + e^{-1}$$

$$P(x>10) = e^{-1} \simeq 0.368$$

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Rayleigh probability Density function :—

The Rayleigh probability density function of a random variable X is defined as

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) \cdot e^{-(x-a)^2/b}, & x \geq a \\ 0, & x < a. \end{cases}$$

where 'a' and 'b' are real constants, $-\infty < a < \infty$. and $b > 0$.

The distribution function is $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$F_X(x) = \int_a^x \frac{2}{b}(x-a) \cdot e^{-(x-a)^2/b} \cdot dx.$$

let us take $y = \frac{(x-a)^2}{b}$

$$dy = \frac{2(x-a)}{b} \cdot dx$$

$$\therefore F_X(x) = \int_a^x e^{-(x-a)^2/b} \cdot \frac{2(x-a)}{b} \cdot dx$$

$$F_X(x) = \int_a^x e^{-y} \cdot dy = \left[-e^{-y}\right]_a^x$$

$$= \left[-e^{-\left(\frac{(x-a)^2}{b}\right)}\right]_a^x$$

$$= -\left[e^{-(x-a)^2/b} - e^0\right]$$

$$\boxed{F_X(x) = 1 - e^{-(x-a)^2/b}}$$

$$\therefore F_X(x) = \begin{cases} 0, & x < a \\ 1 - e^{-(x-a)^2/b}, & x \geq a \\ 1, & x = \infty. \end{cases}$$

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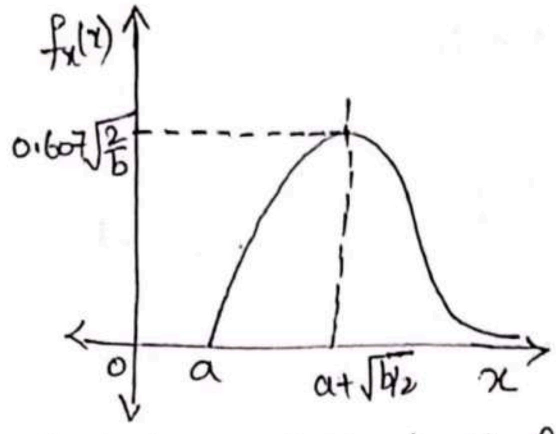
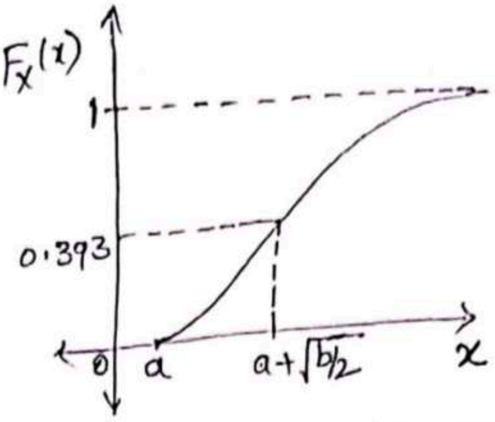


Rayleigh probability density function



Rayleigh distribution function

Problems :-

① A Rayleigh density function is given by $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

ⓐ prove that $f(x)$ satisfies the properties of the pdf (i) $f(x) \geq 0, \forall x$ and (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ⓑ Find the distribution function $F(x)$

ⓒ Find $P(0.5 \leq X \leq 2)$ and $P(0.5 < X < 2)$.

Sol:- Given $f_X(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

ⓐ (i) By substituting x values, for $x \geq 0$, it is observed that $f_X(x)$ is always greater than (or) equal to zero.

(ii) Evaluate $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} x e^{-x^2/2}\,dx = \int_{0}^{\infty} x e^{-x^2/2}\,dx$$

Let $x^2/2 = t \Rightarrow x\,dx = dt$

$$= \int_{0}^{\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{\infty} = -\left[e^{-\infty} - e^{0}\right]$$

$$= 1 - 0$$

$$= 1$$

$$\boxed{\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1}$$

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(b) The distribution function $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \int_0^x x\,e^{-x^2/2}\,dx \qquad \text{let } x^2/2 = t, \quad x\,dx = dt.$$

$$= \int_0^t e^{-t}\,dt = \left[-e^{-t}\right]_0^t = -e^{-t} + e^0$$

$$= 1 - e^{-t}$$

$$= 1 - e^{-x^2/2}$$

$$\therefore \boxed{F_X(x) = 1 - e^{-x^2/2}} \quad \text{for } x \geq 0.$$

Hence $F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2/2}, & x \geq 0 \\ 1, & x = \infty. \end{cases}$

(c) $P(0.5 \leq x \leq 2) = P(x \leq 2) - P(x \leq 0.5)$

$$= F_X(2) - F_X(0.5)$$

$$= 1 - e^{-2^2/2} - 1 + e^{-0.5^2/2} = e^{-1/8} - e^{-2}.$$

$$= 0.7472.$$

(d) $P(0.5 < x < 2) = P(X < 2) - P(X < 0.5)$

$$= P(X \leq 2) - P(X = 2) - P(X \leq 0.5) + P(X = 0.5)$$

$$= F_X(2) - F_X(0.5) + P(X = 0.5) - P(X = 2) \quad \text{--- (1)}$$

$$\therefore P(X = 0.5) = f_X(0.5) = \frac{1}{2} e^{-1/8} = 0.4412$$

$$P(X = 2) = f_X(2) = 2e^{-2} = 0.2706.$$

$$(1) \Rightarrow P(0.5 < x < 2) = e^{-1/8} - e^{-2} + \frac{1}{2} e^{-1/8} - 2e^{-2}$$

$$= \frac{3}{2} e^{-1/8} - 3e^{-2}$$

$$= \frac{1}{2}\left[3 e^{-1/8} - 6 e^{-2}\right] = 0.9178.$$

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(2) The probability density function of a random variable X is given by $f_X(x) = \begin{cases} K, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$, where '$K$' is a constant. (i) Determine the value of K.

(ii) Let $a=1$, $b=2$, Calculate $P(|x| \le c)$ for $c = 0.5$.

Sol:-

Given that $f_X(x) = \begin{cases} K, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

(i) If the function is a valid pdf, then $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\Rightarrow \int_a^b k \cdot dx = 1$$

$$\Rightarrow k \cdot [x]_a^b = 1$$

$$\Rightarrow k \cdot (b-a) = 1 \Rightarrow \boxed{k = \frac{1}{b-a}}$$

(ii) Given $a=1$ and $b=2$ $\Rightarrow k = \frac{1}{2-1} = \frac{1}{1} = 1$

Then $f_X(x) = \begin{cases} 1, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

$\therefore P(|x| \le c)$, at $c = 0.5$,

$$P(|x| \le 0.5) = P\{-0.5 \le x \le 0.5\}$$

$$= \int_{-0.5}^{0.5} f_X(x)\,dx$$

$$= \int_{-0.5}^{0.5} 1\,dx = [x]_{-0.5}^{0.5}$$

$$= 0.5 + 0.5$$

$$= 1. //$$

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(3) If the probability density of a random variable is given by $f_X(x) = \begin{cases} c\,e^{-x/4}, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

find the value that 'c' must have and evaluate $F_X(0.5)$.

Sol:- Given probability density function,

$$f_X(x) = \begin{cases} c\,e^{-x/4}, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

If $f_X(x)$ is a valid density function, then $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

$$\Rightarrow \int_0^1 f_X(x)\,dx = 1$$

$$\int_0^1 c\,e^{-x/4}\,dx = 1$$

$$c\left[-4e^{-x/4}\right]_0^1 = 1 \Rightarrow c(-4)\left[e^{-1/4} - e^{0}\right] = 1$$

$$\Rightarrow -4c\left[e^{-1/4} - 1\right] = 1$$

$$\Rightarrow c = \frac{1}{4\left[1 - e^{-1/4}\right]} = 1.13$$

$$\boxed{c = 1.13}$$

$$\therefore F_X(0.5) = \int_0^{0.5} 1.13 \cdot e^{-x/4}\,dx = 1.13\left[-4e^{-x/4}\right]_0^{0.5}$$

$$= -4.52\left[e^{-0.5/4} - e^{0}\right]$$

$$\boxed{F_X(0.5) = 0.5311}$$

(4) A random variable 'X' has a pdf. $f_X(x) = \begin{cases} c(1-x^4), & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

(i) find c, (ii) find $P[|x| < 1/2]$

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Unit - 2

Operations on random Variables

Mathematical Expectation: The mean or average value of a random variable X Probability distribution function of a random variable X is called mathematical expectation of X. It is denoted as $E[X]$ or $\bar{x}$.

Expected value of a random variable: If X is a continuous random variable with a valid probability density function $f_X(x)$, then the expected value of X is defined as $E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$

If X is a discrete random variable with set of elements $\{x_1, x_2 ---- x_n\}$ and a set of corresponding probabilities $\{P(x_1), P(x_2), ---- P(x_n)\}$ then the expected value of X is defined as $E[X] = \bar{X} = \sum_{i=1}^{n} x_i P(x_i)$.

Note: If $P(x_1) = P(x_2) = P(x_3) ---- P(x_n) = \frac{1}{n}$ then the expected value is $E[X] = \bar{X} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{n}(x_1 + x_2 + --- + x_n)$

Expected value of a function:

Consider a random variable X with probability density function $f_X(x)$. If $g(x)$ is a real function of x then expected value of $g(x)$ is defined as

For Continuous: $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

For discrete: $E[g(x)] = \sum_{i=1}^{n} g(x_i) P(x_i)$

Properties of expectations:

1. If a random variable x is a Constant i.e., $x=a$ then $E[a]=a$. where a is a Constant

By the definition

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$E[a] = \int_{-\infty}^{\infty} a f_x(x)\, dx$$

$$= a\int_{-\infty}^{\infty} f_x(x)\, dx$$

$$E[a] = a$$

2. If a any Constant then. $E[ax] = a \cdot E[x]$

$$E[ax] = \int_{-\infty}^{\infty} (ax) f_x(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$E[ax] = a \cdot E[x]$$

3. $E[ax+b] = a \cdot E[x] + b$

Similarly

$E[a] = a$

$E[ax] = a \cdot E[x]$

$E[ax+b] = a \cdot E[x] + b$

4. $|E[x]| \leq E|x|$

5. $E[g(x_1)+g(x_2)] = E[g(x_1)] + E[g(x_2)]$

$$= a\int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$E[ax] = a \cdot E[x]$$

3. $E[ax+b] = a\cdot E[x] + b$

Similarly

$$E[a] = a$$

$$E[ax] = a\cdot E[x]$$

$$E[ax+b] = a\cdot E[x] + b$$

4 $|E[x]| \leq E|x|$

5. $E[g(x_1)+g(x_2)] = E[g(x_1)] + E[g(x_2)]$

i) $E[x] = \sum_{i=1} x_i\, P(x_i)$

$$= -2\left[\tfrac{1}{5}\right] + (-1)\tfrac{2}{5} + 0\left(\tfrac{1}{10}\right) + 1\left(\tfrac{1}{10}\right) + 2\left(\tfrac{1}{5}\right)$$

$$= \frac{-2}{5} \cancel{-\frac{2}{5}} + \frac{1}{10} \cancel{+\frac{2}{5}}$$

$$= \frac{-2}{5} + \frac{1}{10}$$

$$= \frac{-4+1}{10} = \frac{-3}{10}$$

ii) $E[2x+3] = 2E[x] + E[3]$

$$= 2\left(\frac{-3}{10}\right) + 3$$

$$= \frac{-3}{5} + 3$$

$$= \frac{-3+15}{5} = \frac{12}{5}$$

iii) $E[x^2] = \sum_{i=1}^{5} x_i^2 \, p(x_i)$

$= x_1^2 \, p(x_1) + x_2^2 \, p(x_2) + x_3^2 \, p(x_3) + x_4^2 \, p(x_4) + x_5^2 \, p(x_5)$

$= 4\left(\frac{1}{5}\right) + 1\left(\frac{2}{5}\right) + 0 + 1\left(\frac{1}{10}\right) + 4\left(\frac{1}{5}\right)$

$= \frac{4}{5} + \frac{2}{5} + \frac{1}{10} + \frac{4}{5}$

$= \frac{21}{10} = 2.1$

Moments :

There are two types of moments for a function of a random Variable x

1. Moments about the origin
2. Moments about the mean (or) Central moments

1. **Moments about the origin:** Let $g(x)$ be a real function of the random Variable x Such that $g(x) = x^n$ for $n = 0, 1, 2, 3 \ldots$ then the expected value of the function $g(x)$ is called the moments about the origin of a random Variable x, it is denoted as 'm_n' Where n indicates the order of the moment mathematically the n^{th} moment is defined as Continuous

$$\boxed{m_n = \int_{-\infty}^{\infty} x^n f_X(x) \, dx = E[x^n]}$$

discrete

$$\boxed{m_n = E[x^n] = \sum_{i=1}^{n} x_i^n \, p(x_i)}$$

i) If $n = 0$, $m_0 = \int_{-\infty}^{\infty} f_X(x) \, dx$

It is the zeroth moment of x, it gives the area of the p.d.f i.e., $\boxed{m_0 = 1}$

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ii) If $n=1$, $m_1 = \int_{-\infty}^{\infty} x f_X(x)\,dx$

It is the first moment of the x is equal to its mean value i.e., $m_1 = E[x] = \bar{x}$

iii) If $n=2$, $m_2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

It is the Second moment of the x, it gives the mean Square value of x.

Moments about the mean:

Let $g(x)$ be the real function of random Variable x Such that $g(x) = (x-\bar{x})^n$ where $n = 0,1,2,3 \dots$ where $\bar{x}$ is the mean of x. Then the expected value of $g(x)$ is called the moments about the mean (or) the Central moments of random Variable X. It is denoted as μ_n. Where n indicates the order of the moment Mathematically the n^{th} moment about the mean is given as for Continuous:

$$\boxed{\mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\,dx}$$

for discrete:

$$\boxed{\mu_n = \sum_{i=1}^{n} (x_i - \bar{x})^n P(x_i)}$$

* If $n=0$, it gives the 0^{th} central moment of x and it gives the area of pdf.

$$\boxed{\mu_0 = \int_{-\infty}^{\infty} f_X(x)\,dx}$$

its total area is zero.

$$\boxed{\mu_0 = 1}$$

* If $n=1$, it gives the 1^{st} Central moment of x and it is the mean Value

$$\mu_1 = \int_{-\infty}^{\infty} (x-\bar{x}) f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx - \int_{-\infty}^{\infty} \bar{x} f_X(x)\, dx$$

$$= \bar{X} - \bar{X} \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= \bar{X} - \bar{X}(1)$$

$$\boxed{\mu_1 = 0}$$

The first Central moment of x is always equal to zero.

Variance: The Variance of the density function $f_X(x)$ for a random Variable X is defined as the Second Central moment μ_2 of X $\mu_2[x]$. It is also denoted as σ_x^2 (or) $var[x]$

$$\mu_2 = E[(x-\bar{x})^2] = \sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\, dx$$

$$\mu_2 = E[(x-\bar{x})^2] = var[x] = \sum_{i=1}^{n} (x_i-\bar{x})^2 P(x_i)$$

Note:-

Origin: $m_n = \int_{-\infty}^{\infty} x^n F_X(x)\, dx$

If $n=0, \Rightarrow m_0 = 1$

$n=1, \Rightarrow m_1 = \bar{x} = E[x]$

$n=2, \Rightarrow m_2 = E[x^2]$

m_3

Central:

$$\mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n f_x(x)\, dx$$

If $n=0$, $\mu_0 = 1$

$n=1$, $\mu_2 = \sigma_x^2$

$n=2$, $\mu_2 = \sigma_x^2$

μ_3

(4)

Find the variance of x for a uniform probability density function.

By the definition of uniform density function

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b$$

Variance $\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\,dx \rightarrow (1)$

Now to find $\bar{x} = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2(b-a)}[b^2-a^2]$$

$$= \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

$$\sigma_x^2 = \int_a^b \left[x-\left(\frac{a+b}{2}\right)\right]^2\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\int_a^b \left[x^2+\left(\frac{a+b}{2}\right)^2 - \frac{2x(a+b)}{2}\right]dx$$

$$= \frac{1}{b-a}\left[\int_a^b x^2\,dx + \frac{(a+b)^2}{4}\int_a^b 1\,dx - (a+b)\int_a^b x\,dx\right]$$

$$= \frac{1}{b-a}\left[\left(\frac{x^3}{3}\right)_a^b + \frac{(a+b)^2}{4}(x)_a^b - (a+b)\left(\frac{x^2}{2}\right)_a^b\right]$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3} + \frac{(a+b)^2(b-a)}{4} - \left(\frac{a+b}{2}\right)(b^2-a^2)\right]$$

$$= \frac{1}{b-a}\left[\frac{(b-a)(b^2+ab+b^2)}{3} + \frac{(a+b)^2(b-a)}{4} - \frac{(a+b)^2(b-a)}{2}\right]$$

$$= \frac{(b-a)}{(b-a)}\left[\frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}\right]$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

Skew and coefficient of Skewness:

The Skew of the density $f_X(x)$ for a random Variable x is defined as the third central moment of x it is denoted as μ_3

$$\mu_3 = E[(x-\bar{x})^3] = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_X(x)\, dx$$

for a discrete random Variable

$$\mu_3 = E[(x-\bar{x})^3] = \sum_{i=1}^{n} (x_i-\bar{x})^3 \, P(x_i)$$

The third Central moment or Skew is a measure of the assymmetry of $f_X(x)$ about its mean. It is the amount of deviation of Symmetry from the mean value.

Coefficient of Skewness:

The normalised thought Centralized moment (or) the ratio of the 3rd central moment to the cube of Standard deviation is called Skewness of the density function (or) the coefficient of Skewness.

It is defined as coefficient of Skewness

$$\frac{\mu_3}{\sigma_x^3} = \frac{E[(x-\bar{x})^3]}{E[(x-\bar{x})^2]^{3/2}}$$

$\sigma_x^2 = E[(x-\bar{x})^2]$

$\sigma_x = [E[(x-\bar{x})^2]]^{1/2}$

$\sigma_x^3 = [E[(x-\bar{x})^2]]^{3/2}$

Properties of variances:

The variance of a constant is zero.

Proof: Let K is a constant i.e., $x = K$

$var[x] = E[(x-\bar{x})^2]$

$var[K] = E[(K-K)^2]$

$var[K] = 0$

if K is a constant then for the random variable x $var[Kx] = K^2 \, var[x]$

Proof: We know that $var[Kx] = E[(x-\bar{x})^2]$

$var[Kx] = E[(Kx - K\bar{x})^2]$

$= E[(K(x-\bar{x}))^2]$

$= E[K^2(x-\bar{x})^2]$

$= K^2 \cdot E[(x-\bar{x})^2]$

$= K^2 \, var[x]$

If x is a random variable and (a, b) are real constants then $var[ax+b] = a^2 \, var[x]$

Proof: $var[(ax+b)] = E[(ax+b) - \overline{(ax+b)})^2] \rightarrow (1)$

We know that $\overline{(ax+b)} = E[ax+b]$

$= a \cdot E[x] + b \rightarrow (2)$

$var[a+b] = E[(ax+b) - a \cdot E[x] + b]^2]$

$= E[(ax+\cancel{b} - a E[x] - \cancel{b})^2]$

$= E[(ax - a\bar{x})^2]$

$= E[a^2(x-\bar{x})^2]$

$= a^2 E[(x-\bar{x})^2]$

$= a^2 \, var[x]$

if two Variables x_1 & x_2 are independent then

$Var[x_1 + x_2] = Var[x_1] + Var[x_2]$

if $var[x_1 - x_2] = var[x_1] + Var[x_2]$.

Relationship between Central moments & moments about the origin.

Let x be a random Variable the n^{th} central moment is given by $\mu_n = \sum_{i=1}^{n} (x_i - \bar{x})^n p(x_i)$

$$\mu_n = E[(x-\bar{x})^n] \rightarrow (1)$$

By the binomial theorem we have

$$(x+y)^n = \sum n_{C_r} \cdot x^r y^{n-r}$$

$$(x-\bar{x})^n = \sum_{K=0}^{n} n_{C_K} x^{n-K} \bar{x}^K (-1)^K$$

Now the expected Value is

$$E[(x-\bar{x})^n] = \sum_{K=0}^{n} n_{C_K} (-1)^K E(x^K)^{n-K} (\bar{x})^K \rightarrow (2)$$

$$= \sum_{n=0}^{n} n_{C_K} (-1)^K E(x)^{n-K} (\bar{x})^K \rightarrow (2)$$

We know that

$$\bar{x} = m_1$$

$$E[x^n] = m_n$$

$$E[x^{n-K}] = m_{n-K}$$

$$2 \rightarrow \boxed{E[(x-\bar{x})^n] = \sum_{K=0}^{n} (-1)^K n_{C_K} m_1^K m_{n-K}}$$

for $n=2$:

$$\sigma_x^2 = (-1)^0 \, 2c_0 \, m_1^0 m_2 + (-1)^1 \, 2c_1 m_1 \cdot m_1 + (-1)^2 \, 2c_2 \, m_1^2 \cdot m_0$$

$$= m_2 - 2m_1 2 + m_1^2$$

$$\boxed{M_2 = \sigma_x^2 = m_2 - m_1^2}$$

for $n=3$:

$$\mu_n = \sum_{K=0}^{n} (-1)^K \, n_{C_K} \, m_1^K \, m_{n-K}$$

$$\mu_n = \sum_{K=0}^{n=3} (-1)^K \, 3c_K \, m_1^K \, m_{3-K}$$

$$= (-1)^0 \, 3c_0 \, m_1^0 \, m_3 + (-1)^1 \, 3c_1 \, m_1^1 \, m_2 + (-1)^2 \, 3c_2 \, m_1^2 \, m_1 + (-1)^3 \, 3c_3 \, m_1^3 \, m_0$$

$$= m_3 - 3m_1 m_2 + 3m_1^2 m_1 - m_1^3$$

$$\mu_3 = m_3 - 3m_1[m_2 - m_1^2] - m_1^3 \quad \text{(or)}$$

$$\mu_3 = m_3 - 3m_1 m_2 + 2m_1^3$$

$$\boxed{\mu_3 = \overline{x^3} - 3\bar{x} \, \sigma_x^2 - (\bar{x})^3}$$

(1.) Find out the skew and skewness of a uniform Probability density function for a random Variable x.

We know that for a uniform probability density function has the mean value $\bar{x} = \frac{a+b}{2}$ and the variance $\sigma_x^2 = \frac{(a-b)^2}{12}$

Now to find $\overline{x^3} = \int_{-\infty}^{\infty} x^3 f_X(x)\, dx$

$E[x^3]$

But $f_X(x) = \frac{1}{b-a} \quad a \le x \le b$

$$= \int_a^b x^3 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \left[\frac{x^4}{4}\right]_a^b$$

$$= \frac{b^4 - a^4}{4(b-a)} = \frac{(b^2)^2 - (a^2)^2}{4(b-a)}$$

$$= \frac{(b^2+a^2)(b^2-a^2)}{4(b-a)}$$

$$= \frac{(a^2+b^2)(a+b)\cancel{(b-a)}}{4\cancel{(b-a)}}$$

$$= \frac{(a+b)(a^2+b^2)}{4}$$

Skewness $\mu_3 = \overline{x^3} - 3\bar{x}\,\sigma_x^2 - (\bar{x})^3$

$$= \frac{(a+b)(a^2+b^2)}{4} - \cancel{3}\left(\frac{a+b}{2}\right)\left(\frac{(a-b)^2}{\cancel{12}_4}\right) - \frac{(a+b)^3}{8}$$

$$= \frac{(a+b)}{4}\left[a^2+b^2 - \frac{(a-b)^2}{2} - \frac{(a+b)^2}{2}\right]$$

$$= \frac{(a+b)}{4}\left[\frac{\cancel{2a^2} + \cancel{2b^2} - \cancel{a^2} - \cancel{b^2} + \cancel{2ab} - \cancel{a^2} - \cancel{b^2} - \cancel{2ab}}{2}\right]$$

$$= 0$$

Functions for Moments:

To calculate the n^{th} moment of a random variable x, two functions are generally used.

1. characteristic function
2. Moment generating function

1. characteristic function: Consider a random variable x with a pdf $f_X(x)$. Then the expected value

of the function $e^{j\omega x}$ is called the characteristic function. It is expressed as $\phi_x(\omega) = E[e^{j\omega x}]$.

It is a function of the real variable, ω lies in b/w $-\infty < \omega < \infty$ where 'j' is an imaginary operator.

$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx \quad [\text{for continuous}]$$

for a discrete random variable

$$\phi_x(\omega) = E[e^{j\omega x}] = \sum_i e^{j\omega x_i} p(x_i)$$

Note:

If $\phi_x(\omega)$ is a characteristic function of a random variable x then the n^{th} moment of x is given by

$$\boxed{m_n = (-j)^n \left.\frac{d^n(\phi_x(\omega))}{d\omega^n}\right|_{at\,\omega=0}}$$

Properties of the characteristic function

(1.) The characteristics function is unity at $\omega=0$ i.e.,

$$\boxed{\phi_x(0) = 1}$$

Proof: By the characteristic function $\phi_x(\omega) = E[e^{j\omega x}]$

$$\phi_x(0) = E[e^{0}]$$
$$= E[1]$$
$$\therefore \phi_x(0) = 1$$

(2.) If $\phi_x(\omega)$ is a characteristic function of a random variable x then $\boxed{\phi_x(\omega) = \phi_{cx}(\omega)}$

where c is a real constant.

Proof: By the characteristic function $\phi_x(\omega) = E[e^{j\omega x}]$

$$\phi_x(\omega) = E[e^{j\omega x}]$$
$$= E[e^{j\omega(cx)}]$$
$$= \phi_{cx}(\omega)$$

(3.) If $\phi_x(\omega)$ is a characteristic function of a random Variable x then the characteristic function $y = ax+b$ is given by $\boxed{\phi_x(\omega) = e^{j\omega b} \cdot \phi_x(a\omega)}$.

<u>Proof:</u> where a, b are real constants

By the characteristic function $\phi_x(\omega) = E[e^{j\omega x}]$

$$= E[e^{j\omega x}]$$

$$= E[e^{j\omega(ax+b)}]$$

$$= E[e^{j\omega ax + j\omega b}]$$

$$= E[e^{j\omega ax} \cdot e^{j\omega b}]$$

$$= E[e^{j(\omega a)x}][e^{j\omega b}]$$

$$= e^{j\omega b} \phi_x(a\omega)$$

(4.) If x_1 & x_2 are two independent random variables then $\boxed{\phi_{x_1+x_2}(\omega) = \phi_{x_1}(\omega) \cdot \phi_{x_2}(\omega)}$

<u>Proof:</u> By the characteristic function

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$\phi_{x_1+x_2}(\omega) = E[e^{j\omega(x_1+x_2)}]$$

$$= E[e^{j\omega x_1} \cdot e^{j\omega x_2}]$$

$$= E[e^{j\omega x_1}] E[e^{j\omega x_2}]$$

$$= \phi_{x_1}(\omega) \cdot \phi_{x_2}(\omega).$$

5. The maximum amplitude of the characteristic function is unity at $\omega = 0$. $|\phi_x(\omega)| \le \phi_x(0)$

(or)

$$|\phi_x(\omega)| \le 1.$$

6. $\phi_X(\omega)$ is a continuous function of ω in the range $-\infty < \omega < \infty$.

7. $\phi_X(-\omega)$ & $\phi_X(\omega)$ are conjugate functions i.e.,

$\phi_X(\omega) = \phi_X^*(\omega)$ (or) $\phi_X^*(-\omega) = \phi_X(\omega)$

1. Find the characteristics function of a uniformly distributed a random variable X in the range [0, 1] and hence find m_1.

By the uniform distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Let the uniform distributed in the range [0, 1]

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

characteristic function, $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$

$$= \int_0^1 e^{j\omega x} (1)\, dx$$

$$= \left[\frac{e^{j\omega x}}{j\omega}\right]_0^1$$

$$= \frac{1}{j\omega}\left[e^{j\omega} - e^0\right]$$

$$= \frac{1}{j\omega}\left[e^{j\omega} - 1\right]$$

n^{th} moment of a characteristic function

$$m_n = (-j)^n \frac{d^n \phi_X(\omega)}{d\omega^n}$$

$$m_1 = (-j)^1 \frac{d(\phi_X(\omega))}{d\omega}$$

$$= (-j) \frac{d}{d\omega}\left[\frac{1}{j\omega}(e^{j\omega}-1)\right]$$

$$= \frac{-d}{d\omega}\left[\frac{e^{j\omega}}{\omega} - \frac{1}{\omega}\right]$$

$$= -\left[e^{j\omega}\left(\frac{-1}{\omega^2}\right) + \frac{1}{\omega} e^{j\omega} j + \frac{1}{\omega^2}\right]$$

$$= -\left[\frac{e^{j\omega}}{\omega^2} + \frac{je^{j\omega}}{\omega} + \frac{1}{\omega^2}\right]$$

$$m_1 = \frac{e^{j\omega}}{\omega^2} - \frac{je^{j\omega}}{\omega} - \frac{1}{\omega^2}$$

$$(m_1) \text{ at } \omega=0 \quad \lim_{\omega \to 0}\left[\frac{e^{j\omega}}{\omega^2} - \frac{je^{j\omega}}{\omega} - \frac{1}{\omega^2}\right]$$

Moment generating function:

The moment generating function of a random Variable is also used to generate the n^{th} Moments about the origin.

Consider a random Variable X with a pdf $f_X(x)$. Then the moment generating function of X is defined as the expected value of the function e^{vx}. It can be expressed as "$M_X(v) = E[e^{vx}]$"

where v is a real variable $-\infty < v < \infty$

∴ The Continuous moment generating function is

$$\boxed{M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx}$$

For discrete, $\boxed{M_X(v) = E[e^{vx}] = \sum_i e^{vx_i} p(x_i)}$

The n^{th} Moment of x can be derived from the moment generating function. Here the operator 'j' does not exist. The main disadvantage of the moment generating function is that it may not exist for all random variables & all values of v. But the characteristic function exists for all values of x and w.

Theorem Statement:

If $M_x(v)$ is a moment generating function of a random Variable x then the n^{th} moment of x is given by $\boxed{m_n = \frac{d^n[M_x(v)]}{dv^n}\Big|_{at\ \omega=0}}$

Properties of the moment generating function;

1. The moment generating function at $v=0$ is unity.

i.e., $\boxed{M_x(0)=1}$

Proof: $M_x(v) = E[e^{vx}]$

$M_x(0) = E[e^0]$

$= E[1]$

$= 1.$

2. Let x be a random variable with Moment generating function $M_x(v)$. then the MGF for $y=ax+b$ is given by $\boxed{M_y(v) = e^{bv} M_x(av)}$

Proof: $M_x(v) = E[e^{vx}]$

$M_y(v) = E[e^{vy}]$

$= E[e^{v(ax+b)}]$

$= E[e^{avx} \cdot e^{vb}]$

$= e^{vb} E[e^{avx}]$

$= e^{vb} M_x(av)$.

3. If $M_x(v)$ is a Moment generating function of a random Variable x then $\boxed{M_x(cv) = M_{cx}(v)}$

Proof: $M_x(v) = E[e^{vx}]$

$M_x(cv) = E[e^{cvx}]$

$= E[e^{v(cx)}]$

$= M_{cx}(v)$.

4. If x_1 & x_2 are two independent Random Variables with Moment generating function. (MGF) $M_{x_1}(v)$ and $M_{x_2}(v)$ then $\boxed{M_{x_1+x_2}(v) = M_{x_1}(v) \cdot M_{x_2}(v)}$

Proof: $M_x(v) = E[e^{vx}]$

$M_{x_1+x_2}(v) = E[e^{v(x_1+x_2)}]$

$= E[e^{vx_1} \cdot e^{vx_2}]$

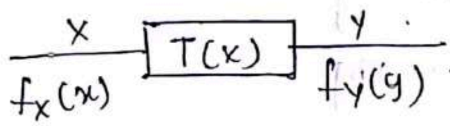
$= E[e^{vx_1}] E[e^{vx_2}]$

$= M_{x_1}(v) \, M_{x_2}(v)$

Transformation of a random Variable:

Transformation is used to Convert a given random Variable x into another random Variable Y. It is denoted as $Y = T(x)$ where T represents the transformation. It may be linear, non-Linear, Stair Case (or) Segmented.

The transformation of x to Y is



$y = 2x + 3$

$\therefore \; y = T(x)$

$y_0 = T(x_0)$

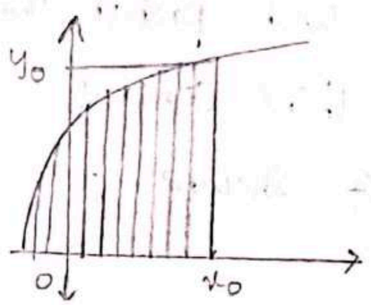
$x_0 = T^{-1}(y_0)$

where T^{-1} represents the inverse of the transformation T.

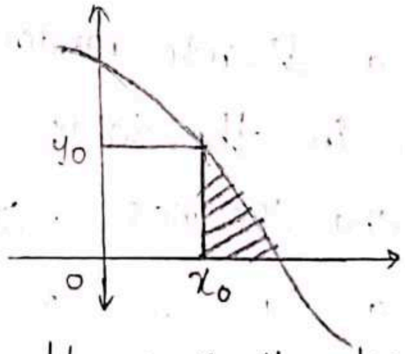
Monotonic transformation of a Continuous random Variable:—

Consider a random Variable 'x'. If the transformation is $T(x_1) < T(x_2)$ for any $x_1 < x_2$ then it is called a monotonically increasing transformation.

For the transformation to be monotonically decreasing the condition is $T(x_1) > T(x_2)$ for any $x_1 < x_2$.



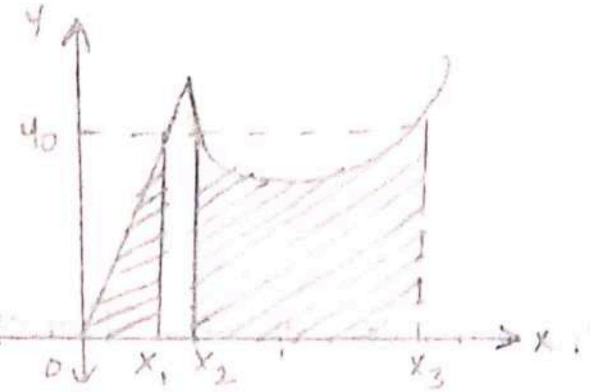
Monotonically increasing function



Monotonically decreasing function

Non-Monotonic Transformation of a Continuous random Variable:—

Consider a random variable y is a non-Monotonic transformation of a random variable x is shown below.



For a given event $\{y \le y_0\}$, there is more than one value of x. It is found that the event $\{y \le y_0\}$ corresponds to the events $\{x \le x_1, \; x_2 \le x \le x_3\}$.

$\therefore$ The probability of the event $\{y \le y_0\}$ is equal to the probability of the event $\{x / y \le y_0\}$ i.e.,

$$F_Y(y_0) = \int_{(x/y \le y_0)} f_X(x)\, dx$$

Problems:

1. Let x be a discrete random variable with probabilities as shown in the table. Find the third central moment and skewness {coefficient of skewness}.

X	0	1	2	3
P(X)	1/3	1/6	1/4	1/4

We know that the mean of the discrete data

$$\text{Mean } \bar{x} = E[x] = \sum_{i=0}^{3} x_i \, P(x_i)$$

$$= x_0 P(x_0) + x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3)$$

$$= 0 + 1(1/6) + 2(1/4) + 3(1/4)$$

$$= \frac{1}{6} + \frac{1}{2} + \frac{3}{4}$$

$$= 1.4167.$$

$$\text{Variance} = \sigma_x^2 = E[(x-\bar{x})^2] \quad (\text{or}) \quad \sum x^2 P(x_i) - \mu^2$$

$= 0 + 1(1/6) + 4(1/4) + 9(1/4) - (1.4167)^2$

$= \frac{1}{6} + 1 + \frac{9}{4} - (1.4167)^2$

$= 1.41$

Let $\overline{x^3} = E[x^3] = \Sigma x^3 p(x_i)$

$= 0 + 1(\frac{1}{6}) + 8(\frac{1}{4}) + 27(\frac{1}{4})$

$= \frac{1}{6} + 2 + \frac{27}{4} = 8.917$

Third central Moment $(M_3) = 8.917 - 3(1.4167)(1.41) - (1.4167)^3$

$= 0.081$

Skewness $= \frac{0.081}{(\sqrt{1.41})^3} = 0.048.$

(2.) Let x be a uniform random Variable with pdf $f_X(x) = \frac{1}{10}$, $-5 \leq x \leq 5$. Then find $E(x)$, $E(x^2)$, $E(2x+5)$, $E(x+1)^2$.

$f_X(x) = \frac{1}{10}, \quad -5 \leq x \leq 5$

1) $E[x] = \bar{x} = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$= \int_{-5}^{5} x\left(\frac{1}{10}\right) dx$

$= \frac{1}{10}\int_{-5}^{5} x\,dx$

$E[x] = 0$

2) $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$= \int_{-5}^{5} x^2\left(\frac{1}{10}\right) dx$

$= \frac{1}{10}\int_{-5}^{5} x^2\,dx$

$= \frac{1}{\cancel{10}_{5}} \cancel{2} \int_{0}^{5} x^2\,dx$

$= \frac{1}{8}\left[\frac{x^3}{3}\right]_0^5$

$= \left[\frac{5^{\cancel{3}}}{3}\right]\frac{1}{\cancel{8}}$

$= \frac{25}{3}$

4) $E[(x+1)^2] = E[x^2+1+2x]$

$= E[x^2]+1+2E[x]$

$= \frac{25}{3}+1+0$

$= \frac{28}{3}$

3) $E[2x+5] = 2E[x]+5$

$= 2(0)+5 = 5$

Workout Problem:

3. Let x be a random Variable with probabilities then find $E[x]$, $E[x^2]$, $E[2x+1]^2$ and σ_x^2.

X	-1	1	2
P(x)	1/6	1/3	1/2

4. Consider a random Variable x with $E[x]=5$ and $\sigma_x^2=3$. Another random Variable is given as $Y=-8x+10$. Find $E[x^2]$, $E[xY]$, $E[Y^2]$, σ_Y^2.

Given that $E[x] = 5 = \bar{x}$

$\sigma_x^2 = 3$

$Y = -8x+10$

1. $E[x^2]$

We know that $\sigma_x^2 = E[x^2]-[E[x]]^2$

$E[x^2] = \sigma_x^2+(E[x])^2$

$E[x^2] = 3+5^2 = 28$

$E[x^2] = 28$

2. $F[xY] = E[x(-8x+10)]$

$= E[-8x^2+10x]$

$= -8E[x^2]+10E[x]$

$= -8[28] + 110[5]$

$= -174$

3. $E[Y^2] = E[(-8X+10)^2]$

$= E[64x^2 + 100 - 160x]$

$= 64E[x^2] + 100 - 160E[x]$

$= 64[28] + 100 - 160[5]$

$= 1092.$

4. $\sigma_Y^2 = E[Y^2] - [E(Y)]^2$

Let $E[Y] = E[-8x+10] = -8E[x]+10 = -40+10 = -30.$

$\sigma_Y^2 = 1092 - [-30]^2 = 1092 - 900 = 192.$

(5.) Find the characteristic function for a random variable with density function $f_x(x) = x$, for $0 \le x \le 1$.

