ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES, TIRUPATI
(AUTONOMOUS)
ELECTRONICS AND COMMUNICATION ENGINEERING (ECE)

Course Code	Year & Sem	SIGNALS AND SYSTEMS	L	T/CLC	P	C
20APC0403	II-I		2	1	0	3

Course Outcomes: After studying the course, Student will be able to:

CO1 **Understand** the representation of continuous time and discrete time signals

CO2 **Analyze** the signals in frequency domain using Fourier series and Fourier Transforms

CO3 **Apply** the Sampling theorem to convert continuous time signals into discrete time signals

CO4 **Analyze** the properties of systems and characteristics of LTI systems

CO5 **Evaluate** Continuous Time and Discrete Time LTI systems by using Laplace and Z-Transforms.

CO	Action Verb	Knowledge Statement	Condition	Criteria	Blooms level
CO1	**Understand**	the representation of continuous time and discrete time signals			L2
CO2	**Analyze**	the signals in frequency domain		Fourier series and Fourier Transforms	L4
CO3	**Apply**	To convert continuous time signals into discrete time signals	Sampling theorem		L3
CO4	**Analyze**	the properties of systems and characteristics of LTI systems			L4
CO5	**Evaluate**	Continuous Time and Discrete Time LTI systems by using		Laplace and Z-Transforms	L5

UNIT - I		21Hrs

SIGNALS

Introduction: Definition of Signals, classification of signals: continuous time and discrete time signals, standard signals: impulse function, step function, ramp function complex exponential and sinusoidal signals, Signum, Sinc and Gaussian functions. Operations on signals and sequences. Analogy between vectors and signals, orthogonal signal space, Signal approximation using orthogonal functions, mean square error, Orthogonality of complex functions.

UNIT - II		16Hrs

FOURIER SERIES AND FOURIER TRANSFORMS

Fourier series: Representation of signals using Fourier Series, Trigonometric Fourier series(TFS) and complex exponential Fourier series (CEFS). Illustrative problems. Continuous Time Fourier Transform, definition, properties, Fourier Transforms of standard signals, complex Fourier spectrum, inverse Fourier Transform. Discrete Time Fourier Transform, definition, properties of Discrete Time Fourier Transform transforms of standard signals. Introduction to Hilbert Transform. Illustrative problems.

UNIT - III		12Hrs

SAMPLING THEOREM

Definition of sampling, types: impulse and pulse sampling. Sampling theorem for band limitedsignals-Graphicalandanalyticalproof,Nyquistcriterion,Reconstructionofsignalfromitssamples,effectofundersampling – Aliasing.SamplingtheoremforBandpasssignals.Illustrativeproblems.

UNIT - IV		12Hrs

SYSTEMS Definition of Systems, Classification of Systems, impulse response, response of a Linear Time Invariant system, Convolution and Correlation: time domain, frequency domain and Graphical representation. Transfer function of a LTI system. Filter characteristics of linear systems. Distortion less transmission through a system, signal bandwidth, system bandwidth, Ideal LPF,HPFandBPFcharacteristics,CausalityandPoly-Wienercriterionforphysicalrealization,relationship between bandwidth and rise time. Illustrative problems.		
UNIT - V		20Hrs
LAPLACE TRANSFORMS & Z TRANSFORMS **Laplace Transforms:** Review of Laplace Transforms, concept of Region of Convergence(ROC) for Laplace Transforms, Inverse Laplace Transform, constraints on ROC for various classes of signals, properties of Laplace Transforms. Analysis of CT-LTI systems using Laplace Transforms: causality and stability. **Z-Transforms**: Review of Z-Transforms, concept of Region of Convergence(ROC) for Z-Transforms, Inverse Z-Transform, constraints on ROC for various classes of signals, properties of Z-Transforms. Analysis of DT-LTI systems using Z- Transforms: causality and stability. Illustrative problems.		
Textbooks:		
1. B.P. Lathi,Signals, Systems&Communications,BSPublications,2003. 2. A.V.Obppenheim,A.S.WillskyandS.H.Nawab,SignalsandSystemsPHI,2ndEdition.2009		
Reference Books:		
1. SimonHaykinandVanVeen,Signals&Systems,Wiley,2ndEdition. 2. John G.Proakis, Dimitris G. Manolakis, Digital Signal Processing, Principles, Algorithms, and Applications, 4 th Edition, PHI, 2007 3. BP Lathi, Principles of Linear Systems and Signals Oxford University Press, 2015.		
Online Learning Resources:		
nptel videos		

Mapping of course outcomes with program outcomes

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PSO1	PSO2
CO1	2	3										2	
CO2	3	3		3								1	
CO3	3	3										2	
CO4	3	3		3								2	
CO5	3	3		3								2	

Correlation matrix

Unit No.	CO					Program Outcome (PO)	PO(s) :Action Verb and BTL(for PO1 to PO11)	Level of Correlation (0-3)
	Lesson plan(Hrs)	%	Correlation	Co's Action verb	BTL			
1	21	28%	3	**Understand**	**L2**	PO1, PO2,	PO1: Apply (L3) PO2: Review(L2)	2 3
2	16	21%	3	**Analyze**	**L4**	PO1,PO2, PO4	PO1: Apply (L3) PO2: Identify (L3) PO4:Analyze(L4)	3 3 3
3	12	**16%**	2	**Apply**	**L3**	PO1,PO2, PO11	PO1:Apply(L3) PO2:Identify(L4)	3 3
4	12	**16%**	2	**Analyze**	**L4**	PO1, PO2,PO4	PO1:Apply(L3) PO2:Identify(L3) PO4:Analyze(L4)	3 3 3

Signals, Systems And Stochastic Processes

Unit - I

Syllabus: Signals And Systems: Basic Definitions and Classification of Signals and Systems (Continuous Time and discrete time), operations on Signals, Concepts of Convolution and Correlation of Signals. Analogy b/w Vectors and Signals — orthogonality — Mean Square error.
Fourier Series: Trigonometric Fourier Series, Wave Symmetry, Even or odd Symmetry, Exponential Fourier Series, problems on Trigonometric Fourier Series and exponential Fourier Series.

Definition of a Signal:

"A Signal is defined as a Single Valued function of one or more independent Variables that carries some information"

If a Signal depends on only one Variable, it is called a one dimensional signal, and if a Signal depends on two independent Variables, it is called a two dimensional Signal.

Ex: Speech Signal
ECG
Electric Current & Voltage etc are familiar examples of Signals.

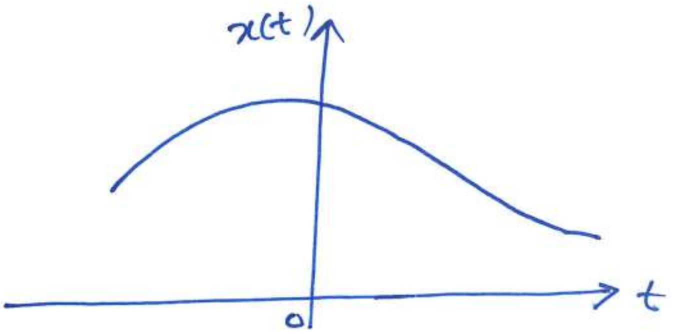
Classification of Signals:

In general Signals are broadly classified into two types.

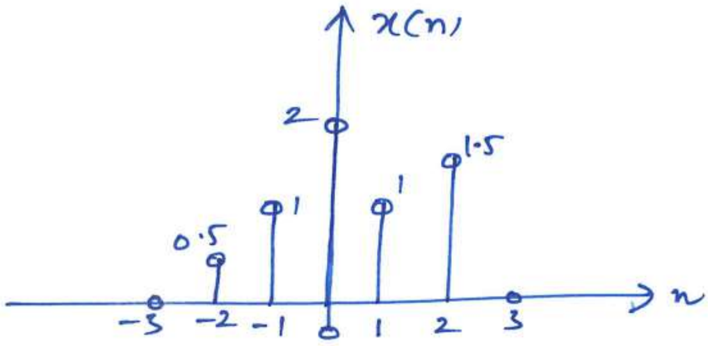
* Continuous time Signals
* Discrete Time Signals.

* A Signal which is defined continuously for all instants of time 't' is called a continuous time Signal. Continuous time Signals are denoted as $x(t)$ $y(t)$ and $z(t)$ etc.

* A Signal which is defined only at discrete instants of time (n) is called a discrete time Signal. These are denoted as $x(n)$, $y(n)$, $z(n)$ etc.



CT Signal



DT Signal.

* Continuous time Signals are represented by a function or a graph.

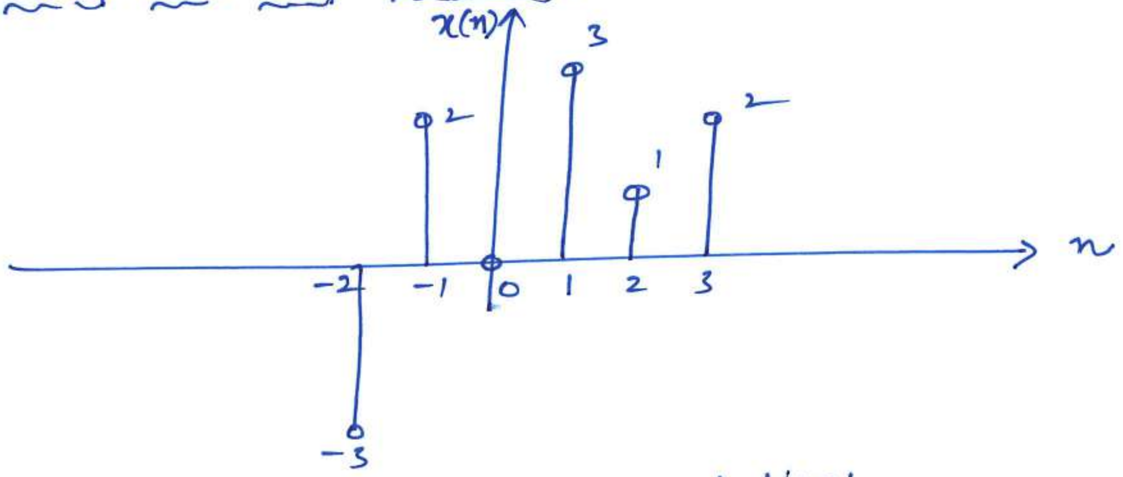
* There are four ways of representing a discrete time Signal.
  - → Graphical representation
  - → Functional representation
  - → Tabular representation
  - → Sequence representation.

Consider a discrete time Signal $x(n)$ with Values

$x(-2) = -3$, $x(-1) = 2$, $x(0) = 0$, $x(1) = 3$,

$x(2) = 1$, $x(3) = 2$.

* Graphical Representation:



* Functional (Analytical) Representation:

1) $x(n) = \begin{cases} 0, & n = 0 \\ 3, & n = 1 \\ 1, & n = 2 \\ 2, & n = -1, 3 \\ -3, & n = -2 \end{cases}$ 2) $x(n) = \begin{cases} 2^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

* Tabular Representation:

n	-2	-1	0	1	2	3
$x(n)$	-3	2	0	3	1	2

* Sequence representation:

$$x(n) = \{-3, 2, \underset{\uparrow}{0}, 3, 1, 2\}$$

The arrow mark ↑ indicates $n=0$ term. when no arrow is indicated the first term corresponds to $n=0$

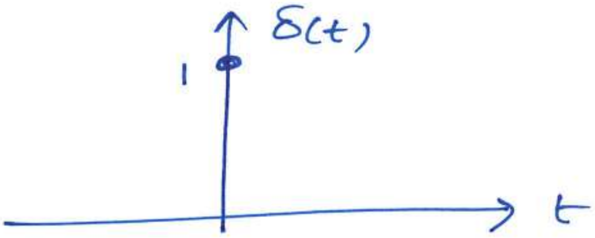
Standard Elementary Signals:

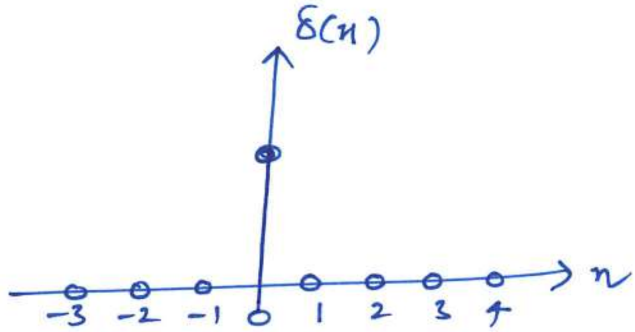
* **Unit impulse Signal**: The Continuous time unit impulse Signal denoted by $\delta(t)$, also called dirac-deltafunction is defined as.

$$\delta(t) = \begin{cases} 1, & t=0 \\ 0, & t \neq 0 \end{cases} \quad \& \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Discrete time Unit impulse Sequence is defined as

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0. \end{cases}$$





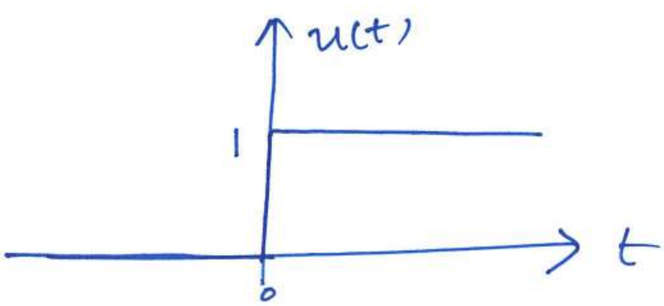
* **Unit Step Signal**:

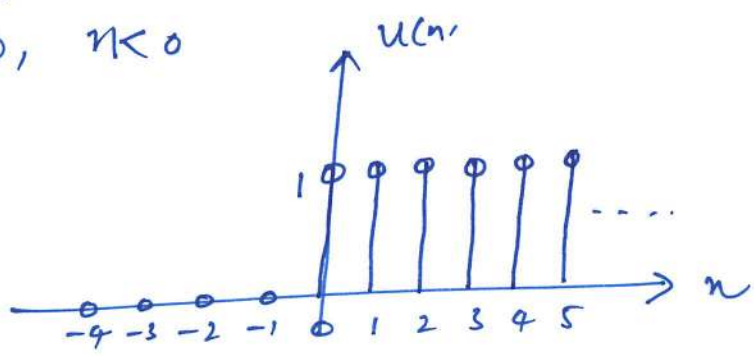
The Continuous time Unit Step Signal is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The Discrete time Unit Step Signal is defined as

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



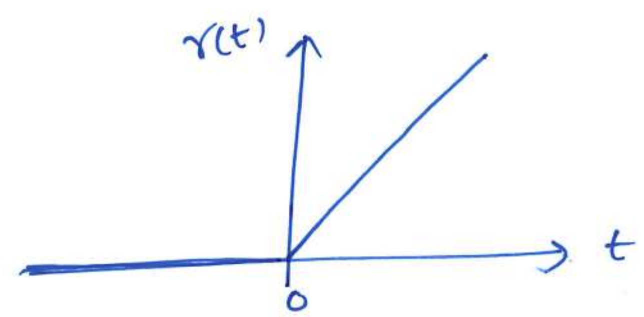


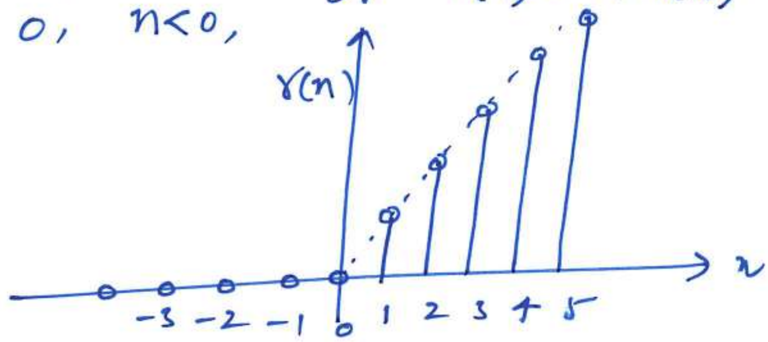
Unit ramp Signal: The Continuous time Unit ramp Signal is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{or} \quad r(t) = t \cdot u(t)$$

The Discrete time Unit ramp Signal is defined as

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0, \end{cases} \quad \text{or} \quad r(n) = n\, u(n)$$



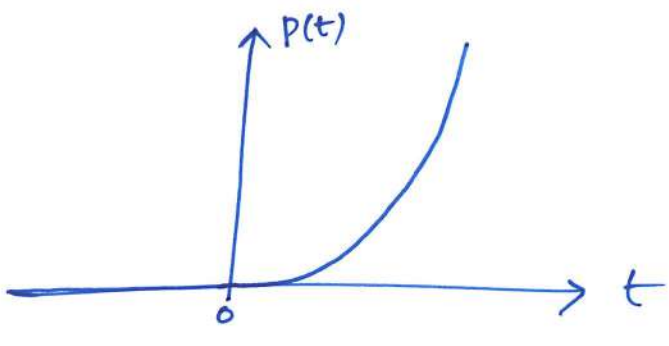


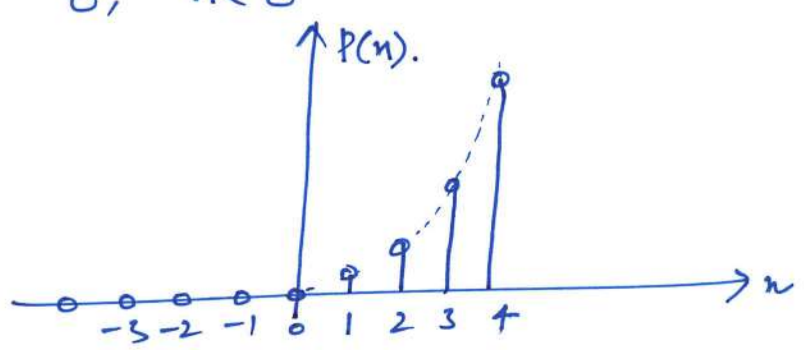
Unit Parabolic Signal: The Continuous time Unit parabolic Signal P(t) is defined as

$$P(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{or} \quad \frac{t^2}{2} u(t)$$

The Discrete time Unit parabolic Signal P(n) is defined as

$$P(n) = \begin{cases} \frac{n^2}{2}, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad \frac{n^2}{2} u(n)$$





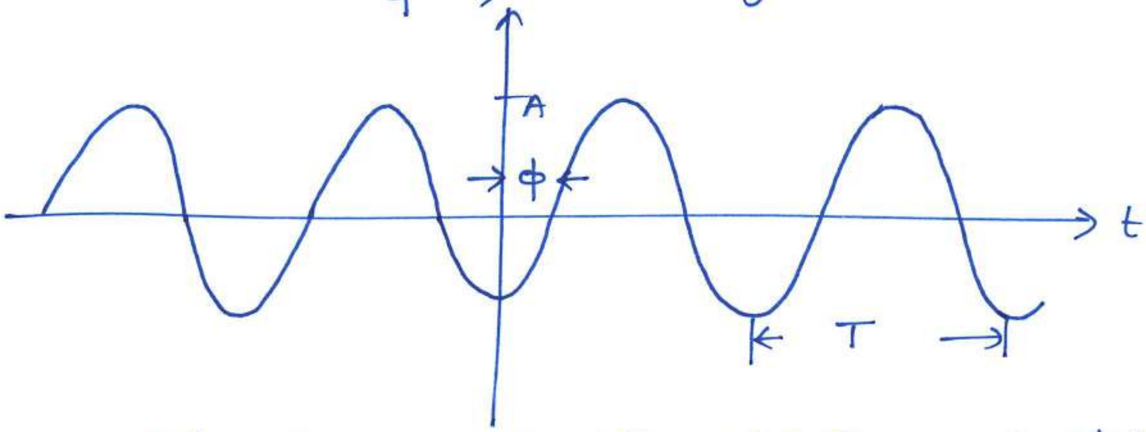
Sinusoidal Signal: A Continuous time Sinusoidal Signal in its most general form is given by

$$x(t) = A \sin(\omega t + \phi)$$

where $A \rightarrow$ Amplitude

$\omega \rightarrow$ Angular freq. in radians

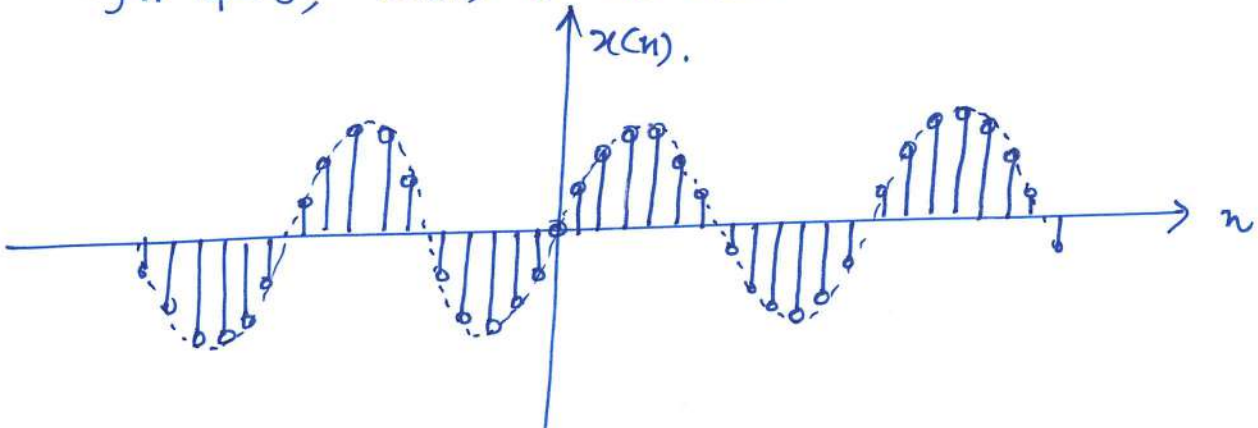
$\phi \rightarrow$ Phase angle in radians.



The Discrete time Sinusoidal Sequence is defined as

$$x(n) = A \sin(\omega n + \phi)$$

for $\phi = 0$, $x(n)$ is as Shown.



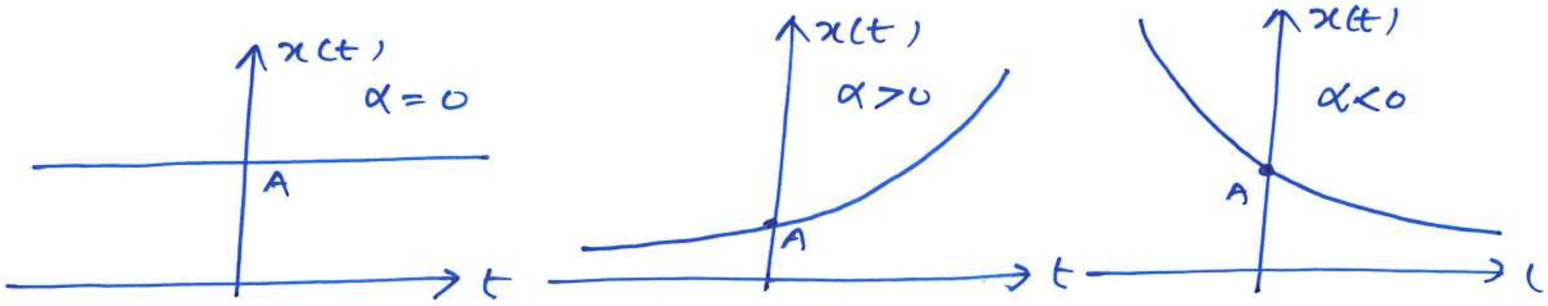
Real Exponential Signal:

A Continuous time real exponential Signal has the general form, as

$$x(t) = A e^{\alpha t}$$

where Both A & α are real.