Characteristic function $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$

$$= \int_0^1 e^{j\omega x}\, x\, dx$$

$$= \left[\frac{xe^{j\omega x}}{j\omega}\right]_0^1 - \int \frac{e^{j\omega x}}{j\omega}$$

$$= \frac{e^{j\omega}}{j\omega} - \left[\frac{e^{j\omega x}}{(j\omega)^2}\right]_0^1$$

$$= \frac{e^{j\omega}}{j\omega} + \left[\frac{e^{j\omega}}{\omega^2} - e^0\right]$$

$$= \frac{e^{j\omega}}{j\omega} + \frac{e^{j\omega}}{\omega^2} - 1$$

(6.) The density function of a random variable is given as $f_x(x) = a\cdot e^{-bx}$, $x \ge 0$. Find the characteristic function and the first two moments.

Sol: $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$

$= \int_{-\infty}^{\infty} e^{j\omega x} a e^{-bx}$

$= \int_{-\infty}^{\infty} a e^{-bx + j\omega x}\,dx$

$= \int_{-\infty}^{\infty} a e^{-(bx - j\omega x)}\,dx$

$= -a\left[\frac{e^{-(b-j\omega)x}}{b-j\omega}\right]_0^{\infty}$

$= \frac{-a}{b-j\omega}\left[e^{-\infty} - e^{0}\right]$

$= \frac{-a}{b-j\omega}\left[0-1\right]$

$$\boxed{\phi_X(\omega) = \frac{a}{b-j\omega}}$$

The moments are; $m_n = (-j)^n \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{at\,\omega=0}$

First Moment:-

$m_1 = -j\,\frac{d\phi_X(\omega)}{d\omega} = -j\frac{d}{d\omega}\left[\frac{a}{b-j\omega}\right]$

$= -aj\cdot\left[\frac{-1}{(b-j\omega)^2}\right](-j)$

$= \frac{-aj^2}{(b-j\omega)^2} = \frac{a}{(b-j\omega)^2} = \frac{a}{b^2}.$

second Moment:-

$m_2 = (-j)^2\,\frac{d^2\phi_X(\omega)}{d\omega^2} = j^2\frac{d}{d\omega}\left[\frac{d\,\phi_X(\omega)}{d\omega}\right]$

$= -1\,\frac{d}{d\omega}\left[\frac{-a}{(b-j\omega)^2}(-j)\right]$

Q) Given the Rayleigh random variable with density function $f(x) = \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} \cdot u(x-a)$. Show that the mean and variance are $E[X] = a + \sqrt{\frac{\pi b}{4}}$ and $\sigma_x^2 = b\left(\frac{4-\pi}{4}\right)$.

Sol:-

Given Rayleigh density function, $f_X(x) = \frac{2}{b}(x-a)\cdot e^{-\frac{(x-a)^2}{b}} \cdot u(x-a)$, $x \geq a$.

$\therefore$ Mean $E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{a}^{\infty} x \cdot \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} \cdot dx$

$$E[X] = \int_a^{\infty} \frac{2(x^2-ax)}{b} e^{-(x-a)^2/b} \cdot dx \quad \text{—— (1)}$$

Let us take $x^2 - ax = x^2 + a^2 - a^2 + 2ax - 2ax - ax$

$= (x^2 - 2ax + a^2) + (ax - a^2)$

$$\boxed{x^2 - ax = (x-a)^2 + a(x-a)}$$

put (x^2-ax) value in eq (1), we get

$$(1) \Rightarrow E[X] = \int_a^{\infty} \frac{2}{b}\left[(x-a)^2 + a(x-a)\right] e^{-(x-a)^2/b} \cdot dx$$

$$E[X] = \int_a^{\infty} \frac{2}{b}(x-a)^2 e^{-(x-a)^2/b} \cdot dx + a \cdot \int_a^{\infty} \frac{2}{b}(x-a) \cdot e^{-(x-a)^2/b} \cdot dx \quad \text{—— (2)}$$

$\therefore$ Now to solve,

$$\int_a^{\infty} \frac{2}{b}(x-a)^2 e^{-(x-a)^2/b} dx = \int_a^{\infty} 2\left(\frac{x-a}{\sqrt{b}}\right)^2 \cdot e^{-\left(\frac{x-a}{\sqrt{b}}\right)^2} \cdot dx$$

Let $\left(\frac{x-a}{\sqrt{b}}\right) = t$. | L.L:- $t = 0$

$dx = \sqrt{b} \cdot dt$ | U.L:- $t = \infty$

$$= 2\int_0^{\infty} t^2 e^{-t^2} \cdot \sqrt{b}\, dt$$

$$= 2\sqrt{b}\int_0^{\infty} t^2 e^{-t^2} dt$$

$$= 2\sqrt{b}\left(\frac{\sqrt{\pi}}{4_2}\right)$$

$$= \frac{\sqrt{\pi b}}{2} \quad \text{——— (3)}$$

$$\left[\because \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right.$$

$$\left.\int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}\right]$$

Next to solve :-

$$\int_a^{\infty} \frac{2}{b}(x-a)\, e^{-(x-a)^2/b} \cdot dx = \int_0^{\infty} e^{-t} \cdot dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -[0-1]$$

$$= 1. \quad \text{——— (4)}$$

[Let $\frac{(x-a)^2}{b} = t$

$\frac{2(x-a)}{b} \cdot dx = dt$

L.L:- $t=0$

U.L:- $t=\infty$]

Substitute (3), (4) in eq (2), we get

$$E[X] = \frac{\sqrt{\pi b}}{2} + a(1) \Rightarrow \boxed{E[X] = a + \sqrt{\frac{\pi b}{4}}}$$

(ii) Variance :- $\sigma_x^2 = E[X^2] - (E[X])^2$ ——— (5)

Now to calculate $E[X^2] = \int_a^{\infty} \frac{2x^2(x-a)}{b} e^{-(x-a)^2/b} \cdot dx$

Let us take $x^2(x-a) = x^2 \cdot (x-a)$

$$= (x-a+a)^2 (x-a)$$

$$x^2(x-a) = \left[(x-a)^2 + a^2 + 2a(x-a)\right](x-a)$$

$$\boxed{x^2(x-a) = (x-a)^3 + 2a(x-a)^2 + a^2(x-a)}$$

$$\therefore E[X^2] = \int_a^{\infty} \frac{2}{b}\left[(x-a)^3 + 2a(x-a)^2 + a^2(x-a)\right] e^{-(x-a)^2/b} \cdot dx$$

$$E[X^2] = \int_a^{\infty} \frac{2}{b}(x-a)^3 e^{-(x-a)^2/b} \cdot dx + 2a\int_a^{\infty} \frac{2}{b}(x-a)^2 e^{-(x-a)^2/b} \cdot dx +$$

$$a^2 \int_a^{\infty} \frac{2}{b}(x-a) e^{-(x-a)^2/b} \cdot dx \text{ ——— } (6)$$

Since from eq's (3) and (11) we have

$$\int_a^{\infty} \frac{2}{b}(x-a)^2 e^{-(x-a)^2/b} \cdot dx = \underline{\frac{\sqrt{\pi b}}{2}} \text{ and } \int_a^{\infty} \frac{2}{b}(x-a) e^{-(x-a)^2/b} \cdot \underline{dx = 1}$$

Now to evaluate, $\int_a^{\infty} \frac{2}{b}(x-a)^3 e^{-(x-a)^2/b} \cdot dx = \int_a^{\infty} \frac{(x-a)^2}{b} \cdot e^{-(x-a)^2/b} \cdot 2(x-a) \cdot dx$

Let $\frac{(x-a)^2}{b} = t$

$\frac{2(x-a)}{b} dx = dt \cdot b$

$\underline{\underline{L.L:}}-\ t=0$

$\underline{\underline{U.L:}}-\ t=\infty$

$$= \int_0^{\infty} t \cdot e^{-t} \cdot b \cdot dt$$

$$= b \cdot \int_0^{\infty} t e^{-t} dt$$

$$= b\left[-te^{-t} - e^{-t}\right]_0^{\infty}$$

$$= -b\left[(0+0)-(0+1)\right] = \underline{b}.$$

$$\therefore (6) \Rightarrow E[X^2] = b + \cancel{2}a\left(\frac{\sqrt{\pi b}}{\cancel{2}}\right) + a^2(1)$$

$$E[X^2] = b + a\cdot\sqrt{\pi b} + a^2$$

$\therefore$ Variance $\sigma_x^2 = E[x^2] - (E[x])^2$

$$\sigma_x^2 = b + a.\sqrt{\pi b} + a^2 - \left(a + \frac{\sqrt{\pi b}}{2}\right)^2$$

$$\sigma_x^2 = b + a^2 + a.\sqrt{\pi b} - \left(a^2 + \frac{\pi b}{4} + \cancel{2}.a.\frac{\sqrt{\pi b}}{\cancel{2}}\right)$$

$$\sigma_x^2 = b + \cancel{a^2} + \cancel{a.\sqrt{\pi b}} - \cancel{a^2} - \frac{\pi b}{4} - \cancel{a.\sqrt{\pi b}}$$

$$\sigma_x^2 = b - \frac{\pi b}{4} = \frac{4b - \pi b}{4} = b\left(\frac{4-\pi}{4}\right)$$

$$\boxed{\therefore\ \sigma_x^2 = b\left(\frac{4-\pi}{4}\right)}$$

Hence it is proved.

Q) Show that the characteristic function of a Gaussian random variable with zero mean and variance (σ^2) is $\phi_X(\omega) = \exp\left(\frac{-\sigma^2\omega^2}{2}\right)$ (or) $e^{-\left(\frac{\sigma^2\omega^2}{2}\right)}$

Sol:-

We know that the Gaussian density function of a random variable is $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$ $\langle \because$ mean $(\mu) = 0 \rangle$

$\therefore$ The characteristic function is $\phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} . f_X(x)\,dx$

$$\text{ie, } \phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} . \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x^2/2\sigma^2)} . dx$$

$$\phi_X(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} . \int_{-\infty}^{\infty} e^{\left(\frac{-x^2}{2\sigma^2} + j\omega x\right)} . dx \quad \text{---} (1)$$

Let us take,

$$\frac{-x^2}{2\sigma^2} + j\omega x = -\left[\frac{x^2}{2\sigma^2} - j\omega x\right]$$

$$= -\left[\left(\frac{x}{\sqrt{2}\sigma}\right)^2 - j\omega x\right]$$

$$= -\left[\left(\frac{x}{\sqrt{2}\sigma}\right)^2 - 2\cdot\left(\frac{x}{\sqrt{2}\sigma}\right)\cdot\left(\frac{\sqrt{2}\,\sigma j\omega}{2}\right)\right]$$

$$= -\left[\left(\frac{x}{\sqrt{2}\sigma}\right)^2 - 2\cdot\left(\frac{x}{\sqrt{2}\sigma}\right)\cdot\left(\frac{\sigma j\omega}{\sqrt{2}}\right) + \left(\frac{\sigma j\omega}{\sqrt{2}}\right)^2 - \left(\frac{j\sigma\omega}{\sqrt{2}}\right)^2\right]$$

$$= -\left[\left(\frac{x}{\sqrt{2}\sigma} - \frac{j\omega\sigma}{\sqrt{2}}\right)^2\right] + \left(\frac{j\sigma\omega}{\sqrt{2}}\right)^2$$

put this in eq ①, we get

$$① \Rightarrow \phi_X(\omega) = \frac{1}{\sqrt{2\pi}\sigma}\cdot\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma} - \frac{j\omega\sigma}{\sqrt{2}}\right)^2 + \left(\frac{j\omega\sigma}{\sqrt{2}}\right)^2} \, dx$$

$$\phi_X(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{\left(\frac{j\omega\sigma}{\sqrt{2}}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma} - j\omega\sigma\right)^2} \cdot dx$$

$$\phi_X(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{\omega^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2\left(\frac{x}{\sigma} - j\omega\sigma\right)^2} \cdot dx \quad \text{——— ②}$$

Let us take $\frac{x}{\sigma} - j\omega\sigma = t$ | L.L:- $t = -\infty$

$\frac{dx}{\sigma} = dt \Rightarrow dx = \sigma \cdot dt$ | U.L:- $t = \infty$

$$② \Rightarrow \phi_X(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{\omega^2\sigma^2}{2}\right)} \int_{-\infty}^{\infty} e^{-t^2/2} \cdot \sigma \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}\cdot\cancel{\sigma}}\cdot\cancel{\sigma}\cdot e^{-\left(\frac{\omega^2\sigma^2}{2}\right)} \cdot \int_{-\infty}^{\infty} e^{-t^2/2} \cdot dt.$$

$$\phi_X(\omega) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\omega^2\sigma^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-t^2/2} \cdot dt. \qquad \left\langle \because \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \right\rangle$$

$$\phi_X(\omega) = \frac{1}{\cancel{\sqrt{2\pi}}}\, e^{-\frac{\omega^2\sigma^2}{2}} \cdot \cancel{\sqrt{2\pi}}$$

$$\boxed{\phi_X(\omega) = e^{-\left(\frac{\omega^2\sigma^2}{2}\right)}.}$$

Hence it is proved.

Q) The characteristic function for a Gaussian random variable X, having a mean value of zero. is $\phi_X(\omega) = e^{-\left(\frac{\sigma^2\omega^2}{2}\right)}$.
Find the first, second and third moments of X, using $\phi_X(\omega)$.

Sol:—

Given that the characteristic function $\phi_X(\omega) = e^{-\left(\frac{\sigma^2\omega^2}{2}\right)}$.

we know that the n^{th} moment. of a characteristic function

is $$m_n = (-j)^n \cdot \frac{d^n[\phi_X(\omega)]}{d\omega^n}\Bigg|_{\omega=0}$$

First moment :—

$\because$ For $n=1$:— $$m_1 = (-j)\, \frac{d}{d\omega}\left[\phi_X(\omega)\right]$$

$$= -j\, \frac{d}{d\omega}\left[e^{-(\sigma^2\omega^2/2)} \right]$$

$$= -j\, e^{-(\sigma^2\omega^2/2)} \cdot \frac{d}{d\omega}\left(-\frac{\sigma^2\omega^2}{2}\right)$$

$$= j\, e^{-(\sigma^2\omega^2/2)} \cdot \frac{\sigma^2 \cdot \cancel{2}\omega}{\cancel{2}}$$

$$m_1 = \left[j\omega\sigma^2 \cdot e^{-(\sigma^2\omega^2/2)} \right]_{at\ \omega=0} \text{ ——— } (1)$$

$\therefore \boxed{m_1 = 0}$

Second Moment:-

For $n=2$:-

$$m_2 = (-j)^2 \frac{d^2}{d\omega^2}\left[\phi_X(\omega)\right]\Big|_{\omega=0}$$

$$m_2 = (-1)\frac{d}{d\omega}\left[\frac{d}{d\omega}\left[\phi_X(\omega)\right]\right]$$

$$m_2 = (-1)\frac{d}{d\omega}\left[j\omega\sigma^2 \cdot e^{-\left(\frac{\sigma^2\omega^2}{2}\right)}\right]$$

$$m_2 = (-1)\, j\sigma^2 \frac{d}{d\omega}\left[\omega \cdot e^{-\left(\frac{\sigma^2\omega^2}{2}\right)}\right]$$

$$m_2 = -j\sigma^2\left[\omega \cdot e^{-\left(\frac{\sigma^2\omega^2}{2}\right)} \cdot \left(-\frac{\sigma^2}{\cancel{2}}\cancel{2}\omega\right) + e^{-\left(\frac{\sigma^2\omega^2}{2}\right)} \cdot 1\right]$$

$$m_2 = -j\sigma^2\left[-\sigma^2\omega^2 + 1\right] e^{-\sigma^2\omega^2/2}$$

$$m_2 = j\sigma^2(\sigma^2\omega^2 - 1)\, e^{-(\sigma^2\omega^2/2)}\Big|_{\omega=0}$$

$$m_2 = j\sigma^2(0-1) \Rightarrow \boxed{m_2 = -j\sigma^2}$$

Third moment:-

For $n=3$:-

$$m_3 = (-j)^3 \cdot \frac{d^3}{d\omega^3}\left[\phi_X(\omega)\right]$$

$$m_3 = -j^2 \cdot j \frac{d}{d\omega}\left[\frac{d^2}{d\omega^2}\left[\phi_X(\omega)\right]\right]$$

$$m_3 = j\frac{d}{d\omega}\left[j\sigma^2(\omega^2\sigma^2 - 1)\, e^{-(\sigma^2\omega^2/2)}\right]$$

$$m_3 = j^2\sigma^2\frac{d}{d\omega}\left[(\sigma^2\omega^2 - 1)\, e^{-(\sigma^2\omega^2/2)}\right]$$

$$m_3 = -\sigma^2\left[(\sigma^2\omega^2 - 1)\, e^{-(\sigma^2\omega^2/2)} \cdot \left(-\frac{\sigma^2}{2} \cdot 2\omega\right) + e^{-(\sigma^2\omega^2/2)} \cdot 2\omega\sigma^2\right]$$

At $\omega = 0$ $\boxed{m_3 = 0}$

Q) Find the characteristic function for a random variable with density function. $f_X(x) = x$, for $0 \le x \le 1$.

Sol:- The characteristic function of 'x' is given by

$$\phi_X(\omega) = E[e^{j\omega x}] = \left[\int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\, dx\right]$$

$$\therefore \phi_X(\omega) = \int_0^1 x\, e^{j\omega x}\, dx$$

let

$u = x$, $u' = 1$, $u'' = 0$ | $dv = e^{j\omega x}$, $v = \left(\frac{1}{j\omega}\right) e^{j\omega x}$, $v_1 = \left(\frac{1}{j^2\omega^2}\right) e^{j\omega x}$

$$= \left[uv - u'v_1\right]_0^1$$

$$= \left[\left(\frac{1}{j\omega}\right) x \cdot e^{j\omega x} - \left(\frac{1}{j^2\omega^2}\right) e^{j\omega x}\right]_0^1$$

$$= \left[\left(\frac{1}{j\omega}\right) e^{j\omega} - \frac{1}{j^2\omega^2}(e^{j\omega})\right] - \left[0 - \left(\frac{1}{j^2\omega^2}\right)\right]$$

$$\phi_X(\omega) = \left[\frac{e^{j\omega}}{j\omega} + \frac{e^{j\omega}}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$\boxed{\phi_X(\omega) = \frac{e^{j\omega}}{j\omega} + \frac{1}{\omega^2}\left[e^{j\omega} - 1\right]}$$

Q) Find the characteristic function for $f_X(x) = e^{-|x|}$.

Sol:- Given that $f_X(x) = e^{-|x|}$, $-\infty \le x \le \infty$

ie, $e^{-|x|} = \begin{cases} e^{x}, & x \le 0 \\ e^{-x}, & x \ge 0 \end{cases}$

$\therefore$ The characteristic function,

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{0} e^{j\omega x} f_X(x)\, dx + \int_0^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$\phi_x(\omega) = \int_{-\infty}^{0} e^{x} \cdot e^{j\omega x}\, dx + \int_{0}^{\infty} e^{-x} e^{j\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{(1+j\omega)x}\, dx + \int_{0}^{\infty} e^{-(1-j\omega)x}\, dx$$

$$= \left[\frac{e^{(1+j\omega)x}}{(1+j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-j\omega)x}}{-(1-j\omega)}\right]_{0}^{\infty}$$

$$= \left(\frac{1}{1+j\omega}\right)\left[e^{0} - e^{-\infty}\right] + \left(\frac{1}{1-j\omega}\right)\left[e^{-\infty} - e^{0}\right] \qquad \langle \because e^{-\infty} = 0,\ e^{0} = 1 \rangle$$

$$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega}$$

$$= \frac{(1-j\omega) + (1+j\omega)}{1 - j^2\omega^2} = \frac{1 - \cancel{j\omega} + 1 + \cancel{j\omega}}{1+\omega^2} \qquad \langle \because j^2 = -1 \rangle$$

$$\therefore \boxed{\phi_x(\omega) = \frac{2}{1+\omega^2}}$$

Work out problems:-

Q) probability mass function of a discrete random variable is given as

x	$-1/2$	$1/2$
$P(x)$	$1/4$	$3/4$

. Find the moment generating function and find the first three moments.

Q) If a pdf of a random variable 'x' is given by $f_X(x) = b e^{-|ax|}$ where a and b are real constants, find the moment generating function, Mean and Variance

Q) Consider that a pdf of a random variable X is

$$f_X(x) = \begin{cases} \frac{1}{k}, & -2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$, and another random variable

$Y = 2X$. Then find a) K value b) $E[X]$, c) $E[Y]$ and d) $E[XY]$.

Q) A random variable 'X' has a pdf $f_X(x) = \begin{cases} \frac{1}{2}\cos x, & -\pi/2 < x < \pi/2 \\ 0, & \text{else where} \end{cases}$

find the Mean of the function $g(x) = 4x^2$.

Q) Find the characteristic function and the two moments m_1 and m_2 for the density function $f_X(x) = a e^{-bx}$, $x \geq 0$.

Note:—

Consider a random variable 'X' and the transformation of 'X' is Y. For a monotonic transformation, either increasing (or) decreasing, the density function of "Y" is

$$\boxed{f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|}$$

problems:—

① A random variable 'θ' is uniformly distributed in the interval (θ_1, θ_2), where θ_1 and θ_2 are real and satisfy $\theta_1 \leq \theta < \theta_2 < \pi$. Find the probability density function of the transformed random variable $Y = \cos\theta$.

Sol:- Given that 'θ' is the uniformly distributed. in (θ_1, θ_2) [a, b]

∴ The uniform density function is $f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$

i.e $f_\theta(\theta) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \leq \theta \leq \theta_2 \\ 0, & \text{otherwise} \end{cases}$ ——— (1)

∴ The transformed variable is $Y = \cos\theta \Rightarrow dy = -\sin\theta \, d\theta$

$\Rightarrow \boxed{\theta = \cos^{-1}(y)}$ $\Rightarrow \frac{d\theta}{dy} = \frac{-1}{\sin\theta}$.

ie, $\boxed{f_\theta(\theta) = \cos^{-1}(y)}$

we know that the pdf of y is $f_y(y) = f_x(x) \cdot \left|\frac{dx}{dy}\right|$

~~put~~ let $x = \theta$, we have

$$f_y(y) = f_\theta(\theta) \cdot \left|\frac{d\theta}{dy}\right| \quad \text{——— (2')}$$

$\therefore f_\theta(\theta) = \cos^{-1}(y) = \begin{cases} \dfrac{1}{\cos^{-1}y_0 - \cos^{-1}y_1}, & y_0 \leq y \leq y_1 \\ 0, & \text{otherwise} \end{cases}$

$$(2') \Rightarrow f_y(y) = \frac{1}{\cos^{-1}y_0 - \cos^{-1}y_1} \cdot \left|\frac{-1}{\sin\theta}\right|$$

$$f_y(y) = \frac{1}{\cos^{-1}y_0 - \cos^{-1}y_1}\left|\frac{1}{\sin(\cos^{-1}y)}\right| \qquad \langle \because \theta = \cos^{-1}y \rangle$$

$$\boxed{f_y(y) = \frac{\operatorname{cosec}(\cos^{-1}y)}{\cos^{-1}(y_0) - \cos^{-1}(y_1)}}$$

(2) Let 'x' be continuous random variable with pdf.

$f_X(x) = \begin{cases} \dfrac{x}{12}, & 1 < x < 5 \\ 0, & \text{else where} \end{cases}$, find the probability density function of $Y = 2x - 3$.

Sol:- Given that $f_X(x) = \begin{cases} \dfrac{x}{12}, & 1 < x < 5 \\ 0, & \text{else where.} \end{cases}$ ——— (1)

Now to find pdf of $Y = 2x - 3$.

$$\Rightarrow 2x = y + 3$$

$$x = \left(\frac{y+3}{2}\right)$$

$$\boxed{x = \left(\frac{y}{2} + \frac{3}{2}\right)}$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{1}{2}}$$

$\therefore$ If $x = \frac{y}{2} + \frac{3}{2}$ then $f_X(x) = \frac{\left(\frac{y}{2}+\frac{3}{2}\right)}{12}$ $\langle \because$ from ①$\rangle$

$$f_X(x) = \left(\frac{y}{24} + \frac{3}{24}\right)$$

$$\boxed{f_X(x) = \frac{1}{24}(y+3)}$$

$\therefore$ The probability density function for Y is

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$f_Y(y) = \frac{1}{24}(y+3) \cdot \left|\frac{1}{2}\right|$$

$$\boxed{f_Y(y) = \frac{1}{48}(y+3)}$$

Limits of y are :- If $x = 1$, $\Rightarrow$ $y = 2x - 3$

$y = 2 - 3 \Rightarrow \boxed{y = -1}$

If $x = 5$, $\Rightarrow$ $y = 2(5) - 3$

$y = 10 - 3 \Rightarrow \boxed{y = 7}$

$\therefore$ pdf of Y is $f_Y(y) = \begin{cases} \frac{1}{48}(y+3), & -1 < y < 7 \\ 0, & \text{otherwise.} \end{cases}$

Work out problems :-

③ The pdf of a random variable 'x' is given by $f_X(x) = \frac{x}{20}$, $2 \le x \le 5$, find the pdf of $y = 3x - 5$.

Chebychev's Inequality :-

Statement :- For a given random variable 'x' with value $\overline{X}$ and variance σ_x^2, it states that $P\{|x-\overline{X}| \geq \epsilon\} \leq (\sigma_x^2/\epsilon^2)$ where 'ϵ' is a very small positive number.

Proof :- we know that the probability density function of a random variable 'x' is given by

$|x| \leq \epsilon$

$-\epsilon \leq x \leq \epsilon$

$|x| \geq \epsilon$

$-\epsilon \geq x \geq \epsilon$

$$P\{X \leq x\} = F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx.$$

Now expand, $P\{|x-\overline{X}| \geq \epsilon\} = P\{(x-\overline{x}) \leq -\epsilon\} + P\{(x-\overline{x}) \geq \epsilon\}$

$\Rightarrow x \leq -\epsilon ;$

$$= P\{x \leq (\overline{x}-\epsilon)\} + P\{x \geq (\overline{x}+\epsilon)\}$$

$$= \int_{-\infty}^{\overline{x}-\epsilon} f_X(x)\, dx + \int_{\overline{x}+\epsilon}^{\infty} f_X(x)\, dx.$$

$$\therefore P\{|x-\overline{x}| \geq \epsilon\} = \int_{|x-\overline{x}| \geq \epsilon}^{\infty} f_X(x)\, dx$$

we know that,

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\overline{X})^2 \cdot f_X(x)\, dx$$

$$\sigma_x^2 = \int_{-\infty}^{|x-\overline{x}| \geq \epsilon} (x-\overline{X})^2 f_X(x)\, dx + \int_{|x-\overline{x}| \geq \epsilon}^{\infty} (x-\overline{X})^2 f_X(x)\, dx$$

ie, $$\sigma_x^2 \geq \int_{|x-\overline{X}| \geq \epsilon}^{\infty} (x-\overline{X})^2 f_X(x)\, dx.$$

If $(x-\overline{X}) = \epsilon$, we have

$$\sigma_x^2 \geq \int_{|x-\bar{x}|\geq \epsilon}^{\infty} \epsilon^2 \cdot f_X(x)\, dx$$

$$\sigma_x^2 \geq \epsilon^2 \int_{|x-\bar{x}|\geq \epsilon}^{\infty} f_x(x)\, dx.$$

$$\Rightarrow \quad \sigma_x^2 \geq \epsilon^2 \, P\{|x-\bar{x}| \geq \epsilon\}$$

i.e, $\boxed{P\{|x-\bar{x}| \geq \epsilon\} \leq \left(\frac{\sigma_x^2}{\epsilon^2}\right)}$

Hence the theorem is proved.

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Multiple Random Variables:-

In this chapter we can discuss the probability function for two random variables. The theory can be extended to multiple random variables.

Joint probability Distribution Function:-

Consider two random variables X and Y, with elements $\{x\}$ & $\{y\}$ in the xy-plane. The ordered pair of numbers $\{x, y\}$ is called a random vector in two-dimensional product space (or) joint sample space. Let two events be $A = \{X \leq x\}$ and $B = \{Y \leq y\}$. Then the joint probability distribution function for the joint event $\{X \leq x, Y \leq y\}$ is defined as

$$\boxed{F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\} = P(A \cap B)}$$

For discrete random variable, if $X = \{x_1, x_2, \dots x_N\}$ and $Y = \{y_1, y_2, \dots y_M\}$ with joint probabilities $P(x_n, y_m)$ then the joint probability distribution function is

$$\boxed{F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) \cdot u(x - x_n) \cdot u(y - y_m)}$$

Properties of Joint Distribution Function:-

① $F_{X,Y}(-\infty, -\infty) = 0$

$F_{X,Y}(x, -\infty) = 0$

$F_{X,Y}(-\infty, y) = 0$

② $F_{X,Y}(\infty, \infty) = 1$

③ $0 \leq F_{X,Y}(x,y) \leq 1$.

④ $F_{X,Y}(x,y)$ is a monotonic and non-decreasing function of both x and y.

⑤ The probability of the joint event $\{x_1 \le X \le x_2, y_1 \le Y \le y_2\}$ is given by

$$\boxed{P\{x_1 \le X \le x_2,\ y_1 \le Y \le y_2\} = F_{X,Y}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)}$$

⑥ The Marginal distribution functions are given by

$$\boxed{F_{X,Y}(x,\infty) = F_X(x)} \quad \text{and} \quad \boxed{F_{X,Y}(\infty,y) = F_Y(y)}$$

problems:—

① Assume that the joint sample space has the probabilities as shown below. Find the distribution function, $F_{X,Y}(x,y)$ and the marginal density functions.

(x,y)	$(0,0)$	$(1,2)$	$(2,3)$	$(3,2)$
$p(x,y)$	0.2	0.3	0.4	0.1

Sol:—

we know that

$$F_{X,Y}(x,y) = \sum_n \sum_m P(x_n,y_m) \cdot u(x-x_n) \cdot u(y-y_m)$$

$$F_{X,Y}(x,y) = P(0,0) \cdot u(x) \cdot u(y) + P(1,2) \cdot u(x-1) \cdot u(y-2) + P(2,3) \cdot u(x-2) \cdot u(y-3) + P(3,2) \cdot u(x-3) \cdot u(y-2)$$

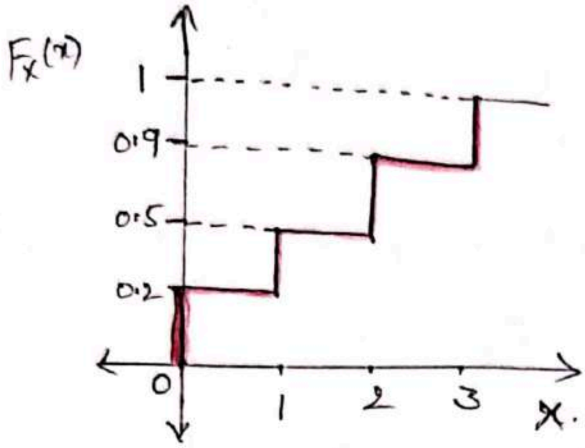
$$F_{X,Y}(x,y) = 0.2\ u(x) \cdot u(y) + 0.3\ u(x-1)\ u(y-2) + 0.4\ u(x-2) \cdot u(y-3) + 0.1\ u(x-3)\ u(y-2)$$

which is the distribution function.

The Marginal distribution functions are.

(1) $F_X(x) = F_{X,Y}(x, \infty)$

$= 0.2\, u(x) + 0.3\, u(x-1) + 0.4\, u(x-2) + 0.1 \cdot u(x-3)$



Marginal distributions of X

Note:-

At $x=0$, $F_X(x) = 0.2$

At $x=1$, $F_X(x) = 0.2+0.3$
$= 0.5$

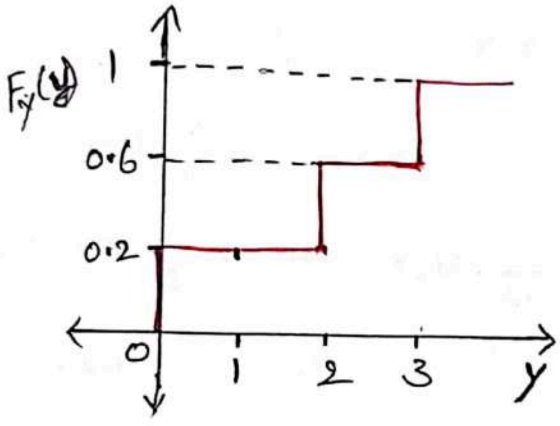
$x=2$, $F_X(x) = 0.2+0.3+0.4$
$= 0.9$

$x=3$, $F_X(x) = 0.2+0.3+0.4+0.1$
$= 1$

(2) $F_Y(y) = F_{X,Y}(\infty, y)$

$= 0.2 \cdot u(y) + 0.3\, u(y-2) + 0.4\, u(y-3) + 0.1\, u(y-2)$

$= 0.2\, u(y) + 0.4\, u(y-2) + 0.4\, u(y-3)$



Marginal distribution of Y.

Note:-

At $y=0$, $F_Y(y) = 0.2$ ✓

$y=1$, $F_Y(y) = 0+0.2$ ∵ value is not there
$= 0.2$

$y=2$, $F_Y(y) = 0.2+0.4$
$= 0.6$

$y=3$, $F_Y(y) = 0.2+0.4+0.4$
$= 1.0$
$= 1$ ✓

(2) The probabilities of the random variables X and Y are given in the table. Find the (a) value of K. (b) the joint distribution function and Marginal distribution functions.

X/Y	-1	0	1
0	3/18	2/18	3/18
1	1/18	K/18	1/18
2	2/18	1/18	2/18

Sol:— If the distribution is a valid probability, then

(a) $\sum_n \sum_m p(x_n, y_m) = 1$ —— ①

ie, we have $(x,y) = \{(0,-1), (0,0), (0,1)$
$(1,-1), (1,0), (1,1)$
$(2,-1), (2,0), (2,1)\}$.

The probabilities are $p(0,-1) = \frac{3}{18}$, $p(0,0) = \frac{2}{18}$, $p(0,1) = \frac{3}{18}$

$p(1,-1) = \frac{1}{18}$, $p(1,0) = \frac{k}{18}$, $p(1,1) = \frac{1}{18}$

$p(2,-1) = \frac{2}{18}$, $p(2,0) = \frac{1}{18}$, $p(2,1) = \frac{2}{18}$.

Eq① $\Rightarrow$ $\frac{3}{18} + \frac{2}{18} + \frac{3}{18} + \frac{1}{18} + \frac{k}{8} + \frac{1}{18} + \frac{2}{18} + \frac{1}{18} + \frac{2}{18} = 1$

$\Rightarrow \frac{15+k}{18} = 1 \Rightarrow k = 18-15 \Rightarrow \boxed{k=3}$

(b) The joint distribution function is,

$$F_{X,Y}(x,y) = \sum_n \sum_m p(x_n, y_m) \cdot u(x-x_n) \cdot u(y-y_m)$$

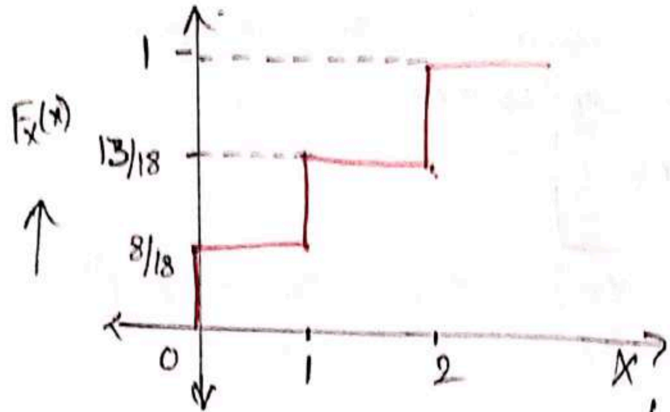
$$F_{X,Y}(x,y) = \frac{3}{18} \cdot u(x) \cdot u(y+1) + \frac{2}{18} \cdot u(x) \cdot u(y) + \frac{3}{18} \cdot u(x) \cdot u(y-1)$$
$$+ \frac{1}{18} u(x-1) \cdot u(y+1) + \frac{3}{18} \cdot u(x-1) \cdot u(y) + \frac{1}{18} \cdot u(x-1) u(y-1)$$
$$+ \frac{2}{18} \cdot u(x-2) \cdot u(y+1) + \frac{1}{18} u(x-2) \cdot u(y) + \frac{2}{18} \cdot u(x-2) \cdot u(y-1)$$

(c) The marginal distribution functions are,

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_X(x) = \frac{3}{18} u(x) + \frac{2}{18} u(x) + \frac{3}{18} u(x)$$
$$+ \frac{1}{18} u(x-1) + \frac{3}{18} u(x-1) + \frac{1}{18} u(x-1)$$
$$+ \frac{2}{18} u(x-2) + \frac{1}{18} u(x-2) + \frac{2}{18} u(x-2)$$

$\therefore F_X(x) = \frac{8}{18} \cdot u(x) + \frac{5}{18} u(x-1) + \frac{5}{18} \cdot u(x-2)$



$\rightarrow$ Marginal distribution for X

$\therefore F_Y(y) = F_{X,Y}(\infty, y)$

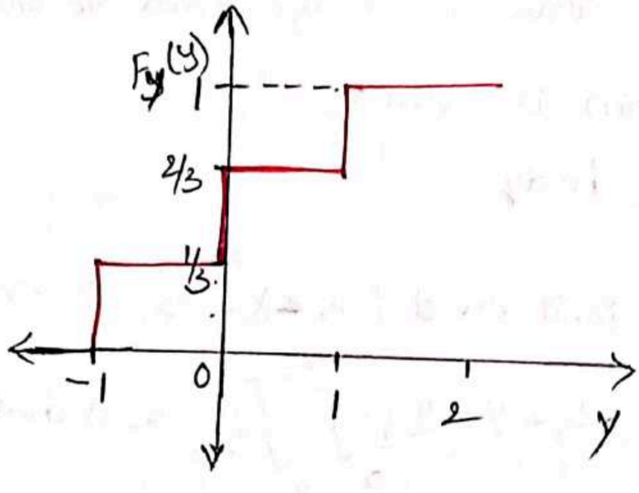
$= \frac{3}{18} \cdot u(y+1) + \frac{2}{18} u(y) + \frac{3}{18} u(y-1)$

$+ \frac{1}{18} u(y+1) + \frac{3}{18} u(y) + \frac{1}{18} \cdot u(y-1)$

$+ \frac{2}{18} u(y+1) + \frac{1}{18} u(y) + \frac{2}{18} u(y-1)$

$F_Y(y) = \frac{6}{18} u(y+1) + \frac{6}{18} u(y) + \frac{6}{18} u(y-1)$

$= \frac{1}{3} u(y+1) + \frac{1}{3} u(y) + \frac{1}{3} u(y-1)$



Marginal distribution for Y

Joint probability density function:—

The joint probability density function of two random variables X and Y is defined as the second derivative of the joint distribution function. It can be expressed as

$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y}}$$

For discrete random variable, $X = \{x_1, x_2, \dots x_n\}$ and $Y = \{y_1, y_2, \dots y_m\}$

the joint density function is

$$f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} p(x_n, y_m) \cdot \delta(x - x_n) \cdot \delta(y - y_m).$$

By direct integration, the joint distribution function can be obtained as $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dx\, dy.$

Properties of Joint Density functions:—

① $f_{X,Y}(x,y) \geq 0$, a joint probability density function is always non-negative.

② $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$, (the area under the density function curve is always equal to one)

③ The joint distribution function is equal to

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dx\, dy$$

④ The probability of the joint event $\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\}$ is given as $P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y)\, dx\, dy$

⑤ The marginal distribution function of X and Y are

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} F_{X,Y}(x,y)\, dx\, dy$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} F_{X,Y}(x,y)\, dx\, dy$$

(5) The Marginal density functions of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \cdot dy = \frac{\partial F(x,\infty)}{\partial x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \frac{\partial F(\infty,y)}{\partial y}.$$

problems :—

(1) The Joint pdf is given as $f_{X,Y}(x,y) = A \cdot e^{-(2x+y)}$, for $x \geq 0$ and $y \geq 0$. find (a) the value of A and (b) the Marginal density functions

Sol:-

Given that $f_{X,Y}(x,y) = A \cdot e^{-(2x+y)}$, for $x \geq 0$ and $y \geq 0$.

The Joint pdf is valid, then, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1.$

ie, $\int_0^{\infty}\int_0^{\infty} A \cdot e^{-(2x+y)} dx\, dy = 1$

$\Rightarrow A \int_0^{\infty}\int_0^{\infty} e^{-2x-y}\, dx\, dy = 1$

$\Rightarrow A \int_0^{\infty} e^{-2x}\left(\int_0^{\infty} e^{-y}\, dy\right) dx = 1$

$A \int_0^{\infty} e^{-2x} \left[\frac{e^{-y}}{-1}\right]_0^{\infty} dx = 1$

$-A \int_0^{\infty} e^{-2x}\left[e^{-\infty} - e^{0}\right] dx = 1$

$A \int_0^{\infty} e^{-2x}\, dx = 1 \Rightarrow A\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$

$\Rightarrow -\frac{A}{2}\left[e^{-\infty} - e^{0}\right] = 1.$

$\Rightarrow \frac{A}{2} = 1 \Rightarrow \boxed{A=2}$

The marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} 2\cdot e^{-(2x+y)}\,dy, \quad x \geq 0$$

$$= 2e^{-2x}\int_0^{\infty} e^{-y}\,dy, \quad x \geq 0$$

$$= 2e^{-2x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty}, \quad x \geq 0$$

$$= -2e^{-2x}\left[e^{-\infty} - e^{0}\right], \quad x \geq 0$$

$$\boxed{f_X(x) = 2e^{-2x}, \quad x \geq 0}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} 2\,e^{-(2x+y)}\,dx, \quad y \geq 0$$

$$= 2e^{-y}\int_0^{\infty} e^{-2x}\,dx, \quad y \geq 0$$

$$= 2e^{-y}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}, \quad y \geq 0$$

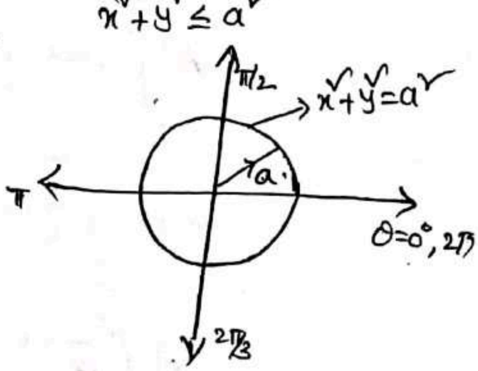
$$= -e^{-y}\left[e^{-\infty} - e^{0}\right], \quad y \geq 0$$

$$\boxed{f_Y(y) = e^{-y}, \quad y \geq 0}$$

② If the joint pdf of X and Y is $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

find the mass in the circle, $x^2+y^2 \leq a^2$.

Sol:- Given that $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$,



$$\therefore P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}\,dx\,dy$$

$$P = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2\sigma^2}\,dx\,dy \quad \text{--- ①}$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r.\,dr\,d\theta$

$\Rightarrow x^2+y^2=r^2$, we have

$$P = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, r\,dr.\,d\theta.$$

$$P = \frac{1}{2\pi\sigma^2}\int_{r=0}^{a} r.\,e^{-r^2/2\sigma^2}.\,dr.\int_{\theta=0}^{2\pi} 1.\,d\theta$$

$$P = \frac{1}{2\pi\sigma^2}\int_{r=0}^{a} r.\,e^{-r^2/2\sigma^2}.\,dr.\,[2\pi-0]$$

$$P = \frac{\cancel{2\pi}}{\cancel{2\pi}\sigma^2}\int_{r=0}^{a} r.\,e^{-r^2/2\sigma^2}.\,dr$$

Let $r^2/2\sigma^2 = t$

$2r\,dr = 2\sigma^2 dt.$

$r\,dr = \sigma^2 dt.$

$t=0$ & $t = a^2/2\sigma^2$

$$P = \frac{1}{\cancel{\sigma^2}}\int_{t=0}^{a^2/2\sigma^2} e^{-t}.\,\cancel{\sigma^2}\,dt$$

$$P = \left[\frac{e^{-t}}{-1}\right]_0^{a^2/2\sigma^2} = -1\left[e^{-a^2/2\sigma^2} - e^{0}\right]$$

$$\boxed{P = 1 - e^{-a^2/2\sigma^2}}$$

(3) Determine a Constant 'b' such that the given function is a valid joint density function. $f_{X,Y}(x,y) = \begin{cases} b(x^2+4y^2), & 0 \le |x| < 1 \text{ and } 0 \le y < 2 \\ 0, & \text{otherwise} \end{cases}$

Sol:-

Given that $f_{X,Y}(x,y) = \begin{cases} b(x^2+4y^2), & 0 \le |x| < 1 \text{ and } 0 \le y < 2 \\ 0, & \text{otherwise.} \end{cases}$

since,

$f_{X,Y}(x,y)$ is a valid Joint density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_{x=-1}^{1}\int_{y=0}^{2} b(x^2+4y^2)\cdot dx\, dy = 1$$

$$b\int_{x=-1}^{1}\int_{y=0}^{2}(x^2+4y^2)\, dy\cdot dx = 1$$

$$b\cdot\int_{x=-1}^{1}\left[x^2 y+4\cdot\frac{y^3}{3}\right]_{y=0}^{2}\cdot dx = 1$$

$$b\int_{x=-1}^{1}\left(2x^2+\frac{32}{3}\right) dx = 1$$

$$b\left[2\frac{x^3}{3}+\frac{32}{3}\cdot x\right]_{-1}^{1} = 1$$

$$b\left[\frac{2}{3}+\frac{32}{3}-\left(-\frac{2}{3}-\frac{32}{3}\right)\right] = 1$$

$$b\left[\frac{2}{3}+\frac{32}{3}+\frac{2}{3}+\frac{32}{3}\right] = 1$$

$$\frac{68b}{3} = 1 \Rightarrow \boxed{b=\frac{3}{68}}$$

(4) The joint density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} a x^2 y, & 0<y<x<1 \\ 0, & \text{else where.} \end{cases}$$

a) Find the value of 'a', if it is a valid pdf.

b) Find the Marginal density functions.

Sol:-

Given that $f_{X,Y}(x,y) = \begin{cases} a x^2 y, & 0<y<x<1 \\ 0, & \text{else where.} \end{cases}$

Since it is a valid density function, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{x} a x^2 y\; dy\cdot dx = 1$$

$$\Rightarrow a\int_{x=0}^{1} x^2\cdot\left[\frac{y^2}{2}\right]_0^x dx = 1$$

$$\Rightarrow \frac{a}{2}\int_{x=0}^{1} x^2 (x^2-0)\,dx = 1$$

$$\Rightarrow \frac{a}{2}\int_{x=0}^{1} x^4\,dx = 1$$

$$\frac{a}{2}\left[\frac{x^5}{5}\right]_0^1 = 1$$

$$\frac{a}{10}[1-0] = 1 \Rightarrow \boxed{a=10}$$

The Marginal density functions are,

$$\therefore f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad 0<x<1$$

$$= \int_{y=0}^{x} 10x^2y\,dy$$

$$= 10x^2\left[\frac{y^2}{2}\right]_0^x = 5x^2[x^2-0] = 5x^4$$

$$\boxed{f_X(x) = 5x^4, \quad 0<x<1}$$

$$\therefore f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx, \quad 0<y<1$$

$$= \int_{x=y}^{1} 10x^2y\,dx$$

$$= 10y\left[\frac{x^3}{3}\right]_y^1$$

$$= 10y\left[\frac{1}{3}-\frac{y^3}{3}\right]$$

$$= \frac{10}{3}(y-y^4)$$

$$\boxed{\therefore f_Y(y) = \frac{10}{3}(y-y^4); \quad 0<y<1}$$

Conditional Distribution and Density functions:—

Point conditioning:— Consider two random variables X and Y. The distribution of a random variable X when the distribution function of a random variable Y is known at some value of 'y' is defined as the Conditional distribution function of X.

It can be expressed as

$$\boxed{F_X(x/Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx}{f_Y(y)}}$$

The Conditional density function 'X' is

$$f_X(x/Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

(or)

$$\boxed{f_X\left(\frac{x}{y}\right) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

Similarly,

The Conditional distribution function of Y is

$$\boxed{F_Y(Y/X=x) = \frac{\int_{-\infty}^{y} f_{x,y}(x,y)\,dy}{f_X(x)}}$$

The Conditional density function of Y is

$$f_Y(y/X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

(or)

$$\boxed{f_Y\left(\frac{y}{x}\right) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

For a discrete random variables:-

Cosider both x and y are discrete random variables with elements $X = \{x_1, x_2, --- x_N\}$ and $Y = \{y_1, y_2, --- y_M\}$ with the corresponding probabilities $p(x_n)$ and $p(y_m)$ then the density functions are
$$f_X(x) = \sum_{n=1}^{N} p(x_n) \cdot \delta(x-x_n)$$
$$f_Y(y) = \sum_{m=1}^{M} p(y_m)\, \delta(y-y_m)$$
$$f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} p(x_n, y_m) \cdot \delta(x-x_n) \cdot \delta(y-y_m).$$

Then,

① The Conditional Distribution function 'X' at a specified value at 'y_K' is
$$F_X(x/y=y_K) = \frac{\sum_{n=1}^{N} P(x_n, y_K) \cdot u(x-x_n)}{P(y_K)}.$$

② The Conditional density function of 'X' is
$$f_X(x/y_K) = \frac{\sum_{n=1}^{N} P(x_n, y_K) \cdot \delta(x-x_n)}{P(y_K)}.$$

③ The Conditional distribution function of 'Y' at a specified value at 'x_K' is
$$F_Y(y/x=x_K) = \frac{\sum_{m=1}^{M} P(x_K, y_m) \cdot u(y-y_m)}{P(x_K)}$$

④ The Conditional density function of 'Y' is
$$f_Y(y/x_K) = \frac{\sum_{m=1}^{M} P(x_K, y_m) \cdot \delta(y-y_m)}{P(x_K)}$$

Internal Conditioning :— Consider the event 'B' is defined in the interval $y_1 < y \le y_2$ for the random variable 'y' ie,

$B = \{y_1 < Y \le y_2\}$. Assume that $P(B) = P(y_1 < Y \le y_2) \ne 0$,

then the conditional distribution function of 'x' is given as

$$F_X\left(x/(y_1 < y \le y_2)\right) = \frac{\int_{y_1}^{y_2}\int_{-\infty}^{x} f_{X,Y}(x,y)\cdot dx\,dy}{\int_{y_1}^{y_2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy}.$$

By differentiating, we get the conditional density function of 'X' is

$$f_X\left(x/(y_1 < y \le y_2)\right) = \frac{\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy}{\int_{y_1}^{y_2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\cdot dy}$$

Similarly

the conditional distribution function of 'Y' is

$$F_Y\left(y/(x_1 < X \le x_2)\right) = \frac{\int_{x_1}^{x_2}\int_{-\infty}^{y} f_{X,Y}(x,y)\cdot dy\cdot dx}{\int_{x_1}^{x_2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\cdot dx}$$

By differentiating, we get the conditional density function of 'Y' is

$$f_Y\left(y/(x_1 < X \le x_2)\right) = \frac{\int_{x_1}^{x_2} f_{X,Y}(x,y)\cdot dx}{\int_{x_1}^{x_2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\cdot dy\,dx}.$$

problems:—

① Find the conditional density functions for the joint density functions $f_{X,Y}(x,y) = 4xy\, e^{-(x^2+y^2)} \cdot u(x)\cdot u(y)$.

Sol:- we know that the conditional density functions are

$$f_X\left(\frac{x}{y}\right) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{and} \quad f_Y\left(\frac{y}{x}\right) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad \text{——} ①$$

∴ The marginal density functions are,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\cdot dy.$$

$$= \int_0^{\infty} 4\,xy\, e^{-(x^2+y^2)} \cdot dy$$

$$= 4x\int_0^{\infty} e^{-x^2}\cdot e^{-y^2} dy$$

$$= 4x\, e^{-x^2}\int_0^{\infty} e^{-y^2} dy$$

Let $y^2 = t$ | L.L $t=0$

$2y\,dy = dt$ | U.L: $t=\infty$.

$y\,dy = \frac{dt}{2}$

$$f_X(x) = \left[4x e^{-x^2}\int_0^{\infty} e^{-t}\, dt/2\right]$$

$$f_X(x) = \frac{4}{2} x\, e^{-x^2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= -2x\, e^{-x^2}\left[e^{-\infty} - e^{0}\right]$$

$$= -2x\, e^{-x^2}[0-1]$$

$$\boxed{f_X(x) = 2x e^{-x^2}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^{\infty} 4\,xy\, e^{-(x^2+y^2)}\, dx$$

$$= 4y\int_0^{\infty} e^{-x^2}\cdot e^{-y^2} dx$$

$$= 4y\, e^{-y^2}\int_0^{\infty} e^{-x^2} dx$$

Let $x^2 = t$ | L.L:- $t=0$

$2x\,dx = dt$ | U.L:- $t=\infty$

$x\,dx = dt/2$

$$f_Y(y) = 4y\, e^{-y^2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}$$

$$f_Y(y) = \frac{4}{2} y\, e^{-y^2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= -2y\, e^{-y^2}\left[e^{-\infty} - e^{0}\right]$$

$$= -2y\, e^{-y^2}[0-1]$$

$$= 2y\, e^{-y^2}$$

$$\boxed{\therefore f_Y(y) = 2y\, e^{-y^2}}$$

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∴ The Conditional density functions are,

$$f_X\left(\frac{x}{y}\right) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = 2x\,e^{-x^2}$$

ie, $\boxed{f_X\left(\frac{x}{y}\right) = 2x\,e^{-x^2}\cdot u(x).}$

$$f_Y\left(\frac{y}{x}\right) = \frac{f_{X,Y}(x,y)}{f_X(x)} = 2y\,e^{-y^2}$$

$\boxed{\text{ie, } f_Y\left(\frac{y}{x}\right) = 2y\,e^{-y^2}\cdot u(y)}$

② Consider that the joint pdf of random variables, X and Y is $f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(x+y) & , 0<x<2,\ 0<y<2 \\ 0 & , \text{otherwise} \end{cases}$

find the ⓐ Conditional density functions and ⓑ $P\left((0<Y<\tfrac{1}{2})/X=1\right)$

Sol:-

Given that the joint density function, $f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(x+y), & 0<x<2, 0<y<2 \\ 0, & \text{otherwise} \end{cases}$

ⓐ The Marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\cdot dy$$

$$= \int_0^2 \frac{1}{8}(x+y)\,dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{8}\left[2x + \frac{4}{2} - 0\right]$$

$$= \frac{\cancel{2}}{\cancel{8}_4}(x+1)$$

$$\boxed{f_X(x) = \frac{1}{4}(x+1),\ 0<x<2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_0^2 \frac{1}{8}(x+y)\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} + xy\right]_0^2$$

$$= \frac{1}{8}\left(\left[\frac{2^2}{2} + 2y\right] - [0]\right)$$

$$= \frac{1}{8}(2+2y) = \frac{\cancel{2}}{\cancel{8}_4}(1+y)$$

$$\boxed{\therefore f_Y(y) = \frac{1}{4}(y+1),\ 0<y<2}$$

$\therefore$ The Conditional density functions are.

$$\Rightarrow f_{X,Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{(x+y)/8}{(y+1)/4} = \frac{(x+y)}{\cancel{8}_2} \times \frac{\cancel{4}}{(y+1)}$$

$$= \frac{x+y}{2(y+1)}$$

$$\boxed{\text{ie, } f_{X,Y}\left(\frac{x}{y}\right) = \begin{cases} \frac{x+y}{2(y+1)}, & 0<x<2,\ 0<y<2 \\ 0, & \text{otherwise.} \end{cases}}$$

$$\Rightarrow f_{X,Y}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y)/8}{(x+1)/4} = \frac{(x+y)}{\cancel{8}_2} \times \frac{\cancel{4}}{(x+1)}$$

$$= \frac{x+y}{2(x+1)}$$

$$\boxed{\text{ie, } f_{X,Y}\left(\frac{y}{x}\right) = \begin{cases} \frac{x+y}{2(x+1)}, & 0<x<2,\ 0<y<2 \\ 0, & \text{otherwise.} \end{cases}}$$

Statistical Independence of Random Variables:-

Consider two random variables X and Y with events $A = \{X \le x\}$ and $B = \{Y \le y\}$ for two real numbers x and y.

The two random variables are said to be statistically independent if and only if the joint probability is equal to the product of the individual probabilities.

$$\boxed{P\{X \le x, Y \le y\} = P\{X \le x\} \cdot P\{Y \le y\}}$$

$\Rightarrow$ The Joint distribution function is,

$$\boxed{F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)}$$

$\Rightarrow$ The Joint density function is,

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

The above functions give the condition for two random variables X and Y to be statistically independent.

$\Rightarrow$ The Conditional distribution functions for independent random variables are given by

$$F_X\left(\frac{x}{Y\le y}\right) \text{ (or) } F_X\left(\frac{x}{y}\right) = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x)\cdot \cancel{F_Y(y)}}{\cancel{F_Y(y)}} = F_X(x)$$

$$\therefore \boxed{F_X\left(\frac{x}{y}\right) = F_X(x)}$$

Similarly, $\boxed{F_Y\left(\frac{y}{x}\right) = F_Y(y)}$

$\Rightarrow$ The conditional density functions for independent random variables are given by

$$f_X\left(\frac{x}{y}\right) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

$$\therefore \boxed{f_X\left(\frac{x}{y}\right) = f_X(x)}$$

Similarly, $\boxed{f_Y\left(\frac{y}{x}\right) = f_Y(y)}$

UNIT - 3

operations on Multiple Random variables

Function of Joint Random Variables:-

If $g(x,y)$ is a function of two random variables X and Y with Joint density function $f_{X,Y}(x,y)$ then the expected value of the function $g(x,y)$ is given by

$$\bar{g} = E[g(x,y)]$$

i.e., $$\bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

similarly, for N random variables $X_1, X_2, X_3 \ldots X_N$ with joint density function $f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots x_N)$ then the expected value of the function $g(x_1, x_2 \ldots x_N)$ is given as

$$\bar{g} = E[g(x_1, x_2, \ldots x_N)]$$

i.e., $$\bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2 \ldots x_N)\, f_{X_1 X_2 \ldots X_N}(x_1, x_2 \ldots x_n)\, dx_1\, dx_2 \ldots dx_n$$

Joint Moments:

Joint Moments about the origin:

The Joint moments about the origin for two random variables X, Y is the expected values of the function. $g(x,y) = x^n Y^k$. It is denoted as m_{nk}

Mathematically

$$\boxed{m_{nk} = E[x^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\, dx\, dy}$$

where n and k are positive integers.

* The sum $n+k$ is called the order of Moments.
* If $k=0$, then $m_{n0} = E[X^n]$ are called moments of X.
* If $n=0$, then $m_{0k} = E[Y^k]$ are called moments of Y.

first order moments are:

$$m_{10} = E[X] = \bar{X} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx\, dy$$

$$m_{01} = E[Y] = \bar{Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dx\, dy$$

second order moments are:

$$m_{20} = E[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\, dx\, dy$$

$$m_{02} = E[Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{X,Y}(x,y)\, dx\, dy$$

$$m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx$$

similarly for N random variables, we have joint moments about the origin.

$$m_{n_1 n_2 \ldots n_N} = E\left[X_1^{n_1} X_2^{n_2} \ldots X_N^{x_N}\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots x_N)\, dx_1\, dx_2 \cdots dx_N$$

where $n_1, n_2, \ldots n_N$ are all positive integers.

Note :-

the point coordinate $(\bar{X}, \bar{Y})$ is the centre of gravity of the function $f_{X,Y}(x,y)$

correlation:-

consider two random variables X and Y. the second order Joint Moment "m_{11}" is called correlation of X and Y. It is denoted by R_{XY}

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

for discrete random variables :

$$R_{XY} = \sum_m \sum_n x_m y_n P_{XY}(x_m, y_n).$$

* properties of correlation :

1) If two random variables X and Y are statistically independent. then X and Y are said to be uncorrelated.

i.e., $E[XY] = E[X].E[Y]$

proof :-

consider X,Y are two random variables with Joint density function $f_{X,Y}(x,y)$ and marginal density functions $f_X(x)$ and $f_Y(y)$.

If X and Y are statistically independent. then we know that

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \rightarrow (1)$$

The correlation is

$$R_{X,Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$R_{X,Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy \quad [\text{from } (1)]$$

$$R_{X,Y} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$R_{xy} = E[X] \cdot E[Y]$$

$$\boxed{E[XY] = E[X]\, E[Y]}$$

2) If the random variables X and Y are orthogonal, then their correlation is zero

i.e., $R_{xy} = 0$

Proof :-

consider two random variables X and Y. with the density functions $f_x(x)$ and $f_y(y)$.

If X and Y are said to be orthogonal, then their joint occurance is zero.

i.e., $f_{x,y}(x,y) = 0 \rightarrow (1)$

The correlation is

$$R_{xy} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{x,y}(x,y)\, dx\, dy$$

$$R_{xy} = E[X,Y] = 0$$

$$\boxed{R_{xy} = 0}$$

problems :

* consider two random variables X and Y such that $Y = -4X + 20$. The mean value and the variance of X and Y are 4 and 2 respectively. find the correlation and comment on result.

sol: Given that

$Y = -4X + 20$ $\quad E[X] = 4$ and $\sigma_x^2 = 2$.

we know that

$$E[XY] = E[X(-4X + 20)]$$

$$E[XY] = E[-4X^2 + 20X]$$

$$\boxed{E[XY] = -4\,E[X^2] + 20\,E[X]} \rightarrow (1)$$

we know that

$$\sigma_x^2 = E[X^2] - [E[X]]^2$$

$$E[X^2] = \sigma_x^2 + [E[X]]^2$$

$$E[X^2] = 2 + 4^2$$

$$\boxed{E[X^2] = 18}$$

from (1) $\Rightarrow$ $E[XY] = -4(18) + 20(4)$

$$E[XY] = -72 + 80$$

$$\boxed{E[XY] = 8}$$

The correlation $E[XY] = 8 \rightarrow (2)$

and $E[Y] = E[-4X + 20]$

$$E[Y] = -4\,E[X] + 20$$

$$E[Y] = -16 + 20$$

$$\boxed{E[Y] = 4}$$

$\therefore$ $E[X].E[Y] = 4 \times 4 = 16 \rightarrow (3)$

from (2) & (3)

$$\boxed{E[XY] \neq E[X].\,E[Y]}$$

$\therefore$ The random variables are neither statistically independent and nor orthogonal.

*) The joint density function of X and Y is $f_{X,Y}(x,y) = \begin{cases} \frac{1}{100}, & 0<x<5 \\ & 0<y<20 \\ 0, & \text{otherwise} \end{cases}$ find the expected value of the functions:

a) XY b) X^2Y c) $(XY)^2$

Sol:- a) $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{x=0}^{5}\int_{y=0}^{20} xy\left(\frac{1}{100}\right)dy\,dx$$

$$= \frac{1}{100}\int_{x=0}^{5} x\left[\frac{y^2}{2}\right]_0^{20} dx$$

$$= \frac{1}{200}\int_{x=0}^{5} x\left[(20)^2-0\right]dx$$

$$= \frac{400}{200}\int_{x=0}^{5} x\,dx$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^5 dx$$

$$\boxed{E[XY] = 25}$$

b) $E[X^2Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y\, f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{x=0}^{5}\int_{y=0}^{20} x^2y\left[\frac{1}{100}\right]dy\,dx.$$

$$= \frac{1}{100}\int_{x=0}^{5} x^2\left[\frac{y^2}{2}\right]_0^{20} dx$$

$$= \frac{1}{200}\int_{x=0}^{5} x^2\left[20^2-0\right]dx$$

$$= \frac{400}{2}\left[\frac{x^3}{3}\right]_0^5.$$

$$= \frac{2}{3}[5^3 - 0]$$

$$= \frac{250}{3}$$

$$\boxed{E[X^2Y] = 83.33}$$

c) $$E[(XY)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y^2 f_{XY}(x,y)\,dx\,dy$$

$$= \int_{x=0}^{5}\int_{y=0}^{20} x^2y^2\left(\frac{1}{100}\right)dy\,dx$$

$$= \frac{1}{100}\int_{x=0}^{5} x^2\left[\frac{y^3}{3}\right]_0^{20} dx$$

$$= \frac{1}{300}\int_{x=0}^{5} x^2(20^3)\,dx.$$

$$= \frac{800}{300}\left[\frac{x^3}{3}\right]_0^5$$

$$= \frac{80}{9}[5^3 - 0]$$

$$= \frac{80 \times 125}{9}$$

$$\boxed{E[(XY)^2] = 1111.111}$$

* The density function of two random variables x and Y $f_{X,Y}(x,y) = 4e^{-2(x+y)}u(x)u(y)$ find the mean value of the function $e^{-(x+y)}$.

Sol: Given that

$$f_{X,Y}(x,y) = 4e^{-2(x+y)}\,u(x).u(y)$$

$$E[e^{-(x+y)}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x+y)} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x+y)}\,4e^{-(x+y)}\,dx\,dy$$

$$= 4 \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-3x} e^{-3y} \, dy \, dx$$

$$= 4 \int_{x=0}^{\infty} e^{-3x} \left[\frac{e^{-3y}}{-3} \right]_0^{\infty} dx$$

$$= \frac{-4}{3} \int_{x=0}^{\infty} e^{-3x} \, dx$$

$$= \frac{-4}{3} \left[\frac{e^{-3x}}{-3} \right]_0^{\infty}$$

$$\boxed{E\left[e^{-(x+y)}\right] = \frac{4}{9}.}$$

* consider two independent random variables x and Y in two experiments their probabilities are shown below.

X	0	1
P(X)	1/3	2/3

Y	1	0
P(Y)	1/3	2/3

then find i) $E[2x+3y]$ ii) $E[2x^2 y^2]$ iii) $E[XY]$

sol: Given

i) $E[2X+3Y] = 2E[X] + 3E[Y]$

$$= 2\left[\Sigma x_i P(x_i)\right] + 3\left[\Sigma y_i P(y_i)\right]$$

$$= 2\left[0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{2}{3}\right)\right] + 3\left[1\left(\tfrac{1}{3}\right) + 0\left(\tfrac{2}{3}\right)\right]$$

$$= 2\left[\tfrac{2}{3}\right] + 3\left[\tfrac{1}{3}\right]$$

$$= \frac{4}{3} + 1$$

$$\boxed{E[2X+3Y] = \frac{7}{3}}$$

ii) $E[2X^2 - Y^2] = 2E[X^2] - E[Y^2]$

Now

$E[X^2] = \Sigma x_i^2 P(x_i)$ $\quad\quad$ $E[Y^2] = \Sigma y_i^2 P(y_i)$

$E(X^2) = 0 + \frac{2}{3}$ $\quad\quad$ $E[Y^2] = \frac{1}{3} + 0$

$E[X^2] = \frac{2}{3}$ $\quad\quad$ $E[Y^2] = \frac{1}{3}$

Now

$$E[2X^2 - Y^2] = 2E(X^2] - E[Y^2]$$

$$= 2\left(\frac{2}{3}\right) - \frac{1}{3}$$

$$= \frac{4}{3} - \frac{1}{3}$$

$$\boxed{E[2X^2 - Y^2] = 1}$$

iii) $E[XY] = E[X] \cdot E[Y]$

$$= \frac{2}{3} \times \frac{1}{3}$$

$$\boxed{E[XY] = \frac{2}{9}}$$

* The joint density function for X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{xy}{9}, & 0<x<2 \\ & 0<y<3 \\ 0, & \text{otherwise} \end{cases}$$

i) show that X and Y are statistically independent.

ii) show that X and Y are uncorrelated.

Sol: Given that

$$f_{X,Y}(x,y) = \begin{cases} \frac{xy}{9} & 0<x<2 \\ & 0<y<3 \\ 0, & \text{otherwise} \end{cases}$$

we know that

a) X and Y are statistically independent

we have

$$f_{X,Y}(x,y) = f_X(x).\ f_Y(y)$$

∴ The Marginal density function are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$f_X(x) = \int_0^3 \frac{xy}{9}\, dy \qquad f_Y(y) = \int_0^2 \frac{xy}{9}\, dx$$

$$f_X(x) = \frac{x}{9}\left[\frac{y^2}{2}\right]_0^3 \qquad f_Y(y) = \frac{y}{9}\left[\frac{x^2}{2}\right]_0^2$$

$$f_X(x) = \frac{x}{9}\left[\frac{9}{2}\right] \qquad f_Y(y) = \frac{y}{9}\left[\frac{4}{2}\right]$$

$$\boxed{f_X(x) = \frac{x}{2};\ 0<x<2} \qquad \boxed{f_Y(y) = \frac{2y}{9}\ ;\ 0<y<3}$$

Now

$$f_X(x).\ f_Y(y) = \frac{x}{2} \times \frac{2y}{9}$$

$$= \frac{xy}{9}$$

$$\boxed{f_X(x).\ f_Y(y) = f_{X,Y}(x,y)}$$

X and Y are statistically Independent.

b)
$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ .f(x,y)\, dx\, dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} xy\left(\frac{xy}{9}\right) dy\ dx$$

$$= \frac{1}{9}\int_{x=0}^{2}\int_{y=0}^{3} x^2 y^2\ dy\ dx$$

$$= \frac{1}{9}\int_{x=0}^{2} x^2\left[\frac{y^3}{3}\right]_0^3 dx$$

$$= \frac{1}{9} \times 9 \left[\frac{x^3}{3}\right]_0^2$$ ⑥

$$E[XY] = \frac{8}{3}$$

Now

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(X) = \int_0^2 x\left[\frac{x}{2}\right] dx$$

$$E(X) = \int_0^2 \frac{x^2}{2}\, dx$$

$$E(X) = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$E[X] = \frac{1}{2}\left[\frac{8}{3}\right]$$

$$E[X] = \frac{4}{3}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$E[Y] = \int_0^3 y\left[\frac{2y}{9}\right] dy$$

$$E[Y] = \frac{2}{9}\int_0^3 y^2\, dy$$

$$E(Y) = \frac{2}{9}\left[\frac{y^3}{3}\right]_0^3$$

$$E[Y] = \frac{2}{9}\left[\frac{27}{3}\right]$$

$$E(Y) = 2$$

$$E[XY] = E[X].E(Y)$$

$$= \frac{4}{3} \times 2$$

$$= \frac{8}{3}$$

$$\therefore \boxed{E[XY] = E[X].E[Y]}$$

HW The joint density function of two random variables X and Y is $f(x,y) = \begin{cases} \frac{(x+y)^2}{40} & -1<x<1,\ -3<y<3 \\ 0, & \text{otherwise} \end{cases}$

find

i) the covariances of X and Y

ii) verify the correlation.

Joint central Moments:

consider two random variables X, Y then the expected values of the function $g(x,y) = (X-\bar{X})^n (Y-\bar{Y})^k$ are called Joint central moments.

$$E[g(x,y)] = E\left[(X-\bar{X})^n (Y-\bar{Y})^k\right]$$

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X-\bar{X})^n (Y-\bar{Y})^k f_{X,Y}(x,y)\, dx\, dy$$

$$\mu_{nk} = E\left[(X-\bar{X})^n (Y-\bar{Y})^k\right]$$

$$\mu_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X-\bar{X})^n (Y-\bar{Y})^k f_{X,Y}(x,y)\, dx\, dy$$

zeroth order central Moments:-

zeroth order central Moments is

$$\mu_{00} = E\left[(X-\bar{X})^0 (Y-\bar{Y})^0\right]$$

$$\mu_{00} = E[1]$$

$$\boxed{\mu_{00} = 1}$$

The first order:-

The first order central Moments are

$$\mu_{10} = E[(X-\bar{X})] = E[X] - \bar{X} = \bar{X} - \bar{X}$$

$$\mu_{10} = 0$$

$$\mu_{01} = E[(Y-\bar{Y})] = E[Y] - \bar{Y} = \bar{Y} - \bar{Y}$$

$$\mu_{01} = 0$$

The second order:-

The second order central Moments are

$$\mu_{20} = E[(X-\bar{X})^2] = \sigma_x^2$$

$$\boxed{\mu_{20} = \sigma_x^2.}$$

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$$\mu_{02} = E[(Y-\bar{Y})^2] = \sigma_y^2$$

$$\boxed{\mu_{02} = \sigma_y^2}$$

$$\mu_{11} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$\boxed{\mu_{11} = \sigma_{xy}}$$

* For N random variables $X_1, X_2 \dots X_N$. The joint central moments are defined as

$$\mu_{n_1 n_2 \dots n_N} = E[(X_1-\bar{X}_1)^{n_1}(X_2-\bar{X}_2)^{n_2} \dots (X_N-\bar{X}_N)^{n_N}]$$

$$\mu_{n_1 n_2 \dots n_N} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (X_1-\bar{X})^{n_1}(X_2-\bar{X}_2)^{n_2} \dots (X_N-\bar{X}_N)^{n_N} f(x_1, x_2 \dots x_N)\, dx_1\, dx_2 \dots dx_N$$

co-variance :

consider two random variables X and Y. The second order joint central moment 'μ_{11}' is called co-variance of X and Y. It can be expressed as $cov(x,y)$ or C_{xy}.

$$\sigma_{xy} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$\sigma_{xy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X-\bar{X})(Y-\bar{Y}) f(x,y)\, dx\, dy$$

for Discrete Random variables

$$\sigma_{xy} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$\sigma_{xy} = \Sigma \Sigma (X-\bar{X})(Y-\bar{Y})\, P(x_n, y_k)$$

Correlation coefficient :-

The two random variables X and Y the normalised second order central moment is called correlation coefficient. It is denoted as "ρ" or $\rho(x,y)$ or $\rho(x,y)$

$$\rho = \frac{cov(x,y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}$$

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(or)

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}.\mu_{02}}}$$

Note :

The range of correlation coefficient is $-1 \le \rho \le 1$. If x and Y are independent. If the correlation between x and y is perfect. $\rho = 1$

* If x and Y are equal then $\rho = 1$

properties of co-variance :-

* If x and Y are two random variables then the co-variance

$$C_{xy} = R_{xy} - \bar{X}\bar{Y}$$

$$C_{xy} = E[XY] - E[X].E[Y]$$

* If two random variables x and Y are Independent. then the co-variance is 0 i.e., $C_{xy} = 0$ but the converse is not True.

* for two random variables x and Y

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2C_{xy}$$

$$\text{var}(X-Y) = \text{var}(X) + \text{var}(Y) - 2C_{xy}.$$

* for two random variables x and Y the inequality $|\sigma_{xy}| \le \sigma_x . \sigma_y$ is true.

* If x and Y are two random variables then the co-variances of $x+a$, $Y+b$ where a and b are constants.

$$\text{cov}(X+a, Y+b) = \text{cov}(x, y)$$

* If x and y are two random variables (8)

Then the $cov(ax, by) = ab\, cov(x, y)$

Note :-

$$C_{xy} = R_{xy} - \overline{X}\,\overline{Y}$$

$$C_{xy} = E[XY] - E[X].E[Y]$$

Problems :-

* find the coefficient of correlation between x and y from the data.

X	1	2	3	4
Y	2	4	8	10

Sol:- coefficient of correlation

$$\rho = \frac{cov(x, y)}{\sqrt{\sigma_x^2 - \sigma_y^2}}$$

$$\sigma_x^2 = E[x^2] - [E(x)]^2$$

$$\sigma_y^2 = E[Y^2] - [E(Y)]^2$$

$$cov(x, y) = R_{xy} - \overline{X}.\overline{Y} = E[XY] - E[X]\, E[Y]$$

$$E[X] = \overline{X} = \frac{1+2+3+4}{4} = 2.5$$

$$E[Y] = \overline{Y} = \frac{2+4+8+10}{4} = 6$$

$$E[X^2] = \frac{1+4+9+16}{4} = 7.5$$

$$E[Y^2] = \frac{4+16+64+100}{4} = 46$$

$$E[XY] = \frac{2+8+24+40}{4} = 18.5$$

$\sigma_x^2 = 7.5 - (2.5)^2 = 1.25$

$\sigma_y^2 = 46 - (6)^2 = 10$

$cov(x,y) = 18.5 - (2.5)6 = 3.5$

The coefficient of correlation is

$$\rho = \frac{3.5}{\sqrt{1.25 \times 10}}$$

$$\rho = 0.989$$

The coefficient correlation is 0.989.

* Find the correlation coefficient between x and y from the data.

x	1	2	3	4
P(x)	0.2	0.4	0.1	0.3

y	1	2	3	4
P(y)	0.25	0.25	0.15	0.35

x,y	(1,1)	(2,2)	(3,3)	(4,4)
P(x,y)	1/2	1/8	1/4	1/8

Sol: we know that

$E[X] = \bar{X} = \Sigma x_i P(x_i)$

$= 1(0.2) + 2(0.4) + 3(0.1) + 4(0.3$

$E(X) = 2.5$

$E[Y] = \bar{Y} = \Sigma y_i p(y_i)$

$= 1(0.25) + 2(0.25) + 3(0.15) + 4(0.35)$

$E[Y] = 2.6$

$E[x^2] = \Sigma x_i^2 p(x_i)$

$= 0.2 + 4(0.4) + 9(0.1) + 16(0.3$

$E(x^2) = 7.5$

* $E[Y^2] = \Sigma y_i^2 P(y_i)$

$= 0.25 + 4(0.25) + 9(0.15) + 16(0.35)$

$E[Y^2] = 8.2$

* $E[XY] = \Sigma\Sigma x_i y_i \, P(x_i y_i)$

$= 1(1)(\frac{1}{2}) + 2(2)(\frac{1}{8}) + 3(3)(\frac{1}{4}) + 4(4)(\frac{1}{8})$

$E[XY] = 5.25$

* $\sigma_x^2 = E[X^2] - [E[X]]^2$

$\sigma_x^2 = 1.25$

* $\sigma_y^2 = E[Y^2] - [E[Y]]^2$

$= 8.2 - (2.6)^2$

$\sigma_y^2 = 1.44$

$cov(x,y) = R_{xy} - \bar{X}\bar{Y}$

$= E(XY) - E(X).E(Y)$

$= 5.25 - 2.5 \times 2.6$

$cov(x,y) = -1.25$

the coefficient of correlation is

$$\rho = \frac{cov(x,y)}{\sqrt{\sigma_x^2 . \sigma_y^2}}$$

$$\rho = \frac{-1.25}{\sqrt{1.25 \times 1.44}}$$

$$\boxed{\rho = -0.9317}$$

* If the Joint density function is

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

find the correlation coefficient

Sol: Given that
$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\rho = \frac{cov(x,y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}$$

$$cov(x,y) = E[XY] - E[X]\,E[Y]$$

$$\sigma_x^2 = E[X^2] - [E[X]]^2$$

$$\sigma_y^2 = E[Y^2] - [E[Y]]^2$$

The marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_X(x) = \int_0^1 (x+y)\,dy$$

$$f_X(x) = \left[x[y] + \frac{y^2}{2}\right]_0^1$$

$$f_X(x) = x[1] + \frac{1}{2}$$

$$\boxed{f_X(x) = x + \frac{1}{2}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$f_Y(y) = \int_0^1 (x+y)\,dx$$

$$f_Y(y) = \left[y[x] + \frac{x^2}{2}\right]_0^1$$

$$f_Y(y) = \frac{1}{2} + y$$

$$\boxed{f_Y(y) = y + \frac{1}{2}}$$

Now

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_0^1 x\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx$$

$$= \left[\frac{x^3}{3} + \frac{1}{2}\cdot\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$\boxed{E[X] = \frac{7}{12}}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

$$= \int_0^1 y\left[y + \frac{1}{2}\right]dy$$

$$= \left[\frac{y^3}{3} + \frac{1}{2}\cdot\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$\boxed{E[Y] = \frac{7}{12}}$$

(10)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$
$$= \int_0^1 x^2\left[x + \frac{1}{2}\right]dx$$
$$= \int_0^1 \left[x^3 + \frac{x^2}{2}\right]dx$$
$$= \left[\frac{x^4}{4} + \frac{1}{2}\,\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{4} + \frac{1}{6}$$
$$\boxed{E[X^2] = \frac{5}{12}}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f(y)\,dy$$
$$= \int_0^1 y^2\left[y + \frac{1}{2}\right]dy$$
$$= \int_0^1 \left[y^3 + \frac{y^2}{2}\right]dy$$
$$= \left[\frac{y^4}{4} + \frac{1}{2}\,\frac{y^3}{3}\right]_0^1$$
$$= \frac{1}{4} + \frac{1}{6}$$
$$\boxed{E[Y^2] = \frac{5}{12}}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$
$$= \int_{x=0}^{1}\int_{y=0}^{1} xy(x+y)\,dy\,dx$$
$$= \int_{x=0}^{1}\left[\int_{y=0}^{1} x^2y + xy^2\,dy\right]dx$$
$$= \int_{x=0}^{1}\left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_0^1 dx$$
$$= \int_{x=0}^{1}\left[\frac{x^2}{2} + \frac{x}{3}\right]dx$$
$$= \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{2}{6}$$
$$\boxed{E[XY] = \frac{1}{3}}$$

* $\sigma_{x}^2 = E[X^2] - \left[E[X]\right]^2$
$$= \frac{5}{12} - \frac{49}{144}$$
$$\boxed{\sigma_x^2 = \frac{11}{144}}$$

* $\sigma_y^2 = E[Y^2] - \left[E[Y]\right]^2$
$$= \frac{5}{12} - \frac{49}{144}$$
$$\boxed{\sigma_y^2 = \frac{11}{144}}$$

* $\mathrm{cov}_{xy} = E[XY] - E[X].E[Y]$
$$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12}$$
$$= \frac{1}{3} - \frac{49}{144}$$
$$\boxed{\mathrm{cov}_{xy} = \frac{-1}{144}}$$

* the coefficient of correlation is

$$\rho = \frac{-1/144}{\sqrt{\frac{11}{144}\cdot\frac{11}{144}}} \Rightarrow \boxed{\rho = -0.0909}$$

* Two random variables X & Y have density function $f(x,y) = \begin{cases} \frac{2}{43}(x+0.5y)^2 & 0<x<2, \\ & 0<y<3 \\ 0, & \text{otherwise} \end{cases}$ *

i) find first and second order moments

ii) find the co-variance

iii) are X and Y uncorrelated.