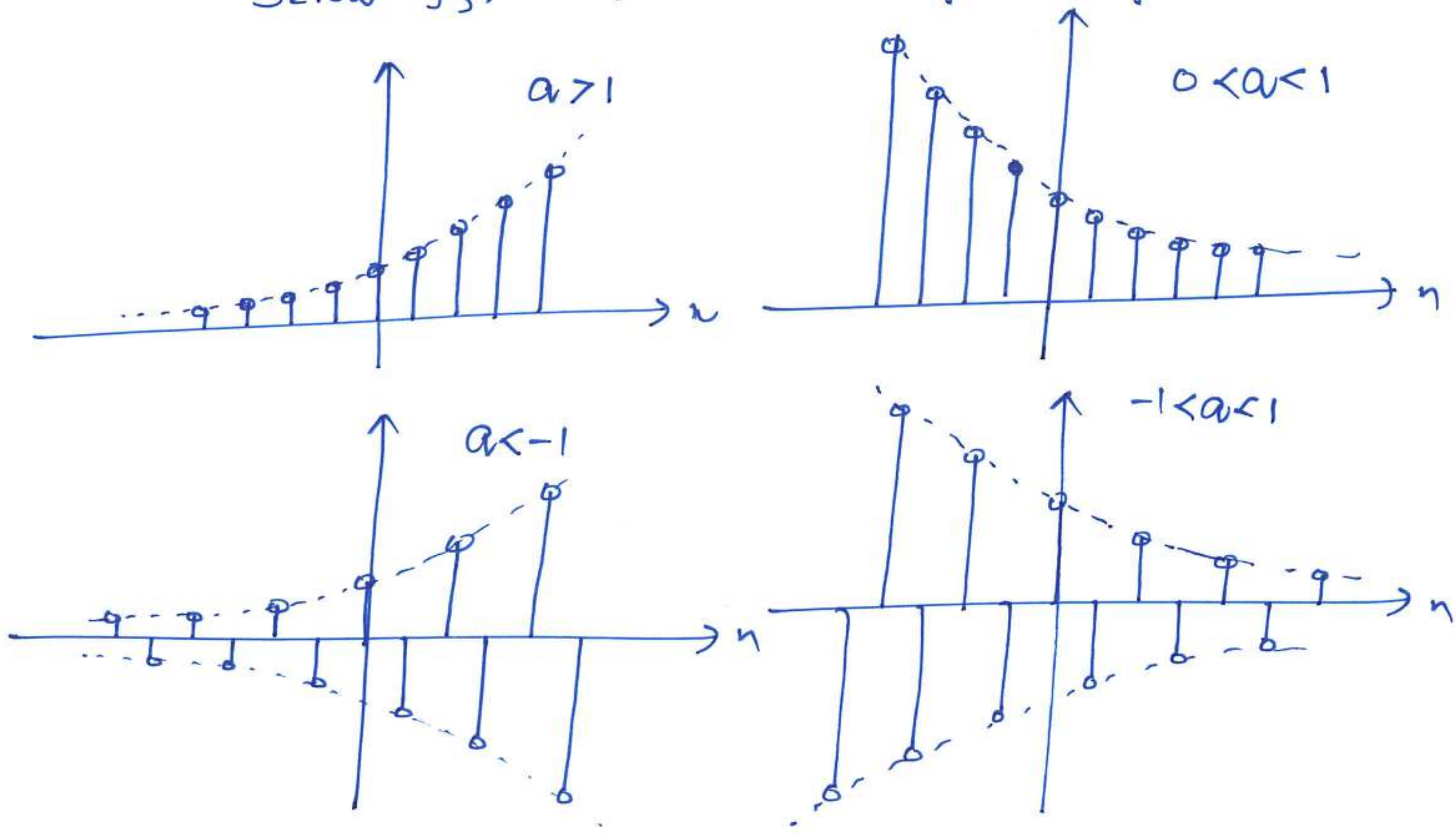
* $\alpha = 0$, $x(t) = A$ is a Constant at All times 't'
* $\alpha > 0$, $x(t) = A e^{\alpha t}$ is a growing exponential Signal
* $\alpha < 0$, $x(t) = A e^{\alpha t}$ is a decaying exponential Signal.



The Discrete time real exponential Sequence is of the form,

$$x(n) = a^{n} \quad \forall n.$$

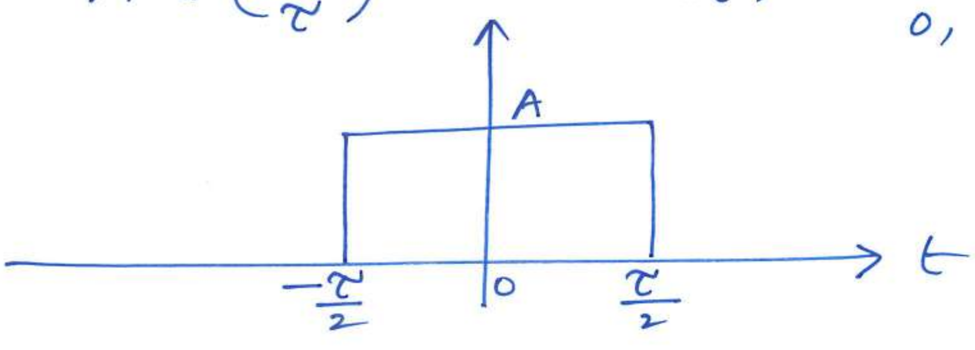
Below fig. illustrates Diff. types of exponential Sequences.



* Rectangular Pulse Signal:

A Symmetrical rectangular pulse Signal or Gate function is defined as

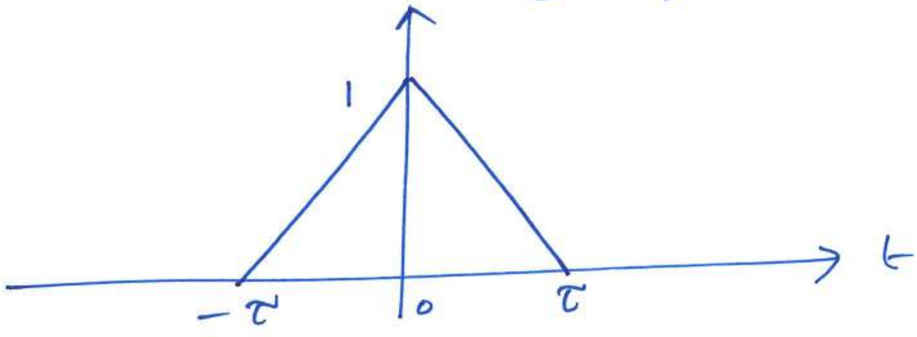
$$A\,\Pi\left(\frac{t}{\tau}\right) \text{ or } A\,\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} A, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$



* **Triangular Pulse Signal:**

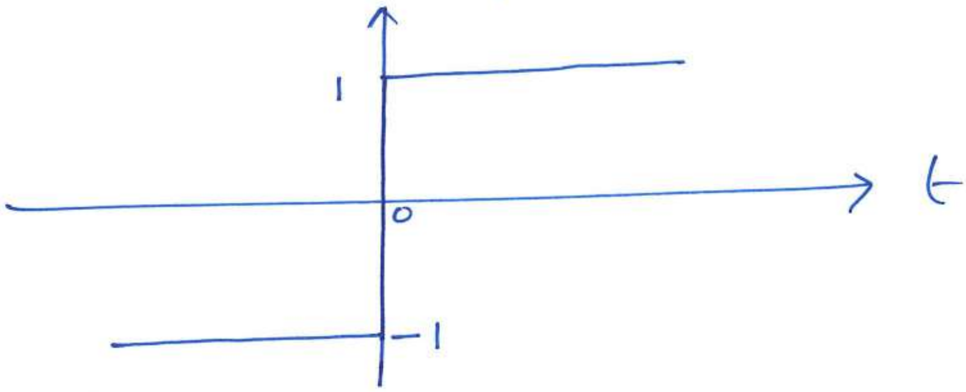
The Symmetrical Unit triangular pulse Signal is defined as,

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau}, & |t| \leq \tau \\ 0, & |t| > \tau \end{cases}$$



* **Signum Signal:** The Signum Signal is defined as the Sign of the independent Variable 't' and is defined as

$$Sgn(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

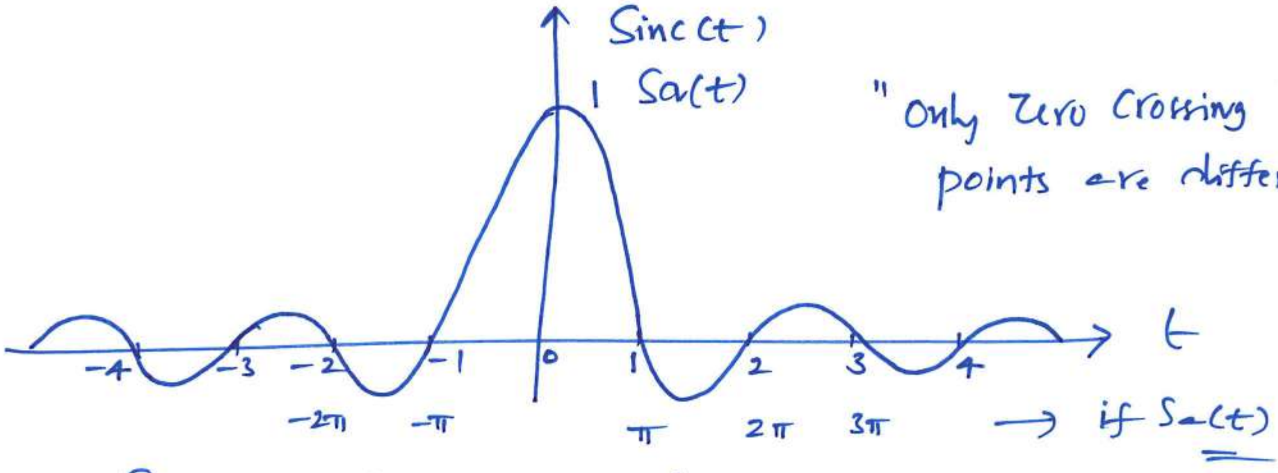


* **Sinc Signal:** The Sinc Signal is defined as

$$Sinc(t) = \frac{Sin \pi t}{\pi t}, \quad -\infty \leq t \leq \infty$$

$$Sin(0) = 1,$$

$$Sin(t) = 1, \quad t = \pm 1, \pm 2, \pm 3 \cdots\cdots$$



Sampling function is defined as

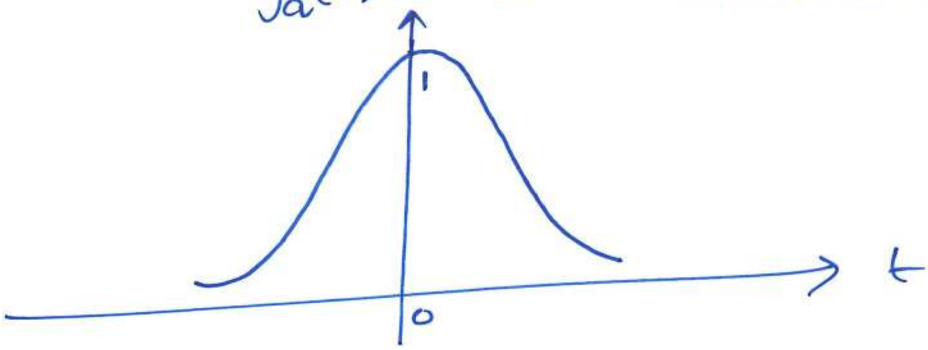
$$Sa(t) = \frac{\sin t}{t}, \quad -\infty \le t \le \infty$$

$$Sa(0) = 1,$$

$$Sa(t) = 1, \quad t = \pm n\pi$$

* Gaussian function: The gaussian function $g_a(t)$ is defined as

$$g_a(t) = e^{-a^2 t^2} \quad -\infty \le t \le \infty$$



Basic Operations on Signals and Sequences:

The Basic Operations on Signals & Sequences are as follows.

- * Amplitude Scaling
- * Time Scaling
- * Time Shifting
- * Time Reversal
- * Addition of Signals
- * Multiplication of Signals.

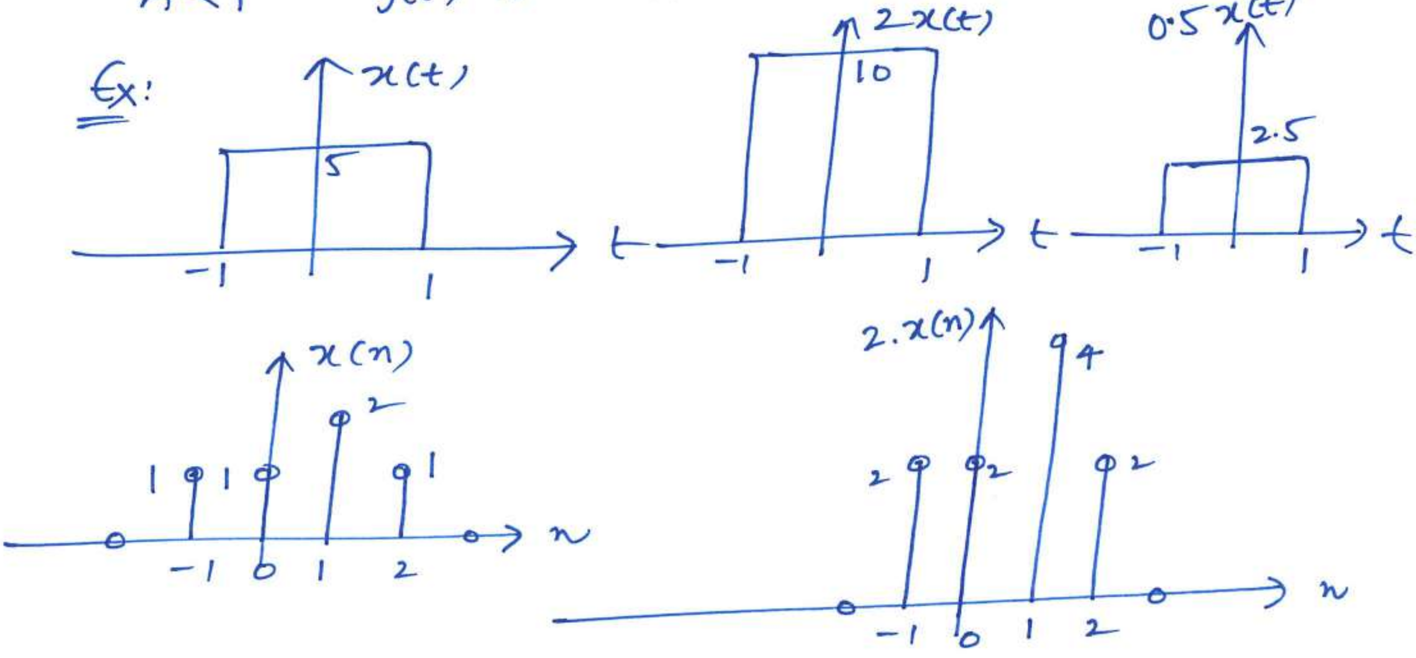
* **Amplitude Scaling**: The Amplitude Scaling of a CT Signal can be represented by

$$y(t) = A \cdot x(t). \quad \text{or} \quad y(n) = A\, x(n)$$

$A > 1$ $\quad y(t)$ is Amplified Signal

$A < 1$ $\quad y(t)$ is Attenuated Signal.

Ex:



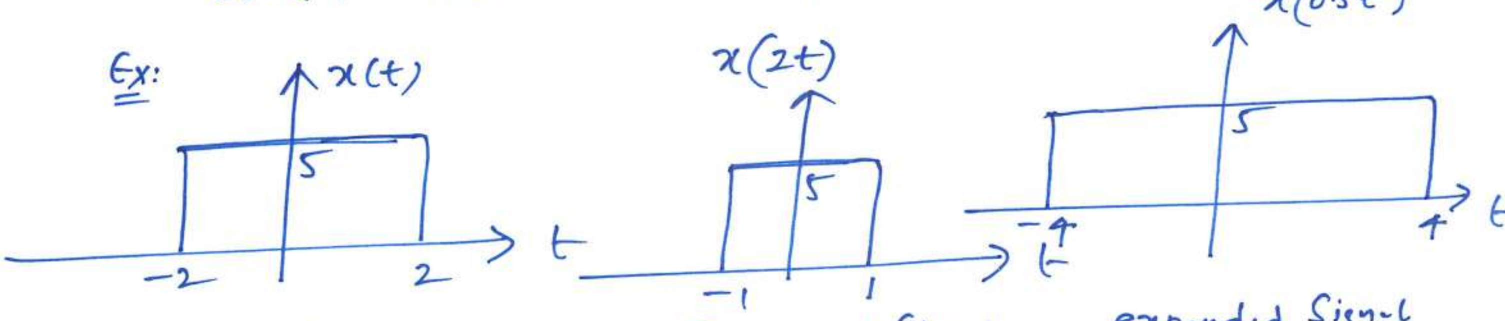
* **Time Scaling**: Time Scaling may results in time expansion or time compression. Time Scaling of a CT Signal can be represented as,

$$y(t) = x(at)$$

$a > 1$ results in compression of a Signal

$a < 1$ results in expansion of a Signal.

Ex:



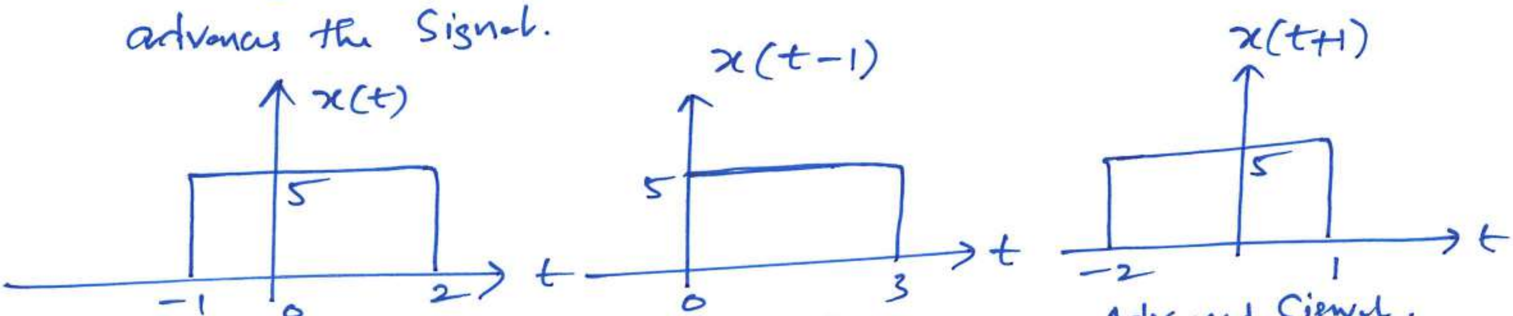
Original Signal $\quad$ Compressed Signal $\quad$ expanded Signal

* **Time Shifting**: Time shifting of a signal $x(t)/x(n)$ can be represented by

$$y(t) = x(t-t_0) \quad \text{or}$$
$$y(n) = x(n-n_0)$$

Time shifting of a signal may result in time delay (right shift) or time advance (left shift).

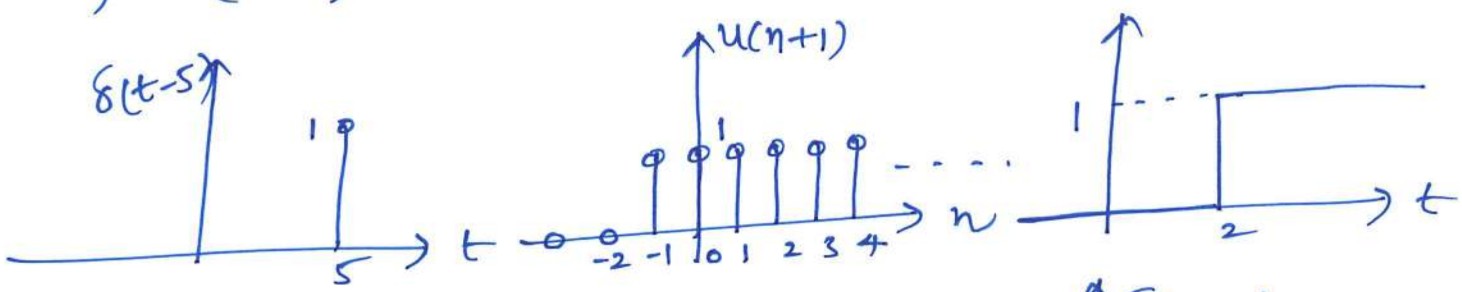
If t_0/n_0 is positive, the time shifting delays the signal and if t_0/n_0 is negative, the time shifting advances the signal.

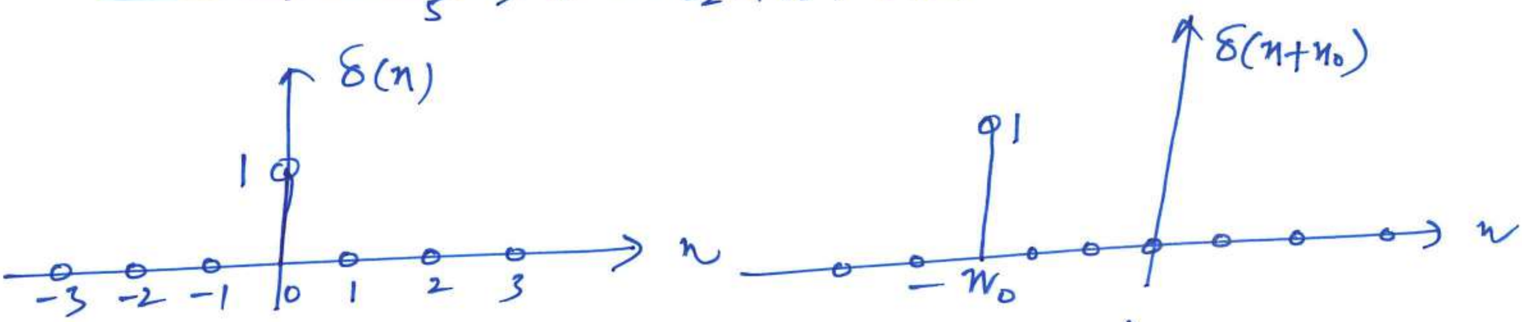


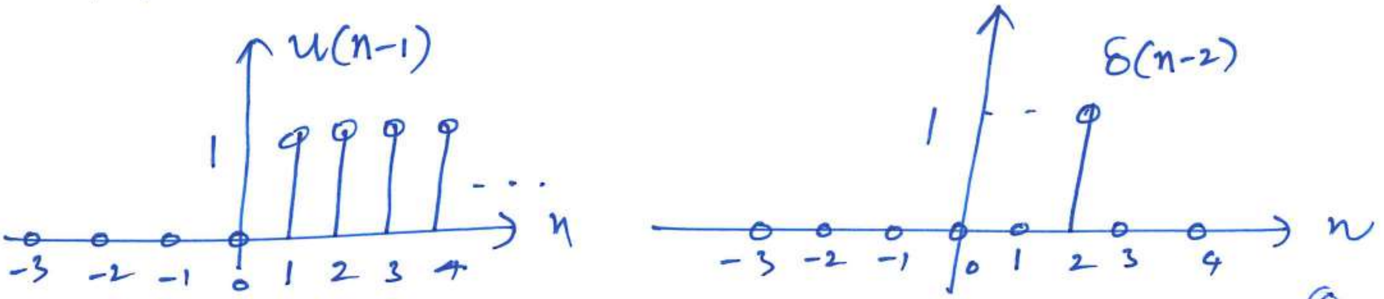
Original Signal   Delayed Signal   Advanced Signal.

Sketch the following:

a) $\delta(t-5)$ b) $u(n+1)$ c) $u(t-2)$ d) $\delta(n+n_0)$

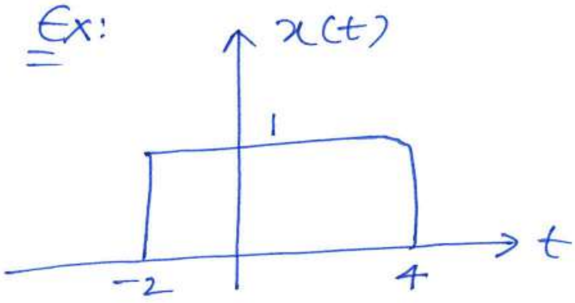




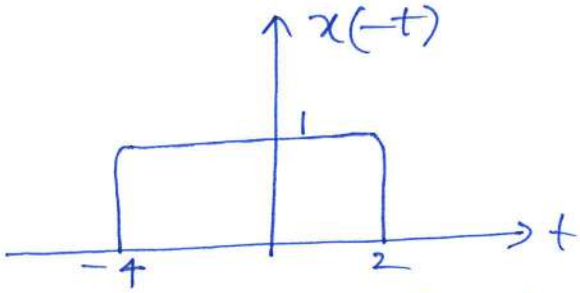


* **Time Reversal of a Signal:** The time reversal or time folding of a signal can be obtained by folding the signal about $t = 0$. It is denoted by $x(-t)$ or $x(-n)$

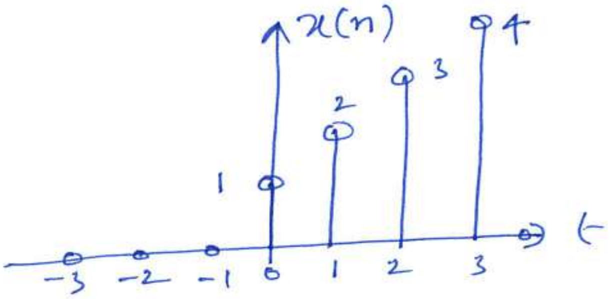
Ex:

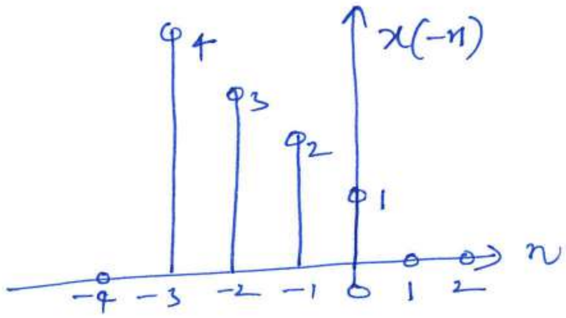


Original Signal



Time Reversal Signal





Addition of Signals:

The sum of two signals $x_1(t)$ and $x_2(t)$ is obtained by adding their amplitudes at every instant of time as,

$$x_3(t) = x_1(t) + x_2(t)$$

Similarly $x_3(n) = x_1(n) + x_2(n)$.