Sol:- Given that

$$f(x,y) = \begin{cases} \frac{2}{43}(x+0.5y)^2, & 0<x<2 \\ & 0<y<3 \\ 0, & \text{otherwise} \end{cases}$$

$$m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f(x,y)\, dx\, dy$$

$$m_{nk} = \frac{2}{43}\int_{x=0}^{2}\int_{y=0}^{3} x^n y^k \left[x+\frac{y}{2}\right]^2 dx\, dy$$

$$m_{nk} = \frac{2}{43}\int_{x=0}^{2}\int_{y=0}^{3} x^n y^k \left[x^2+\frac{y^2}{4}+xy\right] dx\, dy$$

$$m_{nk} = \frac{2}{43}\int_{x=0}^{2}\int_{y=0}^{3} x^{n+2}y^k + \frac{1}{4}x^n y^{k+2} + x^{n+1}y^{k+1}\Big]\, dy\, dx$$

$$m_{nk} = \frac{2}{43}\int_{x=0}^{2}\int_{y=0} x^{n+2}\left[\frac{y^{k+1}}{k+1}\right] + \frac{x^n}{4}\left[\frac{y^{k+3}}{k+3}\right] + x^{n+1}\left[\frac{y^{k+2}}{k+2}\right]\Bigg]_0^{1} dx$$

$$m_{nk} = \frac{2}{43}\int_{x=0}^{2}\left[x^{n+2}\left[\frac{3^{k+1}}{k+1}\right] + \frac{x^n}{4}\left[\frac{3^{k+3}}{k+3}\right] + x^{n+1}\left[\frac{3^{k+2}}{k+2}\right]\right] dx$$

$$m_{nk} = \frac{2}{43}\left[\frac{x^{n+3}}{n+3}\left[\frac{3^{k+1}}{k+1}\right] + \frac{x^{n+1}}{4(n+1)}\left[\frac{3^{k+3}}{k+3}\right] + \frac{x^{n+2}}{n+2}\left[\frac{3^{k+2}}{k+2}\right]\right]_0^2$$

$$m_{nk} = \frac{2}{43}\left[\frac{2^{n+3}\cdot 3^{k+1}}{n+3(k+1)} + \frac{2^{n+1}(3^{k+3})}{4(n+1)(k+3)} + \frac{2^{n+2}(3^{k+2})}{n+2(k+2)}\right]$$

The first order moments are

$$* \; m_{10} = \frac{2}{43}\left[\frac{2^{1+3}\cdot 3^{0+1}}{(1+3)(0+1)} + \frac{2^{1+1}\cdot 3^{0+3}}{4(1+1)(0+3)} + \frac{2^{1+2}\cdot 3^{0+2}}{(1+2)(0+2)}\right]$$

$$m_{10} = \frac{2}{43}[3+18+4]$$

$$\boxed{m_{10} = 1.325}$$

* $m_{01} = \frac{2}{43}\left[\frac{2^3.3^2}{3\times2} + \frac{2.3^4}{4.4} + \frac{2^2.3^3}{2.3}\right]$

$m_{01} = \frac{2}{43}\left[12 + \frac{81}{8} + 18\right]$

$\boxed{m_{01} = 1.8663}$

The second order Moments are

* $m_{20} = \frac{2}{43}\left[\frac{2^5.3}{5.1} + \frac{2^3.3^3}{2.3} + \frac{2.3^2}{4.2}\right]$

$m_{20} = \frac{2}{43}\left[\frac{2^5\times3}{5} + \frac{2\times3^2}{3} + 18\right]$

$\boxed{m_{20} = 2.0093}$

* $m_{02} = \frac{2}{43}\left[\frac{2^3\times3^3}{3\times3} + \frac{2\times3^5}{4\times5} + \frac{2^2\times3^4}{2\times4}\right]$

$m_{02} = \frac{2}{43}\left[24 + \frac{243}{10} + \frac{81}{2}\right]$

$\boxed{m_{02} = 4.1302.}$

* $m_{11} = \frac{2}{43}\left[\frac{2^4.3^2}{4.2} + \frac{2^2.3^4}{8.4} + \frac{2^3.3^3}{3.3}\right]$

$m_{11} = \frac{2}{43}\left[18 + \frac{81}{8} + 24\right]$

$\boxed{m_{11} = 2.4244}$

ii) $cov(x,y) = E[XY] - E[X]\,E[Y]$

we know that

$m_{10} = E[x^1y^0] = E[x] = 1.325$

$m_{01} = E[x^0y^1] = E[Y] = 1.866$

$m_{11} = E[x^1y^1] = E[XY] = 242$

$cov(x,y) = 242 - (1.325)(1.866)$

$cov(x,y) = 242 - (1.325)(1.866)$

$\boxed{cov(x,y) = -0.0524}$

iii) since the covariance $c_{xy} \neq 0$, then the x and y are not uncorrelated [correlated].

6) Show that two random variables X_1 & X_2 with joint pdf $f_{X_1X_2}(x_1,x_2) = \begin{cases} \frac{1}{16}, & |x_1|<4,\ 2<x_2<4 \\ 0, & \text{otherwise} \end{cases}$ are independent & orthogonal.

Sol:- The Marginal density functions are

$$f_X(x) = \int_{-\infty}^{\infty} f(x_1,x_2)\,dx_2 \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x_1,x_2)\,dx_1$$

$$= \int_2^4 \frac{1}{16}\,dx_2 \qquad = \int_{-4}^{4} \frac{1}{16}\,dx_1$$

$$= \frac{1}{16}\left[x_2\right]_2^4 \qquad = \frac{1}{16}\left[x_1\right]_{-4}^{4}$$

$$= \frac{1}{16}\left[4-2\right] \qquad = \frac{1}{16}\left[4+4\right]$$

$$\boxed{f_X(x) = \frac{1}{8}} \qquad \boxed{f_Y(y) = \frac{1}{2}}$$

consider

$$f_{X_1}(x_1)\cdot f_{X_2}(x_2) = \frac{1}{8}\times\frac{1}{2}$$

$$= \frac{1}{16}$$

$$\therefore f_{X_1}(x_1)\,f_{X_2}(x_2) = f_{X_1X_2}(x_1,x_2)$$

$\therefore X_1$ and X_2 are Independent

ii) we know that X_1 and X_2 are said to be orthogonal. If it has $E[X_1 \cdot X_2] = 0$

$$E[X_1X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f(x_1,x_2)$$

$$= \int_{x_1=-4}^{4}\int_{x_2=2}^{4} x_1x_2\left(\frac{1}{16}\right)dx_2\,dx_1$$

$$= \frac{1}{16}\int_{x_1=-4}^{4}\left[x_1\left[\frac{x_2^2}{2}\right]_2^4\right]dx_1$$

$$= \frac{1}{16}\int_{x_1=-4}^{4} x_1\left[\frac{4^2-2^2}{2}\right]dx_1$$

$$= \frac{1}{16}\int_{x_1=-4}^{4} x_1\left[\frac{12}{2}\right]dx_1$$

$$= \frac{6}{16}\left[\frac{x_1^2}{2}\right]_{-4}^{4}$$

$$= \frac{6}{16}\left[\frac{(4)^2-(-4)^2}{2}\right]$$

$$= \frac{6}{16}[0]$$

$$\boxed{E[x_1x_2] = 0}$$

$\therefore$ x_1 and x_2 are orthogonal.

* for a random variables x and Y the Joint density function is $f_{x,y}(x,y) = 0.15\ \delta(x+1)\ \delta(y) + 0.1\ \delta(x)\ \delta(y) + 0.1\ \delta(x)\ \delta(y-2) + 0.4\ \delta(x-1)\ \delta(y+2) + 0.2\ \delta(x-1)\ \delta(y-1) + 0.05\ \delta(x-1)\ \delta(y-3)$. Then find

a) the correlation

b) the correlation coeff. of X & Y

c) are X & Y either uncorrelated or orthogonal.

Sol:

$x\ y$	$(-1,0)$	$(0,0)$	$(0,2)$	$(1,-2)$	$(1,1)$	$(1,3)$
$p(x,y)$	0.5	0.1	0.1	0.4	0.2	0.05

correlation: $R_{xy} = E[XY]$

$$\rho = \frac{cov(x,y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}$$

$$cov(x,y) = E[XY] - E[X]\,E[Y]$$

$$\sigma_x^2 = E[X^2] - \left[E[X]\right]^2$$

$$\sigma_y^2 = E[Y^2] - \left[E[Y]\right]^2$$

* $E[X] = \Sigma x_i\, p(x_i)$

$$= -1(0.15) + 0(0.1) + 0(0.1) + 1(0.4) + 1(0.2) + 1(0.05)$$

$$\boxed{E[X] = 0.5}$$

* $E[Y] = \Sigma y_i\, p(y_i)$

$= 0(0.15) + 0(0.1) + 2(0.1) + (-2)(0.4) + 1(0.2) + 3(0.05)$

$\boxed{E[Y] = -0.25}$

* $E[x^2] = 1(0.15) + 0(0.1) + 0(0.1) + 1(0.4) + 1(0.2) + 1(0.05)$

$\boxed{E[x^2] = 0.8}$

* $E[Y^2] = \Sigma y_i^2 p(y_i)$

$= 0(0.15) + 0(0.1) + 0(0.1) + 4(0.4) + 1(0.2) + 9(0.05)$

$\boxed{E[Y^2] = 2.65}$

* $E[XY] = \Sigma\Sigma x_i y_i \, p[x_i y_i]$

$= 0(0.15) + 0(0.1) + 0(0.1) - 2(0.4) + 1(0.2) + 3(0.05)$

$\boxed{E[XY] = -0.45}$

$cov(x,y) = E[XY] - E[X]\,E[Y]$

$= 0.45 - (0.5)(-0.25)$

$\boxed{cov(x,y) = -0.325}$

$\sigma_x^2 = E[x^2] - [E(x)]^2$

$= 0.8 - (0.5)^2$

$\boxed{\sigma_x^2 = 0.55}$

$\sigma_y^2 = E[y^2] - [E[Y]]^2$

$\boxed{\sigma_y^2 = 2.587}$

i) correlation : $R_{xy} = E[xy]$

$\boxed{R_{xy} = -0.45}$

ii) correlation coefficient $\rho = \dfrac{cov(x,y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}} = \dfrac{-0.325}{\sqrt{0.55(2.587)}}$

$\boxed{\rho = -0.272}$

iii) uncorrelated.

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Joint characteristic function :-

The Joint characteristic equation of two random variables x and Y is defined as the expected value of the Joint function $g(x,y) = e^{j\omega_1 x} \cdot e^{j\omega_2 y}$. It can be expressed as

$$\phi_{x,y}(\omega_1,\omega_2) = E\left[e^{j\omega_1 X} e^{j\omega_2 Y}\right]$$

$$= E\left[e^{j(\omega_1 X + \omega_2 Y)}\right]$$

$$\phi_{x,y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\omega_1 x + \omega_2 y)} f_{x,y}(x,y)\, dx\, dy$$

where $\omega_1 \omega_2$ are real variables

for discrete data, the characteristic function is

$$\phi_{x,y}(\omega_1\omega_2) = E\left[e^{j\omega_1 x} e^{j\omega_2 Y}\right]$$

$$\phi_{xy}(\omega_1\omega_2) = \sum_m \sum_n e^{j\omega_1 x_m} e^{j\omega_2 y_n} P(x_m, y_n)$$

Properties of Joint characteristic equation.

* Marginal characteristic functions can be obtained from a Joint characteristic function as

$$\phi_{x,y}(\omega_1,0) = E\left[e^{j\omega_1 x}\right] = \phi_x(\omega_1)$$

$$\phi_{x,y}(0,\omega_2) = E\left[e^{j\omega_2 Y}\right] = \phi_Y(\omega_2)$$

$$\phi_{x,y}(0,0) = E[1] = 1$$

* If two random variables x and Y are independent then the characteristic function is equal to the product of their individual characteristic function. i.e,

$$\phi_{x,y}(\omega_1\omega_2) = \phi_x(\omega_1)\cdot \phi_Y(\omega_2)$$

* If x and y are independent random variables then

$$\phi_{X+Y}(\omega) = \phi_X(\omega)\cdot\phi_Y(\omega)$$

* If x & y are two random variables the joint moments can be derived from the joint characteristic function as

$$m_{nk} = (-j)^{n+k}\cdot\frac{\partial^{n+k}}{\partial\omega_1^n\,\partial\omega_2^k}\left[\phi_{XY}(\omega_1\omega_2)\right]\Big|_{\text{at }\omega_1=0,\ \omega_2=0}$$

* the joint characteristic function for n random variable $x_1, x_2, \dots x_N$ is defined as

$\phi_{x_1x_2x_3\dots x_N}(\omega_1\omega_2,\dots\omega_N) = E\left[e^{j\omega_1x_1}\,e^{j\omega_2x_2}\dots e^{j\omega_Nx_N}\right]$ and the joint moment is defined as

$$m_{n_1n_2\dots n_N} = (-j)^{n_1+n_2+\dots n_N}\ \frac{\partial^{n_1+n_2+\dots+n_N}\left[\phi_{x,y}(\omega_1,\omega_2\dots\omega_N)\right]}{\partial\omega_1^{n_1}\,\partial\omega_2^{n_2}\dots\partial\omega_N^{n_N}}$$

where $n_1+n_2+\dots+n_N$ is the order of the joint moments.

* Two random variables x and Y have the joint characteristic function is $\phi_X(\omega_1,\omega_2) = \exp(-2\omega_1^2-8\omega_2^2)$. i) show that x and y are 0 mean random variables. ii) Are x and y correlated.

Sol:- Given function

$$\phi_X(\omega_1,\omega_2) = \exp(-2\omega_1^2-8\omega_2^2)$$

By the joint moment characteristic function.

$$m_{nk} = (-j)^{n+k}\ \frac{\partial^{n+k}}{\partial\omega_1^n\,\partial\omega_2^k}\left[\phi_{x,y}(\omega_1,\omega_2)\right]$$

we know that

$$m_{10} = \phi_X(\omega_1) = E[X]$$

$$m_{10} = (-j)\frac{\partial}{\partial \omega_1}\left[e^{(-2\omega_1^2 - 8\omega_2^2)}\right]$$

$$m_{10} = (-j)\frac{\partial}{\partial \omega_1}\left[e^{-2\omega_1^2} e^{-8\omega_2^2}\right]$$

$$m_{10} = (-j)\, e^{-8\omega_2^2}\left[e^{-2\omega_1^2}\frac{\partial}{\partial \omega_1}\left[-2\omega_1^2\right]\right)$$

$$m_{10} = j\, 2 e^{-8\omega_2^2} e^{-2\omega_1^2}\, 2\omega_1$$

$$\therefore m_{10}\Big|_{\substack{at\,\omega_1=0\\ \omega_2=0}} = 0$$

$$\boxed{E[X] = \overline{X} = 0}$$

$$m_{01} = \phi_{X,Y}(0,\omega_2) = (-j)\frac{\partial}{\partial \omega_2}\left[e^{(-2\omega_1^2 - 8\omega_2^2)}\right]$$

$$m_{01} = -j\frac{\partial}{\partial \omega_2}\left[e^{-2\omega_1^2} e^{-8\omega_2^2}\right]$$

$$m_{01} = -j\, e^{-2\omega_1^2}\left[\frac{\partial}{\partial \omega_2} e^{-8\omega_2^2}\right]$$

$$m_{01} = -j\, e^{-2\omega_1^2}\left[e^{-8\omega_2^2}(-16)\right)$$

$$m_{01} = j\, 16\, e^{-2\omega_1^2} e^{-8\omega_2^2}$$

$$m_{01}\Big|_{\substack{\omega_1=0\\ \omega_2=0}} = 0 \qquad \therefore X \,\&\, Y \text{ are zero mean}$$

ii) $$m_{11} = (-j)^2\frac{\partial^2}{\partial \omega_1 \partial \omega_2}\left[e^{(-2\omega_1^2 - 8\omega_2^2)}\right]$$

$$m_{11} = -\frac{\partial}{\partial \omega_1}\left[\frac{\partial}{\partial \omega_2}\left(e^{-2\omega_1^2} e^{-8\omega_2^2}\right)\right]$$

$$m_{11} = -e^{-2\omega_1^2}(-4\omega_1)e^{-8\omega_2^2}(-16\omega_2)$$

$$m_{11}\Big|_{\substack{\omega_1=0\\ \omega_2=0}} = 0$$

$$E[XY] = 0$$

$$\text{cov}(x,y) = E[XY] - E[X]E(Y)$$

$$\boxed{\text{cov}(x,y) = 0} \qquad \rho = E[XY] = 0$$

$\therefore$ X & Y are uncorrelated

Gaussian Random variables

If two random variables X and Y are said to be jointly gaussian, The joint density function is given as

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left[\frac{-\left(\frac{(x-\bar{X})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_x \cdot \sigma_y} + \frac{(y-\bar{Y})^2}{\sigma_y^2}\right)}{2(1-\rho^2)}\right]$$

This is also called as bivariate gaussian density function, where

$$\bar{X} = E[X] \text{ and } \bar{Y} = E[Y]$$

$$\sigma_x^2 = E[X^2] - [E[X]]^2$$

$$\sigma_y^2 = E[Y^2] - [E[Y]]^2$$

$$\boxed{\rho = \frac{\text{cov}(x,y)}{\sqrt{\sigma_x^2 \cdot \sigma_y^2}}}$$

N random variables gaussian density function:-

consider N random variables $X_1, X_2, \dots X_N$ there are said to be jointly Gaussian density function is given by

$$f_{X_1 X_2 \dots X_N}(x_1 x_2 \dots x_N) = \frac{1}{(2\pi)^{\frac{\pi}{2}} \left|[C_x]\right|^{1/2}} \exp\left(\frac{-[x-\bar{X}]^t [C_x]^{-1} (x-\bar{X})^{-1}}{2}\right)$$

where

$[C_x]$ is a covariance Matrix of a N random variables.. i.e.,

$$[C_x] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

$(x-\bar{x})$ is variable of column matrix.

$$(x-\bar{x}) = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_N - \bar{x} \end{bmatrix}$$

$[x-\bar{x}]^t \rightarrow$ Transpose of $[x-\bar{x}]$

$|[C_x]| \rightarrow$ Determinant of $[C_x]$

$[C_x]^{-1} \rightarrow$ Inverse of $[C_x]$

Note:-

The elements of the co-variance matrix $[C_x]$ is given by

$$C_{ij} = E[(x_i - \bar{x_i})(x_j - \bar{x_j})]$$

$$C_{ij} = \begin{cases} \sigma_{x_i}^2, & i=j \\ \sigma_{x_i x_j}, & i \neq j \end{cases}$$

Example:-

Let us consider, joint density function of two Gaussian Random variables x_1 & x_2

i.e, $N=2$.

$\therefore$ The covariance matrix for $N=2$ is

$$C_x = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & C_{12} \\ C_{21} & \sigma_{x_2}^2 \end{bmatrix} \rightarrow (1)$$

we know that

$$\rho = \frac{C_{xy}}{\sigma_x \cdot \sigma_y}$$

$$(1) \Rightarrow [C_x] = \begin{bmatrix} \sigma_{x_1}^2 & \rho\sigma_{x_1}\sigma_{x_2} \\ \rho\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{bmatrix}$$

$$|C_x| = \begin{vmatrix} \sigma_{x_1}^2 & \rho\,\sigma_{x_1}\sigma_{x_2} \\ \rho\,\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{vmatrix}$$

$$|[C_x]| = \sigma_{x_1}^2 \cdot \sigma_{x_2}^2 - \rho^2 \sigma_{x_1}^2 \sigma_{x_2}^2$$

$$|[C_x]| = \sigma_{x_1}^2 \cdot \sigma_{x_2}^2 (1-\rho^2) \rightarrow (2)$$

Now to find $[C_x]^{-1}$

we have that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore [C_x]^{-1} = \frac{1}{\sigma_{x_1}^2 \sigma_{x_2}^2 (1-\rho^2)} \begin{bmatrix} \sigma_{x_2}^2 & \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}} \\ \frac{-\rho}{\sigma_{x_1}\sigma_{x_2}} & 1/\sigma_{x_2}^2 \end{bmatrix} \rightarrow (3)$$

and we have

$$[X-\bar{X}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix} \Rightarrow$$

$$[X-\bar{X}]^t = [x_1 - \bar{x}_1 \quad x_2 - \bar{x}_2] \rightarrow (4)$$

from (2), (3), (4).

Let

$$[X-\bar{X}]^t [C_x]^{-1} [X-\bar{X}] = [x_1-\bar{x}_1 \quad x_2-\bar{x}_2] \frac{1}{1-\rho^2} \begin{bmatrix} 1/\sigma_{x_1}^2 & -\rho/\sigma_{x_1}\sigma_{x_2} \\ -\rho/\sigma_{x_1}\sigma_{x_2} & 1/\sigma_{x_2}^2 \end{bmatrix} \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho^2} [x_1-\bar{x}_1 \quad x_2-\bar{x}_2] \begin{bmatrix} 1/\sigma_{x_1}^2 & -\rho/\sigma_{x_1}\sigma_{x_2} \\ -\rho/\sigma_{x_1}\sigma_{x_2} & 1/\sigma_{x_2}^2 \end{bmatrix} \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho^2} \Bigg[[x_1-\bar{x}_1 \quad x_2-\bar{x}] \begin{bmatrix} \frac{1}{\sigma_{x_1}^2}(x_1-\bar{x}_1) + \frac{(-\rho(x_2-\bar{x}_2))}{\sigma_{x_1}\sigma_{x_2}} \\ -\frac{\rho(x_1-\bar{x}_1)}{\sigma_{x_1}\sigma_{x_2}} + \frac{x_2-\bar{x}_2}{\sigma_{x_2}} \end{bmatrix} \Bigg]$$

$$= \frac{1}{1-\rho^2} \Bigg[[x_1-\bar{x}_1 \quad x_2-\bar{x}] \begin{bmatrix} \frac{x_1-\bar{x}_1}{\sigma_{x_1}^2} - \frac{\rho(x_2-\bar{x}_2)}{\sigma_{x_1}\sigma_{x_2}} \\ -\frac{\rho(x_1-\bar{x}_1)}{\sigma_{x_1}\sigma_{x_2}} + \frac{x_2-\bar{x}_2}{\sigma_{x_2}^2} \end{bmatrix} \Bigg]$$

$$= \frac{1}{1-\rho^2}\left[\frac{(x_1-\overline{x_1})^2}{\sigma_{x_1}^2} - \frac{\rho(x_2-\overline{x_2})(x_1-\overline{x_1})}{\sigma_{x_1}\sigma_{x_2}} - \frac{\rho(x_1-\overline{x_1})(x_2-\overline{x_2})}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\overline{x_2})^2}{\sigma_{x_2}}\right]$$

$$= \frac{1}{1-\rho^2}\left[\frac{(x_1-\overline{x_1})^2}{\sigma_{x_1}^2} - \frac{2\rho(x_1-\overline{x_1})(x_2-\overline{x_2})}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-\overline{x_2})^2}{\sigma_{x_2}^2}\right]$$

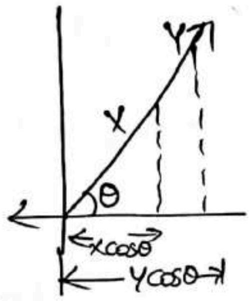
$\therefore$ The joint density function for two random variables is given by

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi[C_X]^{1/2}} \exp\left[-\frac{[x-\overline{x}]^t [C_x]^{-1}[x-\overline{x}]}{2}\right]$$

Problems :-

* consider two random variables X_1 and Y_1 related to two random variables X and Y by the co-ordinate relation $X_1 = X\cos\theta + Y\sin\theta$ $Y_1 = Y\cos\theta + X\sin\theta$, where θ is the co-ordinate rotation angle as shown in figure.

If X_1 and Y_1 are gaussian random variables, independent and uncorrelated, then show that the angle of co-ordinate rotation is



Rotation of X & Y

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right)$$

Sol:- Given that

$$X_1 = X\cos\theta + Y\sin\theta$$

$$Y_1 = Y\cos\theta - X\sin\theta$$

$\therefore$ The covariance of $X_1 Y_1$ is $C_{X_1Y_1} = E[(X_1-\overline{X_1})(Y_1-\overline{Y_1})]$

i.e., $C_{X_1Y_1} = E\left[\{(X\cos\theta + Y\sin\theta) - (\overline{X\cos\theta + Y\sin\theta})\}\{(Y\cos\theta - X\sin\theta) - (\overline{Y\cos\theta - X\sin\theta})\}\right]$

$$= E[\{x\cos\theta + y\sin\theta - \bar{x}\cos\theta - \bar{y}\sin\theta\}\{y\cos\theta - x\sin\theta - \bar{y}\cos\theta + \bar{x}\sin\theta\}]$$

$$= E[\{(x-\bar{x})\cos\theta + (y-\bar{y})\sin\theta\}\{(y-\bar{y})\cos\theta - (x-\bar{x})\sin\theta\}]$$

$$= E[(x-\bar{x})(y-\bar{y})\cos^2\theta - (x-\bar{x})^2\sin\theta\cos\theta + (y-\bar{y})^2\cos\theta\sin\theta - (x-\bar{x})(y-\bar{y})\sin^2\theta]$$

$$= E[(x-\bar{x})(y-\bar{y})(\cos^2\theta - \sin^2\theta) - [(x-\bar{x})^2 - (y-\bar{y})^2]\sin\theta\cos\theta]$$

$$C_{x_1y_1} = E[(x-\bar{x})(y-\bar{y})]\cos 2\theta - E[(x-\bar{x})^2 - (y-\bar{y})^2]\left[\frac{\sin 2\theta}{2}\right] \rightarrow (1)$$

we know that

$$C_{xy} = E[(x-\bar{x})(y-\bar{y})] \rightarrow (2)$$

and $\rho = \dfrac{C_{xy}}{\sigma_x \sigma_y} \Rightarrow C_{xy} = \rho\,\sigma_x \sigma_y \rightarrow (3)$

$$\therefore E[(x-\bar{x})^2] = \sigma_x^2 \rightarrow (4)$$

$$E[(y-\bar{y})^2] = \sigma_y^2 \rightarrow (5)$$

Put 2,4,5 in eq(1) we have

$$C_{x_1y_1} = C_{xy}\cos 2\theta - (\sigma_x^2 - \sigma_y^2)\frac{\sin 2\theta}{2}$$

$$C_{x_1y_1} = C_{xy}\cos 2\theta - \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\sin 2\theta$$

$$C_{x_1y_1} = \rho\,\sigma_x\sigma_y\cos 2\theta - \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\sin 2\theta \rightarrow (2)$$

since $x_1 y_1$ are uncorrelated:

$$C_{x_1y_1} = 0$$

$$\rho\,\sigma_x\sigma_y\cos 2\theta - \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\sin 2\theta = 0$$

$$\frac{1}{2}(\sigma_x^2 - \sigma_y^2)\sin 2\theta = \rho\,\sigma_x \sigma_y \cos 2\theta$$

$$(\sigma_x^2 - \sigma_y^2)\sin 2\theta = 2\rho\,\sigma_x\sigma_y\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2\rho\,\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

$$\tan 2\theta = \frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

$$2\theta = \tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}\right)$$

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}\right)}$$

* Two Gaussian random variables X and Y have the variances $\sigma_x^2 = 9$ and $\sigma_y^2 = 4$ respectively. It is know that a co-ordinate rotation by an angle $\pi/8$ results in new random variables Y_1 & Y_2 that are uncorrelated random variables. What is the correlation coefficient of X & Y.

sol:- Given that

$\sigma_x^2 = 9$ $\sigma_y^2 = 4$ $\theta = \pi/8$

$\sigma_x = 3$ $\sigma_y = 2.$

we know that the angle of rotation for which the new random variables are uncorrelated is

$$\tan 2\theta = \frac{2\rho\sigma_x.\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

where 'ρ' is the correlation coefficient of X and Y is

$$\tan\left(\frac{2\pi}{8}\right) = \frac{2\times\rho\times3\times2}{9-4}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{12\rho}{5}$$

$$\rho = \frac{5}{12} \Rightarrow \boxed{\rho = 0.4167}$$

Unit-IV

Complex Variable - Differentiation

Complex Variable :— Let 'z' is a complex variable and it can be written as $z = x+iy$, where x, y are real and $i=\sqrt{-1}$

$\Rightarrow$ Let 'f' is a function of 'z', ie, $w=f(z)$ is a complex number.

we write $w=f(z)=u+iv$, where u, v are real.

ie, $w=f(z)=u(x,y)+iv(x,y)$

Real part of $f(z)$ is $u(x,y)$

Imaginary part of $f(z)$ is $v(x,y)$

Note :-

If $z=x+iy$ be a complex number, and $\bar{z}=x-iy$ be the complex conjugate of 'z'.

Limit of a function :—

Let 'f' be a function of z, ie, $w=f(z)$ is said to be a limit 'l' then there exist a positive value $\delta>0$ such that $|f(z)-l|<\epsilon$ for $0<|z-z_0|<\delta$ then,

$$\lim_{z\to z_0} f(z)=l.$$

Continuity of a function :—

A function $f(z)$ is said to be continuous at $z=z_0$, if corresponding to each positive number $\epsilon>0$ then there exist a number δ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$ then $\lim_{z\to z_0} f(z)=f(z_0)$

Derivative of f(z) :—

Let $w=f(z)$ be a given function defined for all z in a neighbourhood of z_0. If $\lim_{\Delta z\to 0}\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists, the function $f(z)$ is said to be derivable at z_0 and the limit is denoted by $f'(z_0)$. ie, $f'(z_0)$ exists is called the derivative of $f(z)$ at z_0.

Taking $z-z_0=\Delta z$, $f'(z_0)=\lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$,

Analytic functions :-

Let a function $f(z)$ be derivable at every point z in an ϵ neighbourhood of z_0 ie, $f'(z)$ Exists for all z such that $|z-z_0|<\epsilon$ where $\epsilon>0$. Then, $f(z)$ is said to be analytic at z_0.

Note:-

① The terms Regular (or) Holomorphic are synonyms of analytic.

② If $f(z)$ is analytic at z_0,

(i) $f'(z_0)$ exists and

(ii) $f'(z)$ exists at every point z in a neighbourhood of z_0

⇒ Let 'D' be a domain of Complex numbers.

If $f(z)$ is analytic at every $z\in D$, $f(z)$ is said to be analytic in the "domain-D".

Entire function:- If $f(z)$ is analytic at every point 'z' on the Complex plane, $f(z)$ is said to be an entire function. (or) integral function.

Singular point:- If $f'(z_0)$ does not exist then $z=z_0$ is called a singular point of $f(z)$.

Isolated singular point:- If $f'(z)$ exists at every point in a neighbourhood of z_0 but $f'(z_0)$ does not exist, then z_0 is said to be an isolated singular point of $f(z)$.

Ex:- $f(z)=\frac{1}{z}$ is analytic at every point $z\neq 0$.

At $z=0$, $f'(z)$ does not exist.

$z=0$ is an isolated singular point of $f(z)$.

Properties of Analytic functions:—

① If $f(z)$ and $g(z)$ are analytic functions, then $f \pm g$, $f \cdot g$ and $\frac{f}{g}$ are also analytic functions, provided $g(z) \neq 0$.

② Analytic function of an analytic function is analytic.

③ An entire function of an entire function is entire.

④ Derivative of an analytic function is itself analytic.

Cauchy – Riemann Equations (CR Equations):—

[It is used to test the analyticity of a Complex function].

Statement:— The necessary and sufficient condition for the derivative of the function $f(z) = w = u(x,y) + iv(x,y)$ to exist for all values of z in domain R are.

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are Continuous functions of x and y in R.

(ii) $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}}$

The above relations are known as Cauchy - Riemann equations.

Polar form of Cauchy - Riemann equations:—

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and $f(z)$ is derivable at $z_0 = r_0 e^{i\theta_0}$ then. $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$

Harmonic functions – Laplace Equation:—

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

respectively in D, and have Continuous second order partial derivatives in D.

Note :—

① To obtain analytic function at the origin, the given function satisfies the following cases.

(i) Along x-axis. [put $y=0$]

(ii) Along y-axis. [put $x=0$]

(iii) Along the line $y=mx$.

(iv) Along $y=mx^2$.

(v) Along $x=my^2$.

② To verify CR-equations at the origin :—

(i) $$\left(\frac{\partial u}{\partial x}\right)_{at\,(0,0)} = \underset{x \to 0}{Lt} \frac{u(x,0)-u(0,0)}{x}$$

(ii) $$\left(\frac{\partial u}{\partial y}\right)_{at\,(0,0)} = \underset{y \to 0}{Lt} \frac{u(0,y)-u(0,0)}{y}.$$

(iii) $$\left(\frac{\partial v}{\partial x}\right)_{at\,(0,0)} = \underset{x \to 0}{Lt} \frac{v(x,0)-v(0,0)}{x}$$

(iv) $$\left(\frac{\partial v}{\partial y}\right)_{at\,(0,0)} = \underset{y \to 0}{Lt} \frac{v(0,y)-v(0,0)}{y}.$$

problems :—

(1) show that $f(z) = xy + iy$ is every where continuous but is not analytic.

Sol:- $f(z) = xy + iy$, where $z = x + iy$

$\Rightarrow f(z_0) = x_0 y_0 + i y_0$ is well defined for any $z_0 = x_0 + i y_0$

$$\therefore \lim_{z \to z_0} f(z) = \lim_{z \to z_0} (xy + iy)$$

$$= \lim_{\substack{x \to x_0 \\ y \to y_0}} (xy + iy)$$

$$= x_0 y_0 + i y_0$$

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$$

$\therefore f$ is continous every where and $f(z) = u + iv$.

$f(z) = xy + iy$

$\Rightarrow u = xy$	$v = y$
$\frac{\partial u}{\partial x} = y$	$\frac{\partial v}{\partial x} = 0$
$\frac{\partial u}{\partial y} = x$	$\frac{\partial v}{\partial y} = 1$

since CR-equations are not satisfied, 'f' is not analytic.

(2) show that $f(z) = z^3$ is analytic for all z.

Sol:- Let $z = x + iy$

$$\Rightarrow z^3 = (x+iy)^3 = x^3 + (iy)^3 + 3 \cdot x^2 (iy) + 3x(iy)^2$$

$$= x^3 - iy^3 + i\,3x^2 y - 3xy^2$$

$$f(z) = z^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3)$$

Now comparing with $f(z) = u + iv$.

$\therefore u = x^3 - 3xy^2$ | $v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ | $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy$ | $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$.

Cauchy-Riemann equations are satisfied and hence $f(z) = z^3$ is analytic.

③ If $w = \log z$, find $\frac{dw}{dz}$ and determine where 'w' is non-analytic.

Sol:- Let $w = u + iv$.

$\therefore$ we have $z = x + iy = re^{i\theta}$.

where $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$.

$\therefore w = \log(z) = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log(r) + i\theta \cdot \log_e e$ $\langle \because \log_e e = 1 \rangle$

$= \log(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x)$

$= \log(x^2+y^2)^{1/2} + i\tan^{-1}(y/x)$

$w = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$u = \frac{1}{2}\log(x^2+y^2)$ | $v = \tan^{-1}(y/x)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)(2x) = \frac{x}{x^2+y^2}$ | $\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\,\frac{1}{x^2+y^2}\,(2y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right)(y)$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $W = \log z$ satisfies Cauchy-Riemann equations.

$\therefore$ W is analytic every where except at $z=0$.

$$\therefore \frac{dW}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{\cancel{(x-iy)}}{(x+iy)\cancel{(x-iy)}} = \frac{1}{x+iy} = \frac{1}{z}.$$

$$\therefore \frac{dW}{dz} = \frac{1}{z}. \quad (z \neq 0)$$

④ show that $f(z) = z + 2\bar{z}$ is not analytic any where in the Complex plane.

<u>Sol</u>:- $f(z) = u + iv = z + 2\bar{z}$

$$= (x+iy) + 2(x-iy)$$

$$= (x+2x) + i(y-2y)$$

$$f(z) = 3x - iy$$

ie, $u = 3x$ and $v = -y$.

$$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$\therefore$ $f(z)$ is not analytic any where since Cauchy-Riemann Conditions are not satisfied.

⑤ Find where the function (i) $w = \frac{1}{z}$ (ii) $w = \frac{z}{z-1}$ (iii) $w = z^3 - 4z + 1$ ceases (fails) to be analytic.

Sol:-

(i) Given $f(z) = w = \frac{1}{z}$

$$f'(z) = \frac{-1}{z^2}, \text{ if } z \neq 0, \text{ it is analytic } \forall z.$$

Here the function $f'(z)$ does not exist at $z=0$, so that $f(z)$ is not analytic at $z=0$, is a singular point of $f(z)$.

(ii) Given $f(z) = \frac{z}{z-1}$

$$f'(z) = \frac{(z-1)\cdot 1 - z(1-0)}{(z-1)^2}$$

$$f'(z) = \frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} = \frac{-1}{(z-1)^2}$$, if $z \neq 1$ it is analytic for all 'z'.