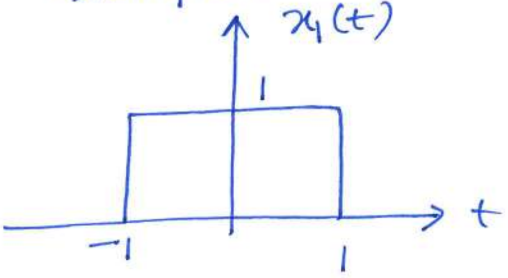
Also subtraction can also be performed

$$x_3(t) = x_1(t) - x_2(t) = x_1(t) + (-x_2(t))$$

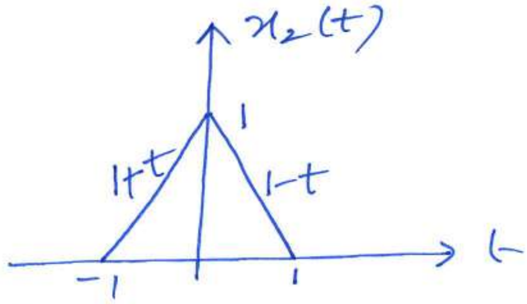
$$x_3(n) = x_1(n) - x_2(n)$$

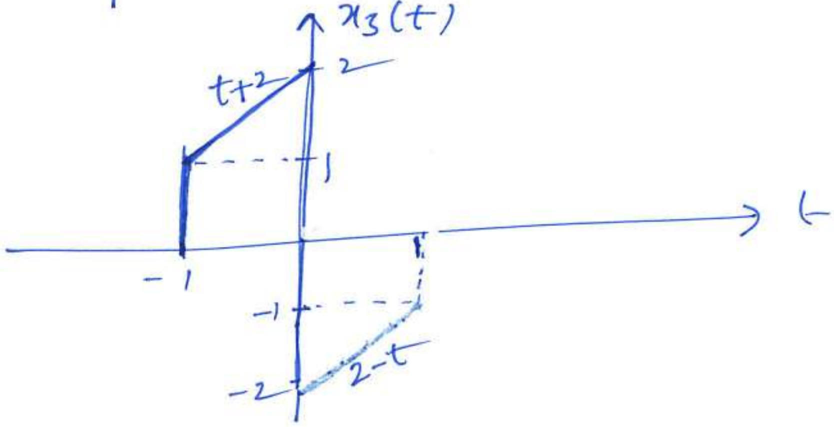
$$= x_1(n) + (-x_2(n))$$

Examples:

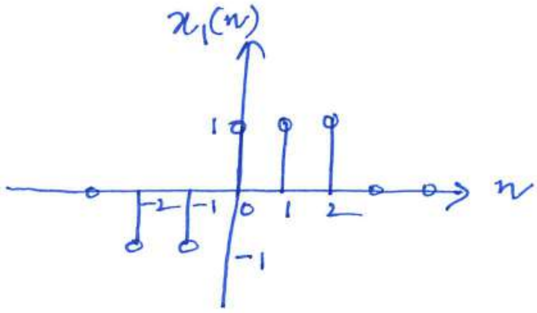


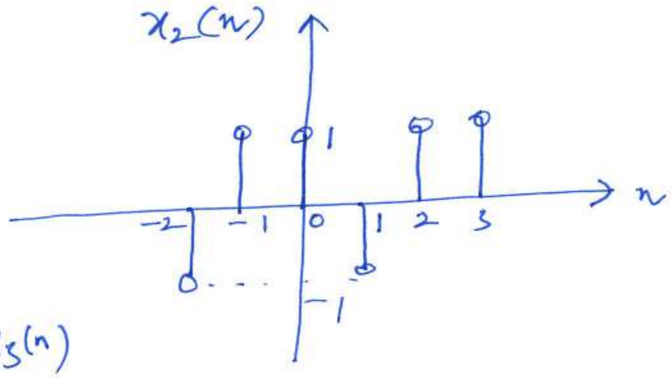
(+)

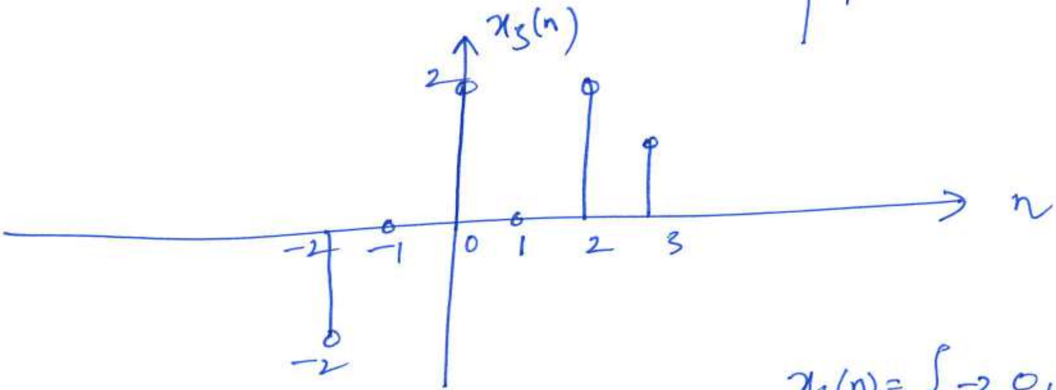




$$x_3(t) = \begin{cases} t+2, & -1 \le t \le 0 \\ 2-t, & 0 \le t \le 1 \end{cases}$$







$$x_3(n) = \{-2, 0, \underset{\uparrow}{2}, 0, 2, 1\}$$

$x_3(0) = x_1(0) + x_2(0) = 1+1 = 2$

$x_3(1) = x_1(1) + x_2(1) = 1-1 = 0$

$x_3(2) = x_1(2) + x_2(2) = 1+1 = 2$

$x_3(3) = x_1(3) + x_2(3) = 0+1 = 1$

$x_3(-1) = x_1(-1) + x_2(-1) = -1+1 = 0$

$x_3(-2) = x_1(-2) + x_2(-2) = -1-1 = -2$

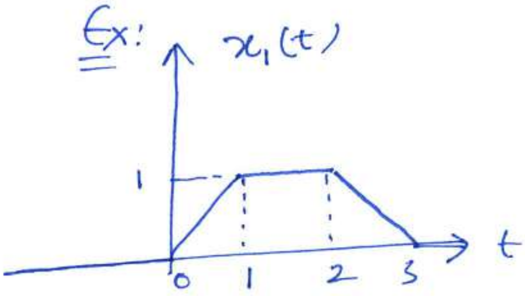
Multiplication of Signals:

The multiplication of two signals can be performed by multiplying their amplitudes at their respective time instants.

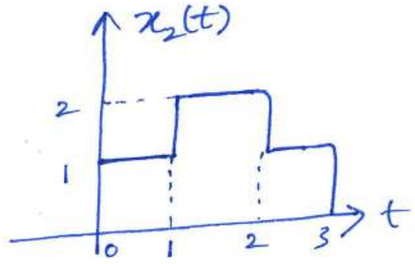
$$x_3(t) = x_1(t) \cdot x_2(t) \quad \text{(or)}$$

$$x_3(n) = x_1(n) \cdot x_2(n).$$

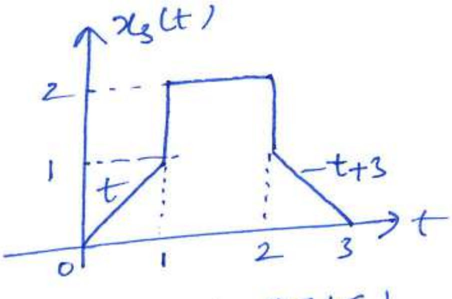
Ex:



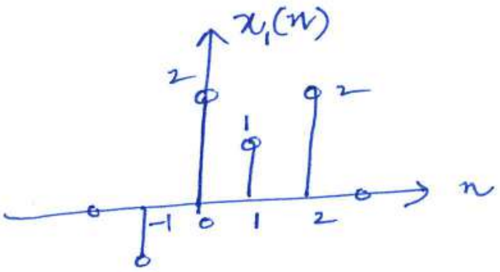
$$x_1(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 \le t \le 2 \\ -t+3, & 2 \le t \le 3 \end{cases}$$



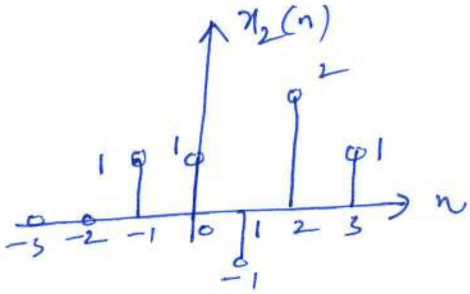
$$x_2(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 2, & 1 \le t \le 2 \\ 1, & 2 \le t \le 3 \end{cases}$$



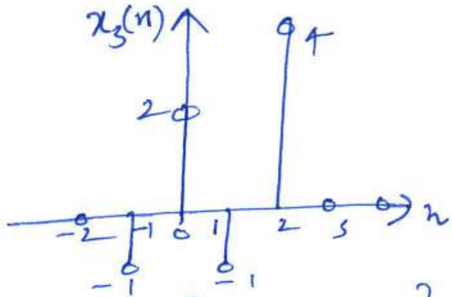
$$x_3(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2, & 1 \le t \le 2 \\ -t+3, & 2 \le t \le 3 \end{cases}$$



$$x_1(n) = \{-1, \underset{\uparrow}{2}, 1, 2\}$$



$$x_2(n) = \{1, \underset{\uparrow}{1}, -1, 2, 1\}$$



$$x_3(n) = \{-1, \underset{\uparrow}{2}, -1, 4, 0\}$$

Classification of Signals (Continuous & Discrete Time):

Signals are classified according to their characteristics as,

- → Deterministic / Non-Deterministic (Random) Signals
- → periodic / Non periodic (Aperiodic) Signals
- → Symmetric (Even) / Asymmetric (odd) Signals
- → Energy / Power Signals
- → Causal / Non-Causal / Anti Causal Signals.

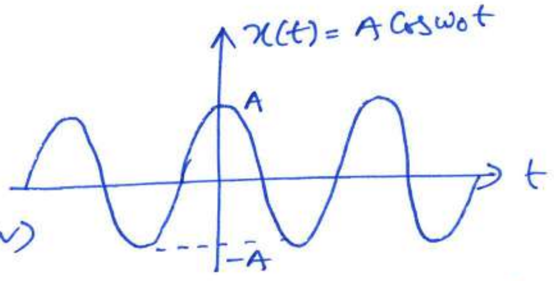
Deterministic / Non-Deterministic (Random) Signals:

A Signal which exhibits no uncertainity about its magnitude and phase at any given instant of time is called deterministic Signal. A Deterministic Signal can be Completely represented by a mathematical equation.

Ex: $x(t) = A \sin w_0 t$

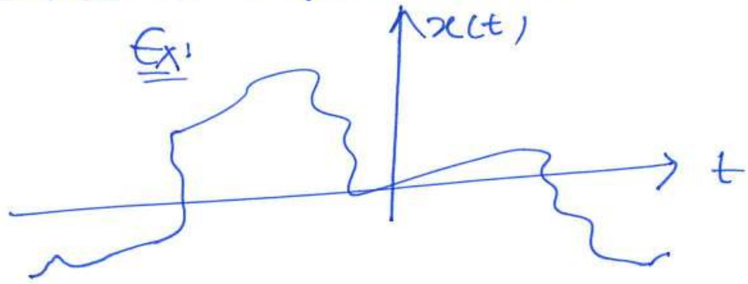
$x(t) = A\, u(t)$

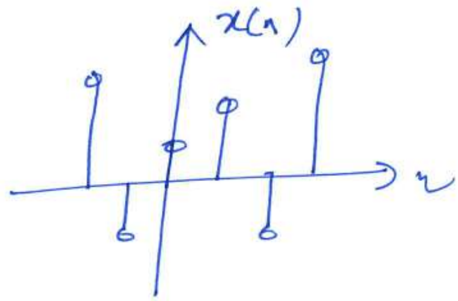
$x(n) = \left(\frac{1}{2}\right)^n u(n)$



A Signal which is characterized by uncertainity about its occurence is called a random Signal. A random Signal cannot be represented by a mathematical equation.

Ex:





⇒ Periodic / Nonperiodic (Aperiodic) Signals:

A Signal which repeats again and again at regular intervals of time is called a periodic Signal.

Mathematically a Signal $x(t)/x(n)$ is periodic if and only if,

$$\left.\begin{aligned} x(t \pm T) &= x(t) \quad \forall t \\ x(n \pm N) &= x(n) \quad \forall n. \end{aligned}\right\} \rightarrow (1)$$

A Signal which do not Satisfy the above Condition is Said to be aperiodic Signal.

In Eqn (1) the minimum non zero Value of T/N for which Eqn (1) is true is Called the fundamental period of the Signal $x(t)/x(n)$.

Note:

① The Sum of two Continuous time periodic Signals $x_1(t)$ & $x_2(t)$ with periods T_1 and T_2 may or may not be periodic depending on the relation b/w T_1 & T_2.

If the ratio of T_1 & T_2 is a rational number (p/q form) then the sum is periodic & the resultant Signal period is given by $T = \text{LCM of } (T_1, T_2)$.

② The Sum of two Discrete time periodic Signals $x_1(n)$ & $x_2(n)$ with periods N_1 & N_2 is always periodic. and the resultant Signal period is

$$N = \text{Lcm of } (N_1, N_2).$$

③ Continuous time Sinusoidal Signals & Complex exponential Signals are always periodic with period $T = \frac{2\pi}{\omega_0}$

$$A \sin(\omega_0 t + \theta) \qquad A e^{j(\omega_0 t + \theta)}$$

$$A \cos(\omega_0 t + \theta) \qquad A e^{-j(\omega_0 t + \theta)}$$

let $x(t) = A \cos(\omega_0 t + \theta)$

$$x(t+T) = A \cos(\omega_0 t + \omega_0 T + \theta)$$

$$= A \cos(\underline{\omega_0 T} + \omega_0 t + \theta)$$

$$= A \cos(\omega_0 t + \theta) = x(t) \text{ if}$$

$$\omega_0 T = 2\pi(n). \quad \text{or} \quad T = \frac{2\pi}{\omega_0}(n)$$

∴ The fundamental period (for $n=1$) is given by

$$\boxed{T = \frac{2\pi}{\omega_0}}$$

Same is proved for remaining three Signals also

④ Unlike CT Signals, Discrete time Sinusoidal Signals and Complex exponential Sequences are not always periodic.

$$A\cos(\omega_0 n+\theta) \qquad A e^{j(\omega_0 n+\theta)}$$

$$A\sin(\omega_0 n+\theta) \qquad A e^{-j(\omega_0 n+\theta)}$$

let $x(n) = A e^{-j(\omega_0 n+\theta)}$

$$x(n+N) = A e^{-j(\omega_0 n+\omega_0 N+\theta)}$$

$$= A e^{-j(\omega_0 n+\omega_0 N+\theta)}$$

$$= A e^{-j(\omega_0 n+\theta)} = x(n) \text{ if and only if}$$

$$\omega_0 N = 2\pi(m)$$

$$\omega_0 = 2\pi\left(\frac{m}{N}\right) \quad m \to \text{integer},\ N \to \text{integer}$$

i.e $x(n)$ is periodic if ω_0 is 2π times a rational number. Otherwise $x(n)$ is aperiodic.

Period is now given by

$$N = \frac{2\pi}{\omega_0}(m).$$

The Smallest Value of 'm' for which 'N' should be an integer is called the fundamental period of $x(n)$

prob: Check whether the following Signals are periodic or not. If So Determine its fundamental period.

a) $x(t) = 2\cos\left(\frac{6\pi}{5}t - \pi/3\right)$

b) $x(t) = 5\sin\left(\sqrt{2}t + \pi/6\right)$

c) $x(t) = e^{-jt}$

d) $x(t) = e^{j(2\pi t + \pi/2)}$

e) $x(n) = 2\sin\pi n$

f) $x(n) = 2e^{-j\left(\frac{2\pi}{3}n - \pi/3\right)}$

g) $x(n) = 5\cos\left(\sqrt{2}\pi n + \pi/2\right)$

h) $x(n) = 2e^{j\frac{\pi}{2}n}$

i) $x(t) = e^{-|t|}$

j) $x(t) = 2 + \cos 2\pi t$

Soln: a) Given the Signal

$$x(t) = 2\cos\left(\frac{6\pi}{5}t - \frac{\pi}{3}\right)$$

Continuous time Sinusoidal Signal is always periodic.

$\therefore$ $x(t)$ is periodic. Compare the given Signal with Standard Signal $A\cos(\omega_0 t + \theta)$, we have.

$$\omega_0 = \frac{6\pi}{5}$$

$$\therefore \text{ fundamental Period } T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{6\pi}{5}} = \frac{5}{3}$$

b) Given the Signal

$$x(t) = 5\sin\left(\sqrt{2}t + \frac{\pi}{6}\right)$$

Continuous time Sinusoidal Signal is always periodic. So. $x(t)$ is periodic. Compare with Standard form, we have

$$\omega_0 = \sqrt{2}$$

$$\therefore \text{ fundamental Period } T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi.$$

c) Given Signal,

$$x(t) = e^{jt}$$

Continuous time Complex exponential Signal is always periodic. $\therefore$ $x(t)$ is periodic.

Compare with Standard form, $Ae^{j(\omega_0 t + \theta)}$

$$\omega_0 = 1 \Rightarrow \text{fundamental Period } T = \frac{2\pi}{\omega_0}$$

$$T = \frac{2\pi}{1} = 2\pi$$

d) Given $x(t) = e^{-j(2\pi t + \pi/2)}$

Given Signal $x(t)$ is periodic

$$\omega_0 = 2\pi \Rightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$$

e) The given Signal $x(n)$ is given by

$$x(n) = 2 \sin \pi n$$

$$\omega_0 = \pi = 2\pi\left(\cancel{\pi}\tfrac{1}{2}\right) = 2\pi(\text{rational number})$$

$\therefore$ $x(n)$ is periodic.

$\therefore$ fundamental periodic $N = \frac{2\pi}{\omega_0}(n)$

$$N = \frac{2\cancel{\pi}}{\cancel{\pi}}(n) = 2n$$

$\therefore$ for $n = 1$, $\boxed{N = 2}$

f) Given $x(n) = 2e^{-j\left(\frac{2\pi}{3}n - \frac{\pi}{3}\right)}$

Compare with Standard form

$$\omega_0 = \frac{2\pi}{3} = 2\pi\left(\tfrac{1}{3}\right) = 2\pi(\text{rational number})$$

$\therefore$ $x(n)$ is periodic

$\therefore$ fundamental Period $N = \frac{2\pi}{\omega_0}(n)$

$$N = \frac{\cancel{2\pi}}{\frac{\cancel{2\pi}}{3}}(n) = 3n$$

$\therefore$ for $n = 1$, $\boxed{N = 3}$

g) $x(n) = 5\cos\left(\sqrt{2}\pi n + \pi/2\right)$

Compare with Standard form,

$\omega_0 = \sqrt{2}\pi$, $2\pi\left(\frac{\sqrt{2}}{2}\right) \rightarrow$ not a rational number

$\therefore$ $x(n)$ is aperiodic.

h) $x(n) = 2e^{j\pi/2\, n}$

Compare with Standard form

$$\omega_0 = \pi/2, = 2\pi\left(\frac{1}{4}\right) = 2\pi(\text{rational number})$$

$\therefore$ $x(n)$ is periodic

$\therefore$ period is $N = \frac{2\pi}{\omega_0}(n) = \frac{2\pi}{\pi/2}(n) = 4n$

$\therefore$ for $n = 1$ $\boxed{N = 4}$

(i) Given the Signal

$$x(t) = e^{-|t|}$$

$$x(t+T) = e^{-|t+T|}$$

$$= e^{-|t|} \cdot e^{-|T|} \neq x(t) \text{ for any Value}$$

Of 'T'. $\therefore$ $x(t)$ is Aperiodic.

(j) $x(t) = 2 + \cos 2\pi t$.

$\underbrace{2}_{\text{D.C}} + \underbrace{\cos 2\pi t}_{\text{Continuous time Sinusoidal Sigl.}}$

$\therefore$ $x(t) =$ $\cos 2\pi t$ Shifted upward by 2

$\therefore$ $x(t)$ is also periodic. $\omega_0 = 2\pi$, $T = \frac{2\pi}{\omega_0}$

$$T = \frac{2\pi}{2\pi} = 1$$

$\Rightarrow$ Symmetric (Even) / Asymmetric (Odd) Signals:

A Signal $x(t)/x(n)$ which is Symmetrical about the Vertical axis is Called an even Signal.

Mathematically if

$x(-t) = x(t)\ \forall t$ or

$x(-n) = x(n)\ \forall n$ then $x(t)/x(n)$

is Said to be an even Signal.

A Signal $x(t)/x(n)$ which is asymmetrical about the Vertical axis is called an odd Signal.

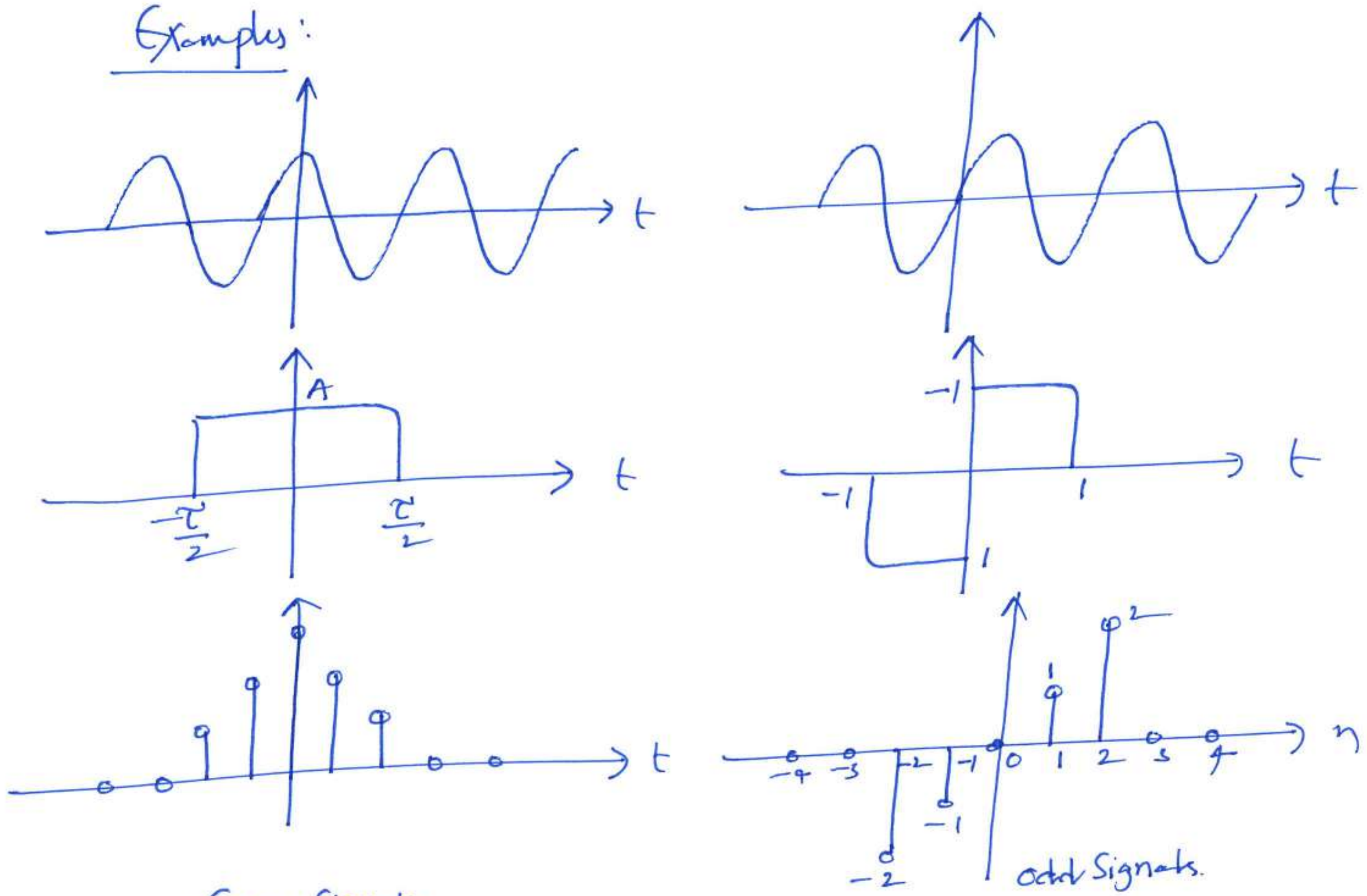
Mathematically if

$x(-t) = -x(t)\ \forall t$ or

$x(-n) = -x(n)\ \forall n$, then $x(t)/x(n)$

is Said to be an odd Signal.