Here the function $f'(z)$ does not exist at $z=1$, so that it is not analytic at $z=1$, '1' is a singular point of $f(z)$.

(iii) Given $f(z) = w = z^3 - 4z + 1$

$$f'(z) = 3z^2 - 4$$

∴ Here $f'(z)$ is exist at all points of z in the complex plane.

Hence $f(z)$ is analytic every where in the complex plane.

6Q) prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$ where $w = f(z)$ is analytic.

Sol:- Let $f(z) = u + iv$ then Real $f(z) = u$.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = u_x + iv_x \qquad \left\langle \because \text{By the CR equations } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \right\rangle$$

$$f'(z) = u_x - iu_y$$

$$|f'(z)| = \sqrt{u_x^2 + u_y^2} \Rightarrow |f'(z)|^2 = u_x^2 + u_y^2 \quad \text{——— (1)}$$

Now, L.H.S $= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^2 = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}$ ——— (2)

$$\therefore \frac{\partial^2 u^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u^2}{\partial x}\right] = \frac{\partial}{\partial x}\left[2u\cdot\frac{\partial u}{\partial x}\right]$$

$$= 2\left[u\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= 2\left[u\cdot\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$\therefore \frac{\partial^2 u^2}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u^2}{\partial y}\right] = \frac{\partial}{\partial y}\left[2u\cdot\frac{\partial u}{\partial y}\right]$$

$$= 2\left[u\cdot\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

(2) $\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^2 = 2\left[u\cdot\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$\therefore$ By the Laplace Equation, we have $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real of } f(z)|^2 = 2\left[0 + u_x^2 + u_y^2\right] = 2\cdot|f'(z)|^2 \quad \langle \because \text{from (1)} \rangle$$

(10)

7 Q1) Show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$, where $f(z)$ is an analytic function.

Sol:- we know that $z=x+iy \Rightarrow \bar{z}=x-iy$

$$f(z)=u(x,y)+iv(x,y)$$

$\therefore$ By taking $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}\times\frac{i}{i}$

$$y=\frac{i(z-\bar{z})}{2i^2}=\frac{-i}{2}(z-\bar{z})$$

$\Rightarrow$ $x=\frac{z+\bar{z}}{2}$, $y=\frac{-i}{2}(z-\bar{z})$

$\frac{\partial x}{\partial z}=\frac{1}{2}$ $\quad\frac{\partial y}{\partial z}=\frac{-i}{2}(1)=-i/2$

$\frac{\partial x}{\partial \bar{z}}=\frac{1}{2}$ $\quad\frac{\partial y}{\partial \bar{z}}=\frac{-i}{2}(-1)=i/2$

$$\therefore\ \frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial}{\partial y}\left(-i/2\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial \bar{z}}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial \bar{z}}\right)=\frac{\partial}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial}{\partial y}\left(i/2\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$\therefore\left(\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial \bar{z}}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z\,\partial \bar{z}}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\,\partial \bar{z}} \quad\text{—— (1)}$$

$$\therefore\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=4\cdot\frac{\partial^2}{\partial z\,\partial \bar{z}}\left(\log|f'(z)|\right)$$

$$=4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log\left(|f'(z)|^2\right)\right]$$

$$=4\cdot\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\left|\log\left(f'(z)\cdot f'(\bar{z})\right)\right|\right]$$

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$$= \frac{4}{2} \frac{\partial^2}{\partial z \partial \bar{z}} \left[\log(f'(z)) + \log f'(\bar{z})\right]$$

$$= 2 \cdot \left\{ \frac{\partial}{\partial \bar{z}}\left[\frac{\partial}{\partial z}(\log(f'(z)))\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}(\log f'(\bar{z}))\right] \right\}$$

$$= 2 \cdot \left\{ \frac{\partial}{\partial \bar{z}}\left[\frac{f''(z)}{f'(z)}\right] + \frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right] \right\}$$

$$= 2\{0+0\} = 0.$$

8Q) If $f(z)$ is regular function, of z prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$$

Sol:- we know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \cdot \left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)$. [write the proof]

$\because z \cdot \bar{z} = |z|^2$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left(|f(z)|^2\right)$$

$$= 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left(f(z) \cdot f(\bar{z})\right)$$

$$= 4 \cdot \frac{\partial}{\partial z}(f(z)) \cdot \frac{\partial}{\partial \bar{z}}\left(\overline{f(\bar{z})}\right)$$

$$= 4 \cdot f'(z) \cdot \overline{f'(z)} = 4 \cdot |f'(z)|^2.$$

9Q) Find all values of K, such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic.

Sol:- Let $f(z) = u + iv = e^x(\cos ky + i \sin ky)$

$$f(z) = e^x \cos ky + i e^x \sin ky$$

ie, $u(x,y) = e^x \cos ky$ and $v(x,y) = e^x \sin ky$.

$\therefore \dfrac{\partial u}{\partial x} = e^x \cos ky$

$\dfrac{\partial u}{\partial y} = e^x(-\sin ky)\cdot k$

$= -ke^x \sin ky$

$\dfrac{\partial v}{\partial x} = e^x \sin ky$

$\dfrac{\partial v}{\partial y} = e^x(\cos ky)\cdot k$

$= ke^x \cos ky.$

$\therefore$ Given that $f(z)$ is analytic, ie, it satisfies CR-equations

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ --- (1) and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ --- (2)}$$

from (1) $\Rightarrow e^x \cos ky = ke^x \cos ky$

$$\Rightarrow k = \frac{e^x \cancel{\cos ky}}{e^x \cancel{\cos ky}} \Rightarrow \boxed{k=1}$$

Note :—

(1) To obtain analytic function at the origin, the given function satisfies the following cases.

(i) Along x-axis. [put $y=0$]

(ii) Along y-axis. [put $x=0$]

(iii) Along the line $y=mx$.

(iv) Along $y=mx^2$.

(v) Along $x=my^2$.

(2) To verify CR-equations at the origin :—

(i) $\left(\dfrac{\partial u}{\partial x}\right)_{at(0,0)} = \underset{x\to 0}{Lt} \dfrac{u(x,0)-u(0,0)}{x}$

(ii) $\left(\dfrac{\partial u}{\partial y}\right)_{at(0,0)} = \underset{y\to 0}{Lt} \dfrac{u(0,y)-u(0,0)}{y}.$

(iii) $\left(\frac{\partial v}{\partial x}\right)_{at(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0) - v(0,0)}{x}$

(iv) $\left(\frac{\partial v}{\partial y}\right)_{at(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y) - v(0,0)}{y}$.

problems :—

① Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although Cauchy Riemann equations are satisfied at that point.

Sol:- Given function $f(z) = \sqrt{|xy|}$

∴ It is Comparing with $f(z) = u + iv$.

$\Rightarrow u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

(i) CR- equations at the origin:—

$$\left(\frac{\partial u}{\partial x}\right)_{at(0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0) - u(0,0)}{x} = \underset{x\to 0}{Lt} \left(\frac{0-0}{x}\right) = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{at(0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y) - u(0,0)}{y} = \underset{y\to 0}{Lt} \left(\frac{0-0}{y}\right) = 0$$

∴ $f(z)$ satisfies CR- equations at the origin.

(ii) Analytic function at the origin:—

Along x-axis:— $f'(z) = \underset{z\to 0}{Lt} \frac{f(z) - f(0)}{z - 0}$

$$= \underset{x\to 0}{Lt} \frac{\sqrt{|xy|} - 0}{(x+iy)} \quad \langle \because \text{put } y=0 \rangle$$

$$= \underset{x\to 0}{Lt} \left(\frac{0-0}{x}\right) = 0$$

Along y-axis :— $f'(z) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$

$$f'(z) = \underset{y\to 0}{Lt} \frac{\sqrt{|xy|} - 0}{(x+iy)} \quad (\because \text{put } x=0)$$

$$= \underset{y\to 0}{Lt} \left(\frac{0-0}{iy}\right) = 0$$

Along $y=mx$:— $f'(z) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$

$$= \underset{\substack{x\to 0 \\ y\to mx}}{Lt} \frac{\sqrt{|xy|} - 0}{(x+iy)}$$

$$= \underset{x\to 0}{Lt} \frac{\sqrt{|x\cdot mx|}}{(x+imx)}$$

$$= \underset{x\to 0}{Lt} \frac{\sqrt{x^2\cdot m}}{x(1+im)}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x}\cdot\sqrt{m}}{\cancel{x}(1+im)} = \frac{\sqrt{m}}{(1+im)}$$

Here the derivative of $f(z)$ along $y=mx$ is not unique and this limit of a function is depending on 'm'.

$\therefore$ $f'(z)$ does not exist at $y=mx$.

$\therefore$ $f(z)$ is not analytic at the origin but CR-equations are satisfied at that point.

(2) prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } z\neq 0 \\ 0, & \text{if } z=0 \end{cases}$$ is continuous and

the Cauchy-Riemann equations are satisfied at the origin yet $f'(0)$ does not exist.

(11)

Sol: Given that $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

First to show that $f(z)$ is continuous at $z=0$.

We have $\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \left(\dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}\right)$

$$= \lim_{y\to 0} \left[\frac{-y^3(1-i)}{y^2}\right]$$

$$= \lim_{y\to 0} -y(1-i) = 0$$

and $\lim_{z\to 0} f(z) = \lim_{\substack{y\to 0\\ x\to 0}} \left(\dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}\right)$

$$= \lim_{x\to 0} \frac{x^3(1+i)}{x^2}$$

$$= \lim_{x\to 0} x(1+i) = 0.$$

$\therefore \lim_{z\to 0} f(z) = f(0)$; it is continuous.

Now, let $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \dfrac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$

$$f(z) = \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}$$

It is comparing with $f(z) = u + iv$.

i.e, $u(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ and $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

CR-equations at the origin:

$$\left(\frac{\partial u}{\partial x}\right)_{at\,(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\left[\frac{\left(\frac{x^3}{x^2}\right) - 0}{x}\right].$$

$$= \lim_{x\to 0}\left(\frac{x^3}{x^3}\right) = 1.$$

$$\left(\frac{\partial u}{\partial y}\right)_{at\,(0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\left[\frac{\left(\frac{-y^3}{y^2}\right)-0}{y}\right]$$

$$= \underset{y\to 0}{Lt}\left(-\cancel{y^3}/\cancel{y^3}\right) = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{at\,(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt} \frac{\left(x^3/x^2\right)}{x} = \underset{x\to 0}{Lt}\left(\cancel{x^3}/\cancel{x^3}\right) = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{at\,(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt} \frac{\left(y^3/y^2\right)}{y} = \underset{y\to 0}{Lt}\left(\cancel{y^3}/\cancel{y^3}\right) = 1.$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ It satisfies CR-equations at the origin.

Analytic function at the origin:—

Along x-axis:— $f'(z) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$ $\langle \because z = x+iy$

$$= \underset{x\to 0}{Lt} \frac{\left(\frac{x^3(1+i)-0}{x^2}\right)}{x}$$ $\langle \because$ put $y=0$ in $f(z)\rangle$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x^3}(1+i)}{\cancel{x^3}} = \underline{\underline{(1+i)}}$$

Along y-axis:—

$$f'(z) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$$

$$= \underset{y\to 0}{Lt} \frac{\left(\frac{-y^3(1-i)}{y^2}\right)}{iy}$$ $\langle \because$ put $x=0$ in $f(z)\rangle$

$$= \underset{y\to 0}{Lt} \frac{-\cancel{y^3}(1-i)}{i\cancel{y^3}} = \frac{-(1-i)}{i} = \frac{-i(1-i)}{i^2}$$

$$= \frac{\cancel{-}i(1-i)}{\cancel{-}1} = i - i^2 = \underline{\underline{i+1}}$$

Along $y=mx$:—

$$f'(z) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{x\to 0}\left[\left(\frac{x^3(1+i) - m^3x^3(1-i)}{x^2+m^2x^2}\right) \Big/ (x+i(mx))\right]$$

$$= \lim_{x\to 0} \frac{x^3(\cdot 1+i-m^3(1-i))}{x^2(1+m^2)\cdot x(1+im)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left[1+i-m^3+im^3\right]}{\cancel{x^3}(1+m^2)(1+im)} = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}.$$

$\therefore$ $f'(z)$ is depending on 'm' and hence it is not unique.

$\therefore$ $f'(z)$ does not exist at $(0,0)$ along $y=mx$.

$\therefore$ $f(z)$ is not analytic at the origin.

Q) Determine 'p' such that the function $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Sol:- Given that $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$.

It is of the form $f(z)=u+iv$

ie, $u=\frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\left(\frac{1}{x^2+y^2}\right)\cdot \cancel{2}x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\cdot 2y = -$$

$v=\tan^{-1}\left(\frac{px}{y}\right)$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\cdot\left(\frac{p}{y}\right) = \frac{py}{y^2+p^2x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{-px}{y^2}\right) = \frac{-px}{y^2+p^2x^2}$$

$\therefore f(z)$ is an analytic and it satisfies CR-equations, (18)

i.e, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow \frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2} \Rightarrow \boxed{p=-1}$

$\Rightarrow \therefore f(z)$ is analytic when $p=-1$ only.

Q) Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Sol:- Given that $u = e^{-x}(x\sin y - y\cos y)$

Now to show that 'u' is harmonic, i.e, to show that

$$\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} (1)$$

$\therefore u = e^{-x}(x\sin y - y\cos y)$

partially Differentiating w.r.to x and y we get

$$\frac{\partial u}{\partial x} = e^{-x}[\sin y \cdot 1 - 0] + (x\sin y - y\cos y)\, e^{-x}(-1)$$

$$= e^{-x}\sin y - e^{-x}\cdot x\sin y + y\, e^{-x}\cos y.$$

$$= e^{-x}[\sin y - x\sin y + y\cos y].$$

$$= e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(-1)\sin y - \left[e^{-x}(\sin y + 0) + (x\sin y - y\cos y)\, e^{-x}(-1)\right]$$

$$= -e^{-x}\sin y - [e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y]$$

$$= -e^{-x}\sin y - e^{-x}\sin y + e^{-x}\cdot x\sin y - e^{-x}\cdot y\cos y.$$

$$\frac{\partial u}{\partial y} = e^{-x}\left[x\cos y - (y(-\sin y) + \cos y \cdot (1))\right]$$

$$= e^{-x}\left[x\cos y + y\sin y - \cos y\right]$$

$$= e^{-x}\cdot x\cos y + e^{-x}(y\sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}\cdot x(-\sin y) + e^{-x}\left[y\cos y + \sin y\cdot 1 - (-\sin y)\right]$$

$$= -e^{-x}\cdot x\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}\sin y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{-x}\cancel{\sin y} - e^{-x}\cancel{\sin y} + e^{-x}\cdot\cancel{x\sin y} - e^{-x}\cancel{y}\cos y$$
$$- e^{-x}\cdot\cancel{x\sin y} + e^{-x}y\cancel{\cos y} + e^{-x}\cancel{\sin y} + e^{-x}\cancel{\sin y} = 0$$

$$\boxed{\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\therefore$ 'u' is harmonic.

Q) prove that if $u = x^2 - y^2$, $v = \dfrac{-y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but $u + iv$ is not a regular (analytic) function of z.

Sol:- Given that $u = x^2 - y^2$ $\qquad v = \dfrac{-y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = 2x. \text{ and } \frac{\partial u}{\partial y} = -2y \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \text{ and } \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial v}{\partial x} = -y\left[\frac{-1}{(x^2+y^2)^2}\right](2x) = \frac{+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2\cdot 2y - 2xy\, 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\cancel{(x^2+y^2)}\left[2x^2y + 2y^3 - 8x^2y\right]}{(x^2+y^2)^{\cancel{4}\,3}} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$ (20)

Simillarly

$$\frac{\partial v}{\partial y} = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2=0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{-6x^2y+2y^3}{(x^2+y^2)^3} + \frac{6x^2y-2y^3}{(x^2+y^2)^3} = \frac{\cancel{-6x^2y}+\cancel{2y^3}+\cancel{6x^2y}-\cancel{2y^3}}{(x^2+y^2)^3}$$

i.e, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

$\therefore$ u and v satisfies the harmonic function.

but $f(z) = u+iv$ is not analytic because.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}.$$

Conjugate Harmonic function :-

If $f(z) = u+iv$ is analytic and if 'u' and 'v' satisfy Laplace's equation, then 'u' and 'v' are called conjugate harmonic functions.

problems :—

① verify that $u = x^2 - y^2 - y$ is harmonic in the whole Complex plane and find a Conjugate harmonic function v of u.

Sol:- Given that $u = x^2 - y^2 - y$.

first ~~to show that 'u' is harmonic~~ ie, it is enough to show that $\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \quad u = x^2 - y^2 - y$

$\frac{\partial u}{\partial x} = 2x$ | $\frac{\partial u}{\partial y} = -2y - 1$

$\frac{\partial^2 u}{\partial x^2} = 2$ | $\frac{\partial^2 u}{\partial y^2} = -2$.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, 'u' is harmonic.

Now to find the harmonic Conjugate function 'v'.

By the CR-Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore \quad 2x = \frac{\partial v}{\partial y}$

$\partial v = 2x \, \partial y$

$\int_v \partial v = 2x \int_y 1 \, dy$

$\boxed{v = 2xy + c}$

$-2y - 1 = -\frac{\partial v}{\partial x}$

$(2y+1)\, \partial x = \partial v$

$\int_v 1 \, dv = \int_x (2y+1)\, dx$

$v = (2y+1) \cdot x + c$

$\boxed{v = 2xy + x + c}$

$\therefore$ The conjugate function $v = 2xy + x + c$ //

② Show that the function $u(x,y) = e^x \cos y$ is harmonic. Determine its harmonic Conjugate $v(x,y)$ and the analytic function $f(z) = u+iv$. (22)

Sol:- Given that $u(x,y) = e^x \cos y$.

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = e^x(-\sin y) = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{e^x \cos y} - \cancel{e^x \cos y} = 0.$$

$\therefore u(x,y)$ is a harmonic function.

Now to find the harmonic Conjugate function $v(x,y)$,

$\therefore$ By the CR-equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^x \cos y = \frac{\partial v}{\partial y}$$

$$\partial v = e^x \cos y \, \partial y$$

$$\int_v dv = \int_y e^x \cos y \, dy$$

$$v = e^x \sin y + c$$

$$-e^x \sin y = -\frac{\partial v}{\partial x}$$

$$e^x \sin y = \frac{\partial v}{\partial x}$$

$$\partial v = e^x \sin y \, \partial x$$

$$\int_v dv = \int_x e^x \sin y \, dx$$

$$v = e^x \sin y + c$$

$$\boxed{\therefore v(x,y) = e^x \sin y + c.}$$

$\therefore$ Analytic function $f(z) = u + iv = e^x \cos y + i(e^x \sin y + c)$

$$= e^x \cos y + i e^x \sin y + ic$$

$$= e^x [\cos y + i \sin y] + ic$$

$$= e^x \cdot e^{iy} + ic = e^{x+iy} + ic = \underline{\underline{e^z + ic}}$$

③ Show that $u(x,y) = x^3 - 3xy^2$ is harmonic and find its harmonic Conjugate and the Corresponding analytic function $f(z)$ in terms of z.

Sol:- Given $u(x,y) = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

$u(x,y)$ is a harmonic function.

Now to find the harmonic Conjugate $v(x,y)$.

By the CR-Equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$\int_v dv = \int_y (3x^2 - 3y^2)\, dy$$

$$v = 3x^2 y - \cancel{3}y^3/\cancel{3} + c$$

$$v = 3x^2 y - y^3 + c$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\cancel{-}6xy = \cancel{-}\frac{\partial v}{\partial x}$$

$$\int_v dv = \int_x 6xy\, dx$$

$$v = {}^{3}\cancel{6}y \cdot x^2/\cancel{2} + c$$

$$v = 3x^2 y + c.$$

$$\therefore \boxed{v(x,y) = 3x^2 y - y^3 + c}$$

Analytic function $f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$

$$= x^3 - 3xy^2 + i3x^2y - iy^3 + ic$$

$$= x^3 - iy^3 - 3xy^2 + i3x^2y + ic$$

$$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2 + ic$$

$$= (x+iy)^3 + ic$$

$\therefore\ f(z) = (x+iy)^3 + k \quad \langle \because \text{ where } ic = k \rangle$

$$\boxed{f(z) = z^3 + K.}$$

④ Find 'k' such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate.

Sol:- We have $f(x,y) = x^3 + 3kxy^2$.

$$\frac{\partial f}{\partial x} = 3x^2 + 3ky^2 \quad \Big| \quad \frac{\partial f}{\partial y} = 6kxy$$

$$\frac{\partial^2 f}{\partial x^2} = 6x. \quad \Big| \quad \frac{\partial^2 f}{\partial y^2} = 6kx.$$

$\therefore$ The function $f(x,y)$ may be harmonic then it satisfies the condition $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$$\Rightarrow 6x + 6kx = 0$$

$$\Rightarrow 6kx = -6x$$

$$k = \frac{-6\cancel{x}}{6\cancel{x}} = -1 \Rightarrow \boxed{k=-1}$$

$\therefore$ Hence $f(x,y) = x^3 + 3(-1)xy^2 = x^3 - 3xy^2$.

Let us consider $u(x,y) = f(x,y)$

i.e, $u(x,y) = x^3 - 3xy^2 \Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \frac{\partial u}{\partial y} = -6xy$

Now to find the harmonic conjugate $v(x,y)$.

By the CR-equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$(3x^2 - 3y^2) = \frac{\partial v}{\partial y}$$

$$\int_v dv = \int_y (3x^2 - 3y^2)\,dy$$

$$v = 3x^2y - \cancel{3}y^3/\cancel{3} + c$$

$$v = 3x^2y - y^3 + c$$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\cancel{-}6xy = \cancel{-}\frac{\partial v}{\partial x}$$

$$\int_v dv = \int_x 6xy\,dx$$

$$v = {}^3\cancel{6}y \cdot x^2/\cancel{2} + c$$

$$v = 3x^2y + c.$$

$$\boxed{\therefore\ v(x,y) = 3x^2y - y^3 + c}$$

Milne-Thomson Method:-

① Find the analytic function whose real part is $u = x^2 - y^2 - x$.

Sol:- Given that $u(x, y) = x^2 - y^2 - x$. is a real part of $f(z)$.

$$\therefore \frac{\partial u}{\partial x} = 2x - 0 - 1 = 2x - 1 \text{ --- ①}$$

$$\frac{\partial u}{\partial y} = 0 - 2y - 0 = -2y \text{ --- ②}$$

we know that $f(z) = u + iv$.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \langle \because \text{Differentiating w.r.to } x \rangle$$

$$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) \quad \langle \because \text{By CR-equations } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rangle$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = 2x - 1 - i(-2y)$$

$$f'(z) = 2x - 1 + i2y \text{ --- ③}$$

By the Milne-Thomson method $f'(z)$ can be expressed in terms of z by replacing 'x' by 'z' and 'y' by '0' in eq ③

we get $f'(z) = 2z - 1 + i2(0)$

$$f'(z) = 2z - 1$$

Integrating on both sides

$$\int f'(z)\,dz = \int (2z-1)\,dz$$

$$f(z) = \cancel{2} \cdot \frac{z^2}{\cancel{2}} - z + c.$$

$\boxed{f(z) = z^2 - z + c}$, where '$c$' is the complex constant.

$\therefore$ Analytic function $f(z) = z^2 - z + c$.

(2) Find the analytic function whose real part is

$u(x,y) = e^{2x}(x\cos 2y - y\sin 2y)$. (26)

Sol:- Given that $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$\Rightarrow \frac{\partial u}{\partial x} = e^{2x}[\cos 2y - 0] + e^{2x}(\cos 2y \cdot x - y\sin 2y)\cdot 2$$

$$\frac{\partial u}{\partial x} = e^{2x}\cos 2y + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}(2x+1)\cos 2y - 2\cdot e^{2x} y\sin 2y$$

Again p.D.w.r.to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \cos 2y\left[e^{2x}\cdot(2) + (2x+1)e^{2x}\cdot 2\right] - 2y\sin 2y \cdot e^{2x}\cdot 2$$

$$= \cos 2y\left[2e^{2x} + (4x+2)e^{2x}\right] - 4e^{2x}\cdot y\sin 2y$$

$$= 2e^{2x}\cos 2y + 4e^{2x}\cdot x\cos 2y + 2e^{2x}\cos 2y - 4e^{2x} y\sin 2y$$

$$= 4e^{2x}\cos 2y + 4e^{2x}\cdot x\cos 2y - 4e^{2x} y\sin 2y \quad \text{---} (1)$$

Now $$\frac{\partial u}{\partial y} = e^{2x}\left[x(-\sin 2y)\cdot 2 - [y\cos 2y\cdot 2 + \sin 2y\cdot 1]\right]$$

$$= e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]$$

$$= e^{2x}[(-2x-1)\sin 2y - 2y\cos 2y]$$

$$= -e^{2x}(2x+1)\sin 2y - 2e^{2x} y\cos 2y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^{2x}(2x+1)\cos 2y\cdot 2 - 2e^{2x}[y(-\sin 2y)\cdot 2 + \cos 2y\cdot 1]$$

$$= -2e^{2x}(2x+1)\cos 2y + 4e^{2x} y\sin 2y - 2e^{2x}\cos 2y$$

$$= -4xe^{2x}\cos 2y - 2e^{2x}\cos 2y + 4e^{2x} y\sin 2y - 2e^{2x}\cos 2y$$

$$= -4e^{2x}\cos 2y - 4e^{2x} x\cos 2y + 4e^{2x} y\sin 2y \quad \text{---} (2)$$

$$eq(1) + eq(2) \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 4e^{2x}\cancel{\cos 2y} + 4e^{2x}\cdot \cancel{x\cos 2y} - 4e^{2x}\cancel{y\sin 2y}$$
$$- 4e^{2x}\cancel{\cos 2y} + 4e^{2x}\cancel{y\sin 2y} - 4e^{2x}\cancel{x\cos 2y}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ $u(x,y)$ is harmonic.

we know that $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\left(\frac{-\partial u}{\partial y}\right) \qquad \left(\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right.$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}. \qquad \left.\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$$

$$\therefore f'(z) = \left[e^{2x}(2x+1)\cos 2y - 2e^{2x}\cdot y\sin 2y\right] - i\left[-e^{2x}(2x+1)\sin 2y - 2e^{2x}y\cos 2y\right].$$

By the Milne-Thomson method, replace <u>x by z</u> and <u>y by 0.</u>

we get $f'(z) = \left(e^{2z}(2z+1)\cos 0 - 0\right) - i\left(-e^{2z}(2z+1)\sin(0) - 0\right)$

$$f'(z) = e^{2z}(2z+1)\cdot 1 - i(0)$$

$$f'(z) = e^{2z}(2z+1)$$

$$f'(z) = e^{2z}\cdot 2z + e^{2z} \qquad \left(\because \frac{d}{dz}(ze^{2z}) = z\cdot e^{2z}\cdot 2 + e^{2z}\cdot 1 = 2ze^{2z} + e^{2z}.\right)$$

$$\therefore f'(z) = \frac{d}{dz}(ze^{2z})$$

Integrating on both sides

$$\boxed{f(z) = ze^{2z} + C.}$$

$\therefore$ Hence it is the analytic function.

(3) Show that the function $f(x,y) = x^3y - xy^3 + xy + x + y$. Can be the imaginary part of an analytic function of $z = x+iy$

Sol:- Given that $f(x,y) = x^3y - xy^3 + xy + x + y$ is the imaginary part of $f(z) = u + iv$

ie, let us consider $v(x,y) = f(x,y)$

$$\therefore v(x,y) = x^3y - xy^3 + xy + x + y.$$

Now to show that $v(x,y)$ is a harmonic ie, to show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

$\therefore \frac{\partial v}{\partial x} = 3x^2y - y^3 + y + 1$ | $\frac{\partial v}{\partial y} = x^3 - 3xy^2 + x + 1$

$\frac{\partial^2 v}{\partial x^2} = 6xy$ | $\frac{\partial^2 v}{\partial y^2} = -6xy$.

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6xy - 6xy = 0$.

$\therefore v(x,y)$ is a harmonic and it is imaginary of $f(z)$.

~~[illegible]~~

Now to find the analytic function $f(z)$:

$\therefore$ we know that $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \left(\because \text{By the CR equation } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right)$$

$$f'(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y + 1)$$

By the Milne-Thomson Method, we can express $f'(z)$ in terms of 'z' by replacing x by z and y by 0, we get

$f'(z) = (z^3 - 0 + z + 1) + i(0+0+0+1)$

$f'(z) = z^3 + z + 1 + i$

Integrating on both sides,

$f(z) = \int (z^3 + z + 1 + i)\,dz$

$f(z) = z^4/4 + z^2/2 + z + iz + c.$

$f(z) = \left(\frac{z^4}{4} + \frac{z^2}{2} + z\right) + iz + c.$ //.

where 'c' is a complex constant.

(4) Find the imaginary part whose real part is $e^x(x\cos y - y\sin y)$.

(5) Find $f(z) = u + iv$ given that $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Sol. Given $f(z) = u + iv \longrightarrow (1)$

$if(z) = i(u+iv) = iu - v \longrightarrow (2)$

$(1) + (2) \Rightarrow (1+i)f(z) = u + iv + iu - v$

$= (u-v) + i(u+v)$

Let $(1+i)f(z) = F(z)$; $u - v = U$ and $u + v = V$

we obtain $F(z) = U + iV$.

Given $V = u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{dV}{dx} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{dV}{dy} = \sin 2x \frac{\partial}{\partial y}\left(\frac{1}{\cosh 2y - \cos 2x}\right) = \frac{-2\sin 2x \cdot \sinh y}{(\cosh 2y - \cos 2x)^2}$$

Now $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$ (By C-R equations)

By Milne-Thomson's method, we express $F(z)$ in terms of z by
Putting $x=z$ and $y=0$.

$$F'(z) = \frac{i\left[2\cos 2z(1-\cos 2z) - 2\sin^2 2z\right]}{(1-\cos 2z)^2} = \cancel{\frac{i2(\cos 2z-1)}{(1-\cos 2z)^2}}$$

$$= \frac{i\left(2\cos 2z - 2\cos^2 2z - 2\sin^2 2z\right)}{(1-\cos 2z)^2}$$

$$= \frac{i\left(2\cos 2z - 2(\cos^2 2z + \sin^2 2z)\right)}{(1-\cos 2z)^2}$$

$$= \frac{i(2\cos 2z - 2)}{(1-\cos 2z)^2}$$

$$= \frac{i2(\cos 2z - 1)}{(1-\cos 2z)^2}$$

$$= \frac{2i}{\cos 2z - 1} = \frac{2i}{-2\sin^2 z} = -i\,\mathrm{cosec}^2 z$$

Integrating, $F(z) = i\cot z + C$.

$$(1+i) f(z) = i\cot z + C$$

(or)

$$f(z) = \frac{i}{1+i}\cot z + \frac{C}{1+i}$$

⑥ Construct an analytic function whose imaginary part is

$$e^{-x}(x\cos y + y\sin y)$$

Sol: Given $V = e^{-x}(x\cos y + y\sin y) \longrightarrow (1)$

Differentiating (1) w.r.t x.

$$\frac{\partial v}{\partial x} = e^{-x}(\cos y + 0) + (x\cos y + y\sin y)\, e^{-x}\cdot(-1)$$

$$= e^{-x}\left[\cos y - x\cos y - y\sin y\right]$$

$$= e^{-x}\left[(1-x)\cos y - y\sin y\right]$$

Diff eq① w.r.t y (31)

$$\frac{\partial v}{\partial y} = e^{-x}\left(x(-\sin y) + y\cos y + \sin y\right)$$

$$= e^{-x}\left((1-x)\sin y + y\cos y\right)$$

But $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= e^{-x}\left((1-x)\sin y + y\cos y\right) + i e^{-x}\left((1-x)\cos y - y\sin y\right)$$

By Milne-Thomson method Put $x = z$ and $y = 0$

$$f'(z) = e^{-z}(0+0) + ie^{-z}((1-z)-0)$$

$$= e^{-z} + ie^{-z}(1-z)$$

$$= e^{-z}(1 + i(1-z))$$

By Integrating

$$\int f'(z) = \int e^{-z}dz + i\left[\int e^{-z}dz - \int e^{-z}z\,dz\right]$$

$$f(z) = \frac{e^{-z}}{-1} + i\left[\frac{e^{-z}}{-1} - z\left(\frac{e^{-z}}{-1}\right) - \int \frac{e^{-z}}{-1}dz\right]$$

$$= \frac{e^{-z}}{-1} + i\left[\frac{e^{-z}}{-1} + ze^{-z} + \frac{e^{-z}}{-1}\right]$$

$$= -e^{-z} + i\left(-e^{-z} + ze^{-z} - e^{-z}\right)$$

$$= -e^{-z} + i\left[-2e^{-z} + ze^{-z}\right] + C.$$

$$f(z) = e^{-z}\left[-1 + i(z-2)\right] + C.$$

$$f(z) = -e^{-z} + i(z-2)\cdot e^{-z} + C.$$

where 'C' is Complex Constant.

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Unit-V
Complex Integration

The advantage of complex integration is that certain complicated real integrals can be evaluated. In this chapter we consider Cauchy's integral theorem which is one of the fundamental theorems of complex function theory. It gives sufficient conditions for a line integral around a simple closed curve to be zero. An important consequence of this theorem is the Cauchy's integral formula in which the value $f(z_0)$ of an analytic function at z_0 is completely determined by an integral of $f(z)$ on any simple closed curve enclosing z_0.

Line integral in complex plane:-

If $f(z) = u + iv$ is a function of complex variable define at every point on the curve 'C' then the integral over the curve from the point A to B is called complex line integration and it is defined as

$$\int_C f(z)\,dz \text{ (or) } \oint_C f(z)\,dz$$

W.K.T $z = x + iy$ and $f(z) = u + iv = u(x,y) + iv(x,y)$

$dz = dx + i\,dy$

$$\therefore \int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$\boxed{\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)}$$

Note: 1. **Linearity**: $\int_C [K_1 f(z) + K_2 g(z)]\,dz = K_1 \int_C f(z)\,dz + K_2 \int_C g(z)\,dz$

2. Sense reversal: $\int_A^B f(z)\,dz = -\int_B^A f(z)\,dz$

3. Partitioning of path: $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, 'C' consists of the curves C_1 and C_2

Problems :-

1) Evalutate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths a) $y = x$ and b) $y = x^2$

Sol:- The given integral is $\int_{z=0}^{1+i} (x^2 - iy)\,dz$

W.K.T $z = x + iy \Rightarrow dz = dx + i\,dy$

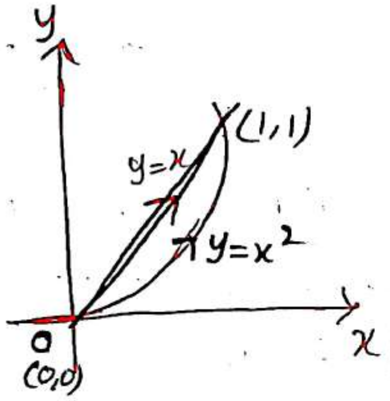
Here $z = 0 \Rightarrow z = 0 + i(0) \longrightarrow (x,y) = (0,0)$

& $z = 1 + i(1) \Rightarrow (x,y) = (1,1)$

a) The integral value along the line $y = x$

$$dy = dx$$

$$dz = dx + i\,dy = dx + i\,dx = (1+i)\,dx.$$



$$\therefore \int_{z=0}^{1+i} (x^2 - iy)\,dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)\,(1+i)\,dx.$$

$$= \int_0^1 (x^2 - ix)\,(1+i)\,dx$$

$$= (1+i)\int_0^1 (x^2 - ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\,\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} - \frac{i}{2}\right] = \frac{1}{3} + \frac{i}{3} - \frac{i}{2} - \frac{i^2}{2}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{2i - 3i}{6}$$

$$= \frac{2+3}{6} + \frac{-i}{6}$$

$$\boxed{\int_{z=0}^{1+i} (x^2 - iy)\,dz = \frac{5}{6} - \frac{i}{6} = \frac{1}{6}(5 - i)}$$

b) The integral value along $y = x^2$ (parabola)

$$dy = 2x\,dx$$

then $dz = dx + i\,dy = dx + i\,2x\,dx = (1 + i2x)dx$

and $x = 0$ to $x = 1$

$$\therefore \int_{z=0}^{1+i} (x^2 - iy)dz = \int_0^1 (x^2 - ix^2)(1 + i2x)dx$$

$$= \int_0^1 (1-i)\,x^2(1 + i2x)dx$$

$$= (1-i)\int_0^1 (x^2 + i2\,x^3)dx$$

$$= (1-i)\left[\frac{x^3}{3} + 2i\,\frac{x^4}{4}\right]_0^1$$

$$= (1-i)\left[\frac{1}{3} + \frac{i}{2}\right]$$

$$= \frac{1}{3} - \frac{i}{3} + \frac{i}{2} - \frac{i^2}{2}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{i}{2} - \frac{i}{3}$$

$$= \frac{2+3}{6} + \frac{3i - 2i}{6}$$

$$= \frac{5}{6} + \frac{i}{6}$$

$$\boxed{\int_0^{1+i} (x^2 - iy)dz = \frac{1}{6}(5 + i)}$$

2) Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$, along the two paths

a) $x = t+1$, $y = 2t^2 - 1$ b) The straight line joining $1-i$ and $2+i$

Sol: The given integral is $\int_{z=1-i}^{2+i} (2x + iy + 1)dz$

W.K.T $z = x + iy \Rightarrow dz = dx + i\,dy$.

a) Along the path $x = t+1$ and $y = 2t^2-1$

$dx = dt$, $dy = 4t\,dt$

then $dz = dx + i\,dy = dt + i4t\,dt = (1+i4t)dt$

the limits of given integral is

$z = 1 - i(1) \longrightarrow (x,y) = (1,-1)$

& $z = 2 + i(1) \longrightarrow (x,y) = (2,1)$

put $x=1$ in $x = t+1 \Rightarrow t+1 = 1$
$\boxed{t=0}$

put $x=2$ in $x = t+1 \Rightarrow t+1 = 2$
$\boxed{t=1}$

put $y=-1$ in $y = 2t^2-1$
$2t^2 = -1+1$
$\boxed{t=0}$

put $y=1$ in $y = 2t^2-1$
$2t^2 = 1+1$
$t^2 = 1$
$\boxed{t=1}$

$$\therefore \int_{z=1-i}^{2+i} (2x+iy+1)\,dz = \int_{t=0}^{1} [2(t+1) + i(2t^2-1)+1]\,(1+i4t)\,dt$$

$$= \int_0^1 [2t+2+i(2t^2-1)+1]\,(1+i4t)\,dt$$

$$= \int_0^1 [2t+3+2it^2-i]\,(1+4it)\,dt$$

$$= \int_0^1 [(2t+2it^2)+(3-i)]\,(1+4it)\,dt$$

$$= \int_0^1 [(2t+2it^2)+(3-i)+(2t+2it^2)(4it)+(3-i)4it]\,dt$$

$$= \int_0^1 [2t+2it^2+(3-i)t + 8it^2 + 8t^3 + (3-i)4it]\,dt$$

$$= \left[\cancel{2}\frac{t^2}{\cancel{2}} + 2i\frac{t^3}{3} + (3-i)t + 8i\frac{t^3}{3} + \cancel{8}^{2}\frac{t^4}{\cancel{4}} + (3-i)\cancel{4}^{2}i\frac{t^2}{\cancel{2}}\right]_0^1$$

$$= \left[1 + \frac{2i}{3} + (3-i) + \frac{8i}{3} + 2 + (3-i)2i\right]$$

$$= (1+3+\cancel{2}+\cancel{2}) + \frac{2i}{3} - i + \frac{8i}{3} + 6i$$

$$\boxed{\int_{z=1-i}^{2+i} (2x+iy+1)\,dz = 4 + \frac{10i+15i}{3} = 4 + \frac{25i}{3}}$$

b) Along the straight line joining $1-i$ to $2+i$

Here the points of a line Eq^n are $A(1,-1)$ and $B(2,1)$

$\therefore$ The eq^n of the line joining the points AB is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y+1}{1+1} = \frac{x-1}{2-1}$$

$$y+1 = 2(x-1)$$

$$\boxed{y = 2x-3} \Rightarrow dy = 2dx$$

then $dz = dx + idy = dx + i2dx = (1+2i)dx$

$$\therefore \int_{z=1-i}^{2+i} (2x+iy+1)\,dz = \int_{x=1}^{2} (2x + i(2x-3)+1)\ (1+2i)\,dx$$

$$= \int_{x=1}^{2} [(1+i)2x' + (1-3i)]\ (1+2i)\,dx$$

$$= (1+2i)\left[(1+i)\cancel{2}\cdot\frac{x^2}{\cancel{2}} + (1-3i)\,x'\right]_1^2$$

$$= (1+2i)\left[(1+i)\,(2^2-1^2) + (1-3i)\,(2-1)\right]$$

$$= (1+2i)\ [3+\cancel{3i}+1-\cancel{3i}]$$

$$= 4(1+2i) = 4+8i$$

$$\boxed{\int_{z=1-i}^{2+i} (2x+iy+1)\,dz = 4+8i}$$

3) Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along (i) the line $y = x/2$

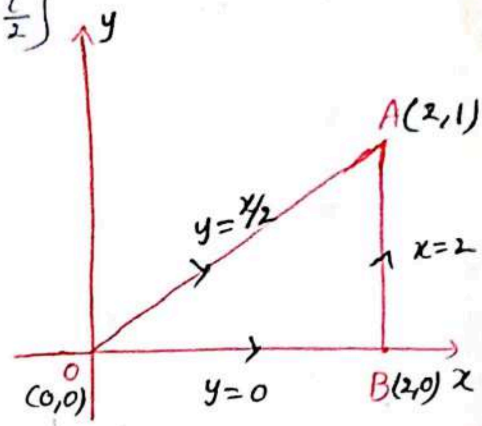
(ii) the real axis to '2' and then vertically to $2+i$

Sol:- The given integral is $\int_{z=0}^{2+i} (\bar{z})^2 dz$

W.K.T $z = x+iy \Rightarrow dz = dx+idy$ and $\bar{z} = x-iy$

a) along the line $y = x/2 \Rightarrow dy = \frac{1}{2}dx$

then $dz = dx + i\,dy = dx + i\cdot\frac{1}{2}dx = dx\left(1+\frac{i}{2}\right)$



$$\therefore \int_{z=0}^{2+i} (\bar{z})^2 dz = \int_{z=0+i0}^{2+i} (x-iy)^2\left(1+\frac{i}{2}\right)dx$$

$$= \left(1+\frac{i}{2}\right)\int_{x=0}^{2} \left(x - i\,\frac{x}{2}\right)^2 dx$$

$$= \left(1+\frac{i}{2}\right)\left(1-\frac{i}{2}\right)^2 \int_{x=0}^{2} x^2 dx$$

$$= \left(1^2 - \frac{i^2}{2^2}\right)\left(1-\frac{i}{2}\right)\left[\frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{\cancel{4}+1}{\cancel{4}}\right)\left(\frac{2-i}{\cancel{2}}\right)\left[\frac{\cancel{2^3}}{3}\right]$$

$$\boxed{\int_{z=0}^{2+i}(\bar{z})^2 dz = \frac{5}{3}(2-i)}$$

b) Along the real axis to '2' and then vertically to '2+i'

i.e from ~~OB~~ O to B and B to A

Here the points of the path OBA are

$O = (0,0)$, $B(2,0)$ and $A = (2,1)$

$\therefore$ The given integral divided into two paths

$$\text{i.e} \quad \int_{z=0}^{2+i} (\bar{z})^2 dz = \int_{OB} (\bar{z})^2 dz + \int_{BA} (\bar{z})^2 dz \longrightarrow (1)$$

Along OB:- Here $O = (0,0)$ and $B = (2,0)$

from the points x varies from '0' to '2'

and $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{OB} (\bar{z})^2 dz = \int_{OB} (x-iy)^2 (dx + i\,dy)$$

$$\int_{OB} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{2^3}{3} = \frac{8}{3} \longrightarrow (2)$$

Along BA :- Here $B(2,0)$ and $A(2,1)$

from the points y varies 0 to 1

and $x = 2 \Rightarrow dx = 0$

$$\therefore \int_{BA} (\bar{z})^2 dz = \int_{BA} (x - iy)^2 (dx + i\,dy)$$

$$= \int_{y=0}^{1} (2 - iy)^2 \, i\,dy$$

$$= i \int_0^1 (4 + y^2 - 4iy)\,dy$$

$$= i \left[4y + \frac{y^3}{3} - \cancel{4}^{2} i \frac{y^2}{\cancel{2}} \right]_0^1$$

$$= i \left[4 + \frac{1}{3} - 2i \right]$$

$$= i \left[\frac{11 - 6i}{3} \right]$$

$$\int_{BA} (\bar{z})^2 dz = \frac{1}{3} [11i + 6] \longrightarrow (3)$$

$\therefore$ from Eq^n (1), (2) & (3)

$$\int_{z=0}^{2+i} (\bar{z})^2 dz = \frac{8}{3} + \frac{1}{3}(11i + 6)$$

$$= \frac{1}{3} [8 + 11i + 6]$$

$$\boxed{\int_{z=0}^{2+i} (\bar{z})^2 \, dz = \frac{1}{3}(14 + 11i)}$$

4) Find the value of $\int_0^{1+i} (x - y + ix^2)\,dz$

(i) along the straight line from $z=0$ to $z=1+i$

(ii) along real axis from $z=0$ to $z=1$ and then along a line parallel to the imaginary axis from $z=1$ to $z=1+i$.

Sol:-

The given integral is $\int_0^{1+i} (x-y+ix^2)\,dz$

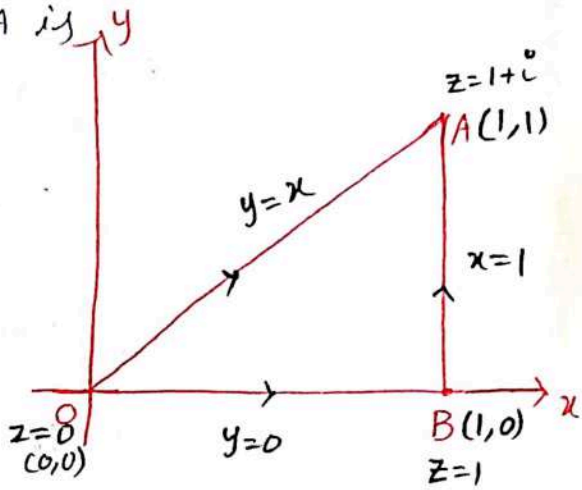
W.K.T $z = x+iy \Rightarrow dz = dx + i\,dy$.

a) Along the straight line $z=0$ to $z=1+i$

Here $z = 0+i(0) \rightarrow (x,y)=(0,0) = O$

$z = 1+i(1) \rightarrow (x,y)=(1,1) = A$

$\therefore$ The Eq^n of the line for the points OA is



$$y-y_1 = \frac{x-x_1}{x_2-x_1}(y_2-y_1)$$

$$y-0 = \frac{x-0}{1-0}(1-0)$$

$$\boxed{y=x} \Rightarrow dy = dx$$

and x varies from $x=0$ to $x=1$

$$\therefore \int_0^{1+i}(x-y+ix^2)\,dz = \int_{OA}(x-y+ix^2)(dx+i\,dy)$$

$$= \int_{x=0}^{1}(x-x+ix^2)(dx+i\,dx)$$

$$= \int_{x=0}^{1} ix^2(1+i)\,dx$$

$$= (1+i)\,i\left[\frac{x^3}{3}\right]_0^1 = (i-1)\frac{1}{3}$$

$$\boxed{\int_0^{1+i}(x-y+ix^2)\,dz = \frac{1}{3}(i-1)}$$

b) Along the real axis $z=0$ to $z=1$ and $z=1$ to $z=1+i$

Here the points on the path OBA are

$z = 0+i0 \rightarrow (x,y)=(0,0) = O$

$z = 1+i(0) \rightarrow (x,y)=(1,0) = B$

$z = 1+i(1) \rightarrow (x,y)=(1,1) = A$

∴ Here the given integral divided into two paths

i.e $$\int_{z=0}^{1+i} (x-y+ix^2)\,dz = \int_{OB} (x-y+ix^2)\,dz + \int_{BA} (x-y+ix^2)\,dz \longrightarrow (1)$$

Along OB: Here $O=(0,0)$ and $B=(1,0)$

from the points x varies from $x=0$ to $x=1$

and $y=0 \Rightarrow dy=0$

$$\therefore \int_{OB} (x-y+ix^2)\,dz = \int_{OB} (x-y+ix^2)(dx+i\,dy)$$

$$= \int_{x=0}^{1} (x+ix^2)\,(dx)$$

$$= \left[\frac{x^2}{2} + i\,\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{i}{3}$$

$$\int_{OB} (x-y+ix^2)\,dz = \frac{1}{6}(3+2i) \longrightarrow (2)$$

Along BA: Here $B=(1,0)$ and $A=(1,1)$

from the points y varies from $y=0$ to $y=1$

and $x=1 \Rightarrow dx=0$

$$\therefore \int_{BA} (x-y+ix^2)\,dz = \int_{BA} (x-y+ix^2)(dx+i\,dy)$$

$$= \int_{y=0}^{1} (1-y+i)\; i\,dy$$

$$= i\left[(1+i)y - \frac{y^2}{2}\right]_0^1 = i\left[1+i-\frac{1}{2}\right] = i\left(\frac{2+2i-1}{2}\right)$$

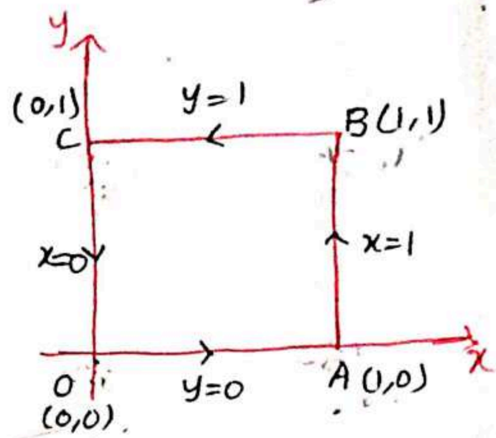
$$\int_{BA} (x-y+ix^2)\,dz = \frac{i}{2}(1+2i) = \frac{1}{2}(i-2) \longrightarrow (3)$$

∴ from Eq^n (1), (2) & (3) $\Rightarrow$ $$\int_{OBA} (x-y+ix^2)\,dz = \frac{1}{6}(3+2i) + \frac{1}{2}(i-2)$$

$$= \frac{1}{6}(3+2i+3i-6)$$

$$\boxed{\therefore \int_{z=0}^{1+i} (x-y+ix^2)\cdot dz = \frac{1}{6}(5i-3)}$$

5) Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.



Sol:- The given integral is $\oint_C |z|^2 dz$ and

the points of a square $O(0,0)$, $A(1,0)$ $B(1,1)$ and $C(0,1)$

Here our path is in closed form

i.e $OABC$

$$\therefore \oint_C |z|^2 dz = \oint_{OABC} |z|^2 dz$$

$$= \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz \longrightarrow (1)$$

Along OA:- Here the points are $O=(0,0)$ and $A=(1,0)$

from the points x varies from $x=0$ to $x=1$

and $y=0 \Rightarrow dy=0$

then $dz = dx+idy = dx$ and $z = x+iy$, $|z| = \sqrt{x^2+y^2}$, $|z|^2 = x^2+y^2$

$$\therefore \int_{OA} |z|^2 dz = \int_{OA} (x^2+y^2)\,dx = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \longrightarrow (2)$$

Along AB:- The points are $A=(1,0)$ and $B=(1,1)$.

from the points y varies from $y=0$ to $y=1$

and $x=1 \Rightarrow dx=0$

$$\therefore \int_{AB} |z|^2 dz = \int_{AB} (x^2+y^2)(\underset{0}{dx}+idy) = \int_{y=0}^{1} (1+y^2)\, i\,dy = i\left(y+\frac{y^3}{3}\right)_0^1 = i\left(1+\frac{1}{3}\right)$$

$$\int_{AB} |z|^2 dz = \frac{4i}{3} \longrightarrow (3)$$

Along BC:- The points are $B=(1,1)$ and $C=(0,1)$.

Here x varies from $x=1$ to $x=0$ and $y=1 \Rightarrow dy=0$

$$\therefore \int_{BC} |z|^2 dz = \int_{BC} (x^2+y^2)(dx+idy) = \int_{x=1}^{0} (x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)_1^0$$

$$\int_{BC} |z|^2 dz = -\left(\frac{1}{3}+1\right) = -\frac{4}{3} \longrightarrow (4)$$

Along CO: The points are $C=(0,1)$ and $O=(0,0)$

Here y varies from $y=1$ to $y=0$ and $x=0 \Rightarrow dx=0$

$$\therefore \int_{CO} |z|^2 dz = \int_{CO} (x^2+y^2)(dx+i\,dy) = \int_{y=1}^{0} y^2\, i\,dy = i\left[\frac{y^3}{3}\right]_1^0 = -\frac{i}{3} \longrightarrow (5)$$

$\therefore$ from Eqn's (1), (2), (3), (4) & (5)

$$\oint_C |z|^2 dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = \frac{1-4}{3} + \frac{4i-i}{3}$$

$$= \frac{-\cancel{3}}{\cancel{3}} + \frac{\cancel{3}i}{\cancel{3}}$$

$$\boxed{\oint_C |z|^2 dz = i-1}$$

Problems:-

1) Integrate $f(z) = Re(z)$
(i) along straight line joining $z=0$ to $z=1+2i$
(ii) along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+2i$

2) Evaluate $\int_{1-i}^{2+3i} (z^2+z)dz$ along the line joining the points $(1,-1)$ & $(2,3)$.

3) Evaluate $\oint_C (y-x-3x^2 i)dz$ where 'c' is the straight line from $z=0$ to $z=1+i$

4) Show that $\oint_C (z+1)dz = 0$ where 'c' is the boundary of the square whose vertices are at the points $z=0$, $z=1$, $z=(1+i)$ & $z=i$

5) Integrate $f(z) = x^2+ixy$ from $A(1,1)$ to $B(2,4)$ along $x=t$, $y=t^2$

6) Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along the curve given by $z=t^2+it$

Note:- The circle in z plane with centre z_0 is $|z-z_0|=r$ and it is in polar form $(z-z_0)=re^{i\theta}$

$$z = z_0 + re^{i\theta} \quad , \text{where } 0 \le \theta \le 2\pi$$

Prob:- Evaluate $\int_C \frac{dz}{z-a}$ where 'C' represents the circle $|z-a|=r$

Sol:- Given that $\int_C \frac{dz}{z-a}$ and

$$z-a = re^{i\theta}$$

$|z-a| = r$ then $z = a + re^{i\theta}$

$z \neq a$ $\quad dz = r e^{i\theta} . i . d\theta$ and $0 \le \theta \le 2\pi$

Hence $\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{i r e^{i\theta} d\theta}{r . e^{i\theta}} = i\int_0^{2\pi} 1 . d\theta = i(\theta)_0^{2\pi}$

$$\boxed{\therefore \int_C \frac{dz}{z-a} = 2\pi i}$$

Prob:- Evaluate $\oint_C (z-a)^n dz$ (n is an integer $\neq -1$), where 'c' is the circle $|z-a| = r$.

Sol:- Given that $\oint_C (z-a)^n dz$

and $|z-a| = re^{i\theta} \Rightarrow z = a + re^{i\theta}$

$$dz = i r e^{i\theta} d\theta \quad \text{and} \quad 0 \le \theta \le 2\pi$$

Hence $\oint_C (z-a)^n dz = \int_0^{2\pi} (re^{i\theta})^n i . r e^{i\theta} d\theta$

$$= i . r^{n+1} \int_0^{2\pi} e^{in\theta} . e^{i\theta} d\theta$$

$$= i . r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta \qquad \left(\because e^{i(n+1)2\pi} = \cos 2(n+1)\pi + i \sin 2(n+1)\pi = 1 + i(0) \right)$$

$$= i . r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} = \frac{r^{n+1}}{n+1} \left[e^{i(n+1)2\pi} - e^0 \right]$$

$$\boxed{\oint_C (z-a)^n dz = \frac{r^{n+1}}{n+1} [1-1] = 0}$$

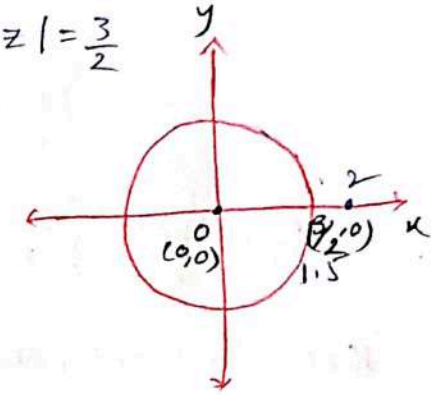
Cauchy's integral Theorem :-

Statement: If a function $f(z)$ is analytic and $f'(z)$ is continuous at each point inside and on a closed curve 'c' then

$$\oint_C f(z)dz = 0$$

Prob:- Evaluate $\int_C \frac{z^2+5z+6}{z-2} dz$ where the curve C is $|z| = \frac{3}{2}$.

Sol:- Given that $\int_C \frac{z^2+5z+6}{z-2} dz$ and c is $|z| = \frac{3}{2}$



Here $f(z) = \frac{z^2+5z+6}{z-2}$

$\therefore f(z)$ is not analytic at $z = 2$, which lies outside the circle $|z| = \frac{3}{2}$

Since $f(z)$ is analytic with in and on the closed curve 'c'

$\therefore$ By cauchy's integral theorem, we have $\int_C f(z)dz = 0$

$$\boxed{\int_C \frac{z^2+5z+6}{z-2} dz = 0}$$

Prob:- Evaluate the following integrals by applying cauchy's integral theorem in case applicable.

a) $\oint_C \cos z \, dz$ b) $\oint_C \sec z \, dz$ c) $\oint_C \frac{dz}{z^2-5z+6} dz$

where 'c' is the circle $|z| = 1$

Sol:- a) Given that $\oint_C \cos z \, dz$ and $|z| = 1$

Here the integrand $f(z) = \cos z$ is analytic for all 'z' and also $f'(z) = \sin z$ is continuous everywhere and hence on and inside the circle 'c' also

$\therefore$ By cauchy's integral theorem, we have $\oint_C f(z) = 0$

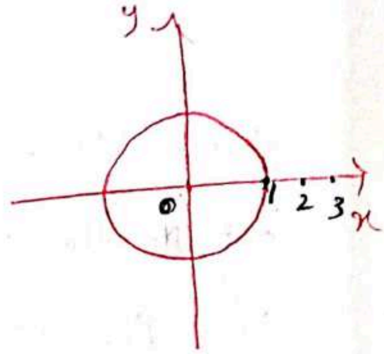
$$\boxed{\oint_C \cos z \, dz = 0}$$

b) Given that $\oint_c \sec z \, dz$ and c is $|z|=1$

Here the integrand $f(z) = \sec z = \frac{1}{\cos z}$ is not analytic at the points $z = \pm \pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the circle $|z|=1$. Hence $f(z)$ is analytic and $f'(z)$ is continous in and on 'c'

$\therefore$ By cauchy's integral theorem

$$\boxed{\int_c \sec z \, dz = 0}$$



c) Given that $\oint_c \frac{dz}{z^2-5z+6}$ and 'c' is $|z|=1$

~~Here~~ The integrand $f(z) = \frac{1}{z^2-5z+6} = \frac{1}{z^2-2z-3z+6}$

$$f(z) = \frac{1}{(z-2)(z-3)}$$

Here $f(z)$ is analytic everywhere except at $z=2,3$ and the points which lie outside the unit circle $|z|=1$. Hence $f(z)$ is analytic and $f'(z)$ is continuous in and on the curve 'c'

$\therefore$ By cauchy's integral theorem, we have

$$\oint_c f(z) = 0$$

i.e $\boxed{\oint_c \frac{1}{z^2-5z+6} dz = 0}$

Singular points:- Singular points means, the points where the function is not an analytic (not differentiable)

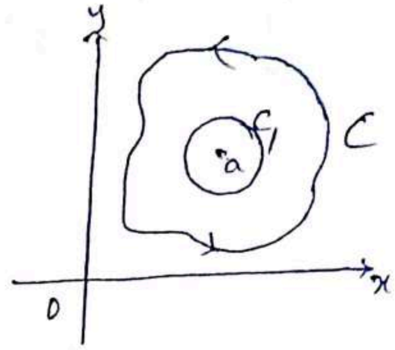
Eg:- $\int \frac{z^2+2}{z-2} dz$

$z-2=0 \Rightarrow z=2$

at $z=2$ the function is not defined (not analytic), So $z=2$ is a singular point.

Cauchy's Integral formula :-

Statement :- If $f(z)$ is analytic with in and on a closed curve 'c' and if 'a' is any point with-in 'c', then



$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)} dz$$

(or)

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$$

Generalisation of cauchy's Integral formula :-

Statement : If $f(z)$ is analytic with in and on a simple closed curve 'c' then n^{th} derivative of $f(z)$ is also analytic in 'c' and is given by $f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$, where 'a' lies inside 'c'.

$$\Rightarrow \int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

Problems :-

1) Evaluate $\oint_c \frac{e^{tz}}{z^2+1} dz$, where 'C' is the circle $|z|=3$.

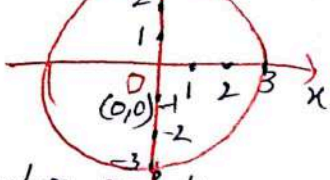
Sol:- Given integral is $\oint_c \frac{e^{tz}}{z^2+1} dz$

To find singular points, equate the denominator to zero'

i.e $z^2+1 = 0 \Rightarrow z^2 = -1$.

$z^2 = i^2 \Rightarrow z = \pm i$

$z = 0+i(1)$, $z = 0-i(1)$ are singular points.



Given circle is $|z|=3$, means it is a circle with centre $(0,0)$ and radius '3'

$\therefore$ The point $z = i$ and $z = -i$ lies inside the circle. $|z| = 3$

Now To find $\oint_c \frac{e^{tz}}{z^2+1} dz$ by using cauchy's integral formula by considering $f(z)$ as

$$\frac{f(z)}{z-a} = \frac{e^{tz}}{z^2+1} = \frac{e^{tz}}{(z-i)(z+i)}$$

$$= \frac{\left(\frac{e^{tz}}{z-i}\right)}{z+i} + \frac{\left(\frac{e^{tz}}{z+i}\right)}{z-i}$$

Taking integration on both sides.

$$\oint_C \frac{f(z)}{z-a} = \oint_C \frac{\left(\frac{e^{tz}}{z-i}\right)}{z+i}\,dz + \oint_C \frac{\left(\frac{e^{tz}}{z+i}\right)}{z-i}\,dz$$

By cauchy's integral formula, $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$

$$\therefore \oint \frac{e^{tz}}{(z^2+1)}\,dz = \oint_C \frac{\left(\frac{e^{tz}}{z-i}\right)}{z+i}\,dz + \oint_C \frac{\left(\frac{e^{tz}}{z+i}\right)}{z-i}\,dz$$

Here $a=-i$ $\quad$ $a=i$

$$= 2\pi i\, f(-i) + 2\pi i\, f(i)$$

$$= 2\pi i \left[\frac{e^{tz}}{z-i}\right]_{at\ z=-i} + 2\pi i \left[\frac{e^{tz}}{z+i}\right]_{z=i}$$

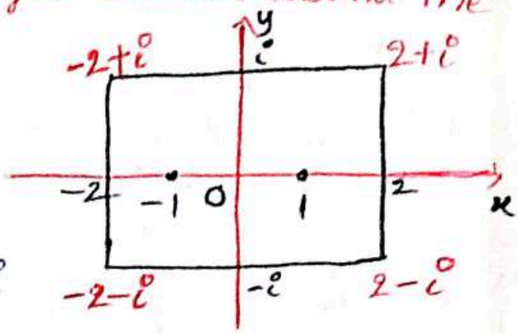
$$= 2\pi i \left[\frac{e^{-ti}}{-i-i}\right] + 2\pi i \left[\frac{e^{t\times i}}{i+i}\right]$$

$$= 2\pi i \left[\frac{e^{-ti}}{-2i} + \frac{e^{ti}}{2i}\right]$$

$$= 2\pi i \left[\frac{e^{it} - e^{-it}}{2i}\right]$$

$$\boxed{\oint_C \frac{e^{tz}}{z^2+1}\,dz = 2\pi i \sin t}$$

2) Evaluate $\oint_C \frac{\cos\pi z}{z^2-1}\,dz$, using cauchy's integral formula around the rectangle with vertices $2 \pm i, -2 \pm i$



Sol:- Given integral is $\oint_C \frac{\cos\pi z}{z^2-1}\,dz$ and the vertices of rectangle is $2+i, 2-i, -2+i, -2-i$

Here $f(z) = \cos\pi z$ is analytic in the given region.

To find singular points equate the denominator to zero.

i.e $z^2 - 1^2 = 0 \Rightarrow (z-1)(z+1) = 0$

$z=1, z=-1$ are the two singular points and which lies inside the rectangle.

$\therefore$ By cauchy's integral formula

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]$$

$$\Rightarrow \oint_C \frac{\cos\pi z}{z^2-1}\,dz = \oint_C \frac{\cos\pi z}{(z-1)(z+1)}\,dz$$

$$= \oint_C \cos\pi z \left[\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\right] dz$$

$$= \frac{1}{2}\oint_C \frac{\cos\pi z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{\cos\pi z}{z+1}\,dz$$

$a=1 \qquad a=-1$

$$= \frac{1}{\cancel{2}}\,\cancel{2}\pi i\, f(1) - \frac{1}{\cancel{2}}\,\cancel{2}\pi i\, f(-1)$$

$$= \pi i \cos\pi - \pi i \cos(-\pi)$$

$$= \pi i(-1) - \pi i(-1)$$

$$\boxed{\oint_C \frac{\cos\pi z}{z^2-1}\,dz = 0}$$

3) Evaluate $\oint_C \dfrac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$ where 'c' is the circle $|z|=3$

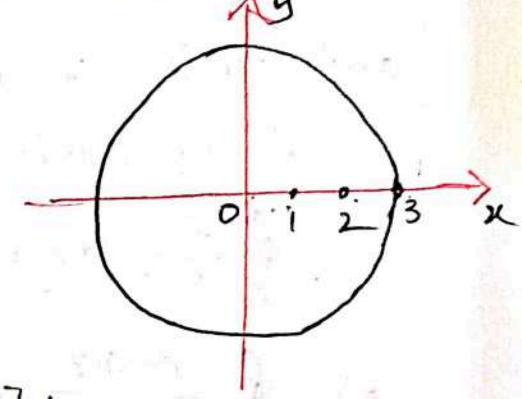
Sol:- Given integral is $\oint_C \dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz$

To find singular points equate the denominator to zero

i.e $(z-1)(z-2)=0 \Rightarrow z=1, z=2$

Here $|z|=3$ is the circle with centre zero $(0,0)$ and radius '3'

$\therefore$ $z=1,2$ are the singular points and which lies inside the circle $|z|=3$



By Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)$$

$$\therefore \oint_C \frac{f(z)}{(z-1)(z-2)}\,dz = \oint_C (\sin(\pi z^2)+\cos\pi z^2)\left[\frac{1}{(z-1)(z-2)}\right]dz$$

$$= \oint_C (\sin\pi z^2+\cos\pi z^2)\left[\frac{1}{z-2} - \frac{1}{z-1}\right]dz$$

$$= \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\,dz - \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\,dz$$

$$= 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 2\pi i\,[\sin\pi 4 + \cos\pi 4] - 2\pi i\,(\sin\pi + \cos\pi)$$

$$= 2\pi i\,[0+1] - 2\pi i\,[0-1]$$

$$= 2\pi i + 2\pi i = 4\pi i$$

$$\boxed{\therefore \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz = 4\pi i}$$

4) Evaluate $\int_C \frac{z^2-z+1}{z-1}\,dz$, where 'c' is the circle (i) $|z|=1$ (ii) $|z|=\frac{3}{2}$

$\longrightarrow 2\pi i \qquad \longrightarrow 0$

5) Evaluate $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$ where 'c' is the circle $|z|=1$

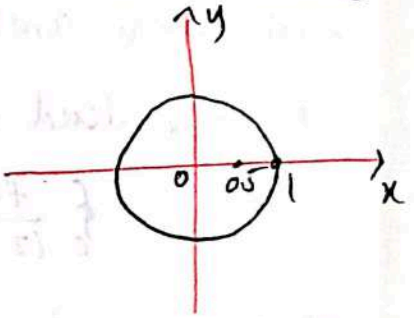
Sol:- The given integral is $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$

and The $|z|=1$ is the circle with centre (0,0) and radius is '1'

To find singular points Equate the denominator to zero

i.e $(z-\pi/6)^3 = 0 \Rightarrow z = \pi/6$ (0.5 appro)

Here $z=\pi/6$ is the singular point and which lies inside the circle $|z|=1$



By cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

$$\oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{n!} f^n(a)$$

$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^{2+1}}\,dz = \frac{2\pi i}{2!} f''(\pi/6)$$

Here $f(z) = \sin^2 z \Rightarrow f'(z) = 2.\sin z.(\cos z)$

$$f'(z) = \sin 2z$$

$$f''(z) = 2.\cos 2z$$

put $z=\pi/6 \Rightarrow f''(\pi/6) = 2\cos 2\pi/6 = 2\cos \pi/3 = 2 \times 1/2 = 1$

$$\boxed{\therefore \oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2} \times 1 = \pi i}$$

6) Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, where 'c' is $|z|=4$.

Sol: Given integral is $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ and 'c' is $|z|=4$

To find singular points equate the denominator to zero

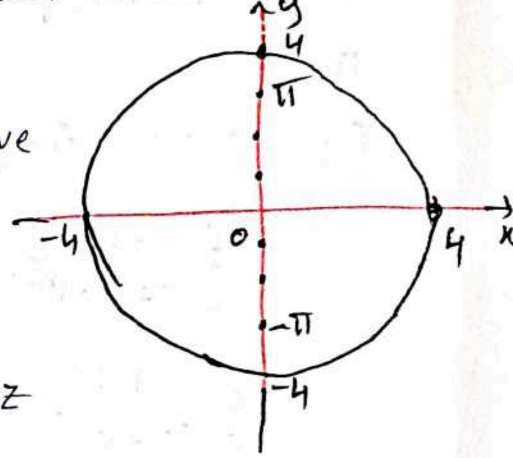
$$(z^2+\pi^2)^2=0$$

$$(z^2-i^2\pi^2)^2=0$$

$$(z+\pi i)^2(z-\pi i)^2=0 \qquad \pi=3.14$$

$$z=-\pi i,-\pi i, z=\pi i,\pi i$$

$\therefore$ $z=\pi i, -\pi i$ are singular points and which lies inside the circle with centre $(0,0)$ and radius '4'



By generalised Cauchy's ingral formula, we have

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

Now

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \oint_C \frac{e^z}{(z+\pi i)^2\,(z-\pi i)^2}\,dz$$

$$= \oint_C \frac{\left[\frac{e^z}{(z+\pi i)^2}\right]}{(z-\pi i)^2}\,dz + \oint_C \frac{\left[\frac{e^z}{(z-\pi i)^2}\right]}{(z+\pi i)^2}\,dz$$

$a=\pi i,\ n=1 \qquad a=-\pi i,\ n=1$

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{2\pi i}{1!} f'(\pi i) + \frac{2\pi i}{1!} f'(-\pi i) \longrightarrow (1)$$

From the 1st integral

$$f(z)=\frac{e^z}{(z+\pi i)^2}$$

$$f'(z)=\frac{(z+\pi i)^2 e^z - e^z 2(z+\pi i)}{(z+\pi i)^4}$$

$$f'(\pi i)=\frac{(\pi i+\pi i)^2 e^{\pi i}-2e^{\pi i}(\pi i+\pi i)}{(\pi i+\pi i)^4} = \frac{\left[(2\pi i)^2-4\pi i\right]}{(2\pi i)^4} e^{\pi i} = \frac{(-4\pi^2-4\pi i)e^{\pi i}}{16\pi^4} = \frac{-(\pi+i)e^{\pi i}}{4\pi^3}$$

from the Second integral

$$f(z) = \frac{e^z}{(z-\pi i)^2} \Rightarrow f'(z) = \frac{(z-\pi i)^2 e^z - e^z\, 2(z+\pi i)}{(z-\pi i)^4}$$

$$f'(z) = \frac{\cancel{(z-\pi i)}\,[z-\pi i-2]\,e^z}{(z-\pi i)^{\cancel{4}\,3}}$$

put $z = -\pi i \Rightarrow f'(-\pi i) = \dfrac{[-\pi i-\pi i-2]\,e^{-\pi i}}{(-\pi i-\pi i)^3}$

$$= \frac{[-2\pi i-2]\,e^{-\pi i}}{(-2\pi i)^3}$$

$$= \frac{-\cancel{2}(\pi i+1)\,e^{-\pi i}}{-\cancel{8}_{4}\,\pi^3 . i^3} = \frac{-(\pi i+1)}{4\pi^3 i}\,e^{-\pi i}$$

$$f'(-\pi i) = \frac{i(\pi i+1)}{4\pi^3}\,e^{-\pi i}$$

Sub^n $f'(\pi i)$ and $f'(-\pi i)$ in Eq^n 1, we get

$$\oint_c \frac{e^z}{(z^2+\pi^2)^2}\,dz = \cancel{2}\pi i\,\frac{-(\pi+i)}{\cancel{4}_{2}\pi^{\cancel{3}\,2}}\,e^{\pi i} + \cancel{2}\pi i\,\frac{(i-\pi)}{\cancel{4}_{2}\pi^{\cancel{3}\,2}}\,e^{-\pi i}$$

$$= \frac{i}{2\pi^2}\left[-\pi e^{\pi i} - i e^{\pi i} + i e^{-\pi i} - \pi e^{-\pi i}\right]$$

$$= \frac{i}{2\pi^2}\left[-\pi\left(e^{\pi i} + e^{-\pi i}\right) - i\left(e^{i\pi} - e^{-i\pi}\right)\right]$$

$$= \frac{i}{2\pi^2}\left[-\pi\ 2\cos\pi - i\ 2i\sin\pi\right]$$

$$= \frac{i}{\cancel{2}\pi^{\cancel{2}}}\left[-\cancel{\pi}\,\cancel{2}\,(-1)\right] = \frac{i}{\pi}$$

$$\boxed{\therefore \oint_c \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}}$$

7) Evaluate, using cauchy's integral formula $\oint_C \frac{e^{2z}}{(z+i)^4}\,dz$ where 'c' is the circle $|z|=3$ $\longrightarrow \frac{8\pi i}{3} e^{-2}$

8) Evaluate $\oint_C \frac{e^z}{z^2+1}\,dz$, where 'c' is the circle $|z| = 1/2$

9) Evaluate $\oint_C \frac{dz}{z^2+9}$ where 'c' is the circle

(i) $|z-3i|=4$ (ii) $|z+3i|=2$ (iii) $|z|=5$

Sol:- The given integral is $\oint_C \frac{dz}{z^2+9}$

To find singular points, equate the denominato to zero

$$z^2+9=0 \Rightarrow z^2=-9.$$

$$z^2 = i^2 3^2$$

$$z = \pm 3i$$

$$z = 3i, -3i$$

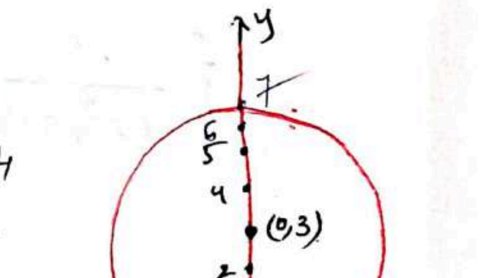
Here $z=3i, -3i$ are singular points

(i) The given circle is $|z-3i|=4 \Rightarrow |x+iy-3i|=4$

$$|x+i(y-3)|=4$$

$$x^2+(y-3)^2=16$$

it is a circle with centre $(0,3)$ and radius 4



Here $z=3i$ lies inside and $z=-3i$ lies outside the circle

By cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\Rightarrow \oint_C \frac{1}{z^2+9}\,dz = \oint_C \frac{1}{(z-3i)(z+3i)}\,dz = \oint_C \frac{\overbrace{\left(\frac{1}{z+3i}\right)}^{f(z)}}{\underbrace{z-3i}_{z-a}}\,dz$$

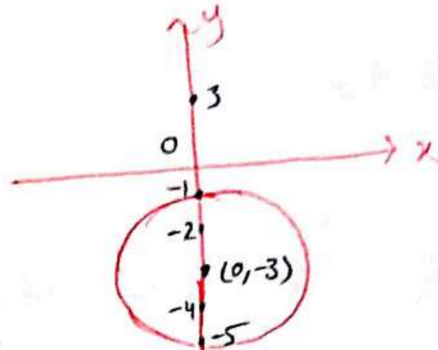
$$= 2\pi i f(3i) = 2\pi i\left(\frac{1}{3i+3i}\right)$$

$$\boxed{\oint_C \frac{dz}{z^2+9} = \pi/3}$$

(ii) Here the circle is $|z+3i| = 2$

$$|x+iy+3i| = 2$$

$$x^2+(y+3)^2 = 4$$



it is a circle with centre $(0,-3)$ and radius '2'

Here $z=3i$ lies outside the circle and $z=-3i$ lies inside the circle

By cauchy's integral formula, we have

$$\oint_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\Rightarrow \oint_c \frac{1}{z^2+9}\,dz = \oint_c \frac{1}{(z-3i)(z+3i)}\,dz = \oint_c \frac{\left(\frac{1}{z-3i}\right)}{z+3i}\,dz$$

$$= 2\pi i\, f(-3i)$$

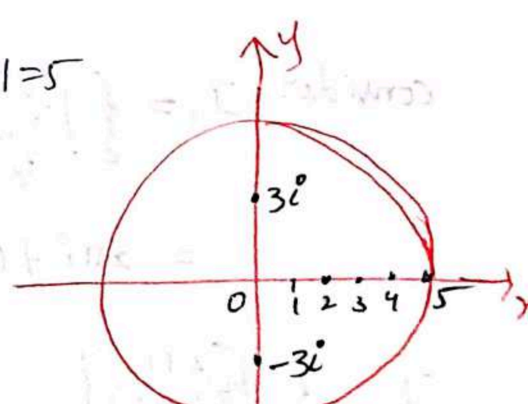
$$= 2\pi i\, \frac{1}{-3i-3i}$$

$$= 2\pi i\, \frac{1}{-6i} = -\pi/3$$

$$\boxed{\therefore \oint_c \frac{1}{z^2+9}\,dz = -\pi/3.}$$

(iii) Here $|z|=5$ is a circle with centre $(0,0)$ and radius '5'

Here $z=3i,-3i$ lies inside the circle $|z|=5$



By cauchy's integral formula

$$\oint_c \frac{1}{z^2+9}\,dz = \oint_c \frac{1}{(z-3i)(z+3i)}\,dz$$

$$= \oint_c \frac{\left(\frac{1}{z-3i}\right)}{z+3i}\,dz + \oint \frac{\left(\frac{1}{z+3i}\right)}{z-3i}\,dz$$

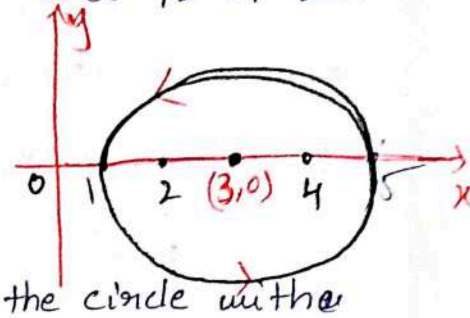
$$= 2\pi i\, f(-3i) + 2\pi i\, f(3i)$$

$$= 2\pi i\left(\frac{1}{-3i-3i}\right) + 2\pi i\,\frac{1}{3i+3i} = -\pi/3 + \pi/3 = 0$$

$$\boxed{\therefore \oint_c \frac{1}{z^2+9}\,dz = 0}$$

10) Evaluate $\oint_c \frac{z+1}{z(z-2)(z-4)^3}\,dz$, where c is the circle $|z-3|=2$ in the counter-clockwise sense.