Examples:



Even Signals   odd Signals.

<u>Note</u>: An arbitrary Signal Can be expressed as Sum of its even and odd Components.

$$x(t) = x_e(t) + x_o(t) \longrightarrow ①$$

Replace t by $-t$ in the above eq.

$$x(-t) = x_e(-t) + x_o(-t)$$

$$x(-t) = x_e(t) - x_o(t) \longrightarrow ②$$

$$① + ② \Rightarrow \quad x(t) + x(-t) = 2x_e(t)$$

$$\therefore \quad x_e(t) = \frac{x(t)+x(-t)}{2} \longrightarrow ③$$

$$① - ② \Rightarrow \quad x(t) - x(-t) = 2x_o(t)$$

$$\therefore \quad x_o(t) = \frac{x(t)-x(-t)}{2} \longrightarrow ④$$

Equations ③ & ④ Can be used to determine even & odd parts of a Signal.

Similarly for discrete case it can be shown that

$$x_e(n) = \frac{x(n)+x(-n)}{2}$$

$$x_o(n) = \frac{x(n)-x(-n)}{2}$$

prob: Determine even and odd components of the following signals.

a) $x(t) = e^{-at}$ b) $x(t) = 1+2t+3t^2+4t^3$

c) $x(t) = \sin 2t + \sin 2t \cos 2t + \cos 2t$ d) $x(t) = e^{j2t}$

Soln: a) Given $x(t) = e^{-at}$

$$x(-t) = e^{at}$$

Even Component $x_e(t) = \frac{x(t)+x(-t)}{2}$

$$x_e(t) = \frac{e^{at}+e^{-at}}{2} = \cosh at$$

odd Component $x_o(t) = \frac{x(t)-x(-t)}{2}$

$$x_o(t) = \frac{e^{-at}-e^{at}}{2} = -\sinh at$$

b). $x(t) = 1+2t+3t^2+4t^3$

$$x(-t) = 1-2t+3t^2-4t^3$$

Even Component $x_e(t) = \frac{x(t)+x(-t)}{2}$

$$= \frac{2+6t^2}{2}$$

$$x_e(t) = 1+3t^2$$

odd Component $x_o(t) = \frac{x(t)-x(-t)}{2}$

$$x_o(t) = \frac{4t+8t^3}{2}$$

$$x_o(t) = 2t+4t^3$$

⟹ Energy / Power Signals:

The Energy of a Continuous time Signal $x(t)$ denoted by 'E' and is defined as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

The Average Power of a Continuous time Signal $x(t)$, denoted by P_{avg} and is defined as

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

The Energy of a Discrete time Signal $x(n)$ denoted by 'E' and is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The Average Power of a Discrete time Signal $x(n)$ denoted by P_{avg} and is defined as

$$P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

" A Signal is Said to be an Energy Signal if it has finite Energy. $(0<E<\infty)$. For Energy Signals, the average power (P_{avg}) is Zero."

∴ for an Energy Signal $E =$ finite, $P_{avg} = 0$

" A Signal is Said to be a power Signal if it has finite average power $(0<P<\infty)$. For power Signals, the Energy is infinity."

∴ for a power Signal, $P_{avg} =$ finite, $E = \infty$.

* All finite duration Signals are Energy Signals and all periodic Signals are Power Signals.
* Signals that donot Satisfy the above Conditions are neither energy, nor power Signals.
* $x(t) = e^{-at} \cdot u(t)$, $a>0$ is a Const. is energy Signal though it has infinite duration but it's magnitude approaches to zero as 't' tends to infinity.

prob: Determine whether the following Signals are Energy Signals or power Signals. and calculate their energy or power.

a) $x(t) = 5\sin\left(\frac{2\pi}{3}t\right)$

b) $x(t) = e^{-j\left(\frac{5\pi}{2}t - \pi/6\right)}$

c) $x(t) = A e^{-at} u(t)$, $a>0$

d) $x(t) = A\,\mathrm{rect}\left(\frac{t}{\tau}\right)$

e) $x(t) = u(t) - u(t-4)$

f) $x(n) = \sin\left(\frac{\pi}{3}n\right)$

g) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

h) $x(n) = n \cdot u(n)$

(i) $x(n) = e^{j\left(\frac{\pi}{3}n + \frac{\pi}{2}\right)}$

Soln:

a) Given the Signal

$$x(t) = 5\sin\left(\frac{2\pi}{3}t\right)$$

Since $x(t)$ is Continuous time periodic Signal it is a power Signal.

Energy of a Signal is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} 25\sin^2\left(\frac{2\pi}{3}t\right) dt$$

$$\therefore \quad E = 25 \int_{-\infty}^{\infty} \left(\frac{1 - \cos \frac{4\pi}{3} t}{2} \right) dt$$

$$= \frac{25}{2} \left[\int_{-\infty}^{\infty} 1 . dt - \int_{-\infty}^{\infty} \cos \frac{4\pi}{3} t \, dt \right]$$

$$= \frac{25}{2} \left(t \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \cos \frac{4\pi}{3} t \, dt \right)$$

$$= \frac{25}{2} \left([\infty - (-\infty)] - \int_{-\infty}^{\infty} \cos \frac{4\pi}{3} t \, dt \right)$$

$$E = \frac{25}{2} \left(\infty - \int_{-\infty}^{\infty} \cos \frac{4\pi}{3} t \, dt \right) = \infty$$

The Average power of a signal $x(t)$ is given by

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} 25 \sin^2 \left(\frac{2\pi}{3} t \right) dt$$

$$= \frac{25}{2} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(1 - \cos \frac{4\pi}{3} t \right) dt$$

$$= \frac{25}{2} \lim_{T \to \infty} \frac{1}{2T} \left(\int_{-T}^{T} 1 . dt - \int_{-T}^{T} \cos \frac{4\pi}{3} t \, dt \right)$$

$$= \frac{25}{2} \lim_{T \to \infty} \frac{1}{2T} \left(t \Big|_{-T}^{T} - \sin \frac{4\pi}{3} t \Big|_{-T}^{T} \right)$$

$$= \frac{25}{2} \lim_{T \to \infty} \frac{1}{2T} \left(T - (-T) - \left[\sin \frac{4\pi}{3} T + \sin \frac{4\pi}{3} T \right] \right)$$

$$= \frac{25}{2} \lim_{T \to \infty} \left(1 - \frac{1}{2T} \left(2 \sin \frac{4\pi}{3} T \right) \right)$$

$$P_{avg.} = \frac{25}{2} (1 - 0) = 12.5 \text{ Watts}$$

Since $P_{avg} = 12.5$ Watts (finite) & $E = \infty$ the given signal is a power signal.

b) Given the Signal

$$x(t) = e^{-j\left(\frac{5\pi}{2}t - \frac{\pi}{6}\right)}$$

Since the given Signal is CT Complex exponential Signal which is periodic always and is a power Signal.

$\therefore$ The Energy of a Signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} \left| e^{-j\left(\frac{5\pi}{2}t - \frac{\pi}{6}\right)} \right|^2 dt$$

$$|e^{j\theta}| = 1$$

$$= \int_{-\infty}^{\infty} 1 \, dt = t \Big|_{-\infty}^{\infty} = \infty - (-\infty) = \infty.$$

$$\therefore \boxed{E = \infty}$$

The Average Power of a Signal $x(t)$ is given by

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e^{-j\left(\frac{5\pi}{2}t - \frac{\pi}{6}\right)} \right|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} 1 \cdot dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} t \Big|_{-T}^{T} = \lim_{T \to \infty} \frac{1}{\cancel{2T}} \left(T - (-T)\right)$$

$$= \lim_{T \to \infty} (1) = 1 \text{ Watt}$$

$\therefore$ $P_{avg} = 1$ Watt (finite) & $E = \infty$ the given Signal is a power Signal.

c) The given Signal is

$$x(t) = A e^{-at} u(t), \quad a>0$$

The Energy of a Signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |A e^{-at} u(t)|^2 dt$$

$$= A^2 \int_{0}^{\infty} e^{-2at} \cdot dt, \qquad u(t) = \begin{matrix} 1, & t \geq 0 \\ 0, & t<0 \end{matrix}$$

$$= A^2 \left. \frac{e^{-2at}}{(-2a)} \right|_0^{\infty}$$

$$E \cdot = \frac{-A^2}{2a}(0-1) = \frac{A^2}{2a} \text{ Joules}$$

The Average Power of a Signal $x(t)$ is given by

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |A e^{-at} u(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T}^{T} e^{-2at} \cdot dt. \qquad u(t) = \begin{matrix} 1, & t \geq 0 \\ 0, & t<0 \end{matrix}$$

$$= \lim_{T\to\infty} \frac{A^2}{2T} \left. \frac{e^{-2at}}{(-2a)} \right|_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{-A^2}{4aT} \left(e^{-2aT} - e^{2aT} \right)$$

$$P_{avg} = 0$$

Since $E = \frac{A^2}{2a}$ Joules (finite) & $P_{avg} = 0$, the given Signal is an Energy Signal.

d) Given Signal

$$x(t) = A \, \mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} A, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Since $x(t)$ is a finite duration Signal it is an Energy Signal.

The Energy of a Signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\tau/2}^{\tau/2} |A|^2 \, dt$$

$$= A^2 \int_{-\tau/2}^{\tau/2} 1 . \, dt = A^2 . \, t \Big|_{-\tau/2}^{\tau/2}$$

$$E = A^2\left(\tau/2 - (-\tau/2)\right) = A^2\tau \text{ Joules}$$

The Average Power of a Signal $x(t)$ is given by

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\tau/2}^{\tau/2} A^2 \, dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2T} \left(t \Big|_{-\tau/2}^{\tau/2} \right)$$

$$= \lim_{T\to\infty} \frac{A^2}{2T} (\tau) = \frac{A^2\tau}{\infty} = 0$$

$\therefore$ Since $E = A^2\tau$ Joules (finite) and $P_{avg} = 0$, the given Signal is an Energy Signal.

(9) Given the Signal,

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

The Energy of a Signal $x(n)$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u(n)\right|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots \infty \text{ terms}$$

$1 + a + a^2 + a^3 + \cdots \infty$ terms $= \frac{1}{1-a}$, $|a| < 1$

$$E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \text{ Joules.}$$

The Average Power of a Signal $x(n)$ is defined as

$$P_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{2}\right)^n u(n)\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots (N+1) \text{ terms}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right)$$

$$P_{avg} = \frac{1}{\infty}\left(\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right) = 0$$

Since $E = \frac{4}{3}$ Joules (finite) & $P_{avg} = 0$, the given Signal is an Energy Signal.

(h) The given Signal is

$$x(n) = n u(n)$$

The Energy of a Signal $x(n)$ is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{\infty} |n\, u(n)|^2$$

$$= \sum_{n=0}^{\infty} n^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + \cdots \infty \text{ terms}$$

$$E = \infty$$

The Average Power of a Signal $x(n)$ is given by

$$P_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |n\, u(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} n^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left(1^2 + 2^2 + 3^2 + 4^2 \cdots \left(\cancel{N+1}\right) \text{ terms} \right)$$

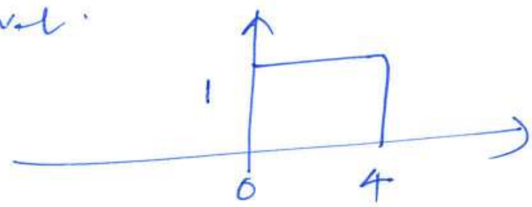
$$= \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \left(\frac{N(N+1)\cancel{(2N+1)}}{6} \right)$$

$$P_{avg} = \frac{\infty}{6} = \infty$$

Since $E = \infty$ & $P_{avg} = \infty$ the given Signal $x(n)$ is neither energy nor power signal.

e). $x(t) = u(t) - u(t-4)$

Energy Signal



(i) $x(n) = e^{j\left(\frac{\pi}{3}n + \pi/2\right)}$ periodic Signal

So power Signal

⇒ Causal / Non-Causal / Anti Causal Signals:

A Signal $x(t)/x(n)$ which is defined only for positive values of time t/n is called a causal Signal. Otherwise the Signal is non-causal. Thus for a Causal Signal we have,

$$x(t) = 0, \ t<0, \quad x(n) = 0, \ n<0$$

A Signal for which $x(t)$ or $x(n) = 0$, for $t>0$ or $n>0$ is called anti causal Signal.

Ex:

$x(t) = e^{-2t} . u(t)$

$x(t) = A \sin \omega_0 t \, u(t)$

$x(n) = u(n)$

$x(n) = \{1, 2, 3, 4\}$

Causal Signals

$x(t) = e^{-|t|}$

$x(t) = \sin \omega_0 t$

$x(n) = (1/2)^n$

$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

Non Causal Signals

$x(t) = e^{-2t} u(-t)$ → Anti Causal Signal.

* Anti causal Signal is also called non-causal but non-causal Signal need not be anti causal Signal.

⇒ Systems:

A System is defined as an entity Or interconnection of Components that gives some output in response to an input Signal.

(or)

Any process that exhibits, Cause and effect relation can be called a System. The output Signal will be transformed Version Of the input Signal.

The diagramatic representation of a System is as Shown.

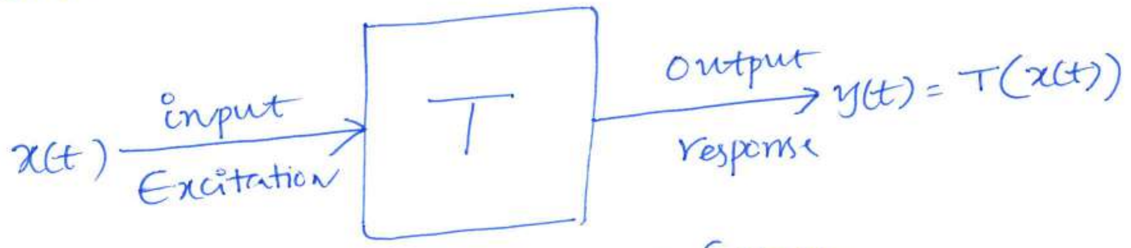


fig: Representation of a System.

Classification of Systems:

The Systems are classified as follows.

- → Continuous time / Discrete Time Systems
- → Static (Memoryless) / Dynamic (Memory) Systems
- → Linear / Non-Linear Systems
- → Time invariant / Time Variant Systems
- → Linear Time Variant (LTI) / Linear Time Variant (LTV) Systems
- → Stable / Unstable Systems
- → Causal / Non-Causal Systems
- → Finite Impulse response (FIR) / Infinite Impulse Response (IIR) System
- → Invertible / Non-invertible Systems.

⇒ Continuous Time / Discrete Time Systems:

A System which can operates on Continuous time Signals is called as Continuous time System and a System which Can operates on Discrete time Signals is called as Discrete Time System.

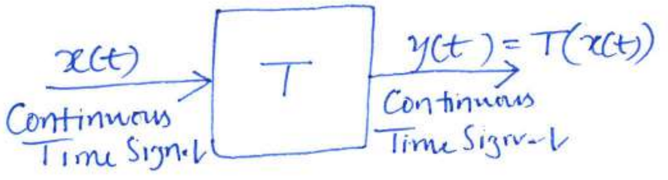


fig: CT System

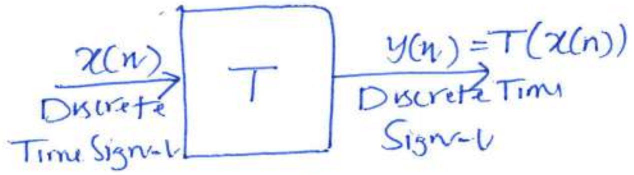


fig: DT System

⟹ Static (Memoryless) / Dynamic (Memory) Systems:

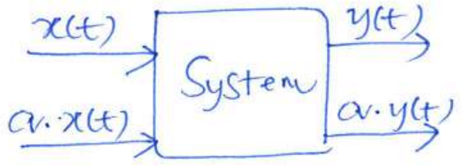
A System is Said to be Static/Memoryless if the response (output) of the System at any given time depends only on the input given at that time only (present inputs). In any other case the System is Said to be dynamic or Said to have memory.

Ex:

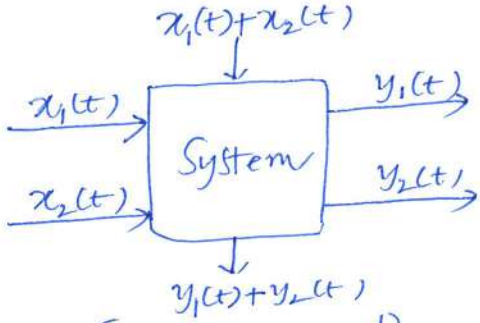
$$\left.\begin{aligned} y(t) &= x(t) + 5 \\ y(t) &= x^2(t) \\ y(n) &= e^{x(n)} \\ y(n) &= 5x^2(n) \end{aligned}\right\} \longrightarrow \text{Static Systems}$$

$$\left.\begin{aligned} y(t) &= x(t^2) \\ y(n) &= x(2n) \\ y(t) &= x(t-2) \\ y(n) &= x(n) + x(n-1) \end{aligned}\right\} \longrightarrow \text{Dynamic Systems.}$$

⟹ Linear/Non-Linear Systems: A System which Satisfies both homogenity and Superposition principles is known as Linear System. i.e



Homogenity Principle



Superposition Principle.

if $x_1(t) \longrightarrow y_1(t)$ ~~$x_2(t)$~~ and

$x_2(t) \longrightarrow$ ~~x_2~~ $y_2(t)$ then

$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$

i.e a System is linear if the o/p due to weighted Sum of inputs is equal to the weighted Sum of outputs.

$|||^{ly}$ for DT Systems

$$\text{if } x_1(n) \longrightarrow y_1(n) \text{ and}$$
$$x_2(n) \longrightarrow y_2(n) \text{ then}$$
$$a \cdot x_1(n) + b \cdot x_2(n) \longrightarrow a\,y_1(n) + b\,y_2(n)$$

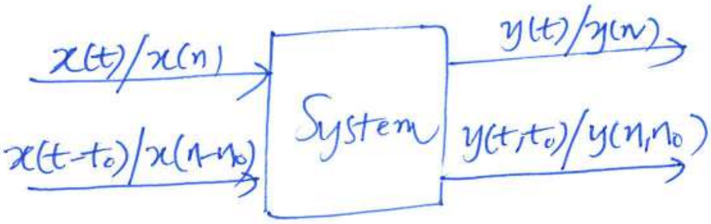
Ex:

$$y(t) = x(t) \longrightarrow \text{Linear}$$
$$y(t) = 2x(t) + 5 \longrightarrow \text{Non-linear}$$
$$y(n) = x(n^2) \longrightarrow \text{Linear}$$
$$y(n) = n \cdot x^2(n) \longrightarrow \text{Non-linear}$$
$$\frac{dy(t)}{dt} + 5\,y(t) = x(t) \longrightarrow \text{Linear.}$$

⇒ Time invariant / Time Variant Systems:

A System is Said to be time invariant or Shift invariant if its input/output Characteristics donot Changes with respect to time. If the output of the System Changes with respect to time then the System is Called time Variant or Shift variant.

In otherwords if the output of the System due to a delayed input is Same as the delayed output then the System is Said to be time invariant other wise the System is Said to be time Variant.



$$\text{if } \quad y(t, t_0) = y(t - t_0) \quad \text{(or)}$$
$$y(n, n_0) = y(n - n_0) \quad \text{then the System is}$$

Time invariant or Shift invariant.

Ex: $y(t) = x(t^2) \longrightarrow$ Time Variant

$y(n) = x(-n) \longrightarrow$ Time Variant

$y(t) = x(t)+5 \longrightarrow$ Time invariant

$y(t) = e^{x(t)} \longrightarrow$ Time invariant

$y(n) = x(n) + x(n-1) \longrightarrow$ Time invariant.

⇒ Linear Time invariant (LTI) / Linear Time Variant (LTV) Systems:

A System which Satisfies both Linearity and time invariance properties is known as an LTI System.

A System which Satisfies both Linearity and time Variance properties is known as an LTV System.

Most of the physically realizable System are Linear time invariant Systems.

Ex: $y(t) = x(t) \longrightarrow$ LTI System.

$y(t) = x(-t) \longrightarrow$ LTV System.

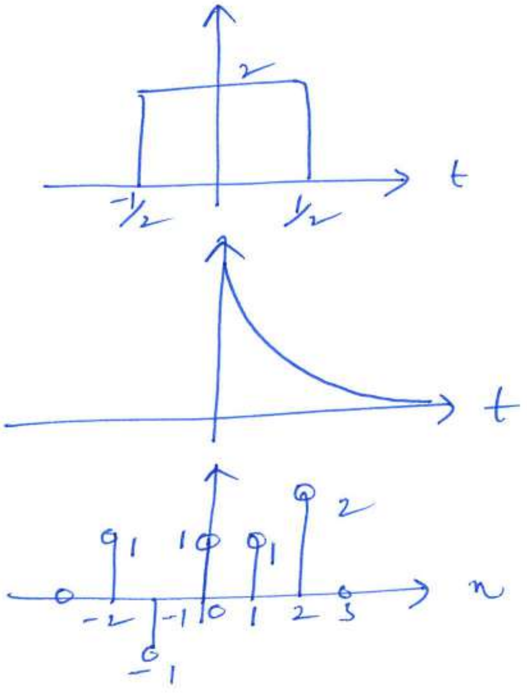
⇒ Stable/Unstable Systems: An arbitrary relaxed System is Said to be BIBO (Bounded i/p Bounded o/p) Stable if and only if every bounded input produces a bounded input. If for any bounded input, if the output is unbounded then the System is Said to be unstable.

A bounded i/p Signal is one for which there exists a Constant M_x Such that,

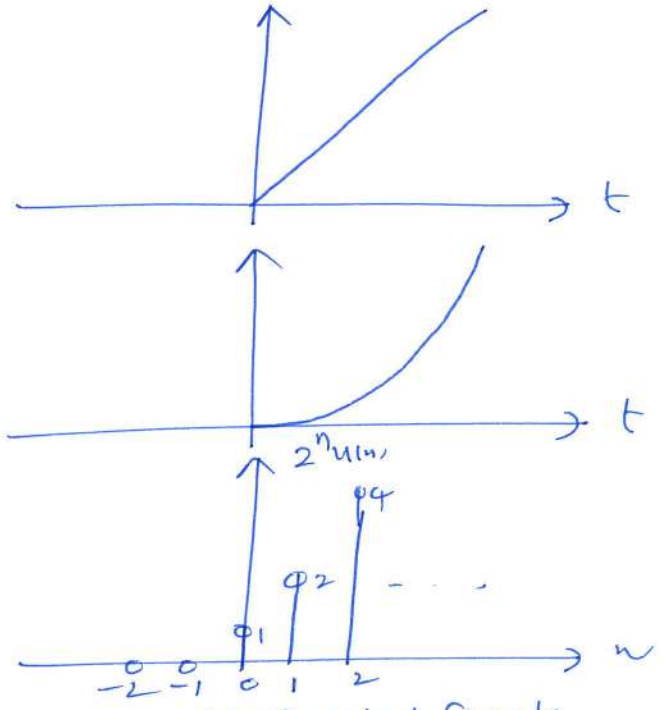
$$|x(t)| \leq M_x < \infty$$

|||ly A bounded output y(t) for which We have

$$|y(t)| \leq M_y < \infty$$



Bounded Signals



Un Bounded Signals.

Causal / Non-Causal Systems:

A System is Said to be Causal if the output of the System at any time t/n depends only on the present and past inputs ~~only~~. but not on future inputs. If at any time the o/p depends on future inputs, then the System is Said to be non-causal System.

Causal Systems are real time Systems and are physically realizable Systems.

Ex:

$$\left.\begin{aligned} y(t) &= x(t) + x(t-1) \\ y(t) &= x^2(t) \end{aligned}\right\} \text{— Causal System}$$

$$\left.\begin{aligned} y(t) &= x(2t) \\ y(n) &= x(n) + x(n+1) \end{aligned}\right\} \text{Non-Causal Systems.}$$

Convolution and Correlation of Signals:

Convolution: Convolution is a mathematical operation that can be performed b/w two signals. Convolution is used to determine the response of an LTI system.

The convolution of two signals $x_1(t)$ and $x_2(t)$ is defined as,

$$\underbrace{x_1(t) * x_2(t)}_{\text{Convolution operator}} = \underbrace{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau}_{\text{Convolution integral.}}$$

properties of Convolution:

- Commutative property:

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

- Associative Property:

$$x_1(t) * \big(x_2(t) * x_3(t)\big) = \big(x_1(t) * x_2(t)\big) * x_3(t)$$

- Distributive Property:

$$x_1(t) * \big(x_2(t) + x_3(t)\big) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Correlation: Correlation is a mathematical operation, similar to convolution. Correlation also uses two signals to form a third signal. Correlation is very useful in communications, and can be used to compare two signals in order to determine the degree of similarity b/w the signals.

Correlation may be auto correlation or cross correlation.

When one signal is correlated with another signal to form a third signal, it is called cross-correlation. and when a signal is correlated with itself to form another signal it is called auto correlation.

Autocorrelation function: The Auto correlation function gives the measure of similarity b/w a signal and its time delayed version.

The Autocorrelation function of an energy signal $x(t)$ is defined as

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

where $\tau \rightarrow$ delay Parameter.

If $x(t)$ is shifted by τ in negative direction, then

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau) \cdot x^*(t)\, dt.$$

The Autocorrelation function of a periodic signal with period 'T' is given by,

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot x^*(t-\tau)\, dt$$

Cross Correlation function: The Cross Correlation function is a measure of similarity b/w one signal and the time delayed version of another signal.

The Cross Correlation function of two energy signals $x_1(t)$ and $x_2(t)$ is defined as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2^*(t)\, dt$$

If two Signals $x_1(t)$ and $x_2(t)$ are real then,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t-\tau)\,dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\,dt$$

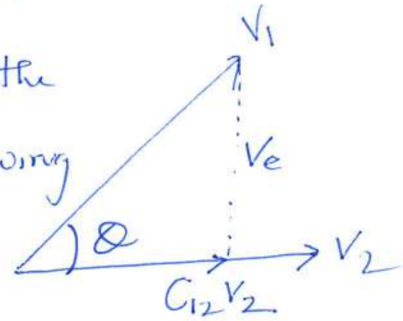
The cross correlation function of two periodic Signals $x_1(t)$ & $x_2(t)$ with Same period 'T' is defined as

$$R_{12}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x_1(t) \cdot x_2^*(t-\tau)\,dt .$$

Analogy b/w Vectors And Signals:

Vectors: A vector is Specified by its magnitude & direction.

Consider two Vectors V_1 & V_2 as shown.



The Component of Vector V_1 along the Vector V_2 can be obtained by drawing a perpendicular from the end of V_1 onto V_2.

Thus the Component of Vector V_1 along Vector V_2 is given by

$$V_1 \simeq C_{12} V_2$$

Where C_{12} is Chosen Such that the error Vector V_e is minimum. Thus the magnitude of C_{12} is an indication of the Similarity of the two Vectors. If $C_{12} = 0$, then Vector V_1 has no Component along another Vector V_2 and hence two Vectors are mutually perpendicular. or orthogonal Vectors. Thus if two Vectors are orthogonal, then

$$C_{12} = 0 .$$

The Component of Vector V_1 along V_2 is given by

$$|V_1| \cos\theta$$

$$= \frac{V_1 \cdot V_2}{\cancel{|V_1|}|V_2|} \cdot \cancel{|V_1|}$$

$$= \frac{V_1 \cdot V_2}{|V_2|} = \frac{V_1 \cdot V_2}{V_2} = C_{12} V_2$$

$$\therefore \quad C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{V_1 \cdot V_2}{V_2 \cdot V_2}$$

when V_1 & V_2 are orthogonal $\Rightarrow$ $V_1 \cdot V_2 = 0 \Rightarrow C_{12} = 0$

Signals: The Concept of Vector orthogonality Can be extended to signals.

Let us consider two Signals $f_1(t)$ and $f_2(t)$.

The Approximation of $f_1(t)$ interms of $f_2(t)$ over a Certain interval $(t_1 < t < t_2)$ is given by

$$f_1(t) \simeq C_{12} f_2(t), \quad t_1 < t < t_2 \longrightarrow (1)$$

For better Approximation C_{12} is Chosen Such that the error between the actual function $(f_1(t))$ and the approximated function $(C_{12} f_2(t))$ is minimum over the interval (t_1, t_2).

Let us define an error function as.

$$f_e(t) = f_1(t) - C_{12} f_2(t) \longrightarrow (2)$$

One possible Criterion for minimizing the error $f_e(t)$ over (t_1, t_2) is to minimize the average Value of the error function $f_e(t)$ over this interval, i.e to minimize

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f_1(t) - C_{12} f_2(t))\, dt \longrightarrow (3)$$

However Above criterion is inadequate because there can be largest positive and negative errors present that may cancel one another in the process of averaging and give false indication that the error is zero. This situation can be corrected if we choose to minimize the average (or mean) of the square of the error instead of the error itself. Let us denote the average of $f_e^2(t)$ by 'ε',

$$\varepsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} f_e^2(t)\,dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[f_1(t) - C_{12}f_2(t)\right]^2 dt \quad \rightarrow (4)$$

To find the value of C_{12}, which will minimize 'ε' we must have,

$$\frac{d\varepsilon}{dC_{12}} = 0$$

$$\frac{d}{dC_{12}}\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[f_1(t) - C_{12}f_2(t)\right]^2 dt\right) = 0$$

$$\frac{d}{dC_{12}}\left(\cancel{\int_{t_1}^{t_2} f_1^2(t)\,dt}^{\,0} - 2\int_{t_1}^{t_2} f_1(t)\,C_{12}f_2(t)\,dt + \int_{t_1}^{t_2} C_{12}^2 f_2^2(t)\,dt\right) = 0$$

$$-2\int_{t_1}^{t_2} f_1(t)\cdot f_2(t)\cdot dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt = 0$$

$$\therefore \cancel{2}\,C_{12}\int_{t_1}^{t_2} f_2^2(t)\,dt = \cancel{2}\int_{t_1}^{t_2} f_1(t)\,f_2(t)\,dt$$

$$\therefore \boxed{C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)\,f_2(t)\,dt}{\int_{t_1}^{t_2} f_2^2(t)\,dt}}$$

∴ By Analogy with Vectors, we Conclude that $f_1(t)$ was a component of waveform $f_2(t)$ and this component has a magnitude C_{12}. If $C_{12} = 0$, then the Signal $f_1(t)$ contains no Component of Signal $f_2(t)$ an we say that the two functions are orthogonal over an interval (t_1, t_2). Thus If two functions $f_1(t)$ & $f_2(t)$ are orthogonal over an interval (t_1, t_2) if and only if

$$\boxed{\int_{t_1}^{t_2} f_1(t) \cdot f_2(t) \cdot dt = 0}$$

prob: A rectangular function $f(t)$ is given by

$$f(t) = \begin{cases} 1, & 0 \leq t \leq \pi \\ -1, & \pi \leq t \leq 2\pi \end{cases}$$

Approximate the above function by a waveform $\sin t$ over the interval $(0, 2\pi)$ Such that the mean Square error is minimum.

Soln: Given the function.

$$f(t) = \begin{cases} 1, & 0 \leq t \leq \pi \\ -1, & \pi \leq t \leq 2\pi. \end{cases}$$

Approximation of $f(t)$ by $\sin t$ over $(0, 2\pi)$ is given by

$$f(t) \simeq C_{12} f_2(t) \simeq C_{12} \sin t.$$

$$\because f_2(t) = \sin t$$

The Optimum Value of C_{12} Such that the mean Square error 'ε' is minimum is given by

$$C_{12} = \frac{\int_0^{2\pi} f(t) \sin t \cdot dt}{\int_0^{2\pi} \sin^2 t \cdot dt}$$

$$I = -\frac{1}{2}\left[\frac{1}{(n+m)\omega_0}\left(\cos\left[(n+m)\omega_0 t_0 + 2\pi(n+m)\right] - \cos(n+m)\omega_0 t_0\right) + \frac{1}{(n-m)\omega_0}\left(\cos\left[(n-m)\omega_0 t_0 + 2\pi(n-m)\right] - \cos(n-m)\omega_0 t_0\right)\right]$$

$$I = -\frac{1}{2}\left[\frac{1}{(n+m)\omega_0}\left(\cancel{\cos(n+m)\omega_0 t_0} - \cancel{\cos(n+m)\omega_0 t_0}\right) + \frac{1}{(n-m)\omega_0}\left(\cancel{\cos(n-m)\omega_0 t_0} - \cancel{\cos(n-m)\omega_0 t_0}\right)\right]$$

$\underline{\underline{\cos(2\pi n + \theta) = \cos\theta}}$

$$I = 0.$$

$\therefore$ The functions $\sin n\omega_0 t$ & $\cos m\omega_0 t$ are orthogonal over the interval $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$

<u>Note</u>: The functions $e^{jn\omega_0 t}$ & $e^{jm\omega_0 t}$ are orthogonal over the interval $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$.

2) A Set of functions $g_1(t), g_2(t), \dots g_r(t)$ mutually orthogonal over the interval (t_1, t_2) is said to be a complete or closed set if there exists no function $x(t)$ for which it is true that,

$$\int_{t_1}^{t_2} x(t)\, g_k(t)\, dt = 0 \text{ for } k = 1,2,3,4 \dots$$

3) Representation of a function $f(t)$ by a set of infinite mutually orthogonal functions is called generalized Fourier Series representation of $f(t)$.

prob: Show that the functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ are orthogonal over any interval, $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$ for integer values of m & n.

Soln: Two signals $f_1(t)$ & $f_2(t)$ are ~~orthogonal~~ over an interval (t_1, t_2) is given by

$$\int_{t_1}^{t_2} f_1(t) \cdot f_2(t) \cdot dt = 0$$

$$I := \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

$$I = \frac{1}{2} \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \left[\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t\right] dt$$

$$= \frac{1}{2} \left[-\frac{\cos(n+m)\omega_0 t}{(n+m)\omega_0} - \frac{\cos(n-m)\omega_0 t}{(n-m)\omega_0} \right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{-1}{2} \left[\frac{1}{(n+m)\omega_0} \left(\cos(n+m)\omega_0 \left(t_0 + \frac{2\pi}{\omega_0}\right) - \cos(n+m)\omega_0 t_0 \right) + \frac{1}{(n-m)\omega_0} \left(\cos(n-m)\omega_0 \left(t_0 + \frac{2\pi}{\omega_0}\right) - \cos(n-m)\omega_0 t_0 \right) \right]$$

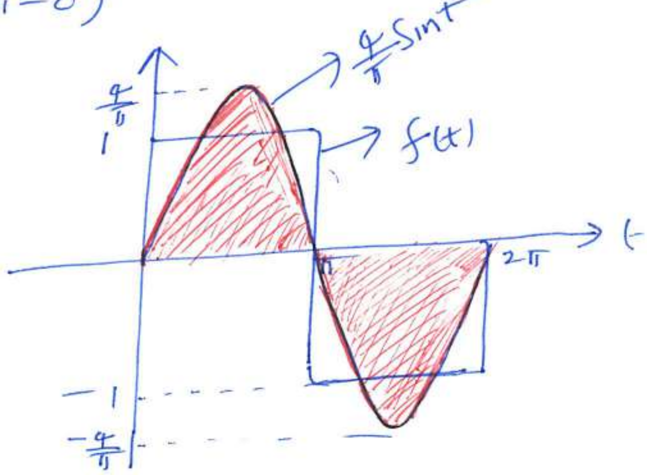
$$C_{12} = \frac{\int_0^{\pi} 1 \cdot \sin t \, dt - \int_{\pi}^{2\pi} 1 \cdot \sin t \, dt}{\int_0^{2\pi} \frac{1-\cos 2t}{2} \, dt}$$

$$C_{12} = \frac{-\cos t \Big|_0^{\pi} + \cos t \Big|_{\pi}^{2\pi}}{\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)_0^{2\pi}}$$

$$= \frac{-(-1-1) + (1-(-1))}{\frac{1}{2}(2\pi - 0)}$$

$$C_{12} = \frac{4}{\pi}$$

$$\therefore \quad f(t) \simeq \frac{4}{\pi} \sin t$$



$\Rightarrow$ Approximation of a function by a Set of Mutually Orthogonal functions:

Let us Consider a Set of 'n' functions $g_1(t)$, $g_2(t), g_3(t) \ldots g_n(t)$ which are orthogonal to one another over an interval (t_1, t_2), i.e

$$\int_{t_1}^{t_2} g_i(t) \cdot g_j(t) \, dt = 0, \quad i \neq j \quad \longrightarrow (1)$$

and let $\int_{t_1}^{t_2} g_j^2(t) \, dt = K_j$

Let an arbitrary function $f(t)$ be approximated over an interval (t_1, t_2) by a linear Combination of these 'n' mutually orthogonal functions as

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \cdots + C_K g_K(t) + \cdots + C_n g_n(t)$$

$$\therefore \quad f(t) \simeq \sum_{r=1}^{n} C_r g_r(t) \longrightarrow ②$$

for better Approximation, we must find the Values of the constants, $C_1, C_2 \ldots C_n$ such that the mean of the square of $f_e(t)$ i.e 'ε' is minimized.

By definition,

$$f_e(t) = f(t) - \sum_{r=1}^{n} C_r g_r(t) \text{ and}$$

$$\varepsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} f_e^2(t)\,dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[f(t) - \sum_{r=1}^{n} C_r g_r(t)\right]^2 dt \longrightarrow ③$$

Above equation is a function of the constants, $C_1, C_2 \ldots C_n$ and to minimize 'ε' we must have,

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots \frac{\partial \varepsilon}{\partial C_j} = \cdots \frac{\partial \varepsilon}{\partial C_n} = 0$$

let us consider the equation,

$$\frac{\partial \varepsilon}{\partial C_j} = 0$$

$$\therefore \frac{\partial}{\partial C_j}\left(\int_{t_1}^{t_2}\left[f(t) - \sum_{r=1}^{n} C_r g_r(t)\right]^2 dt\right) = 0 \quad \text{from Eqn ③} \longrightarrow ④$$

After expanding the above expression, all the terms of the form $\int g_i(t).g_j(t)\,dt$ are zero. Also, the derivative with respect to C_j of all the terms that do not contain C_j are zero, i.e

$$\frac{\partial}{\partial C_j}\left(\int_{t_1}^{t_2} f^2(t)\,dt\right) = \frac{\partial}{\partial C_j}\left(\int_{t_1}^{t_2} C_r^2 g_r^2(t)\,dt\right) = \frac{\partial}{\partial C_j}\left(\int_{t_1}^{t_2} C_r f(t) g_r(t)\,dt\right) = 0$$

This leaves only two non zero terms in Eqn ④ as,

$$\frac{\partial}{\partial C_j}\left(\int_{t_1}^{t_2}\left[-2C_j f(t) g_j(t) + C_j^2 g_j^2(t)\right] dt\right) = 0$$

After changing the order of integration & Differentiation, we get,

$$\cancel{2}\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt = \cancel{2} C_j \int_{t_1}^{t_2} g_j^2(t)\, dt$$

$$\therefore \quad C_j = \frac{\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} \qquad \text{(or)}$$

$$\boxed{C_j = \frac{1}{K_j} \int_{t_1}^{t_2} f(t)\, g_j(t)\, dt}$$

Mean Square Error:

The value of the Mean Square error 'ε' when the optimum value of the coefficients $C_1, C_2, C_3 \dots C_n$ are chosen according to the equation,

$$C_j = \frac{1}{K_j} \int_{t_1}^{t_2} f(t)\, g_j(t)\, dt \longrightarrow (1) \quad \text{is}$$

given by,

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^{n} C_r\, g_r(t) \right]^2 dt$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{r=1}^{n} \int_{t_1}^{t_2} C_r\, f(t)\, g_r(t)\, dt + \int_{t_1}^{t_2} \sum_{r=1}^{n} C_r^2\, g_r^2(t)\, dt \right]$$

$$\therefore \varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t)\, dt - 2\sum_{r=1}^{n} C_r \int_{t_1}^{t_2} f(t) \cdot g_r(t)\, dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt \right]$$

$$\varepsilon = \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{r=1}^{n} C_r . C_r K_r + \sum_{r=1}^{n} C_r^2 K_r\right]$$

$$= \frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{r=1}^{n} C_r^2 K_r + \sum_{r=1}^{n} C_r^2 K_r\right)$$

$$\varepsilon = \frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{r=1}^{n} C_r^2 K_r\right) \quad \text{(or)}$$

$$\varepsilon = \frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2} f^2(t)\,dt - \left(C_1^2 K_1 + C_2^2 K_2 + C_3^2 K_3 + \cdots + C_n^2 K_n\right)\right) \Longrightarrow (2)$$

Note: 1) In equation (2) if we increase the Value of 'n'
i.e if we approximate a function by a large no. of
orthogonal functions, the error will become smaller. Since
'ε' is a positive quantity, hence in the limit as the no. of terms
in the approximation made infinity, the sum $\sum_{r=1}^{n} C_r^2 K_r$
may converge to the integral $\int_{t_1}^{t_2} f^2(t)\,dt$ and 'ε' vanishes.

Thus,
$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{r=1}^{n} C_r^2 K_r$$

Under these conditions, $f(t)$ is represented by the infinite series as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots \longrightarrow (3)$$

Now the series on the RHS side of Eqn (3) converges to $f(t)$ and the representation of $f(t)$ is exact.

Fourier Series

Representation of Signals over a Certain interval of time interms of Linear Combination of orthogonal functions is called Fourier Series. Fourier Analysis is Some times Called Harmonic Analysis. Fourier Series is applicable for only periodic Signals.

There are mainly two types of Fourier Series.

⇒ Trigonometric Fourier Series

⇒ Exponential Fourier Series

Existence Of Fourier Series (Dirchlet's Conditions) :

The Conditions under which a periodic Signal Can be represented by a Fourier Series are known as Dirchlet's Conditions. They are as follows,

1) → The function $x(t)$ must be a Single Valued function

2) → The function $x(t)$ must have finite no. of maxima & minima

3) → The function $x(t)$ has finite no. of discontinuities.

4) → The function is Absolutely integrable over one period.

i.e $$\int_0^T |x(t)|\,dt < \infty$$

Above Conditions are Sufficient but not necessary Conditions for the existence of Fourier Series of a periodic Signal $x(t)$.

Condition ④ is known as Weak Dirchlet Condition.

Conditions ② & ③ are known as Strong Dirchlets Conditions

Trigonometric Fourier Series (TFS):

We know that the set of functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ (where m & n are integers) i.e,

$\sin\omega_0 t$, $\sin 2\omega_0 t$, $\sin 3\omega_0 t$ ..... $\sin n\omega_0 t$ .....

1, $\cos\omega_0 t$, $\cos 2\omega_0 t$, $\cos 3\omega_0 t$ ..... $\cos n\omega_0 t$ ..... form

a complete set of orthogonal functions.

" Representation of a periodic signal by a Linear Combination of infinite number of Sine and Cosine terms is known as trigonometric fourier series"

i.e. if $x(t \pm T) = x(t) \;\forall t$ then

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

Also,

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_T x(t)\cos n\omega_0 t\,dt \text{ and}$$

$$b_n = \frac{2}{T}\int_T x(t)\sin n\omega_0 t\,dt \text{ are}$$

trigonometric Fourier Series coefficients.

Wave Symmetry :

If the periodic Signal $x(t)$ exhibits some type of Symmetry, then some of the TFS coefficients may become zero and reduces the computation time.

There are Four types of Symmetry a Signal $x(t)$ can have,

1. Even Symmetry
2. odd Symmetry
3. Half Wave Symmetry
4. Quarter Wave Symmetry.

Even Symmetry : A function $x(t)$ is said to have even or mirror Symmetry if

$$x(-t) = x(t) \quad \forall t.$$

for a Signal which exhibits even or mirror Symmetry, the TFS coefficients are as follows.

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt.$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = 0.$$

odd Symmetry : A function $x(t)$ is said to have odd Symmetry if

$$x(-t) = -x(t) \quad \forall t,$$

for a Signal which exhibits odd Symmetry, the TFS coefficients are as follows.

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\,dt$$

Half Wave Symmetry:

A periodic Signal, which Satisfies the Condition, $x(t \pm T/2) = -x(t) \ \forall t$, is Said to have half wave Symmetry.

The Fourier Series of this type of functions will have odd harmonics only. i.e $\omega_0, 3\omega_0, 5\omega_0 \ldots\ldots$

i.e When n is even, $a_n = b_n = 0$,

n is odd,

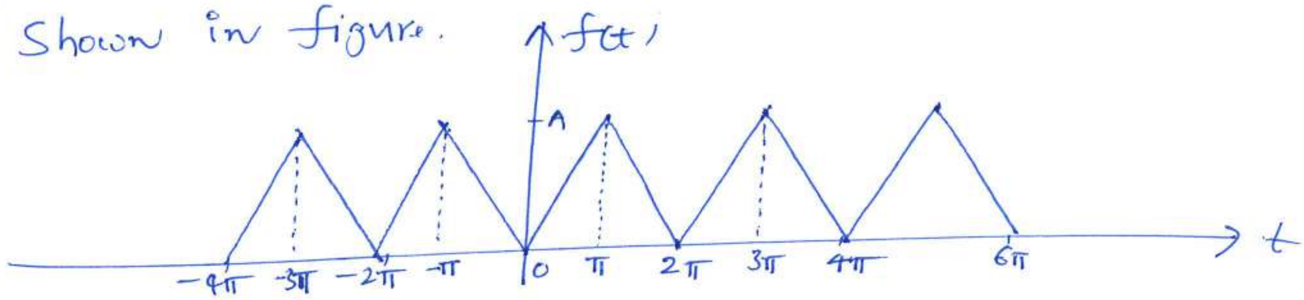
$$a_n = \frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_0 t \, dt$$

&

$$a_0 = 0 \text{ (Always)}$$

prob: Determine the TFS representation for the wave form Shown in figure.



Soln: Given the Signal $f(t)$ is periodic with period, $T = 2\pi$,

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Choose any one Cycle of $f(t)$ i.e from $(-\pi, \pi)$

$$\therefore \quad f(t) = \begin{cases} -\dfrac{At}{\pi}, & -\pi \leq t \leq 0 \\ \dfrac{At}{\pi}, & 0 \leq t \leq \pi \end{cases}$$

Since the given Signal exhibits even Symmetry, i.e

$x(-t) = x(t)$, we have,

$$a_0 = \frac{2}{T}\int_{T/2} x(t)\,dt, \quad a_n = \frac{4}{T}\int_{T/2} x(t)\cos n\omega_0 t, \quad b_n = 0.$$

$$\therefore \quad a_0 = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} \frac{At}{\pi}\,dt$$

$$= \frac{A}{\pi^2}\,\frac{t^2}{2}\Big|_0^{\pi} = \frac{A}{2\cancel{\pi^2}}\left(\cancel{\pi^2} - 0\right) = \frac{A}{2}$$

$$\therefore \quad \boxed{a_0 = \frac{A}{2}}$$

$$a_n = \frac{2\cancel{4}}{\cancel{2}\pi}\int_0^{\pi} \frac{At}{\pi}\cos nt\,dt$$

$$= \frac{2A}{\pi^2}\left[t\,\frac{\sin nt}{n}\Big|_0^{\pi} - \int_0^{\pi}(1)\,\frac{\sin nt}{n}\,dt\right]$$

$$= \frac{2A}{\pi^2}\left[\frac{\pi}{n}\left(\overset{0}{\cancel{\sin n\pi}}\right) - 0 + \frac{1}{n^2}\cos nt\Big|_0^{\pi}\right]$$

$$= \frac{2A}{\pi^2 n^2}\left(\cos n\pi - 1\right) = \frac{2A}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$\therefore \quad a_n = \begin{cases} \dfrac{-4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

∴ The trigonometric Fourier Series is given by

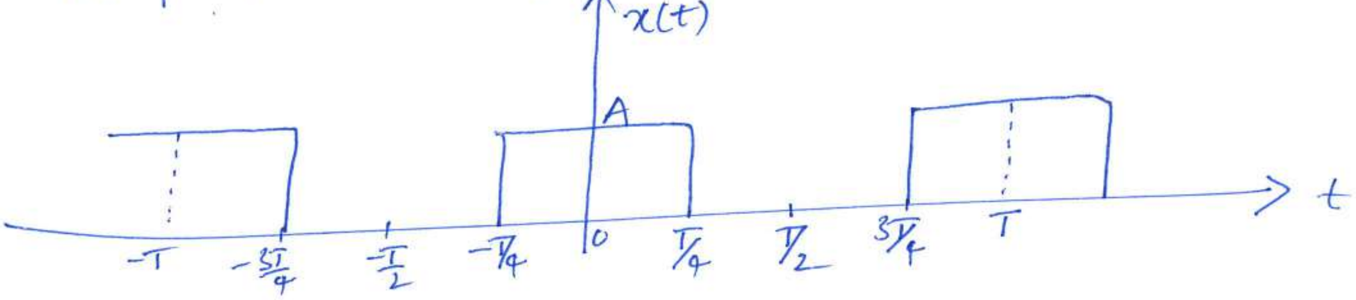
$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-4A}{\pi^2 n^2} \cos nt$$

$$= \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n\ \text{odd}} \frac{\cos nt}{n^2}$$

$$\therefore\ f(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left(\cos t + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots \right)$$

prob. Obtain the trigonometric Fourier Series Components of the periodic rectangular wave form shown in figure.