Sol:- The given integral is $\oint_c \frac{z+1}{z(z-2)(z-4)^3}\,dz$ and $|z-3|=2$



The singular points are $z=0,\ z=2,\ z=4$

$\therefore$ ~~$z \neq 0$ lies~~ Here $|z-3|=2 \Rightarrow |x+iy-3|=2$

$(x-3)^2+y^2=4$ is the circle with the centre $(3,0)$ and radius '2'.

$\therefore$ $z=2,4$ lies inside the circle and $z=0$ lies outside the circle

By cauchy's integral formula.

$$\oint_c \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a) \quad \& \quad \oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$\Rightarrow \oint_c \frac{z+1}{z(z-2)(z-4)^3}\,dz = \underbrace{\oint_c \frac{\left[\frac{z+1}{z(z-4)^3}\right]}{z-2}\,dz}_{I_1} + \underbrace{\oint_c \frac{\left[\frac{z+1}{z(z-2)}\right]}{(z-4)^3}\,dz}_{I_2} \quad \text{—— (1)}$$

consider $I_1 = \oint_c \frac{\left[\frac{z+1}{z(z-4)^3}\right]}{z-2}\,dz$

$$= 2\pi i\, f(2) = 2\pi i \left[\frac{z+1}{z(z-4)^3}\right]_{at\ z=2} = 2\pi i\left[\frac{2+1}{2(2-4)^3}\right] = \frac{-3\pi i}{8}$$

$$I_2 = \oint_c \frac{\left[\frac{z+1}{z(z-2)}\right]}{(z-4)^{2+1}}\,dz = \frac{2\pi i}{2!} f''(4)$$

$$= \pi i\left[\frac{d^2}{dz^2}\left(\frac{z+1}{z^2-2z}\right)\right]_{at\ z=4}$$

$$= \pi i \frac{d}{dz}\left[\frac{(z^2-2z)(1)-(z+1)(2z-2)}{(z^2-2z)^2}\right]_{at\ z=4.}$$

$$I_2 = \pi i \frac{d}{dz}\left[\frac{z^2-2z-2z^2-\cancel{2z}+\cancel{2z}+2}{(z^2-2z)^2}\right]_{at\ z=4}$$

$$= \pi i \frac{d}{dz}\left[\frac{-z^2-2z+2}{(z^2-2z)^2}\right]_{z=4}$$

$$= -\pi i \left[\frac{(z^2-2z)^2(2z+2)-(z^2+2z-2)2(z^2-2z)(2z-2)}{(z^2-2z)^4}\right]_{at\ z=4}$$

$$= -\pi i \left[\frac{(z^2-2z)(2z+2)-2(z^2+2z-2)(2z-2)}{(z^2-2z)^3}\right]_{z=4}$$

$$= -\pi i \left[\frac{(4^2-8)(8+2)-2(16+8-2)(8-2)}{(16-8)^3}\right]$$

$$= -\pi i \left[\frac{8\times 10-2(22)(6)}{8^3}\right]$$

$$I_2 = -\pi i\left[\frac{80-264}{512}\right] = -\pi i\left[\frac{-184}{512}\right] = \frac{\pi i\times 23}{64}$$

$\therefore$ From Eq^n (1) $\oint_C \frac{z+1}{z(z-2)(z-4)^3}dz = I_1 + I_2 = \frac{-3\pi i}{8} + \frac{23\pi i}{64} = \frac{\pi i}{64}$

$$\boxed{\oint_C \frac{z+1}{z(z-2)(z-4)^3}dz = \frac{\pi i}{64}.}$$

11) Evaluate $\oint_C \frac{e^z}{z^3}dz$, where 'c' is the circle $|z|=1$ taken in counter-clockwise. $\longrightarrow \pi i$

12) Evaluate the integral $\oint_C \frac{z^2+1}{z^2-1}dz$, where $C: |z-1|=1 \longrightarrow 2\pi i$

13) Evaluate $\oint_C \frac{z^2+1}{z(2z-1)}dz$, where $C: |z|=1 \longrightarrow \frac{\pi i}{2}$

14) If $f(a) = \oint_C \frac{4z^2+z+5}{z-a}\,dz$ where $C: \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ taken in counter-clockwise; then find $F(3.5)$, $F(i)$, $f'(-1)$ and $f''(-i)$.

Sol:- The given integral is ~~f(a)=~~ $\oint_C \frac{4z^2+z+5}{z-a}\,dz$

and the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

(i) Now $F(3.5) = \oint_C \frac{4z^2+z+5}{z-3.5}\,dz$

Here $z = 3.5$ lies outside the ellipse

$\therefore f(z)$ is analytic everywhere within 'c' and hence by Cauchy's integral theorem. $\oint_C f(z)\,dz = 0$

$$\therefore f(3.5) = \oint_C \frac{4z^2+z+5}{z-3.5}\,dz = 0$$

(ii) $f(i) = \oint_C \frac{4z^2+z+5}{z-i}\,dz$

clearly $z = i$ lies inside the ~~circle~~ ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

By cauchy's integral formula $\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Here $f(z) = 4z^2+z+5$ $\quad$ $f(a)$

put $z = i \Rightarrow f(i) = \cancel{2\pi i}\left[4i^2+i+5\right] = -4+i+5 = 1+i$

$$\therefore \oint \frac{4z^2+z+5}{z-i}\,dz = 2\pi i(1+i) = 2\pi(i-1)$$

(iii) The numerator $f(z) = 4z^2+z+5$ ~~is~~ of the integrand is analytic everywhere in 'c' and $a = -1, -i$ all lie within 'c'

∴ By cauchy's integral theorem

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{4z^2+z+5}{z-a} dz \text{ which gives}$$

$$\oint_c \frac{4z^2+z+5}{z-a} dz = 2\pi i\, f(a) = 2\pi i\,[4z^2+z+5]_{\text{at } z=a}$$

$$f(a) = 2\pi i\,[4a^2+a+5]$$

$$f'(a) = 2\pi i\,[8a+1] \Rightarrow f'(-1) = 2\pi i(-8+1) = -14\pi i$$

$$f''(a) = 2\pi i\,[8] \Rightarrow f''(-i) = 16\pi i$$

15) Evaluate $\oint_c \frac{z^2-2z+1}{(z-i)^2} dz$, where 'c' is $|z|=2$.

16) Evaluate, $\oint_c \frac{e^{-z}}{(z-1)\,(z-2)^2} dz$, where 'c' is $|z|=3$

17) Evaluate, using cauchy's integral formula $\oint_c \frac{z}{z^2-3z+2} dz$ where 'c' is $|z-2| = \frac{1}{2}$

18) Evaluate $\oint_c \frac{\log z}{(z-3)^3} dz$, where 'c' is $|z-1| = \frac{1}{2}$

Complex power Series

Power Series:- An expression of the form $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ (or) in the summation form $\sum_{n=0}^{\infty} a_n (z-a)^n$ is known as power series of the variable z in powers of $(z-a)$ (or) about the point 'a'. The constants $a_0, a_1, a_2, \cdots$ are known as the co-efficients and 'a' is known as centre of the series.

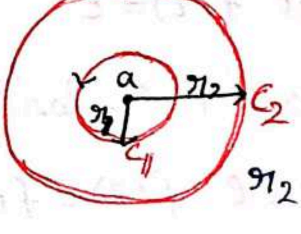
Taylor's Theorem:- If $f(z)$ is analytic at every point with-in a circle 'c' having centre at 'a' then for every point 'z' inside 'c' the function $f(z)$ can be expressed as a power series in positive power of $z=a$ (or) about a point $z=a$ and $|z-a|<r$.

i.e $f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \text{where } a_n = \frac{f^n(a)}{n!} = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}} dz$$

Laurent's theorem: If $f(z)$ is analytic in every point inside the ring shaped region bounded by two concentric circles c_1 & c_2 with centre at 'z=a' and respective radii r_1 and r_2 $(r_2 > r_1)$ then for every point z within 'R', the function $f(z)$ can be expressed as +ve and −ve powers of $(z-a)$ that is

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{principle part}}$$



$r_2 > r_1$

Some standard Expansions:- 1. $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

4. $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots \quad |z|<1$

5. $(1+z)^{-1} = 1-z+z^2-z^3+\cdots \quad |z|<1$

6. $(1-z)^{-2} = 1+2z+3z^2+\cdots \quad |z|<1$

7. $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

8. $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

Prob:- 1) obtain taylor's series expansion of e^{1+z} interms of $z-1$.

Sol:- Let $f(z) = e^{1+z}$

Here we expand $f(z)$ interms of $z-a = z-1$

$$f(z) = e^{z+1} = e^{(z-1)+1+1}$$

$$= e^2 \cdot e^{z-1}$$

$$= e^2\left[1+\frac{(z-1)}{1!}+\frac{(z-1)^2}{2!}+\cdots\right]$$

$$\boxed{e^{z+1} = e^2 \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}}$$

(or)

Let $f(z) = e^{z+1}$, $z = a = 1$

W.K.T Taylor's Series Expansion of $f(z)$ about $z=a$ is

i.e $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$

Now $f(z) = e^{z+1} \Rightarrow f(a) = f(1) = e^{1+1} = e^2$

$f'(z) = e.e^z \Rightarrow f'(a) = f'(1) = e^2$

$f''(z) = e e^z \Rightarrow f''(a) = f''(1) = e^2$

$\therefore f(z) = e^2 + (z-1)e^2 + \frac{(z-1)^2}{2!}e^2 + \cdots$

$f(z) = e^2\left(1+(z-1)+\frac{(z-1)^2}{2!}+\cdots\right)$

2) Find Taylor's series expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point $z=1$ (or) $z-1$.

Sol:- Given $f(z) = \frac{2z^3+1}{z^2+z}$ and $z=1$

Here $f(z)$ expand about $z=1$ (or) interms of $(z-1)$ powers.

Now $f(z) = \frac{2z^3+1}{z^2+z} = (2z-2) + \frac{2z+1}{z(z+1)}$

$$f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}$$

$$= 2(z-1) + \frac{1}{(z-1)+1} + \frac{1}{(z-1)+2}$$

$$= 2(z-1) + [1+(z-1)]^{-1} + \frac{1}{2\left[1+\frac{(z-1)}{2}\right]}$$

$$= 2(z-1) + [1-(z-1)+(z-1)^2 - \cdots] + \frac{1}{2}\left[1+\frac{(z-1)}{2}\right]^{-1}$$

$$= 1 + (z-1) + (z-1)^2 + \cdots + \frac{1}{2}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right]$$

$$f(z) = \frac{3}{2} + \frac{3(z-1)}{2} + \frac{9}{8}(z-1)^2 + \cdots$$

3) obtain the taylor's series expansion of $f(z) = \frac{e^z}{z(z+1)}$ about $z=2$.

Sol:- Given $f(z) = \frac{e^z}{z(z+1)}$ and $z=2 \Rightarrow z-2$

Here we expand $f(z)$ interms of $(z-2)$ powers

Now $f(z) = e^{z+2-2} \cdot \left[\frac{1}{z(z+1)}\right]$

$$= e^2 e^{z-2}\left[\frac{1}{z} - \frac{1}{z+1}\right]$$

$$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2!}+\cdots\right] \cdot \left[\frac{1}{(z-2)+2} - \frac{1}{(z-2)+3}\right]$$

$$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\frac{1}{2\left[1+\frac{(z-2)}{2}\right]} - \frac{1}{3\left[1+\frac{(z-2)}{3}\right]}\right]$$

$$f(z) = e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\frac{1}{2}\left(1+\frac{z-2}{2}\right)^{-1}-\frac{1}{3}\left[1+\left(\frac{z-2}{3}\right)\right]^{-1}\right]$$

$$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\frac{1}{2}\left(1-\frac{(z-2)}{2}+\frac{(z-2)^2}{4}-\cdots\right)\right.$$

$$\left.-\frac{1}{3}\left(1-\left(\frac{z-2}{3}\right)+\frac{(z-2)^2}{9}-\cdots\right)\right]$$

$$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\left(\frac{1}{2}-\frac{1}{3}\right)+(z-2)\left(-\frac{1}{4}+\frac{1}{9}\right)+(z-2)^2\left(\frac{1}{8}-\frac{1}{27}\right)+\cdots\right]$$

$$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\left(\frac{1}{6}\right)+(z-2)\left(\frac{-9+4}{36}\right)+(z-2)^2\left(\frac{27-8}{216}\right)+\cdots\right]$$

$$f(z) = e^2\left[1+(z-2)+\frac{(z-2)^2}{2}+\cdots\right]\left[\frac{1}{6}-\frac{5}{36}(z-2)+\frac{19}{216}(z-2)^2+\cdots\right]$$

4) Expand $\frac{1}{(z-1)(z-2)}$ about $z=0$

Sol: Let $f(z) = \frac{1}{(z-1)(z-2)}$ and $z=0$

Here we expand $f(z)$ in terms of 'z' powers.

i.e $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{(1-z)}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= \frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right] + \left(1+z+z^2+\cdots\right)$$

$$= -\frac{1}{2}+1+z\left(-\frac{1}{4}+1\right)+z^2\left(-\frac{1}{8}+1\right)+\cdots$$

$$f(z) = \frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\cdots$$

5) Expand $f(z) = \frac{1}{z^2-z-6}$ about (i) $z=-1$ (ii) $z=1$

Problems on Laurent's Series:-

1) Expand $\frac{1}{z^2-4z+3}$ in Laurent's Series for $1<|z|<3$

Sol:- Let $f(z) = \frac{1}{z^2-4z+3} = \frac{1}{(z-1)(z-3)}$

$$= \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] \longrightarrow (1)$$

Here we expand $f(z)$ in Laurent's series for $1<|z|<3$

$$\Rightarrow 1<|z| \; ; \; |z|<3$$

$$\frac{1}{|z|}<1 \; ; \; \frac{|z|}{3}<1$$

and we notice that both $|\frac{1}{z}|$ and $|\frac{z}{3}|$ are less than '1', Hence (1) gives an expansion

$$f(z) = \frac{1}{2}\frac{1}{-3(1-\frac{z}{3})} - \frac{1}{2}\frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right] + \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$f(z) = -\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right] + \frac{1}{2}\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

which is a Laurent's Series of $f(z)$ for $1<|z|<3$

Prob:- Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent Series.

Sol:- Given that $f(z) = \frac{e^{2z}}{(z-1)^3}$

Here we expand $f(z)$ in $(z-1)$ powers.

Now $f(z) = \frac{1}{(z-1)^3} e^{2z-2+2} = \frac{e^2}{(z-1)^3} \cdot e^{2(z-1)}$

$$= \frac{e^2}{(z-1)^3}\left[1+2(z-1)+\frac{4(z-1)^2}{2!}+\frac{8(z-1)^3}{3!}+\cdots\right]$$

$$f(z) = e^2\left[\frac{1}{(z-1)^3} + \frac{\cancel{z-1}}{(z-1)^{\cancel{3}\,2}} + 2\frac{(z-1)^2}{(z-1)^3} + \frac{4}{3}\frac{\cancel{(z-1)^3}}{\cancel{(z-1)^3}} + \frac{2}{3}\frac{(z-1)^{\cancel{4}\,1}}{\cancel{(z-1)^3}} + \cdots\right]$$

$$f(z) = e^2\left[\frac{4}{3} + \frac{2}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \frac{2}{3}(z-1) + \cdots\cdots\right]$$

If $|z-1| > 1$ then this series will be convergent.

3) Find the Laurent series expansion of the function f(z)

$f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

Sol:- Given that $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ and the region $3 < |z+2| < 5$

Here we expand $f(z)$ in $(z+2)$ powers.

Now $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \dfrac{A}{z-1} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$ —— (1)

$\Rightarrow \quad z^2-6z-1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$

put $z-1=0 \Rightarrow z=1$, we get $A(1-3)(1+2) = 1-6-1$

$-6A = -6 \Rightarrow \boxed{A=1}$

put $z-3=0 \Rightarrow z=3$, we get $B(3-1)(3+2) = 9-18-1$

$10B = -10 \Rightarrow \boxed{B=-1}$

put $z+2=0 \Rightarrow z=-2$, we get $C(-2-1)(-2-3) = 4+12-1$

$15C = 15 \Rightarrow \boxed{C=1}$

Subn these values in Eqn (1)

$f(z) = \dfrac{1}{z-1} + \dfrac{-1}{z-3} + \dfrac{1}{z+2}$ —— (2)

The given region is $3 < |z+2| < 5 \Rightarrow 3 < |z+2|$ and $|z+2| < 5$

$\dfrac{3}{|z+2|} < 1$ and $\dfrac{|z+2|}{5} < 1$

and we notice that both $\dfrac{3}{|z+2|}$ and $\dfrac{|z+2|}{5}$ are less than 1 Hence

Eqn (2) gives an expansion.

$$f(z) = \frac{1}{(z+2)-3} + \frac{1}{(z+2)-5} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)\left(1-\frac{3}{z+2}\right)} - \frac{1}{-5\left(1-\frac{z+2}{5}\right)} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left[1-\frac{3}{z+2}\right]^{-1} + \frac{1}{5}\left[1-\left(\frac{z+2}{5}\right)\right]^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left[1+\frac{3}{z+2}+\frac{9}{(z+2)^2}+\cdots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{25}+\cdots\right] + \frac{1}{z+2}$$

$$f(z) = \left[\frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{9}{(z+2)^3}+\cdots\right] + \frac{1}{5}\left[1+\frac{(z+2)}{5}+\frac{(z+2)^2}{25}+\cdots\right]$$

which is required Laurents Series in the region $3 < |z+1| < 5$

4) Expand Laurents Series of $\frac{z^2-1}{(z+2)(z+3)}$ for $|z| > 3$

Sol: Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ and $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

Here we expand $f(z)$ in z powers.

Now $f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$ — (1)

$$\Rightarrow z^2-1 = A(z+3) + B(z+2)$$

put $z+2=0 \Rightarrow z=-2$, we get $A(-2+3) = 4-1$

$$\boxed{A = 3}$$

put $z+3=0 \Rightarrow z=-3$, we get $B(-3+2) = 9-1 \Rightarrow -B = 8 \Rightarrow \boxed{B=-8}$

Sub.. these values in Eq^n (1)

$$f(z) = \frac{3}{z+2} - \frac{8}{z+3}.$$

$$= \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z) = \frac{3}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right] - \frac{8}{z}\left[1-\frac{3}{z}+\frac{9}{z^2}+\cdots\right]$$

5) Expand $f(z) = \frac{1}{z^2 - 3z + 2}$ in the region (i) $0 < |z-1| < 1$ (ii) $1 < |z| < 2$

Sol: Given that $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ —— (1)

Here $z = 1, 2$ are the singular points of $f(z)$

(i) The region is $0 < |z-1| < 1$

Here we expand $f(z)$ in $z-1$ powers.

from Eq^n (1)
$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$
$$= \frac{-1}{[1-(z-1)]} - \frac{1}{z-1}$$
$$= -[1-(z-1)]^{-1} - \frac{1}{z-1}$$
$$f(z) = -\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right] - \frac{1}{z-1}$$

(ii) The region is $1 < |z| < 2$
$$\Rightarrow 1 < |z| \ ; \ |z| < 2$$
$$\frac{1}{|z|} < 1 \ ; \ \frac{|z|}{2} < 1$$

Here we expand $f(z)$ in 'z' powers and we notice that both $\left|\frac{1}{z}\right|$ and $\left|\frac{z}{2}\right|$ are less-than '1' Hence Eq^n (1) gives an expansion
$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$
$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$
$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$
$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right] - \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right]$$

which is the required Laurent's Series in the region $1 < |z| < 2$

6) Find Laurent's Series of $\frac{7z-2}{(z+1)z(z-2)}$ in $1<|z+1|<3$

Sol:- Let $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ and the region $1<|z+1|<3$

Here we expand $f(z)$ in 'z+1' powers.

Now $f(z) = \frac{7z-2}{(z+1)z(z-2)} = \frac{A}{z+1} + \frac{B}{z} + \frac{C}{z-2} \longrightarrow (1)$

$\Rightarrow 7z-2 = Az(z-2) + B(z+1)(z-2) + C(z+1)z$

put $z+1=0 \Rightarrow z=-1$, we get $A(-1)(-1-2) = -7-2$

$$3A = -9 \Rightarrow \boxed{A=-3}$$

put $z=0$, we get $B(0+1)(0-2) = -2 \Rightarrow \boxed{B=1}$

put $z-2=0 \Rightarrow z=2$, we get $C(2+1)2 = 14-2 \Rightarrow 6C = 12 \Rightarrow \boxed{C=2}$

Sub these values in Eq^n (1), we get

$$f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} \quad \text{---} (2)$$

The given region is $1<|z+1|<3 \Rightarrow 1<|z+1|$ and $|z+1|<3$

$$\frac{1}{|z+1|}<1 \text{ and } \frac{|z+1|}{3}<1$$

and we notice that both $\frac{1}{|z+1|}$ and $\frac{|z+1|}{3}$ are Less-than '1', Hence Eq^n (2) gives an expansion

$$f(z) = \frac{-3}{z+1} + \frac{1}{(z+1)+1} + \frac{2}{(z+1)-3}$$

$$= \frac{-3}{z+1} + \frac{1}{(z+1)\left[1+\frac{1}{z+1}\right]} + \frac{2}{-3}\left[\frac{1}{\left(1-\frac{z+1}{3}\right)}\right]$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\left[1 - \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right]$$

$$f(z) = -\frac{2}{z+1} - \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\cdots - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right]$$

which is the required Laurent's Series expansion in the region $1<|z+1|<3$

7) obtain Laurents Series expansion for $f(z)=\frac{1}{(z+2)(z+1)^2}$ in

(i) $|z|<2$ (ii) $|1+z|>1$

Sol:- Given that $f(z)=\frac{1}{(z+2)(z+1)^2}$

Now $\frac{1}{(z+2)(z+1)}=\frac{1}{(z+1)}-\frac{1}{z+2}$

$$\frac{1}{(z+1)(z+2)(z+1)}=\frac{1}{(z+1)(z+1)}-\frac{1}{(z+1)(z+2)}$$

$$\frac{1}{(z+2)(z+1)^2}=\frac{1}{(z+1)^2}-\left[\frac{1}{z+1}-\frac{1}{z+2}\right]$$

$$\therefore f(z)=\frac{1}{(z+2)(z+1)^2}=\frac{1}{(z+1)^2}-\frac{1}{z+1}+\frac{1}{z+2} \quad \text{——} (1)$$

(i) $|z|<2 \Rightarrow \frac{|z|}{2}<1$

Here we expand $f(z)$ in 'z' powers.

from Eq^n (1) $f(z)=\frac{1}{(z+1)^2}-\frac{1}{(1+z)}+\frac{1}{2(1+\frac{z}{2})}$

$$=(1+z)^{-2}-(1+z)^{-1}+\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}$$

$$f(z)=(1-2z+3z^2-4z^3+\cdots)-[1-z+z^2-z^3+\cdots]$$
$$+\frac{1}{2}\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right]$$

(ii) $|1+z|>1 \Rightarrow \frac{1}{z+1}<1$

Here we expand $f(z)$ in 'z+1' powers.

from Eq^n (1) $f(z)=\frac{1}{(z+1)^2}-\frac{1}{1+z}+\frac{1}{(z+1)+1}$

$$=\frac{1}{(z+1)^2}-\frac{1}{z+1}+\frac{1}{(z+1)(1+\frac{1}{z+1})}$$

$$=\frac{1}{(z+1)^2}-\frac{1}{z+1}+\frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)\right]^{-1}$$

$$=\frac{1}{(z+1)^2}-\frac{1}{z+1}+\frac{1}{z+1}\left[1-\frac{1}{z+1}+\frac{1}{(z+1)^2}-\frac{1}{(z+1)^3}+\cdots\right]$$

$$f(z) = \cancel{\frac{1}{(z+1)^2}} - \cancel{\frac{1}{z+1}} + \cancel{\frac{1}{z+1}} - \cancel{\frac{1}{(z+1)^2}} + \frac{1}{(z+1)^3} - \frac{1}{(z+1)^4} + \cdots$$

$$f(z) = \frac{1}{(z+1)^3} - \frac{1}{(z+1)^4} + \cdots$$

which is required Laurent's series about $|1+z|>1$

8) Expand $f(z) = \frac{z}{z^2+1}$ for $|z-3|>2$

Sol:- Given that $f(z) = \frac{z}{z^2+1}$ and $|z-3|>2 \Rightarrow \frac{2}{|z-3|}<1$

Here we expand $f(z)$ in 'z-3' powers.

Now $$f(z) = \frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)} = \frac{1}{2}\left[\frac{1}{z+i} + \frac{1}{z-i}\right]$$

$$f(z) = \frac{1}{2}\frac{1}{z+i} + \frac{1}{2}\frac{1}{z-i}$$

$$= \frac{1}{2}\frac{1}{(z-3)+3+i} + \frac{1}{2}\frac{1}{(z-3)+3-i}$$

$$= \frac{1}{2}\frac{1}{(z-3)\left[1+\frac{(3+i)}{z-3}\right]} + \frac{1}{2}\frac{1}{(z-3)\left[1+\frac{3-i}{z-3}\right]}$$

$$= \frac{1}{2(z-3)}\left[1+\left(\frac{3+i}{z-3}\right)\right]^{-1} + \frac{1}{2(z-3)}\left[1+\left(\frac{3-i}{z-3}\right)\right]^{-1}$$

$$f(z) = \frac{1}{2(z-3)}\left[1 - \frac{3+i}{z-3} + \frac{(3+i)^2}{(z-3)^2} - \cdots\right] + \frac{1}{2(z-3)}\left[1 - \frac{3-i}{z-3} + \frac{(3-i)^2}{(z-3)^2} - \cdots\right]$$

which is the required Laurent's series expansion about $|z-3|>2$

9) Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $0<|z-1|<1$

10) Expand the following functions in Laurent's series

a) $f(z) = \frac{1}{z-z^2}$ for $1<|z+1|<2$

b) $f(z) = \frac{z}{(z-1)(z-3)}$ for $|z-1|<2$

Calculus of Residues

Introduction:- In complex analysis we observe the functions which are analytic in a given region. But there are several functions which are not analytic at certain points of its domain. In this unit we discuss such type of functions by using residue theorem and it is called cauchy's residue theorem, which is a powerful tool to evaluate line integrals of analytic functions over closed curves. and its applications are used to compute real integrals and infinite series as well.

Zero's of an analytic function:-

If a function $f(z)$ is analytic in a region 'R' and $f(z)=0$ at a point $z=a$ in R then the function $f(z)$ is said to have a 'zero' at the point '$z=a$'.

Order of zero:-

(i) If $f(a)=0$ and $f'(a)\neq 0$ then the function $f(z)$ is said to have a zero of 1^{st} order at $z=a$.

Eg:- $f(z)=z-3$, Here $z=3$ is a zero of order '1'

(ii) If $f(a)=0$, $f'(a)=0$ and $f''(a)\neq 0$ then the function $f(z)$ is said to have a zero of 2^{nd} order at '$z=a$'.

Eg:- $f(z)=(z-3)^2$, Here $z=3$ is a zero of order '2'

(iii) If $f(a)=0$, $f'(a)=0$, ..., $f^{n}(a)\neq 0$ then the function $f(z)$ is said to have a zero of n^{th} order. at $z=a$.

Eg: $f(z)=(z-4)^n$, Here $z=4$ is a zero of order 'n'.

★ To find zero's of $f(z)$ equate the numerator of $f(z)$ to zero.

Prob:- Find zero's of the function $f(z) = \frac{(z-1)(z-2)^4}{z}$

Sol:- Given $f(z) = \frac{(z-1)(z-2)^4}{z}$

To obtain zero's of $f(z)$, equate the numerator to zero.

i.e, $(z-1)(z-2)^4 = 0$

$z = 1, \; z = 2$

Hence $z=1$ is a zero of order '1' (or) Simple zero.

and $z=2$ is a zero of order '4'

Singularities of an analytic function:-

The point z_0 is called a singular point of f if

* f fails to be analytic at z_0
* but f is analytic at some point in every neighborhood of z_0

Eg:- If $f(z) = \frac{1}{(z-1)(z-2)}$ then $f(z)$ is not analytic at $z=1, 2$ as at these points $f(z) \to \infty$. Then $z=1$ & $z=2$ are called singular points.

Isolated and non-isolated singular points:-

If $z=a$ is a singular point of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z=a$, then $z=a$ is said to be an isolated singularity of the function $f(z)$, otherwise it is called non-isolated singular point.

Eg: Let $f(z) = \frac{z+1}{z(z-2)}$ clearly $z=0, z=2$ are the only singularities of $f(z)$. There is no other singularities of $f(z)$ in the neighborhood of $z=0, z=2$.

Hence $z=0$ and $z=2$ are the isolated singular points of $f(z)$

Types of singularities:- Let $f(z)$ be analytic within a domain D except at $z=z_0$ which is an isolated singular point. then by the expansion of $f(z)$ by Laurent's theorem.

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part of } f(z)} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\text{principal part of } f(z)}$$

(i) If all a_n's are zero ⇒ no term in principal part then $z = z_0$ is 'Removable singularity'.

(ii) If infinite number of terms in principal part then $z = z_0$ is 'isolated essential singularity.'

(iii) If finite number of terms in principal part then $z = z_0$ is called 'pole.'

Order of a pole :- If the principal part of $f(z)$ contains 'm' number of terms at $z = z_0$ then $z = z_0$ is called a pole of order 'm'.

* If $m = 1$ then the pole is called simple pole.

* To find poles of $f(z)$, equate the denominator to zero.

Eg: $f(z) = \frac{e^z}{(z-1)(z-3)^4}$ for finding poles of $f(z)$ equate the denominator to zero.

i.e $(z-1)(z-3)^4 = 0$

$z = 1, \ z = 3$

Here $z = 1$ is a simple pole and $z = 3$ is a pole of order '4.'

Residue :- The co-efficient of $\frac{1}{(z-a)}$ in the expansion of $f(z)$ about the isolated singularity $z = a$ is called the residue of $f(z)$ at that point.

i.e, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} \frac{1}{(z-a)^n} \longrightarrow (1)$$

from (1) the residue of $f(z)$ at $z = a$ is a_{-1} from Laurent Series. We know that the co-efficient a_{-1} is given by

$$a_{-1} = \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$\int_C f(z)\,dz = 2\pi i\, a_{-1} = 2\pi i \left(\text{Res}\, f(z)\right)_{\text{at } z=a}$$

where 'c' is a closed curve containing the point $z = a$ and $f(z)$ is analytic within and on a closed curve 'c'.

Formula for Residues at poles :-

1) **Residue at a simple pole:** If $f(z)$ has a simple pole (i.e pole of order 1) at $z=a$ then

$$\boxed{\operatorname*{Res}_{z=a}\{f(z)\} = \lim_{z\to a}(z-a)f(z)}$$

2) **Residue of $f(z)$ is of the form $\frac{P(z)}{q(z)}$:-**

If $f(z) = \frac{P(z)}{q(z)}$ form, $q(z_0) \neq 0$ then

$$\boxed{\operatorname*{Res}_{z=a} f(z) = \frac{P(z_0)}{q'(z_0)} \text{ (or) } \frac{P(a)}{q'(a)}}$$

3) **Residue of $f(z)$ at pole of order 'm' :-** If $f(z)$ has a pole of order 'm' at $z=a$ then

$$\boxed{\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m \cdot f(z)\right)\right]}$$

4) **Residue at infinity :-** If $f(z)$ has an isolated singularity at $z=\infty$ then

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{-1}{2\pi i}\int_C f(z)\,dz \text{ (or) } -\left[\text{co-eff of } \tfrac{1}{z} \text{ in the expansion of } f(z)\right]$$

(or)

$$\boxed{\operatorname*{Res}_{z=\infty} f(z) = \lim_{z\to\infty}\left[-z\,f(z)\right]}$$

Problems :-

1) Find zero's and poles of the function $\left[\frac{z+1}{z^2+1}\right]^2$

Sol :- Let $f(z) = \left[\frac{z+1}{z^2+1}\right]^2 = \frac{(z+1)^2}{(z^2+1)^2}$

→ To find zero's of the fun" $f(z)$ equate the numerator to zero.

i.e $(z+1)^2 = 0 \Rightarrow (z+1)(z+1) = 0$

$z = -1, -1.$

∴ $z=-1$ is the zero of $f(z)$ with order '2'

→ To find poles of $f(z)$, equate the denominator to 'zero'.

i.e $(z^2+1)^2 = 0 \Rightarrow (z^2+1)(z^2+1) = 0$

$z^2+1 = 0, \quad z^2+1 = 0$

$z^2 = -1, \quad z^2 = -1$

$z^2 = i^2, \quad z^2 = i^2$

$z = \pm i, \quad z = \pm i$

$z = i, i, -i, -i$

∴ $z = i$ is a pole of order '2' and $z=-i$ is a pole of order '2'

2) Find the poles of the function $f(z) = \frac{z^2}{(z-1)(z-2)^2}$ and residues at each pole.

Sol :- given $f(z) = \frac{z^2}{(z-1)(z-2)^2}$

To find poles of $f(z)$, equate the denominator to zero.

i.e $(z-1)(z-2)^2 = 0 \Rightarrow (z-1)(z-2)(z-2) = 0$

$z = 1, 2, 2$

Here $z=1$ is a simple pole and

$z=2$ is a pole of order '2'.

W.K.T The residue of $f(z)$ at a simple pole '$z=a$'

i.e $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z \to a} (z-a) f(z)$

At $z=1=a$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \cancel{(z-1)} \frac{z^2}{\cancel{(z-1)}(z-2)^2}$$

$$\boxed{\operatorname*{Res}_{z=1} f(z) = \frac{1}{(1-2)^2} = 1}$$

Now The Residue of $f(z)$ at $z=a$ is a pole of order 'm'.

i.e $$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

At $z=2=a$ and $m=2$

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-2)^2} \frac{z^2}{(z-1)\cancel{(z-2)^2}}\right]$$

$$= 1 \lim_{z\to 2} \frac{d}{dz}\left[\frac{z^2}{z-1}\right] \frac{u}{v} \quad \left(\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right)$$

$$= \lim_{z\to 2} \left[\frac{(z-1)(2z) - z^2(1)}{(z-1)^2}\right]$$

$$= \frac{(2-1)(2\times 2) - 2^2}{(2-1)^2} = \frac{4-4}{1}$$

$$\boxed{\operatorname*{Res}_{z=2} f(z) = 0}$$

3) Find the residue of $\frac{ze^z}{(z-1)^3}$ at its poles.

Sol:- Let $f(z) = \frac{z.e^z}{(z-1)^3}$

To obtain poles of $f(z)$, equate the denominator to zero.

i.e $(z-1)^3 = 0 \Rightarrow (z-1)(z-1)(z-1) = 0$

$z = 1, 1, 1$

Here $z=1$ is a pole of order '3' and

W.K.T the residue of $f(z)$ at $z=a$ is a pole of 'm'

i.e, $$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

At $z=1=a$ and $m=3$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(3-1)!} \lim_{z\to 1} \frac{d^{3-1}}{dz^{3-1}}\left[\cancel{(z-1)^3}\cdot \frac{z e^z}{\cancel{(z-1)^3}}\right]$$

$$= \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2}\left[\underbrace{z e^z}_{u\cdot v}\right] \qquad \left(\because \frac{d}{dx}(uv) = uv' + vu'\right)$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left[\frac{d}{dz}(\underbrace{z e^z}_{u\,v})\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left[\underbrace{z\cdot e^z + e^z\cdot 1}_{u\,v' + v.u'}\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \left[z\cdot e^z + e^z + e^z\right]$$

$$= \frac{1}{2}\left[1\cdot e^1 + e^1 + e^1\right]$$

$$\boxed{\operatorname*{Res}_{z=1} f(z) = \frac{3e}{2}}$$

4) Find poles and Residues of $f(z) = \tan z$.

Sol:- Given $f(z) = \tan z = \frac{\sin z}{\cos z}$

To find poles equate the denominator to zero

i.e. $\cos z = 0 \Rightarrow \cos z = \cos(2n+1)\pi/2$, $n = 0, \pm1, \pm2, \pm3, \pm4, \text{----}$

$z = (2n+1)\pi/2$, $n = 0, \pm1, \pm2, \pm3, \text{---}$

Here the poles of $f(z)$ are Simple poles and

W.K.T Residue of $f(z)$ when it is of the form $\frac{P(z)}{q(z)}$.

i.e $\operatorname*{Res}_{z=a} f(z) = \frac{P(a)}{q'(a)}$

At $z=(2n+1)\pi/2$

$$\operatorname*{Res}_{z=(2n+1)\pi/2} f(z) = \left(\frac{\cancel{\sin z}}{-\cancel{\sin z}}\right)_{\text{at } z=(2n+1)\pi/2} \qquad \left(\because f(z) = \frac{\sin z}{\cos z},\ P(z) = \sin z,\ q(z) = \cos z,\ q'(z) = -\sin z\right)$$

$$\boxed{\operatorname*{Res}_{z=(2n+1)\pi/2} f(z) = -1}$$

5) Find the residue of $f(z) = \frac{1-e^{2z}}{z^4}$ at its poles.

Sol:- Given $f(z) = \frac{1-e^{2z}}{z^4}$

Here $z^4=0 \Rightarrow z=0$ is a pole of order '4' and

W.K.T Residue of $f(z)$ at $z=a$ is a pole of order 'm'

i.e $\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$

At $z=0=a$ and $m=4$

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{(4-1)!} \lim_{z\to 0} \frac{d^{4-1}}{dz^{4-1}}\left[\cancel{(z-0)^4}\, \frac{1-e^{2z}}{\cancel{z^4}}\right]$$

$$= \frac{1}{3!} \lim_{z\to 0} \frac{d^3}{dz^3}\left[1-e^{2z}\right]$$

$$= \frac{1}{6} \lim_{z\to 0} \frac{d^2}{dz}\left[0-2e^{2z}\right]$$

$$= \frac{1}{6} \lim_{z\to 0} \frac{d}{dz}\left[-2\times 2e^{2z}\right]$$

$$= \frac{1}{6} \lim_{z\to 0} \left(-4\times 2\, e^{2z}\right)$$

$$\boxed{\underset{z=0}{\text{Res}}\, f(z) = \frac{-8}{6}\, e^0 = \frac{-4}{3}}$$

(or)

$f(z) = \frac{1-e^{2z}}{z^4}$, Here $z=0$ is the singular point of $f(z)$.

when expanding $f(z) = \frac{1}{z^4}\left(1-e^{2z}\right)$

$$= \frac{1}{z^4}\left[\cancel{1} - \left(\cancel{1} + \frac{2z}{1!} + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \cdots\right)\right]$$

$$= -\left[\frac{2z}{z^4} + \frac{2z^2}{z^4} + \frac{8}{6}\frac{z^3}{z^4} + \cdots\right]$$

$$f(z) = -\left[\frac{2}{z^3} + \frac{2}{z^2} + \frac{4}{3}\frac{1}{z} + \cdots\right]$$

Here the residue of $f(z)$ at $z=0$ is $-4/3$ (co-efficient of $\frac{1}{z}$)

6) Find the residue at $z=0$ of the function $\frac{1+e^z}{\sin z + z\cos z}$.