Soln: Given the Signal $x(t)$ is periodic with period 'T',
$$\omega_0 = \frac{2\pi}{T}.$$

Consider the Signal over one period i.e from $\left(-\frac{T}{4} \text{ to } \frac{3T}{4}\right)$

$$x(t) = \begin{cases} A, & -T/4 \le t \le T/4 \\ 0, & T/4 \le t \le \frac{3T}{4} \end{cases}$$

Since the given Signal exhibits even Symmetry i.e
$x(-t) = x(t)\ \forall t$, We have

$$a_0 = \frac{2}{T}\int_{T/2} x(t)\, dt, \quad a_n = \frac{4}{T}\int_{T/2} x(t) \cos n\omega_0 t\, dt, \quad b_n = 0$$

$$\therefore \quad a_0 = \frac{2}{T}\int_{\cancel{-T/4}\,0}^{\cancel{T/4}\,T/2} A \cdot dt = \cancel{\frac{2A}{T}\, t\Big|_{-T/4}^{T/4}}$$

$$\cancel{= \frac{2A}{T}\left(\frac{T}{4} - \left(\frac{-T}{4}\right)\right)}$$

$$= \frac{2}{T}\int_0^{T/4} A \cdot dt \qquad \cancel{= \frac{2A}{T}\left(\frac{T}{2}\right)}$$

$$= \frac{2}{T} A\, t\Big|_0^{T/4} = \frac{\cancel{2}A}{\cancel{T}}\left(\frac{\cancel{T}}{\cancel{4}_2}\right) = \frac{A}{2}$$

$$\therefore \quad \boxed{a_0 = \frac{A}{2}}$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t). \cos n\omega_0 t\, dt$$

$$= \frac{4}{T}\int_0^{T/4} A \cos n\omega_0 t\, dt$$

$$= \frac{4A}{T} \left.\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{T/4}$$

$$= \frac{4A}{nT\omega_0}\left(\sin \frac{n\omega_0 T}{4} - 0\right)$$

$$= \frac{2\cancel{4}A}{n\cancel{2}\pi}\left(\sin \frac{n(\cancel{2}\pi)}{\cancel{4}_2}\right)$$

$$a_n = \frac{2A}{n\pi} \sin \frac{n\pi}{2}$$

$$= \frac{2A}{n\pi}, \qquad n = 1, 5, 9, \ldots$$

$$= -\frac{2A}{n\pi}, \qquad n = 3, 7, 11 \ldots$$

$$= 0, \qquad n \text{ even}$$

$\therefore$ The trigonometric Fourier Series is now given as

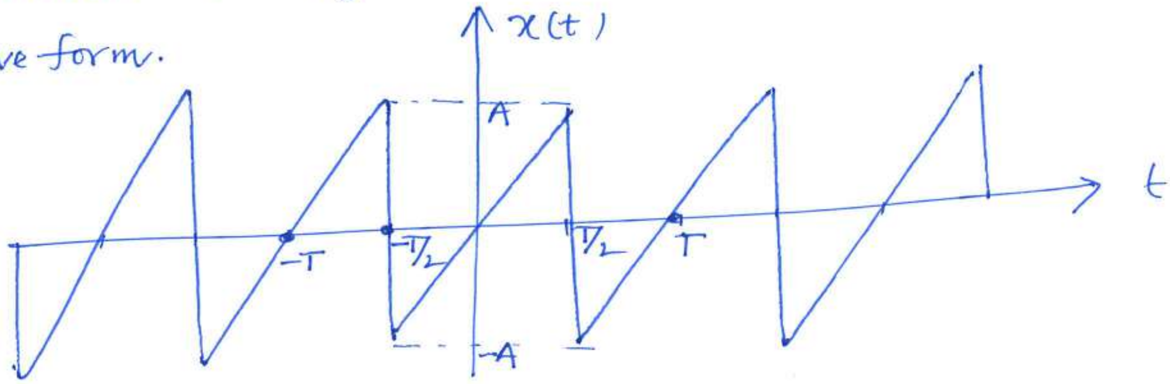
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$x(t) = \frac{A}{2} + \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{2A}{n\pi} \sin\frac{n\pi}{2} \cos n\omega_0 t$$

(or)

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\cos\omega_0 t - \frac{2A}{3\pi}\cos 3\omega_0 t + \frac{2A}{5\pi}\cos 5\omega_0 t - \frac{2A}{7\pi}\cos 7\omega_0 t + \cdots$$

prob: Obtain the trigonometric Fourier Series for the following wave form.



Soln: Given the Signal $x(t)$ is periodic with period 'T'

$$\omega_0 = \frac{2\pi}{T}$$

Since $x(-t) = -x(t)$ the Signal exhibits odd Symmetry.

$\therefore$ $a_0 = 0$

$a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_0 t \cdot dt$$

$(0,0), (T/2, A)$

$y - 0 = \frac{A}{T/2}(x - 0)$

$y = \frac{2Ax}{T}$

$$\therefore x(t) = \frac{2At}{T}, \quad 0 \le t \le \frac{T}{2}$$

$$\therefore\ b_n = \frac{4}{T}\int_0^{T/2} \frac{2At}{T} \sin n\omega_0 t \, dt$$

$$= \frac{8A}{T^2}\left[t \cdot (-)\frac{\cos n\omega_0 t}{n\omega_0}\Big|_0^{T/2} - \int_0^{T/2} (1)\,(-)\frac{\cos n\omega_0 t}{n\omega_0}\, dt \right]$$

$$= \frac{8A}{T^2}\left[\frac{-1}{n\omega_0}\left(\frac{T}{2}\cos n\omega_0 \frac{T}{2} - 0\right) + \frac{1}{(n\omega_0)^2}\sin n\omega_0 t \Big|_0^{T/2} \right]$$

$\omega_0 T = 2\pi$

$\frac{\omega_0 T}{2} = \pi$

$$= \frac{8A}{T^2}\left[-\frac{T}{2n\omega_0}\cos n\pi + \frac{1}{(n\omega_0)^2}\Big(\underbrace{\sin n\frac{\omega_0 T}{2}}_{\sin n\pi} - 0\Big) \right]$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{8A}{T^{\cancel{2}}}\left(\frac{-\cancel{T}}{2n\omega_0}(-1)^n + \frac{1}{(n\omega_0)^2}(0-0) \right)$$

$$= \frac{-8A}{2n\omega_0 T}(-1)^n$$

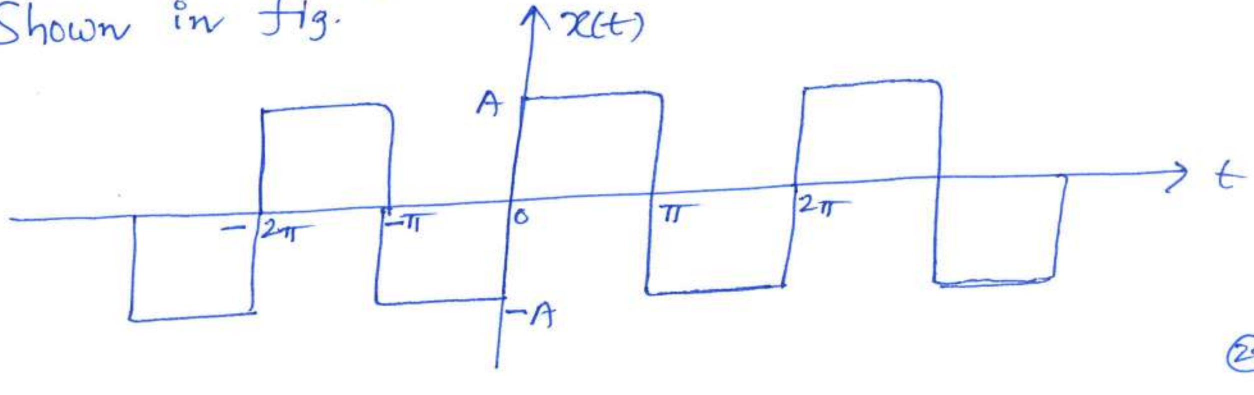
$$b_n = \frac{2\cancel{8}A}{\cancel{4}n\pi}(-1)^{n+1} = \frac{2A}{n\pi}(-1)^{n+1}$$

$\therefore$ The TFS representation is now given by,

$$x(t) = \sum_{n=1}^{\infty} \frac{2A}{n\pi}(-1)^{n+1} \sin n\omega_0 t$$

$$x(t) = \frac{2A}{\pi}\left(\sin\omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots \right)$$

prob: Obtain the trigonometric Fourier Series for the waveform Shown in fig.



Soln:

Given the Signal $x(t)$ is periodic with period

$$T = 2\pi, \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Soln: Since $x(-t) = -x(t) \; \forall t$ and also

$x(t \pm \frac{T}{2}) = -x(t)$. The Signal exhibits both Halfwave Symmetry and odd Symmetry. and hence quarter wave Symmetry.

$$\therefore \quad a_0 = 0, \quad a_n = 0.$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\,\sin n\omega_0 t\,dt$$

$$b_n = \frac{\cancel{2}A}{\cancel{2}\pi}\int_0^{\pi} A\,\sin nt\,.dt$$

$$= \frac{2A}{\pi}\int_0^{\pi}\sin nt\,dt$$

$$= \frac{2A}{\pi}(-)\frac{\cos nt}{n}\Big|_0^{\pi}$$

$$= \frac{-2A}{\pi n}(\cos n\pi - 1)$$

$$b_n = \frac{-2A}{\pi n}((-1)^n - 1)$$

$$\therefore \quad n \text{ even}, \quad b_n = 0$$

$$n \text{ odd}, \quad b_n = \frac{4A}{n\pi}.$$

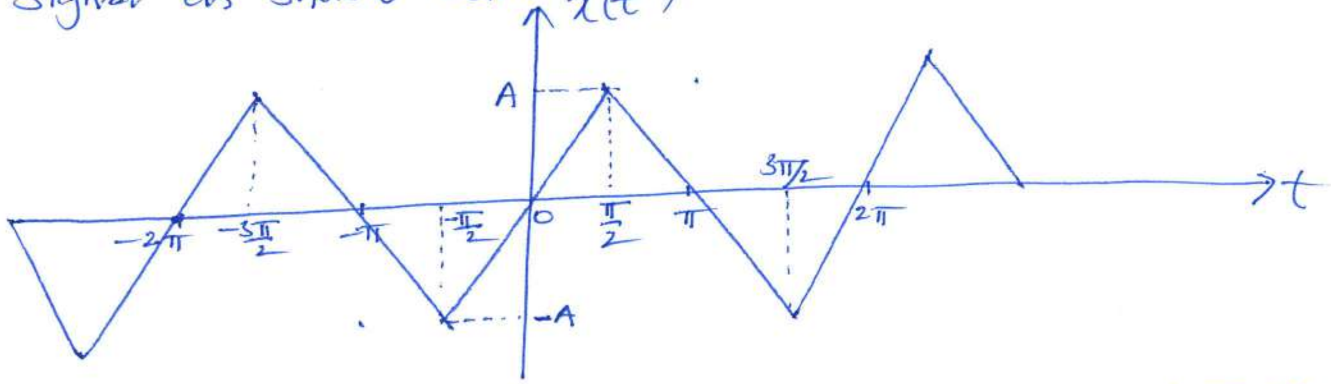
$\therefore$ The trigonometric Fourier Series is now given by

$$x(t) = \sum_{n=1}^{\infty} b_n \cos n\omega_0 t = \sum_{\substack{n=1\\(\text{odd})}}^{\infty} \frac{4A}{n\pi}\cos nt$$

Also

$$x(t) = \frac{4A}{\pi}\left(\cos t + \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t + \cdots\right)$$

prob: Find the trigonometric Fourier Series for the following Signal as Shown below.



Soln: Given the Signal $x(t)$ is periodic with period $T = 2\pi$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$$

Since $x(-t) = -x(t)$ the Signal exhibits odd Symmetry Also, $x(t \pm T/2) = -x(t)$, Halfwave Symmetry also exists.

$$\therefore \quad a_0 = a_n = 0,$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t).\sin n\omega_0 t\, dt$$

$(-\pi/2, -A), (\pi/2, A)$

$y + A = \frac{2A}{\pi}(x + \pi/2)$

$= \frac{2Ax}{\pi} + \cancel{A}$

$y = \frac{2Ax}{\pi}$

$$b_n = \frac{4}{2\pi}\int_{-\pi/2}^{\pi/2} x(t).\sin nt.dt$$

$$x(t) = \frac{2At}{\pi}, \; -\pi/2 \le t \le \frac{\pi}{2}$$

$$\therefore \quad b_n = \frac{4}{\cancel{2}\pi}\int_{-\pi/2}^{\pi/2} \frac{\cancel{2}At}{\pi} \sin nt.\,dt$$

$$= \frac{4A}{\pi^2}\left[t.(-)\frac{\cos nt}{n}\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}(1)(-)\frac{\cos nt}{n}\,dt \right]$$

$$= \frac{4A}{\pi^2}\left[\frac{-1}{n}\left(\frac{\pi}{2}\cos\frac{n\pi}{2} + \frac{\pi}{2}\cos\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin nt\Big|_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{4A}{\pi^2}\left(\frac{-\pi}{n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\left(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right)\right)$$

$$b_n = \frac{4A}{\pi^2}\left(\frac{-\pi}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\right)$$

$$b_n = \frac{4A}{\pi^2 n^2}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd}$$

$$\therefore\ b_n = \frac{4A}{\pi^2 n^2}, \quad n = 1, 5, 9 \ldots$$

$$= \frac{-4A}{\pi^2 n^2}, \quad n = 3, 7, 11 \ldots$$

$\therefore$ The trigonometric fourier series representation is now given by,

$$x(t) = \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} b_n \sin n\omega_0 t,$$

$$= \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}\sin nt, \qquad \omega_0 = 1$$

$$x(t) = \frac{4A}{\pi^2}\left(\sin t - \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t - \frac{1}{7}\sin 7t + \cdots\right)$$

Exponential Fourier Series (EFS):

We know that the set of complex exponential functions ($e^{jn\omega_0 t}$, $n = 0, \pm 1, \pm 2, \pm 3 \ldots$) forms a closed or complete orthogonal set of mutually orthogonal functions. Over the interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$.

" Representation of a periodic function by a Linear Combination of Complex exponential Signals is known as Complex exponential Fourier Series" i.e

if $x(t \pm T) = x(t) \; \forall t$, then.

$$x(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \cdots + F_n e^{jn\omega_0 t} + \cdots$$
$$+ F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \cdots + F_{-n} e^{-jn\omega_0 t} + \cdots$$

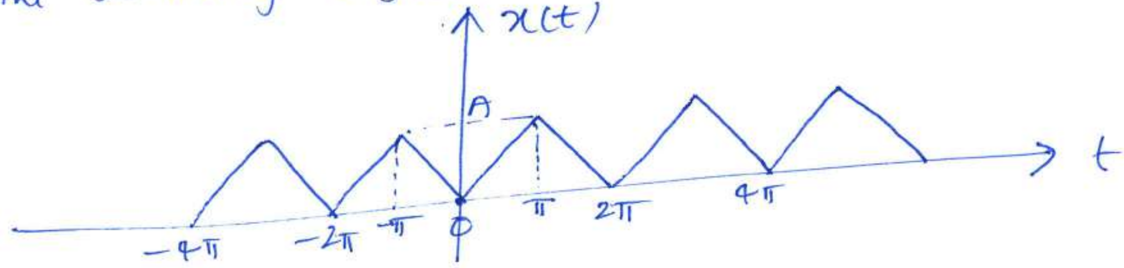
Also,

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow (1)$$

Where $F_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} . dt$ are exponential Fourier Series Coefficients.

Also, $F_0 = \frac{1}{T} \int_T x(t). dt$ is Called the Average or D.C Value of the Signal $x(t)$.

Prob: Determine the exponential fourier Series representation for the following Signal.



Soln: Given the Signal $x(t)$ is periodic with period $T = 2\pi$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

The Signal $x(t)$ over one period $(-\pi \le t \le \pi)$ is given by

$$x(t) = \begin{cases} -\frac{At}{\pi}, & -\pi \le t \le 0 \\ \frac{At}{\pi}, & 0 \le t \le \pi \end{cases}$$

$(-\pi, A), (0,0)$

$(0,0), (\pi, A)$

∴ The EFS Coefficients are thus given by

$$F_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\therefore F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jnt}\, dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\frac{At}{\pi} e^{-jnt}\, dt + \int_0^{\pi} \frac{At}{\pi} e^{-jnt}\, dt\right]$$

$$F_n = \frac{-A}{2\pi^2}\left[t\cdot\frac{e^{-jnt}}{-jn}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}(1)\frac{e^{-jnt}}{-jn}dt - \left(t\cdot\frac{e^{-jnt}}{-jn}\Big|_0^{\pi} - \int_0^{\pi}(1)\frac{e^{-jnt}}{-jn}dt\right)\right]$$

$$F_n = \frac{-A}{2\pi^2}\left[0 - \left(\frac{\pi}{jn} e^{jn\pi}\right) - \frac{1}{n^2} e^{-jnt}\Big|_{-\pi}^{0} - \left(\frac{-\pi}{jn} e^{-jn\pi} - 0 - \frac{1}{n^2} e^{-jnt}\Big|_0^{\pi}\right)\right]$$

$$= \frac{-A}{2\pi^2}\left(\frac{-\pi}{jn}\cancel{(-1)^n} - \frac{1}{n^2}\left(1-(-1)^n\right) + \frac{\pi}{jn}\cancel{(-1)^n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$F_n = \frac{-A}{2\pi^2}\left(\frac{1}{n^2}\left((-1)^n-1\right) - \frac{1}{n^2}\left(1-(-1)^n\right)\right)$$

for n even, $F_n = 0$

n odd, $F_n = \frac{-A}{\cancel{2}\pi^2}\left(\frac{-\cancel{2}}{n^2}\right) = \frac{A}{\pi^2 n^2}$

for $n=0$, $F_0 = \frac{1}{T}\int_T x(t)\cdot dt$

$$F_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\frac{At}{\pi}\,dt + \int_{0}^{\pi}\frac{At}{\pi}\,dt\right)$$

$$F_0 = \frac{1}{2\pi}\left(-\frac{A}{\pi}\left(\frac{t^2}{2}\right)_{-\pi}^{0} + \frac{A}{\pi}\left(\frac{t^2}{2}\right)_{0}^{\pi}\right)$$

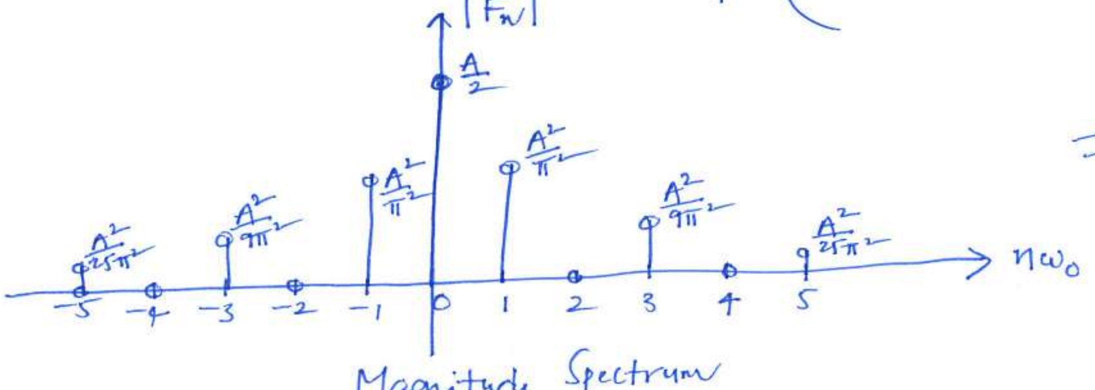
$$= \frac{1}{2\pi}\left(\frac{-A}{2\cancel{\pi}}\left(-\pi^{\cancel{2}}\right) + \frac{A}{2\cancel{\pi}}\left(\pi^{\cancel{2}}\right)\right)$$

$$F_0 = \frac{1}{2\cancel{\pi}}\left(A\cancel{\pi}\right) = \underline{\underline{\frac{A}{2}}}$$

$\therefore$ The exponential series is given by

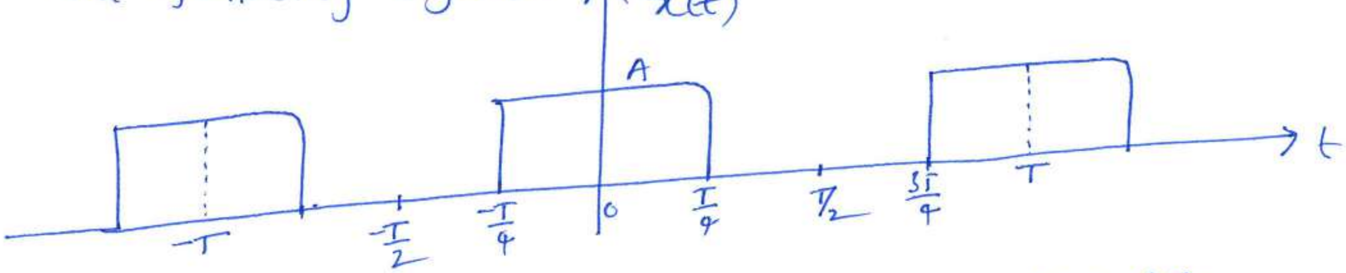
$$x(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{jn\omega_0 t} \qquad \omega_0 = 1$$

$$x(t) = \frac{A}{2} + \frac{A}{\pi^2}\left(e^{jt} + e^{j3t} + e^{j5t} + \cdots\right)$$
$$+ \frac{A}{\pi^2}\left(e^{-jt} + e^{-j3t} + e^{-j5t} + \cdots\right)$$



Magnitude Spectrum

prob: Find the exponential fourier Series representation for the following Signal.



Soln: Given the Signal $x(t)$ is periodic with period 'T'

$$x(t) = \begin{cases} A, & -\frac{T}{4} \le t \le \frac{T}{4} \\ 0, & \frac{T}{4} \le t \le \frac{3T}{4}. \end{cases}$$

$\therefore$ The EFS Coefficients are given by

$$F_n = \frac{1}{T}\int_T x(t).\, e^{-jn\omega_0 t}.\, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{4}}^{T/4} A \cdot \frac{e^{-jn\omega_0 t}}{\cancel{jn\omega_0}}\, dt.$$

$\omega_0 T = 2\pi$

$\frac{\omega_0 T}{4} = \frac{\pi}{2}$

$$= \frac{A}{T}\left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right|_{-T/4}^{T/4}$$

$$= \frac{-A}{jn\omega_0 T}\left(e^{-jn\omega_0 T/4} - e^{jn\omega_0 T/4}\right)$$

$$= \frac{-A}{jn(2\pi)}\left(e^{-jn\pi/2} - e^{jn\pi/2}\right)$$

$$= \frac{A}{n\pi}\left(\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j}\right)$$

$$= \frac{A}{n\pi}\sin\frac{n\pi}{2} = \frac{A}{2}\,\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}$$

$$F_n = \frac{A}{2}\,Sa\left(\frac{n\pi}{2}\right)$$

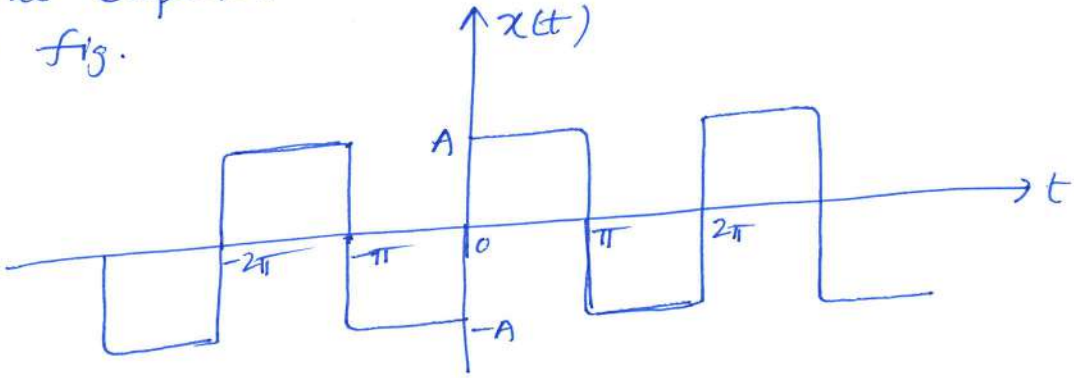
∴ The exponential fourier Series is thus given by

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \frac{A}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{A}{2} Sa\left(\frac{n\pi}{2}\right) e^{jn\omega_0 t}.$$

$F_0 = \frac{A}{2} Sa(0)$
$= \frac{A}{2}$

prob: Obtain the exponential fourier Series for the waveform Shown in fig.



Soln: Given Signal $x(t)$ is periodic with period $T = 2\pi$ is given by

$$x(t) = \begin{cases} A, & 0 \le t \le \pi \\ -A, & \pi \le t \le 2\pi. \end{cases}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$$

∴ The exponential fourier Series Coefficients are given by.

$$F_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A \cdot e^{-jnt} dt + \int_{\pi}^{2\pi} (-A)\, e^{-jnt} \cdot dt\right]$$

$$F_n = \frac{A}{2\pi}\left(\frac{e^{-jnt}}{-jn}\Big|_0^{\pi} - \frac{e^{-jnt}}{-jn}\Big|_{\pi}^{2\pi} \right)$$

$$= \frac{-A}{2\pi jn}\left((-1)^n - 1 - (1 - (-1)^n) \right)$$

$$= \frac{-A}{\cancel{2}\pi jn}\left(\cancel{2}(-1)^n - \cancel{2} \right)$$

$$\therefore \quad F_n = \frac{-A}{\pi j n}\left((-1)^n - 1\right)$$

for n even, $\quad F_n = 0$

n odd, $\quad F_n = \frac{-2A}{\pi j n} = j\,\frac{2A}{\pi n}$

for $n=0$,

$$F_0 = \frac{1}{T}\int_T x(t)\cdot dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\cdot dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} -A\cdot dt\right)$$

$$= \frac{1}{2\pi}\left(A\cdot t\Big|_0^{\pi} - A\cdot t\Big|_{\pi}^{2\pi}\right)$$

$$= \frac{A}{2\pi}\left((\pi - 0) - (2\pi - \pi)\right)$$

$$F_0 = \frac{A}{2\pi}\left(\cancel{\pi} - \cancel{\pi}\right) = 0$$

$\therefore$ The EFS representation is now given by

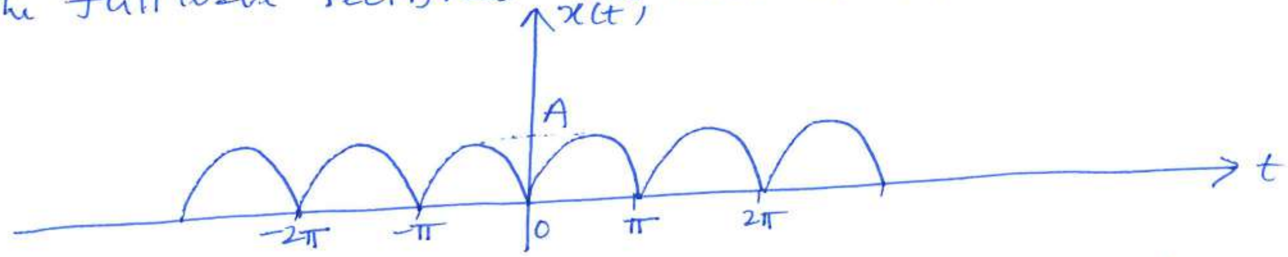
$$x(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} F_n\, e^{jn\omega_0 t}$$

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} j\,\frac{2A}{\pi n}\, e^{jnt}$$

(or)

$$x(t) = \frac{j2A}{\pi}\left(e^{jt} + \frac{e^{j3t}}{3} + \frac{e^{j5t}}{5} + \cdots\cdots\right)$$
$$-\frac{j2A}{\pi}\left(e^{-jt} + \frac{e^{-j3t}}{3} + \frac{e^{-j5t}}{5} + \cdots\cdots\right)$$

prob: Find the exponential fourier Series representation for the full wave rectified Sine wave as shown below.



Soln. Given the Signal $x(t)$ is periodic with period $T = \pi$ is given by, , $w_0 = \frac{2\pi}{T} = 2$

$$x(t) = A \sin t, \quad 0 \le t \le \pi$$

$\therefore$ The exponential Fourier Coefficients are thus given by,

$$F_n = \frac{1}{T}\int_T x(t)\, e^{-jnw_0 t}\, dt$$

$$F_n = \frac{1}{\pi}\int_0^{\pi} A \sin t\, e^{-jn2t}\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi} e^{-jn2t}\, \sin t \cdot dt.$$

$$= \frac{A}{\pi}\int_0^{\pi} e^{-jn2t} \left(\frac{e^{jt} - e^{-jt}}{2j}\right) dt$$

$$= \frac{A}{2\pi j}\left[\int_0^{\pi}\left[e^{j(1-2n)t} - e^{-j(1+2n)t}\right] dt\right]$$

$$= \frac{A}{2\pi j}\left[\frac{e^{j(1-2n)t}}{j(1-2n)}\Bigg|_0^{\pi} - \frac{e^{-j(1+2n)t}}{-j(1+2n)}\Bigg|_0^{\pi}\right]$$

$$= \frac{A}{2\pi j}\left[\frac{e^{j(1-2n)\pi} - e^{0}}{j(1-2n)} + \frac{e^{-j(1+2n)\pi} - e^{-0}}{j(1+2n)}\right]$$

$$F_n = \frac{A}{2\pi j}\left[\frac{\overbrace{e^{j\pi}}^{-1}\cdot \overbrace{e^{-j2n\pi}}^{1} - 1}{j(1-2n)} + \frac{\overbrace{e^{-j\pi}}^{-1}\cdot \overbrace{e^{-j2n\pi}}^{1} - 1}{j(1+2n)}\right]$$

$$= \frac{A}{\cancel{2}\pi j}\left(\frac{-\cancel{2}}{j(1-2n)} - \frac{\cancel{2}}{j(1+2n)}\right)$$

$$= \cancel{\frac{-A}{\pi j^2}}\left(\frac{1}{1-2n} + \frac{1}{1+2n}\right)$$

$$= \frac{A}{\pi}\left(\frac{1+\cancel{2n}+1-\cancel{2n}}{1-4n^2}\right)$$

$$F_n = \frac{2A}{\pi(1-4n^2)}, \quad F_0 = \frac{2A}{\pi}$$

$$F_1 = F_{-1} = \frac{-2A}{3\pi},$$

$$F_2 = F_{-2} = \frac{-2A}{15\pi}$$

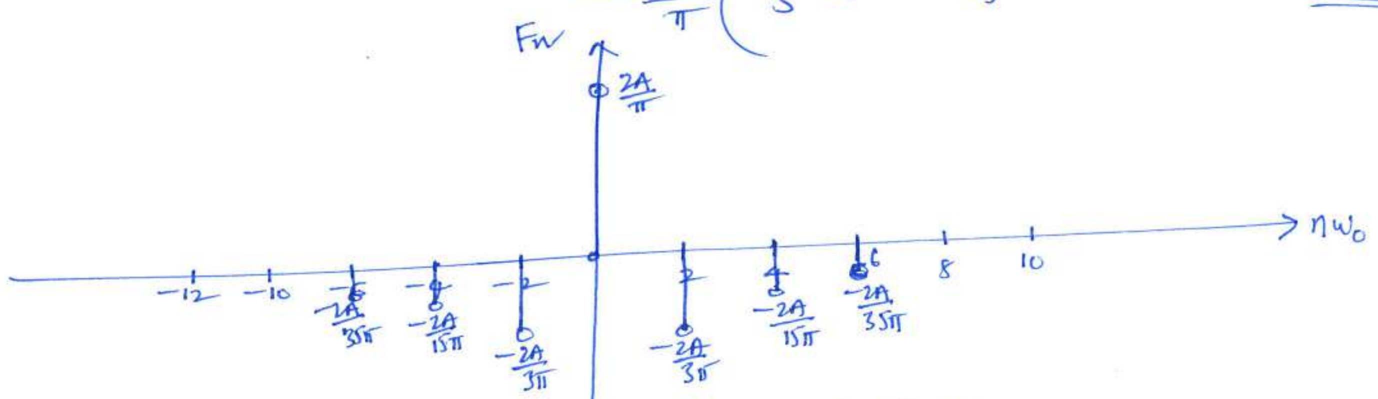
$$F_3 = F_{-3} = \frac{-2A}{35\pi} \quad \cdots$$

$\therefore$ The exponential Fourier Series is thus given by

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{+jnt}$$

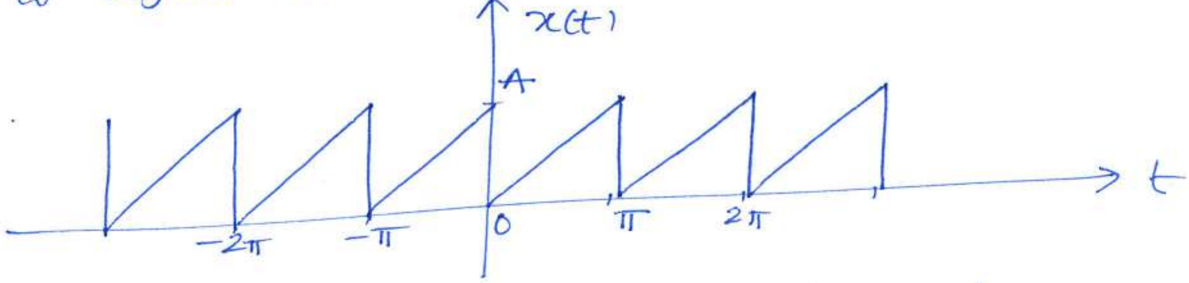
$$= \frac{2A}{\pi} - \frac{2A}{\pi}\left(\frac{1}{3}e^{jt} + \frac{1}{15}e^{j2t} + \frac{1}{35}e^{j3t} + \cdots\right)$$

$$- \frac{2A}{\pi}\left(\frac{1}{3}e^{-jt} + \frac{1}{15}e^{-j2t} + \frac{1}{35}e^{-j3t} + \cdots\right)$$



Magnitude Spectrum

prob: Obtain the exponential fourier Series representation of a Signal as shown below.



Soln: The given Signal $x(t)$ is periodic with period $T = \pi$, $\therefore \omega_0 = \frac{2\pi}{T} = 2$

$$x(t) = \frac{At}{\pi}, \quad 0 \le t \le \pi.$$

$\therefore$ The exponential fourier Series coefficients are given by

$$F_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{\pi}\int_0^\pi \frac{At}{\pi}\, e^{-j2nt}\, dt$$

$$= \frac{A}{\pi^2}\left(t \cdot \frac{e^{-j2nt}}{(-j2n)}\Big|_0^\pi - \int_0^\pi (1)\frac{e^{-j2nt}}{(-j2n)}\, dt \right)$$

$$= \frac{A}{\pi^2}\left(\frac{-1}{j2n}\left(\pi e^{-j2n\pi}\right) + \frac{1}{4n^2} e^{-j2nt}\Big|_0^\pi \right)$$

$$= \frac{A}{\cancel{\pi^2}}\left(\frac{-\cancel{\pi}}{j2n}(1) + \frac{1}{4n^2}\left(\cancel{1-1}\right)\right)$$

$$F_n = \frac{-A}{j2\pi n} \qquad j\frac{A}{2\pi n}$$

$$F_0 = \frac{1}{T}\int_T x(t)\, dt = \frac{1}{\pi}\int_0^\pi \frac{At}{\pi}\, dt$$

$$= \frac{A}{\pi^2}\frac{t^2}{2}\Big|_0^\pi = \frac{A}{\cancel{\pi^2}}\left(\frac{\cancel{\pi^2}}{2} - 0\right)$$

$$F_0 = \frac{A}{2}$$

$\therefore$ The EFS representation is now given by

$$x(t) = \sum_{n=-\infty}^{\infty} F_n \, e^{jn\omega_0 t}$$

$$x(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{jA}{2\pi n} \, e^{j2nt}$$

$$\therefore x(t) = \frac{A}{2} + \frac{jA}{2\pi}\left(e^{j2t} + \frac{1}{2} e^{j4t} + \frac{1}{3} e^{j6t} + \cdots \right)$$
$$- \frac{jA}{2\pi}\left(e^{-j2t} + \frac{1}{2} e^{-j4t} + \frac{1}{3} e^{-j6t} + \cdots \right)$$

* Complex Fourier Spectrum:

The exponential fourier Series Coefficients of a periodic Signal $x(t)$ is given by

$$F_n = \frac{1}{T}\int_T x(t) \, e^{-jn\omega_0 t} \, dt$$

In general F_n is Complex, then

$$F_n = |F_n| \, e^{j\theta_n}$$

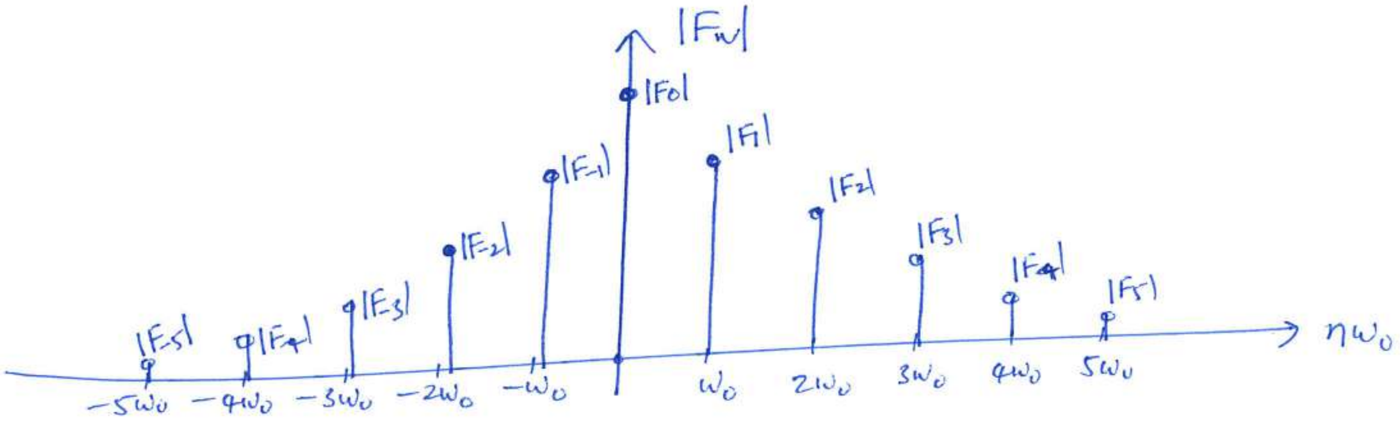
Magnitude Spectrum: plot of $|F_n|$ Vs $n\omega_0$

phase Spectrum : plot of θ_n or $\angle F_n$ Vs $n\omega_0$

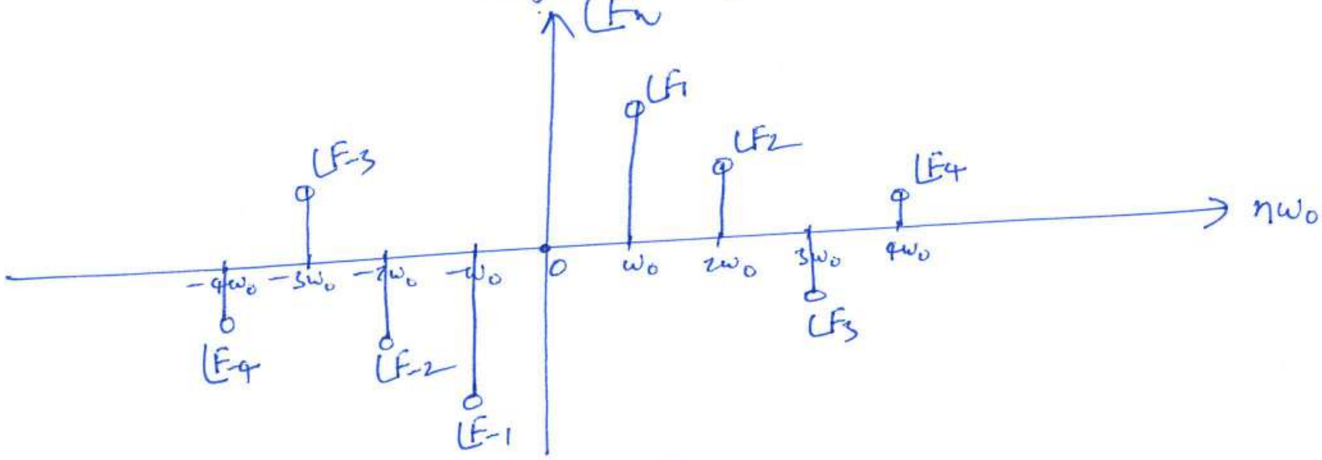
if $x(t)$ is real then $|F_n| = |F_{-n}|$

$$\angle F_n = -\angle F_{-n}$$

i.e Magnitude Spectrum is always even and phase Spectrum is odd



Magnitude Spectrum



phase Spectrum.

→ Fourier Series of periodic Signals consists of discrete frequencies and hence the Spectrum is called discrete line Spectrum

UNIT-III

FOURIER TRANSFORMS

INTRODUCTION: It is possible to represent any periodic function $f(t)$ over the entire interval $(-\infty < t < \infty)$ as a discrete Sum of Exponentials by using Exponential Fourier Series. We can also represent any non periodic function interms of Exponential Functions over any finite interval $(t_0 < t < t_0+T)$. For the purpose of frequency domain analysis, However, we need to represent every type of driving function interms of Exponential functions not over a finite interval but over the entire interval $(-\infty, \infty)$. In this Chapter, we see that a non periodic Signal, in general can be expressed as a Continuous (integral) Sum of Exponential Signals in contrast to the periodic Signals which can be represented by a discrete Sum of Exponential functions.

DERIVING FOURIER TRANSFORM FROM FOURIER SERIES:-

The Exponential form of fourier Series Representation of a periodic Signal is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t} \longrightarrow ①$$

$$\text{where} \quad F_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-jn\omega_0 t}\, dt \longrightarrow ②.$$

In the Fourier Representation of $x(t)$ in Eqn①, the F_n, for Values of 'n' are the Spectral Components of the Signal $x(t)$, Located at discrete intervals of fundamental frequency ω_0. Therefore the frequency Spectrum is discrete in nature.

The Fourier Representation of a Signal using Eqn ① is applicable for periodic Signals. For Fourier Representation of non-periodic Signals, let us Consider the fundamental period 'T' tends to infinity. When the fundamental period tends to infinity, the fundamental frequency ω_0 tends to zero or becomes Very Small. Since the fundamental frequency ω_0 is Very Small, the Spectral Components will Lie Very close to each Other and so the frequency Spectrum becomes Continuous.

In order to obtain the Fourier representation of a non periodic Signal let us Consider that, the fundamental frequency ω_0 is very Small, let $\omega_0 \rightarrow d\omega$

$$\therefore \quad ① \Rightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jnd\omega t} \rightarrow ③$$

On Substituting Equation ② in Eqn ③, (by taking τ as dummy Variable for integration), we get.

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(\tau) \cdot e^{-jnd\omega\tau} d\tau \right] e^{jnd\omega t}$$

Since $\omega_0 = 2\pi F_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{d\omega}{2\pi}$ $\therefore \omega_0 \rightarrow d\omega$

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{d\omega}{2\pi} \int_{-T/2}^{T/2} x(\tau) \cdot e^{-jnd\omega\tau} d\tau \right) e^{jnd\omega t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-T/2}^{T/2} x(\tau) \cdot e^{-jnd\omega\tau} d\tau \right) e^{jnd\omega t} \cdot d\omega$$

For non periodic Signals, the fundamental period 'T' tends to infinity. On letting Limit T tends to infinity in the Above Equation, we get,

$$x(t) = \underset{T \to \infty}{Lt} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} x(\tau) \cdot e^{-jnd\omega\tau} d\tau \right) \cdot e^{jnd\omega t} d\omega$$

When $T \rightarrow \infty$; $\sum \rightarrow \int$ and $d\omega \longrightarrow \omega$

$$\therefore \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t) \cdot e^{-jn\omega t} dt \right) e^{jn\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{jn\omega t} d\omega \longrightarrow ④$$

$$\text{where} \quad X(j\omega) = \int_{-\infty}^{\infty} \cancel{X(j\omega)} \cdot x(t) \cdot e^{-jn\omega t} dt \rightarrow ⑤$$

Equation ⑤ is Fourier Transform of $x(t)$ and Eqn ④ is inverse fourier transform of $x(t)$.

Since the Eqn ⑤ Extracts the frequency Components Of the Signal, transformation using Eqn ⑤ is also Called analysis of the Signal $x(t)$. and Eqn ④ Combines the frequency Components Of the Signal, the inverse transformation using Equation ④ is Called as Synthesis Of the Signal $x(t)$.

REPRESENTATION OF AN ARBITRARY FUNCTION OVER THE ENTIRE INTERVAL $(-\infty, \infty)$: THE FOURIER TRANSFORM :-

There are mainly two approaches for representing an arbitrary function over the entire interval $(-\infty < t < \infty)$. They are,

⟹ Express a function $f(t)$ interms of Exponential functions over a finite interval $(-T/2 < t < T/2)$ and then let T go to infinity.

⟹ We may construct a periodic function of period 'T' so that $f(t)$ Represents the first cycle of this periodic waveform. In the Limit, if we let the period 'T' become infinity, and this periodic function then has only one cycle in the interval $(-\infty < t < \infty)$ and is represented by $f(t)$.

But However the Later Approach is more Convinient, because it allows us to visualize the Limiting process without altering the Shape of the frequency Spectrum. Below Example Shows the Limiting process of a periodic gate function.

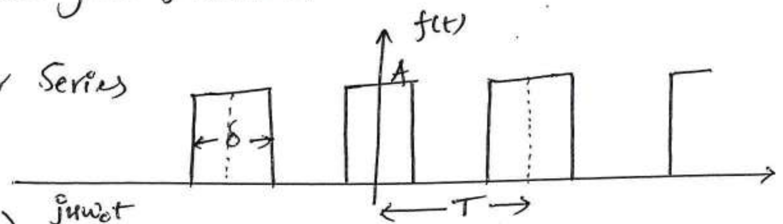


fig: periodic gate function.

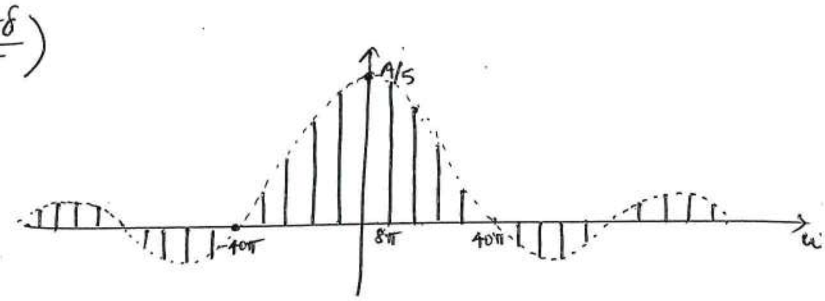
The Exponential Fourier Series of $f(t)$ is given by

$$f(t) = \frac{A\delta}{T} \sum_{n=-\infty}^{\infty} Sa\left(\frac{n\pi\delta}{T}\right) e^{jn\omega_0 t}$$

$$F_n = \frac{A\delta}{T} Sa\left(\frac{n\pi\delta}{T}\right)$$

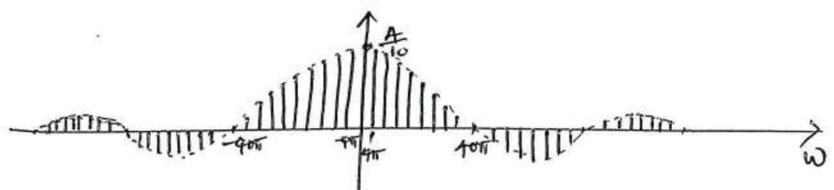
Case – I $\quad \delta = \frac{1}{20}, \quad T = 1/4,$

$$F_n = \frac{A}{5} Sa\left(\frac{n\pi}{5}\right)$$



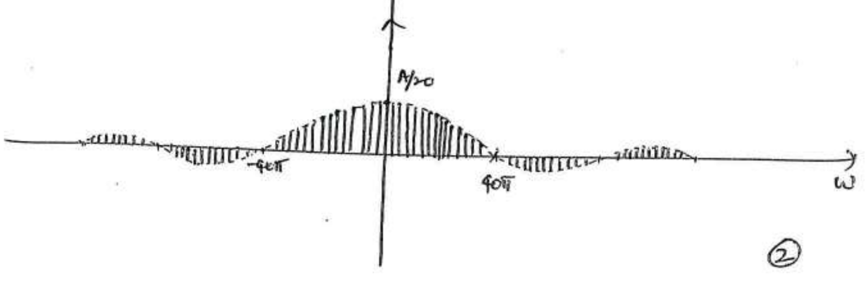
Case – II $\quad \delta = \frac{1}{20}, \quad T = 1/2$

$$F_n = \frac{A}{10} Sa\left(\frac{n\pi}{10}\right)$$



Case – III $\quad \delta = \frac{1}{20}, \quad T = 1.$

$$F_n = \frac{A}{20} Sa\left(\frac{n\pi}{20}\right)$$



From the Above Example it was Observed that as the period 'T' is made Larger, the fundamental frequency becomes Smaller and the frequency Spectrum becomes denser, i.e in a given frequency range there are more frequency Components. But the amplitudes becomes Smaller. The Shape of the frequency Spectrum however remains unaltered.