<u>Sol</u>:- Let $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

W.K.T Residue of $f(z)$, when it is of the form $\frac{P(z)}{q(z)}$

i.e $\underset{z=a}{\text{Res}}\, f(z) = \frac{P(a)}{q'(a)}$

<u>At $z=0$</u>:-

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{P(0)}{q'(0)}$$

Here $f(z) = \frac{1+e^z}{\sin z + z\cos z} = \frac{P(z)}{q(z)}$

$$P(z) = 1+e^z \implies P(0) = 1+e^0 = 2$$

$$q(z) = \sin z + \underset{u}{z}\,\underset{v}{\cos z}$$

$$q'(z) = \cos z + [z.(-\sin z) + \cos z.1]$$

$$q'(z) = \cos z - z\sin z + \cos z = -z\sin z + 2\cos z$$

$$q'(0) = 0 + 2\cos 0 = 2$$

$$\therefore \underset{z=0}{\text{Res}}\, f(z) = \frac{P(0)}{q'(0)} = \frac{\cancel{2}}{\cancel{2}} = 1$$

<u>Problems</u>:-

* Find the poles and the corresponding residues of the following functions.

a) $f(z) = \frac{z}{z^2+1}$ b) $f(z) = \frac{2z+1}{z^2-z-2}$

c) $f(z) = \frac{1+z}{(1-z)^2}$ d) $f(z) = \frac{z+1}{z^2(z-2)}$

e) $f(z) = \frac{z^2}{z^2+a^2}$ f) $f(z) = \frac{4-3z}{z^2-z}$

Prob:- Find the residue of $\frac{z^2}{(z-a)(z-b)(z-c)}$ at $z=\infty$

Sol:- Let $f(z) = \frac{z^2}{(z-a)(z-b)(z-c)}$

W.K.T residue of $f(z)$ at $z=\infty$

i.e $\underset{z=\infty}{\text{Res}}\, f(z) = \lim_{z\to\infty}\left[-z\, f(z)\right]$

$$= \lim_{z\to\infty} -z \frac{z^2}{(z-a)(z-b)(z-c)}$$

$$= \lim_{z\to\infty} \frac{-\cancel{z^3}}{\cancel{z}(1-\frac{a}{z})\cancel{z}(1-\frac{b}{z})\cancel{z}(1-\frac{c}{z})}$$

$$= \lim_{z\to\infty} \frac{-1}{(1-\frac{a}{z})(1-\frac{b}{z})(1-\frac{c}{z})}$$

$$= \frac{-1}{(1-\frac{a}{\infty})(1-\frac{b}{\infty})(1-\frac{c}{\infty})}$$

$$\boxed{\underset{z=\infty}{\text{Res}}\, f(z) = -1}$$

Prob:- find the residue of $\frac{z^3}{z^2-1}$ at $z=\infty$

Sol:- Let $f(z) = \frac{z^3}{z^2-1}$

when we expand this

$$f(z) = \frac{z^{\cancel{3}}}{\cancel{z^2}(1-\frac{1}{z^2})} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right)$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \cdots$$

$\therefore$ Residue at infinity is '1' [co-eff of $\frac{1}{z}$ in expansion of $f(z)$]

Prob:- Evaluate the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z=1, 2, 3$ and ∞.

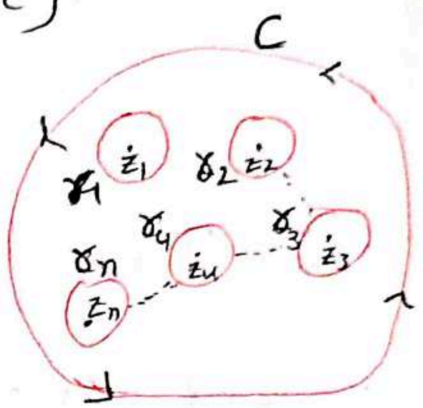
Cauchy's Residue Theorem:-

If $f(z)$ is an analytic with in and on a closed curve 'c' except at a finite number of poles $z_1, z_2, \ldots z_n$ with in 'c' and $R_1, R_2, \ldots R_n$ be the residues of $f(z)$ at these poles then

$$\int_C f(z)\,dz = 2\pi i \,(\text{sum of Residues at poles with in 'c'})$$

(or)

$$= 2\pi i\,(R_1 + R_2 + \cdots + R_n)$$



Proof: Let $\gamma_1, \gamma_2, \ldots \gamma_n$ be the circles with centers $z_1, z_2 \ldots z_n$ respectively and their radii so small that they lie entirely with in closed curve 'c' and do not over lap.

given that $f(z)$ is analytic with in the region enclosed by the curve 'c' between these circles.

By cauchy's theorem for multiple connected region, we have

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

But by the definition of Residue we have

$$\frac{1}{2\pi i}\int_{\gamma_j} f(z)dz = \operatorname{Res}\{f(z) : z = z_j\} = R_j \text{ for } j = 1, 2, \ldots n.$$

$$\boxed{\therefore \int_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \cdots + R_n)}$$

(or)

$$\boxed{\int_C f(z)dz = 2\pi i\,(\text{Sum of the Residues at poles within 'c'})}$$

Problems :-

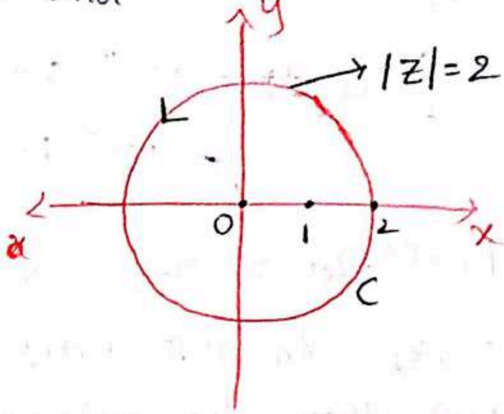
1). Evaluate $\int_C \frac{4-3z}{z^2-z}\,dz$ over the circle $|z|=2$

Sol:- The given integral is $\int_C \frac{4-3z}{z^2-z}\,dz$ and

the circle is $|z|=2$

$|x+iy|=2$

$x^2+y^2=2^2$



Let $f(z)=\frac{4-3z}{z^2-z}=\frac{4-3z}{z(z-1)}$

To find poles of $f(z)$, equate the denominator to zero

i.e $z(z-1)=0$

$z=0,\ z=1$

$\left(\because |0|=|0+i(0)|=0<2\right.$

$\left.|1|=|1+i(0)|=1<2\right)$

Here $z=0$ & $z=1$ are Simple poles and which are inside the circle $|z|=2$

W.K.T Residue of $f(z)$ at a Simple pole $z=a$.

i.e $\underset{z=a}{\operatorname{Res}} f(z)=\lim_{z\to a}(z-a)f(z)$

At $z=0$:-

$$\underset{z=0}{\operatorname{Res}} f(z)=\lim_{z\to 0}(\cancel{z}-0)\cdot\frac{4-3z}{\cancel{z}(z-1)}$$

$$=\frac{4-3(0)}{0-1}$$

$$\underset{z=0}{\operatorname{Res}} f(z)=\frac{4}{-1}=-4=R_1$$

At $z=1$

$$\underset{z=1}{\operatorname{Res}} f(z)=\lim_{z\to 1}\cancel{(z-1)}\frac{4-3z}{z\cancel{(z-1)}}=\frac{4-3}{1}=1=R_2$$

$\therefore$ By cauchy's Residue theorem $\int_C f(z)dz=2\pi i$ (Sum of the Residues)

$$\boxed{\int_C \frac{4-3z}{z^2-z}\,dz=2\pi i(-4+1)=-6\pi i}$$

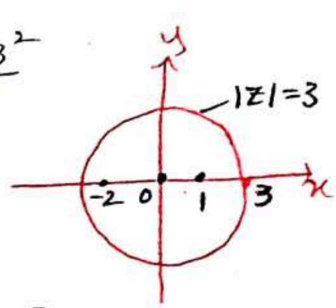
2) Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residues at each pole. Hence evaluate $\int_c \frac{z^2}{(z-1)^2(z+2)} dz$ where $c: |z| = 3$

Sol:- Given $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Then $(z-1)^2(z+2) = 0 \Rightarrow z = 1;\ z = -2$

Here $z=1$ ~~& z/72~~ is a pole of order '2' and $z=-2$ is a pole of order one (or) Simple pole.

Now. observe $|z| = 3 \Rightarrow |x+iy| = 3 \Rightarrow x^2+y^2 = \underline{3^2}$



when $z=1 \Rightarrow |z| = |1| = |1+i(0)| = \sqrt{1^2} = 1 < 3^2$

and $z=-2 \Rightarrow |z| = |-2| = |-2+i(0)| = \sqrt{(-2)^2} = 2 < 3$

$\therefore$ $z=1$ and $z=-2$ are lies inside the circle $|z|=3$.

W.K.T Residue of $f(z)$ at a Simple pole and pole of order 'm'

i.e $\underset{z=a}{\operatorname{Res}} f(z) = \lim_{z\to a}(z-a) f(z)$ and $\underset{z=a}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$

At $z=1=a$ & $m=2$

$$\underset{z=a}{\operatorname{Res}} f(z) = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]$$

$$= \frac{1}{1} \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{z+2}\right] \frac{u}{v}$$

$$= \lim_{z\to 1} \frac{(z+2)(2z) - z^2(1)}{(z+2)^2}$$

$$\underset{z=a}{\operatorname{Res}} f(z) = \frac{(1+2)2(1) - 1^2(1)}{(1+2)^2} = \frac{6-1}{9} = \frac{5}{9} = R_1$$

At $z=-2$ $\underset{z=-2}{\operatorname{Res}} f(z) = \lim_{z\to -2} [z-(-2)] \frac{z^2}{(z-1)^2\cancel{(z+2)}} = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9} = R_2$

$\therefore$ By cauchy's Residue theorem $\int_c f(z) dz = 2\pi i (R_1 + R_2)$

$$\boxed{\int_c \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i\left(\frac{5}{9} + \frac{4}{9}\right) = 2\pi i}$$

3) Evaluate $\int_C \frac{dz}{(z^2+4)^2}$, where $C: |z-i| = 2$

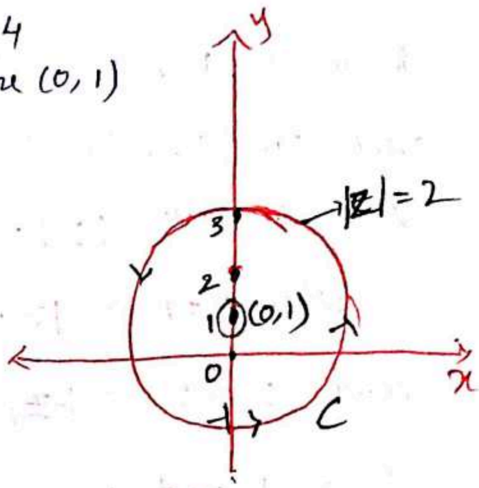
Sol:- Given that $\int_C \frac{dz}{(z^2+4)^2}$ and $|z-i| = 2$

$$|x+iy-i| = 2$$

$$|x+i(y-1)| = 2$$

$$x^2+(y-1)^2 = 4$$

it is a circle with centre $(0,1)$



Let $f(z) = \frac{1}{(z^2+4)^2}$

then $(z^2+4)^2 = 0 \Rightarrow (z^2+4)(z^2+4) = 0$

$$z^2 = -4 \;,$$

$$z^2 = i^2 2^2 \;,$$

$$z = \pm 2i \;, \quad z = \pm 2i$$

$$z = 2i, 2i, -2i, -2i$$

Here $z = 2i$ & $z = -2i$ are poles of order '2'

given circle is $|z-i| = 2$

Now put $z = 2i \Rightarrow |z-i| = |2i-i| = |i| = |0+i(1)| = \sqrt{1^2} = 1 < 2$

put $z = -2i \Rightarrow |z-i| = |-2i-i| = |-3i| = \sqrt{(-3)^2} = 3 > 2$

$\therefore$ $z = 2i$ lies inside the circle $|z-i| = 2$ and

$z = -2i$ lies outside the circle $|z-i| = 2$.

W.K.T the residue of $f(z)$ at a pole of order 'm'

i.e $\operatorname{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$

At $z = 2i$: $\operatorname{Res}_{z=2i} f(z) = \frac{1}{(2-1)!} \lim_{z \to 2i} \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-2i)^2} \frac{1}{\cancel{(z-2i)^2}(z+2i)^2} \right]$

$$= 1 \lim_{z \to 2i} \frac{d}{dz} \left(\frac{1}{(z+2i)^2} \right) = \lim_{z \to 2i} \frac{-2}{(z+2i)^3} = \frac{-2}{(2i+2i)^3}$$

$$\operatorname{Res}_{z=2i} f(z) = \frac{-2}{(4i)^3} = \frac{-2}{-64i} = \frac{1}{32i}$$

$\therefore$ By Residue theorem $\boxed{\int_C \frac{dz}{(z^2+4)^2} = \cancel{2}\pi \cancel{i} \left(\frac{1}{\cancel{32}_{16} \cancel{i}} \right) = \frac{\pi}{16}}$

4) Evaluate $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)}\,dz$, where 'c' is the circle $|z|=3$

Sol: Let $f(z) = \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)}$, $|z| = 3$

$|x+iy| = 3 \Rightarrow x^2+y^2=9$

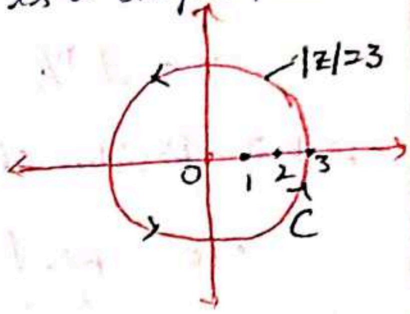
then $(z-1)^2(z-2) = 0 \Rightarrow (z-1)^2=0,\ z-2=0$

$z=1,\ z=2$

Here $z=1$ is a pole of order '2' and $z=2$ is a simple pole.

Now put $z=1$ in $|z| = |1| = \sqrt{1^2} = 1 < 3$

put $z=2 \Rightarrow |z| = |2| = \sqrt{2^2} = 2 < 3$



$\therefore$ $z=1$ & $z=2$ are inside the circle $|z|=3$

W.K.T Residue of $f(z)$ at a simple pole

i.e $\operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z)$

At $z=2$: $\operatorname{Res}_{z=2} f(z) = \lim_{z\to 2} \cancel{(z-2)} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2\cancel{(z-2)}}$ $\left(\begin{array}{l}\sin n\pi = 0\\ \cos n\pi = (-1)^n\end{array}\right)$

$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)^2} = \frac{0+1}{1} = 1 = R_1$

W.K.T Residue of $f(z)$ at a pole of order 'm'

i.e $\operatorname{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$

At $z=1$ and $m=2$:- $\operatorname{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-1)^2}\frac{\sin(\pi z^2)+\cos(\pi z^2)}{\cancel{(z-1)^2}(z-2)}\right]$

$= \frac{1}{1}\lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\right]$

$= \lim_{z\to 1}\left\{\frac{(z-2)\left[\cos(\pi z^2)\cdot 2\pi z - \sin(\pi z^2)\,2\pi z\right] - 1\left[\sin\pi z^2 + \cos\pi z^2\right]}{(z-2)^2}\right\}$

$= \frac{(1-2)\left[(\cos\pi)(2\pi) - (\sin\pi)(2\pi)\right] - \left[\sin\pi + \cos\pi\right]}{(1-2)^2}$

$\operatorname{Res}_{z=1} f(z) = \frac{-1(-2\pi)-(-1)}{1} = 2\pi+1 = R_2$

$\therefore$ By cauchy's Residue theorem $\int_C f(z)\,dz = 2\pi i\,(R_1+R_2)$

$$\boxed{\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz = 2\pi i\,(1+2\pi+1) = 4\pi i(\pi+1)}$$

5) Evaluate $\int \tan z \, dz$ where 'c' is the circle $|z|=2$.

Sol:- Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ and $|z|=2$

then $\cos z = 0 \Rightarrow \cos z = \cos(2n+1)\pi/2$, $n = 0, \pm1, \pm2, \pm3, \cdots$

$$z = (2n+1)\pi/2, \quad n = 0, \pm1, \pm2, \pm3, \cdots$$

Here the poles are simple poles and out of these poles

$$z = \pm \pi/2 \text{ lies inside the circle } |z|=2$$

W.K.T Residue of $f(z)$, when it is of the form $\dfrac{P(z)}{Q(z)}$

i.e $\underset{z=a}{\text{Res}}\, f(z) = \dfrac{P(a)}{Q'(a)}$

At $z = \pi/2$:- $\underset{z=\pi/2}{\text{Res}}\, f(z) = \left(\dfrac{\cancel{\sin} z}{-\cancel{\sin} z}\right)_{z=\pi/2} = -1 = R_1$ $\left(f(z) = \dfrac{\sin z}{\cos z} \right.$

$P(z) = \sin z$

At $z = -\pi/2$:- $\underset{z=-\pi/2}{\text{Res}}\, f(z) = \left(\dfrac{\cancel{\sin} z}{-\cancel{\sin} z}\right)_{z=-\pi/2} = -1 = R_2$ $\quad q(z) = \cos z$

$\left. q'(z) = -\sin z \right)$

$\therefore$ By Cauchy's Residue theorem $\boxed{\int_C \tan z \, dz = 2\pi i(-1-1) = -4\pi i}$

6) Evaluate $\int_C e^{1/z^2} dz$ when 'c' is the positive oriented circle $|z|=1$

Sol:- Let $f(z) = e^{1/z^2}$

Here $1/z^2$ is analytic everywhere except at $z=0$, so $z=0$ is an isolated singular point and which lies inside the circle $|z|=1$

Now the Laurent series expansion of $f(z)$ is

$$f(z) = e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots \quad (0 < |z| < \infty)$$

$\therefore$ The residue of $f(z)$ at $z=0$ is 'zero' (co-efficient of $1/z$)

Hence $\boxed{\int_C e^{1/z^2} dz = 0}$

7) Evaluate $\int_{|z|=1/2} \frac{dz}{(z-1)(z+2)^2}$ using residue theorem.

8) Evaluate $\int_C \frac{2z-3}{z^3+3z^2}\,dz$ where 'c' is $|z|=4$ traversed counterclockwise use Residue theorem.

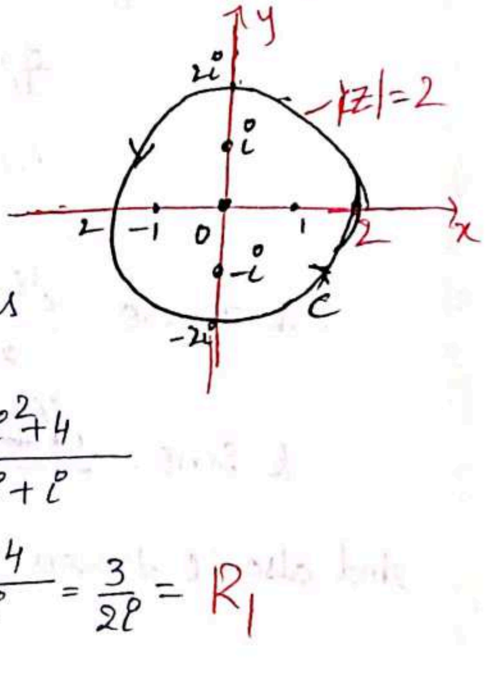
9) Evaluate $\oint_C \frac{dz}{z^2(z+4)}$ where C is $|z+2|=3$

10) Evaluate $\int_C \frac{e^z}{(z-1)(z+2)}\,dz$, where $C: |z-1|=1$

11) Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$, where 'c' is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

12) Show that $\oint_C \frac{z^2+4}{(z-i)(z+i)}\,dz = 0$ where 'c' is the circle $|z|=2$ described in the positive directions.



Sol:- Let $f(z) = \frac{z^2+4}{(z-i)(z+i)}$ and $|z|=2$

then $(z-i)(z+i)=0 \Rightarrow z = i, -i$

Here $z = i, -i$ are simple poles and which lies inside the circle $|z|=2$

At $z=i$:- $\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} \cancel{(z-i)} \frac{z^2+4}{\cancel{(z-i)}(z+i)} = \frac{i^2+4}{i+i}$

$$\operatorname{Res}_{z=i} f(z) = \frac{-1+4}{2i} = \frac{3}{2i} = R_1$$

At $z=-i$:

$$\operatorname{Res}_{z=-i} f(z) = \lim_{z\to -i} \cancel{[z-(-i)]} \frac{z^2+4}{(z-i)\cancel{(z+i)}} = \frac{(-i)^2+4}{(-i-i)} = \frac{-1+4}{-2i} = \frac{-3}{2i} = R_2$$

$\therefore$ By Cauchy's Residue theorem $\int_C f(z)\,dz = 2\pi i(R_1+R_2)$

$$\boxed{\int_C \frac{(z^2+4)}{(z-i)(z+i)}\,dz = 2\pi i\left(\frac{3}{2i} - \frac{3}{2i}\right) = 0}$$

Evaluation of Real definite integral by contour integration:-

contour integration:-

The process of evaluating a definite integral by making the path of integration about a suitable contour (or) curve in the complex plane is called contour integration.

Type-I: Integration around a unit circle:-

An integral of the type $I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

where F is a real, rational function of $\sin\theta$ and $\cos\theta$ which is finite on $0 \le \theta \le 2\pi$

Here, we convert the given integral into a contour integral.

i.e we transform $F(\cos\theta, \sin\theta)$ into a complex function by substituting

$$e^{i\theta} = z \quad \& \quad \frac{1}{z} = e^{-i\theta}$$

Differentiating on both sides w.r.to 'θ' we get

$$\frac{d(e^{i\theta})}{d\theta} = \frac{d(z)}{d\theta}$$

$$i.e^{i\theta} = \frac{dz}{d\theta} \Rightarrow d\theta = \frac{1}{ie^{i\theta}}dz$$

$$d\theta = \frac{1}{i.z}dz$$

w.k.T $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+\frac{1}{z}}{2} = \dfrac{z^2+1}{2z}$

& $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z-\frac{1}{z}}{2i} = \dfrac{z^2-1}{2iz}$

And also 'θ' travels on the entire unit circle $|z| = |e^{i\theta}|$

$$= |\cos\theta + i\sin\theta|$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$\therefore$ The integral $I = \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$. $\quad |z| = 1$

is converted into $I = \int_C f(z)dz$

Hence By cauchy's residue theorem $\int_C f(z)dz = 2\pi i$[Sum of the residues of $f(z)$ at its poles inside the unit circle $|z|=1$]

Problem:

1) Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}$, $(a>b>0)$

Sol:- Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a>b>0$

Now, we convert the given integral into contour integration by substituting $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{iz}$

and $\cos\theta = \frac{z^2+1}{2z}$

substitute these values in the above integral, we get

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz} \quad , \; C: |z|=1$$

$$= \int_C \frac{1}{\frac{2az+bz^2+b}{2z}} \frac{dz}{iz}$$

$$= \int_C \frac{2\cancel{z}}{bz^2+2az+b} \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i}\int_C \frac{1}{b(z^2+\frac{2az}{b}+1)} dz$$

$$I = \frac{2}{ib}\int_C \frac{1}{z^2+\frac{2az}{b}+1} dz$$

It is of the form $\int_C f(z)\,dz$

where $f(z) = \frac{1}{z^2+\frac{2az}{b}+1}$

To find poles of $f(z)$ equate the denominator to zero.

i.e $z^2+\frac{2az}{b}+1=0$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\left(\frac{2a}{b}\right)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1}{2}\left[\frac{-2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}\right] = \frac{1}{2}\left[\frac{-2a}{b} \pm \frac{2\sqrt{a^2-b^2}}{b}\right] = \frac{-a}{b} \pm \frac{\sqrt{a^2-b^2}}{b}$$

$$z = \frac{-a}{b} + \frac{\sqrt{a^2-b^2}}{b}, \quad z = \frac{-a}{b} - \frac{\sqrt{a^2-b^2}}{b}$$

$$z = \alpha \qquad\qquad z = \beta$$

The given condition is $a>b>0$, So that $z=\alpha$ is a simple pole which is inside the circle $|z|=1$

and $z=\beta$ is a simple pole it lies outside the circle $|z|=1$

W.K.T The residue of $f(z)$ at a simple pole

i.e $\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$

At $z=\alpha$: $\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha) \cdot \dfrac{1}{z^2+\frac{2az}{b}+1}$

$$= \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}$$

$$\operatorname*{Res}_{z=\alpha} f(z) = \frac{1}{\alpha-\beta} = R_1$$

$$\alpha-\beta = \frac{-a}{b} + \frac{\sqrt{a^2-b^2}}{b} + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} = \frac{2\sqrt{a^2-b^2}}{b}$$

$$\operatorname*{Res}_{z=\alpha} f(z) = \frac{1}{\alpha-\beta} = \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}} = R_1$$

∴ By cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i\,(R_1)$$

$$= \cancel{2}\pi i \frac{b}{\cancel{2}\sqrt{a^2-b^2}}$$

$$\int_c \frac{1}{\left(z^2+\frac{2az}{b}+1\right)} dz = \frac{\pi i b}{\sqrt{a^2-b^2}}$$

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{ib} \int_c \frac{1}{z^2+\frac{2az}{b}+1} dz$$

$$= \frac{2}{\cancel{i}\cancel{b}} \frac{\pi \cancel{i} \cancel{b}}{\sqrt{a^2-b^2}}$$

$$\boxed{\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}}$$

2) Show that $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{2\pi a^2}{1-a^2}, (a^2<1)$ using residue theorem

Sol:- Given integral is $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = R.P \text{ of } \int_0^{2\pi} \frac{e^{i2\theta}}{1-2a\cos\theta+a^2}\,d\theta$

put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z^2+1}{2z}$

$e^{i2\theta} = z^2$

$e^{i\theta} \quad \cos 2\theta + i\sin 2\theta = x+iy$

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = R.P \text{ of } \int_C \frac{z^{\cancel{2}}}{1-\cancel{2}a\left(\frac{z^2+1}{\cancel{2}z}\right)+a^2}\frac{dz}{i\cancel{z}}$$

$$= R.P \text{ of } \int_C \frac{z}{\frac{z - az^2 - a + a^2 z}{z}} \frac{dz}{i}$$

$$= \frac{1}{i} \text{ Re.P of} \left[\frac{1}{i}\int_C \frac{z^2}{(-az^2+(1+a^2)z-a)}\,dz\right]$$

It is in the form $\int_C f(z)\,dz$

where $f(z) = \frac{z^2}{-az^2+(1+a^2)z-a} = \frac{z^2}{-a\left(z^2 - \left(\frac{1+a^2}{a}\right)z+1\right)}$

To find poles of $f(z)$, $-az^2+(1+a^2)z-a=0$

$$-a\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)=0$$

$$z^2-\frac{(1+a^2)}{a}z+1=0$$

$$z = \frac{\left(\frac{a^2+1}{a}\right) \pm \sqrt{\left(\frac{a^2+1}{a}\right)^2-4(1)(1)}}{2(1)}$$

$$= \frac{1}{2}\left[\frac{(a^2+1)}{a} \pm \sqrt{\frac{a^4+1+2a^2}{a^2}-4}\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{a}+\frac{1}{a} \pm \sqrt{\frac{a^4+1+2a^2-4a^2}{a^2}}\right]$$

$$= \frac{1}{2}\left[\left(a+\frac{1}{a}\right) \pm \sqrt{\frac{a^4-2a^2+1}{a^2}}\right]$$

$$z = \frac{1}{2}\left[\left(a+\frac{1}{a}\right) \pm \frac{a^2-1}{a}\right]$$

$z = \frac{1}{2}\left[a+\cancel{\frac{1}{a}}+a-\cancel{\frac{1}{a}}\right]$ and $z = \frac{1}{2}\left[\cancel{a}+\frac{1}{a}-\cancel{a}+\frac{1}{a}\right]$

$z = a$ and $z = \frac{1}{a}$

Since $a^2<1$, then $z=a$ is a simple pole lies inside the circle $|z|=1$

and $z=\frac{1}{a}$ is lies outside the circle $|z|=1$.

W.K.T the Residue of $f(z)$ at a simple pole.

i.e $\underset{z=a}{\text{Res}} f(z) = \lim_{z\to a} (z-a) f(z)$

At $z=a$:

$$\underset{z=a}{\text{Res}} f(z) = \lim_{z\to a} (z-a) \frac{z^2}{-a\left(z^2-\left(\frac{1+a^2}{a}\right)z+1\right)}$$

$$= \lim_{z\to a} \cancel{(z-a)} \frac{z^2}{-a\,\cancel{(z-a)}\left(z-\frac{1}{a}\right)}$$

$$\underset{z=a}{\text{Res}} f(z) = \frac{a^2}{-a\left(a-\frac{1}{a}\right)} = \frac{-a}{\frac{(a^2-1)}{a}} = \frac{a^2}{1-a^2} = R_1$$

$\therefore$ By cauchy's Residue theorem.

$$\int_C f(z)\,dz = 2\pi i\,(R_1)$$

$$\int_C \frac{z^2}{-az^2+(a^2+1)z-a}\,dz = 2\pi i\left(\frac{a^2}{1-a^2}\right)$$

Hence $$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \text{R.P of}\left[\frac{1}{i}\int_C \frac{z^2}{-az^2+(a^2+1)z-a}\,dz\right]$$

$$= \text{R.P of}\left[\frac{1}{i}\,2\pi i\left(\frac{a^2}{1-a^2}\right)\right]$$

$$\boxed{\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{2\pi a^2}{1-a^2}}$$

3) S.T $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}$

4) S.T $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{\pi}{\sqrt{5}}$

5) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ using residue theorem.

6) S.T $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$

Type - III :- Evaluation of real improper integrals $\int_{-\infty}^{\infty} e^{iax} f(x)\, dx$ by using residues:-

→ The integrals $\int_{-\infty}^{\infty} f(x) \cos ax\, dx$ (or) $\int_{-\infty}^{\infty} f(x) \sin ax\, dx$ can be evaluated by integrating $\int_{-\infty}^{\infty} e^{iax} f(x)\, dx$

around the contour 'c' discussed in type-II.

By residue theorem and similar analysis as in type-II

i.e $\int_{-\infty}^{\infty} f(x)\, e^{iax} dx = 2\pi i \sum Res(f(z) \cdot e^{iaz})$

where the summation extends to all poles of $f(z)\, e^{iaz}$ in the upper half-plane and finally equating the real and imaginary parts of e^{iaz}, we get

$\int_{-\infty}^{\infty} f(x) \cos ax\, dx$ and $\int_{-\infty}^{\infty} f(x) \sin ax\, dx$.

Prob:- Evaluate $\int_{-\infty}^{\infty} \frac{x \sin mx}{1+x^4} dx$

Sol:- $\int_{-\infty}^{\infty} \frac{x}{1+x^4} \sin mx\, dx$.

it is of the form $\int_{-\infty}^{\infty} f(x) \cdot \sin ax\, dx$. $\left(e^{imx} = \cos mx + i \sin mx\right)$

so we can write

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4} \sin mx\, dx = \int_{-\infty}^{\infty} \frac{x}{1+x^4} e^{imx}\, dx$$

Now $\int_{-\infty}^{\infty} \frac{x}{1+x^4} e^{imx} dx = \int_C \frac{z\, e^{imz}}{1+z^4} dz$

it is in the form $\int_C f(z)\, dz$

where $f(z) = \dfrac{z e^{imz}}{z^4+1}$

For finding poles of $f(z)$ equate the denominator to zero

i.e $z^4+1=0 \Rightarrow z^4=-1$

$$z=(-1)^{1/4} = (e^{i\pi})^{1/4} = \left[e^{i\pi(2n+1)}\right]^{1/4}, n=0,1,2,3$$

$$z = e^{\frac{i\pi(2n+1)}{4}}, \quad n=0,1,2,3$$

Here we obtain four roots of 'z' by substituting $n=0,1,2,3$

i.e $n=0 \Rightarrow z_1 = e^{i\pi/4} = \cos\pi/4 + i\sin\pi/4 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$n=1 \Rightarrow z_2 = e^{i3\pi/4} = \cos 3\pi/4 + i\sin 3\pi/4 = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$n=2 \Rightarrow z_3 = e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin 5\pi/4 = \frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$n=3 \Rightarrow z_4 = e^{i7\pi/4} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$\therefore z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ are simple poles and

$z = e^{i\pi/4}, e^{i3\pi/4}$ are lies in the upper-half of the circle

w.k.T the residue of $f(z)$ at a simple pole.

i.e $\underset{z=a}{\text{Res}} f(z) = \lim_{z\to a} (z-a) f(z)$ (or) $\underset{z=a}{\text{Res}} f(z) = \frac{P(a)}{q'(a)}$, when $f(z) = \frac{P(z)}{q(z)}$

At $z = e^{i\pi/4} \Rightarrow \underset{z=a}{\text{Res}} f(z) = \left[\frac{P(z)}{q'(z)}\right]_{at\, z=a}$ $\left[\because f(z) = \frac{z e^{imz}}{z^4+1} = \frac{P(z)}{q(z)}\right]$

$$\underset{z=e^{i\pi/4}}{\text{Res}} f(z) = \left[\frac{\cancel{z} e^{imz}}{4z^{\cancel{3}2}}\right]_{z=e^{i\pi/4}} = \frac{e^{im(e^{i\pi/4})}}{4(e^{i\pi/4})^2} = \frac{e^{im\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4\, e^{i\pi/2}} = \frac{e^{\frac{im}{\sqrt{2}}(1+i)}}{4i}$$

$\left(e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i(1) = i\right)$ $= R_1$

At $z = e^{\frac{3i\pi}{4}}$

$$\operatorname*{Res}_{z=e^{\frac{i\pi 3}{4}}} f(z) = \left[\frac{P(z)}{q'(z)}\right]_{z=e^{i3\pi/4}} = \left[\frac{\cancel{z}\, e^{imz}}{4z^{\cancel{3}2}}\right]_{z=e^{\frac{i3\pi}{4}}}$$

$$= \frac{e^{im(e^{i3\pi/4})}}{4\left(e^{i3\pi/4}\right)^2}$$

$$= \frac{e^{im\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4\,e^{i3\pi/2}} \quad \left(e^{\frac{i3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1),\ e^{\frac{i3\pi}{2}} = -i\right)$$

$$\operatorname*{Res}_{z=e^{\frac{i3\pi}{4}}} f(z) = \frac{e^{\frac{im}{\sqrt{2}}(-1+i)}}{4(-i)} = R_2$$

By cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i \sum \operatorname{Res} f(z)$$

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i\,(R_1 + R_2)$$

Thus $R \to \infty \Rightarrow \int_{C_R} f(z)dz = 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(\cancel{z})d\cancel{z} = 2\pi i\left[\frac{1}{4i}\, e^{\frac{im}{\sqrt{2}}(1+i)} + \frac{1}{-4i}\, e^{\frac{im}{\sqrt{2}}(-1+i)}\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x e^{imx}}{x^4+1}dx = \frac{2\pi \cancel{i}}{\cancel{4}_2 \cancel{i}}\left[e^{\frac{im}{\sqrt{2}} + \frac{i^2 m}{\sqrt{2}}} - e^{\frac{-im}{\sqrt{2}} + \frac{i^2 m}{\sqrt{2}}}\right]$$

$$= \frac{\pi}{2}\left[e^{\frac{im}{\sqrt{2}}} \cdot e^{\frac{-m}{\sqrt{2}}} - e^{\frac{-im}{\sqrt{2}}} \cdot e^{-\frac{m}{\sqrt{2}}}\right]$$

$$= \frac{\pi}{2} e^{-m/\sqrt{2}}\left[\cancel{\cos\frac{m}{\sqrt{2}}} + i\sin\frac{m}{\sqrt{2}} - \cancel{\cos\frac{m}{\sqrt{2}}} + i\sin\frac{m}{\sqrt{2}}\right]$$

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1} e^{imx}\, dx = \frac{\pi}{\cancel{2}} e^{-m/\sqrt{2}} \left[\cancel{2} i \sin\frac{m}{\sqrt{2}}\right].$$

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1} (\cos mx + i \sin mx)\, dx = i\left(\pi e^{-m/\sqrt{2}} \sin\frac{m}{\sqrt{2}}\right)$$

Compare real and imaginary parts on both sides

$$\boxed{\int_{-\infty}^{\infty} \frac{x}{x^4+1} \sin mx\, dx = \pi e^{-m/\sqrt{2}} \sin\frac{m}{\sqrt{2}}}$$

Problem: 1) Evaluate $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \frac{\pi}{2} e^{-a}$

2) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a^2-b^2}\left[\frac{e^{-bm}}{b} - \frac{e^{-am}}{a}\right]$

★ Prove That $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}$

Sol:- Given integral is $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx$

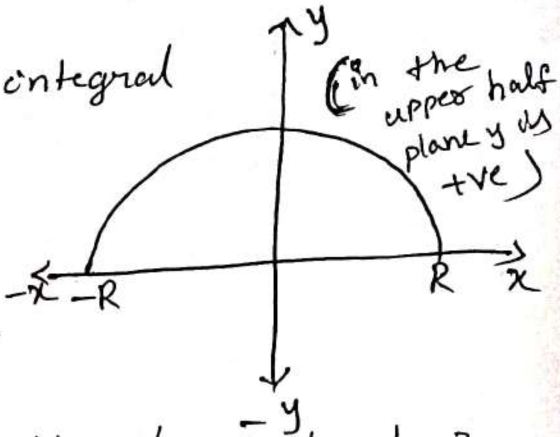
It is in the form $\int_{-\infty}^{\infty} f(x) \cos ax \, dx$

So, we can write

$(\because e^{iax} = \cos ax + i \sin ax)$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \cos ax \, dx = \int_{-\infty}^{\infty} \frac{1}{x^2+1} e^{iax} dx$$

Now sub" 'x' by 'z' in the given integral



i.e $\int_{-\infty}^{\infty} \frac{1}{x^2+1} e^{iax} dx = \int_c f(z) dz$

$$= \int_c \frac{e^{iaz}}{z^2+1} dz$$

for finding poles of $f(z)$ equate the denominator to zero

i.e $z^2+1=0 \Rightarrow z^2=-1$

$z^2 = i^2$

$z = i, -i$

Here $z = i$ is a simple pole and it lies inside the upper half of the circle &

$z=-i$ is a simple pole, it lies outside the upper half of the circle.

W.K.T the residue of $f(z)$ at a simple pole

i.e $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a} (z-a) f(z)$

At $z = i$:- $\underset{z=i}{\text{Res}}\, f(z) = \lim_{z\to i} (z-i) \frac{e^{iaz}}{z^2+1}$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{e^{iaz}}{\cancel{(z-i)}(z+i)}$$

$$= \lim_{z\to i} \frac{e^{iaz}}{z+i}$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{iai}}{i+i}$$

$$= \frac{e^{-a}}{2i} = R_1$$

By cauchy's Residue theorem

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i \,(R_1)$$

$$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \,(R_1)$$

Thus $R \to \infty \Rightarrow \int_{C_R} f(z)\,dz = 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left(\frac{e^{-a}}{2i}\right)$$

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{x^2+1}\,dx = \pi e^{-a} + 0(i)$$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx = \pi e^{-a} + i\,(0)$$

Compare real and imaginary parts on both Sides

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

the function in the integral is an even function.

$$\therefore\ 2\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

$$\boxed{\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}\, e^{-a}}$$