Let us Consider a function $f(t)$ as Shown in fig. 1. Here our attention is to represent this function as a Sum of Exponential functions over the entire interval $(-\infty<t<\infty)$. For this purpose we need to Construct a new periodic function $f_T(t)$ with period 'T' where the function $f(t)$ repeats itself every 'T' Seconds as shown in fig. 2.

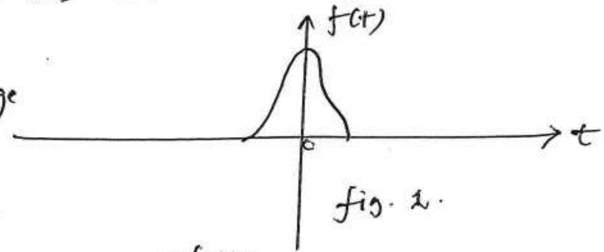


fig. 1.

Here, the period 'T' is made Large enough So that there is no overlap between the pulses of the Shape of $f(t)$.

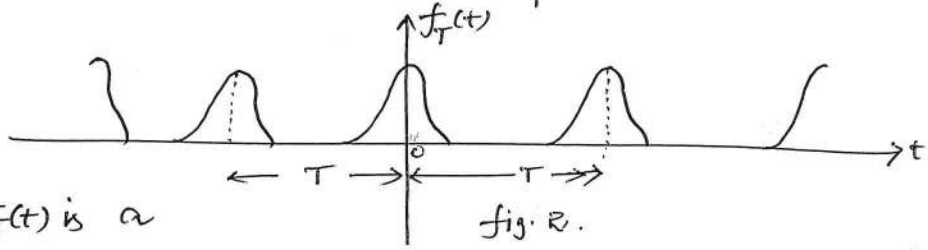


fig. 2.

The new function $f_T(t)$ is a periodic function, and Consequently Can be represented with an Exponential Fourier Series. In the Limit if we let 'T' become infinite then the pulses in the periodic function repeat after an infinite interval. Hence in the Limit $T\to\infty$ $f_T(t)$ and $f(t)$ are identical. i.e

$$\underset{T\to\infty}{Lt} f_T(t) = f(t).$$

Thus the Fourier Series Representing $f_T(t)$ over the entire interval will also represent $f(t)$ over the entire interval if we Let $T\to\infty$ in this Series.

The Exponential fourier Series for $f_T(t)$ can be represented as

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$ and

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\cdot e^{-jn\omega_0 t}\,dt. \rightarrow ①.$$

The term 'F_n' represents the amplitude of the Component of frequency nw_0. We Shall now let T become very Large. As 'T' becomes Larger, w_0 becomes Smaller and the Spectrum becomes denser. Therefore in the Limit when $T \to \infty$, the magnitude of each Component becomes infinitesimally Small, but now there are also an infinite no. of frequency Components. Thus the Spectrum Exists for every Value of w, and is no longer a discrete but a Continuous function of 'w'. To illustrate this point,

$$\text{let } nw_0 = w_n, \text{ then} \rightarrow (2)$$

F_n is a function of w_n and we Shall denote F_n by $F_n(w_n)$. Further let,

$$T \cdot F_n(w_n) = F(w_n) \rightarrow (3)$$

Then,

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n \, e^{jnw_0 t}$$

$$f_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(w_n) \cdot e^{jnw_0 t} \quad \because \text{from Equ (3)} \rightarrow (4)$$

from Equations (1) and (3), we have,

$$F(w_n) = T \cdot F_n = \int_{-T/2}^{T/2} f_T(t) \cdot e^{-jw_n t} \, dt \rightarrow (5)$$

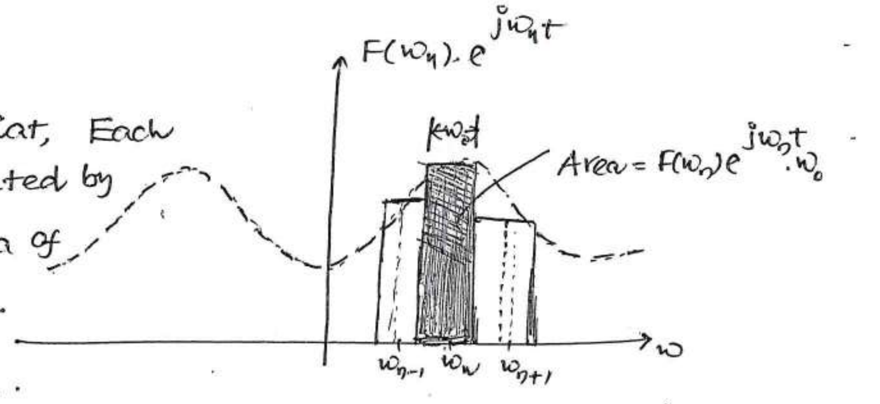
Substitute $T = \frac{2\pi}{w_0}$ in Eq (4) we get.

$$f_T(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(w_n) \cdot e^{jw_n t} \cdot w_0 \rightarrow (6).$$

Equation (6) Shows that, $f_T(t)$ can be expressed as a Sum of Exponential Signals of frequencies $w_1, w_2 \cdots w_n \cdots$ etc. The amplitude of the Component of frequency w_n is $F(w_n) \cdot \frac{w_0}{2\pi}$ $(= F_n)$. Note that here, the amplitude Content in $f_T(t)$ of frequency w_n is not $F(w_n)$ but is proportional to $F(w_n)$.

The graphical Representation of Equ (6) is as Shown in fig. assuming that, the quantity $F(w_n) \cdot e^{jw_n t}$ is real. This function exists only at discrete Values of 'w'; i.e at $w = w_1, w_2 \cdots w_n \cdots$ etc where $w_n = nw_0$.

From figure, it is clear that, Each frequency Component is Seperated by distance w_0. Therefore the area of the Shaded rectangle in fig. is evidently $F(w_n).e^{jw_nt}.w_0$.



Therefore Equation ⑥ Represents the Sum of areas under all Such rectangles corresponding to $w=-\infty$ to $+\infty$. The Sum of rectangular areas represents approximately the area under the dotted Curve. The Approximation becomes better as w_0 becomes Smaller.

In the Limit when $T \to \infty$, w_0 becomes infinitesimally Small and may be represented by dw. Therefore the discrete Sum in Equation ⑥ becomes the integral or the area under this curve. The Curve now is a Continuous function of 'w' and is given by $F(w)e^{jwt}$. Also as $T \to \infty$, the function $f_T(t) \to f(t)$ and Equations ⑤ and ⑥ becomes,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w).e^{jwt}\,dw \rightarrow ⑦$$

where

$$F(w) = \int_{-\infty}^{\infty} f(t).\,e^{-jwt}\,dt. \rightarrow ⑧$$

Eqn.⑦ represents $f(t)$ as a Continuous Sum of Exponential functions with frequencies lying in the interval $(-\infty < w < \infty)$. The amplitude of the Component of any frequency 'w' is proportional to $F(w)$. Therefore $F(w)$ represents the frequency Spectrum of $f(t)$ and is Called the Spectral density function. Note that Here, the frequency Spectrum is Continuous and exists at all values of 'w'. The Spectral density function $F(w)$ can be evaluated from Equation ⑧.

Equations ⑦ and ⑧ are usually referred to as the Fourier Transform pair. Equation ⑧ is known as the Direct fourier transform of $f(t)$, and Equation ⑦ is known as the inverse fourier transform of $F(w)$. Symbolically these transforms are also written as,

$$F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} d\omega.$$

⇒ Time Domain And Frequency domain Representation of a Signal :-

→ The fourier transform is a tool that resolves a given Signal into its exponential Components.

→ The function $F(\omega)$ is the Direct fourier transform of $f(t)$ and represents relative amplitudes of various frequency Components. Therefore $F(\omega)$ is the frequency domain representation of $f(t)$.

→ Time Domain representation Specifies a function at each instant of time, whereas frequency domain representation Specifies the relative amplitudes of the frequency Components of the function.

→ Since, the function $F(\omega)$ is Complex, it needs two plots for its graphical Representation.

$$F(\omega) = |F(\omega)| \cdot e^{j\theta(\omega)}$$

Thus, $F(\omega)$ may be represented by a magnitude plot $|F(\omega)|$ and a phase plot $\theta(\omega)$. In many Cases however, $F(\omega)$ is either real or imaginary and only one plot is necessary.

⇒ Show that for a real function $f(t)$, the magnitude Spectrum is an even function of 'ω' and the phase Spectrum is an odd function of 'ω'.

Soln: The Direct fourier transform of a real function $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt \rightarrow (1).$$

Replace 'ω' by '$-\omega$' in the above Equation, we get

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{+j\omega t} dt.$$

$$F^*(-w) = \int_{-\infty}^{\infty} f^*(t) \cdot e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} f(t) \cdot e^{-jwt} dt \qquad \because f^*(t) = f(t)$$

$$\boxed{F^*(-w) = F(w)} \longrightarrow ②$$

Thus, if $F(w) = |F(w)| \cdot e^{j\theta(w)}$, then

$$F^*(-w) = |F^*(-w)| \, e^{-j\theta(w)}$$

$$= |F(w)| \cdot e^{-j\theta(w)}$$

It is evident from the Above equations that the magnitude Spectrum $F(w)$ is an even function (Symmetrical about the Vertical axis passing through the origin) of 'w' and the phase Spectrum $\theta(w)$ is an odd function of 'w' (i.e Anti-Symmetrical about the Vertical axis passing through the origin).

⟹ Existence of Fourier Transform: Dirchlet Conditions:

The Fourier Transform of $x(t)$ exists if it Satisfies the following Dirchlet Conditions.

→ 1) $x(t)$ Should be absolutely integrable. i.e

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty. \longrightarrow ①$$

→ 2) $x(t)$ Should have a finite no. of maxima and minima within any finite interval.

→ 3) $x(t)$ Should have finite no. of discontinuties within any interval.

Note:→ Absolute integrability of $x(t)$ is a Sufficient Condition but not a necessary Condition for the Existence of the Fourier transform of $x(t)$ because there are such functions which are not absolutely integrable but have fourier transforms.

→ Functions such as $\sin\omega t$, $\cos\omega t$, $u(t)$, etc. do not satisfy the absolute integrable condition, do not posseses the Fourier transform. These functions however do have Fourier transforms in the limit.

Fourier Transform of Some Standard Signals :-

→ **Fourier Transform of Unit impulse Signal :**

By definition, $x(t) = \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$ $\quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$

The Fourier Transform of impulse Signal is then given by

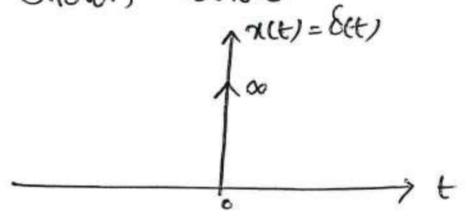
$$\mathcal{F}(x(t)) = X(j\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t}\,dt$$

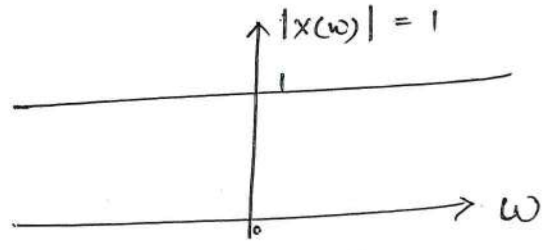
$$= e^{-j\omega t}\Big|_{t=0}$$

$$X(\omega) = 1$$

The plot of impulse Signal and its magnitude Spectrum are as shown below.



a) Unit impulse Signal



b) Magnitude Spectrum

<u>Note</u>: → The fourier Transform of a Unit impulse function is Unity.

→ An impulse function has a Uniform Spectral density over the entire frequency interval. In other words an impulse function contains all frequency components with the same relative amplitudes.

→ Fourier Transform of Single Sided Exponential Signal:

The Single Sided exponential Signal is defined as

$$x(t) = A e^{-at}, \quad t \geq 0 \quad \text{or}$$

$$x(t) = A e^{-at} \cdot u(t).$$

By definition,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} A e^{-at} \cdot e^{-j\omega t} dt$$

$$= A \cdot \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

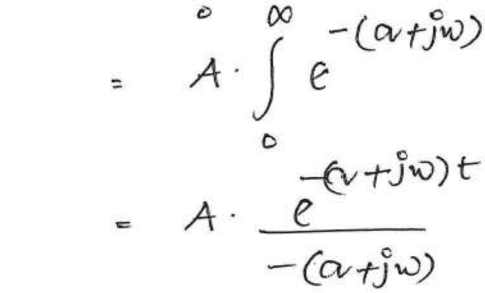
$$= A \cdot \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$= A\left(0 - \left(\frac{-1}{a+j\omega}\right)\right)$$

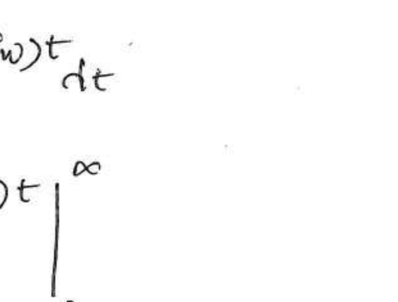
$$X(j\omega) = \frac{A}{a+j\omega} = \frac{A}{\sqrt{a^2+\omega^2}} \angle -\tan^{-1}\left(\frac{\omega}{a}\right) \rightarrow (1).$$

The plot of $x(t)$ and its frequency Spectrum are as follows.

fig: $x(t)$



Magnitude Spectrum



phase Spectrum.

Note: → In the Above Signal if 'a' is −ve then the function doesnot Satisfy the Condition of absolute integrability and Fourier transform doesnot exist. If 'a' is −ve then the function represents a growing exponential, thus

$$\mathcal{F}(Ae^{-at} u(t)) = \frac{A}{a+j\omega} \quad \text{for } a>0.$$

Similarly $$\mathcal{F}^{-1}\left(\frac{A}{a+j\omega}\right) = Ae^{-at} u(t)$$

$$\therefore \quad Ae^{-at} u(t) = \frac{1}{2\pi} \int \frac{1}{a+j\omega} \cdot e^{j\omega t} d\omega.$$

→ It is evident from the above figure that the magnitude Spectrum is an even function, where as the phase Spectrum is an odd function of 'w'.

→ Thus we have expressed the Signal $A e^{-at} u(t)$ interms of a Continuous Sum of eternal exponential functions. These Exponential functions add in Such a way as to yield Zero Value for $t<0$, and addup to e^{-at} for $t>0$.

⟹ Fourier Transform of $f(t) = A t e^{-at} u(t)$.

Soln: By definition.

$$\mathcal{F}(f(t)) = F(jw) = \int_{-\infty}^{\infty} f(t) \cdot e^{-jwt} dt$$

$$F(jw) = \int_{-\infty}^{\infty} A t \cdot e^{-at} u(t) \cdot e^{-jwt} dt$$

$$= A \cdot \int_{0}^{\infty} t \cdot e^{-(a+jw)t} dt \cdot$$

$$= A \cdot \left[\left(t \cdot \frac{e^{-(a+jw)t}}{-(a+jw)} \right)_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-(a+jw)t}}{-(a+jw)} dt \right]$$

$$= A \left((0-0) + \frac{1}{(a+jw)} \cdot \frac{e^{-(a+jw)t}}{-(a+jw)} \Bigg|_0^{\infty} \right)$$

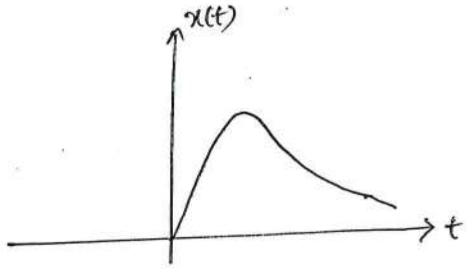
$$= A \left(\frac{1}{(a+jw)} \left(0 - \left(\frac{-1}{a+jw} \right) \right) \right]$$

$$F(jw) = \frac{A}{(a+jw)^2}, \quad a>0.$$

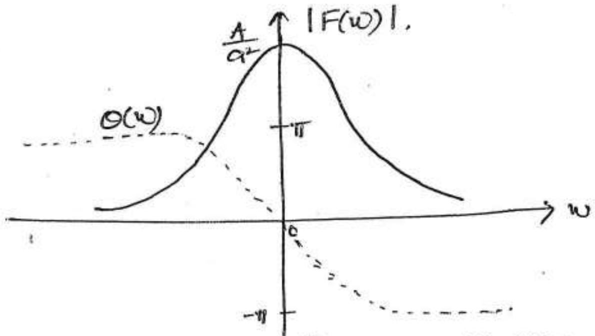
$$\therefore \mathcal{F}^{-1}\left(\frac{A}{(a+jw)^2} \right) = A t e^{-at} u(t).$$

$$F(jw) = \frac{A}{a^2+w^2} \left(-2 \tan^{-1}\left(\frac{w}{a}\right) \right)$$

The time domain and frequency domain representation of $A t e^{-at} u(t)$ are as follows.



Time domain Representation of the Signal $x(t) = Ate^{-at}u(t)$



Frequency Spectrum of $x(t)$.

→ Fourier Transform of Double Sided Exponential Signal:

The double Sided Exponential Signal is defined as.

$$x(t) = A e^{-a|t|}, \quad \forall t.$$

$$\therefore \quad x(t) = \begin{cases} A e^{+at}, & \text{for } t = -\infty \text{ to } 0 \\ A e^{-at}, & \text{for } t = 0 \text{ to } \infty. \end{cases}$$

By definition,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} A e^{at} \cdot e^{-j\omega t} dt + \int_{0}^{\infty} A e^{-at} e^{-j\omega t} dt$$

$$= A \cdot \int_{-\infty}^{0} e^{(a-j\omega)t} dt + A \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= A \cdot \frac{e^{(a-j\omega)t}}{(a-j\omega)} \Bigg|_{-\infty}^{0} + A \cdot \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Bigg|_{0}^{\infty}$$

$$= A\left(\frac{1}{a-j\omega} - 0\right) - \frac{A}{a+j\omega}(0-1)$$

$$= A\left(\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right)$$

$$X[j\omega] = \frac{2aA}{a^2+\omega^2}$$

The plot of double Sided Exponential Signal and its magnitude Spectrum are as follows.

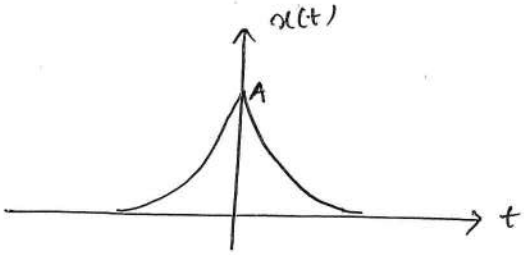
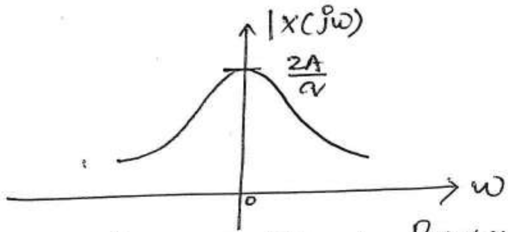


fig: Time Domain Representation of $x(t) = A e^{-a|t|}, \forall t$,

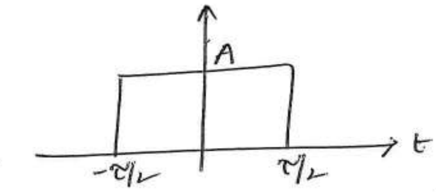


frequency Domain Representation of $x(t)$: Magnitude Spectrum.

→ Fourier Transform Of a Gate function:

A Gate function $G_\tau(t)$ is a rectangular pulse as Shown in fig. and is defined as,

$$G_\tau(t) = \begin{cases} A & |t| < \tau/2 \\ 0 & |t| > \tau/2 \end{cases}$$



By definition,

$$\mathcal{F}(G_\tau(t)) = G_\tau(j\omega) = \int_{-\infty}^{\infty} G_\tau(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\tau/2}^{\tau/2} A \cdot e^{-j\omega t} dt$$

$$= A \cdot \frac{e^{-j\omega t}}{-j\omega} \Bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} \left(e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right)$$

$$= \frac{A\tau}{\frac{\omega\tau}{2}} \left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \right)$$

$$= A\tau \cdot \frac{\sin \omega\tau/2}{\omega\tau/2}$$

$$G_\tau(j\omega) = A\tau \cdot Sa\left(\frac{\omega\tau}{2}\right).$$

Note that here, $G_\tau(j\omega)$ is a real function and hence Can be represented graphically by a Single Curve as Shown

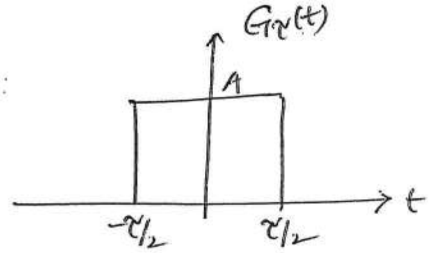


fig (a) Gate function

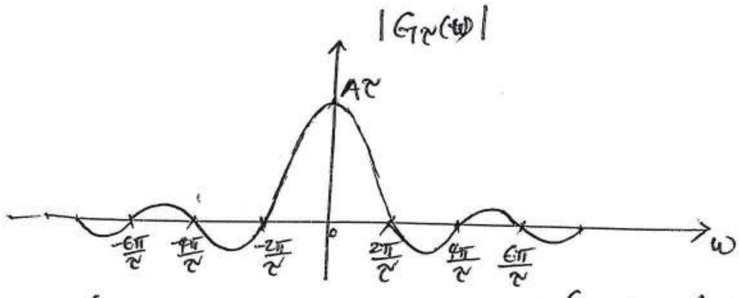


fig (b) Transform (Magnitude Spectrum) of a gate function.

⇒ Singularity functions :

Consider a unit step voltage is applied to a capacitor as shown in fig.1. The current 'i' through the capacitor is given by

$$i = C \cdot \frac{dv}{dt} \rightarrow ①$$

In Eqn ① $\frac{dv}{dt} = 0$, for all values of 't' except at $t=0$, where it is undefined. The derivative at $t=0$, does not exist because the function $u(t)$ is discontinuous at this point. This difficulty arises from the idealization of the source as well as the circuit element.

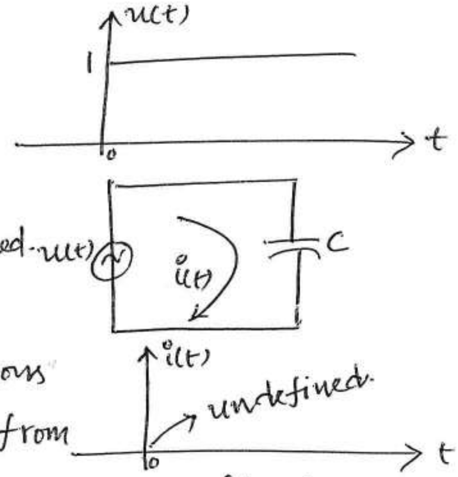


fig. 1.

If either of the source or the capacitor were nonideal the solution would exist. If, for example, the source voltage were as shown in fig.2 instead of that shown in fig. 1. the current through the capacitor would be a pulse of current as shown in fig.2.

The solution to an ideal unit step voltage does not exist, but it is possible to obtain a solution in the limit by assuming an unideal source $u_a(t)$ and then letting 'a' go to zero in the limit.

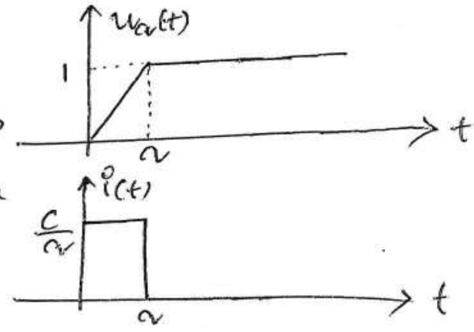


fig. 2.

In the limit when 'a' goes to zero, this voltage, $u_a(t)$ becomes a unit step voltage. The derivative of the function $u_a(t)$ is a rectangular pulse of height $1/a$ and width 'a'. As 'a' Varies, the pulse Shape Varies, but the area Of the pulse Remains Constant. Fig. 3 Shows Sequence Of Such pulses.

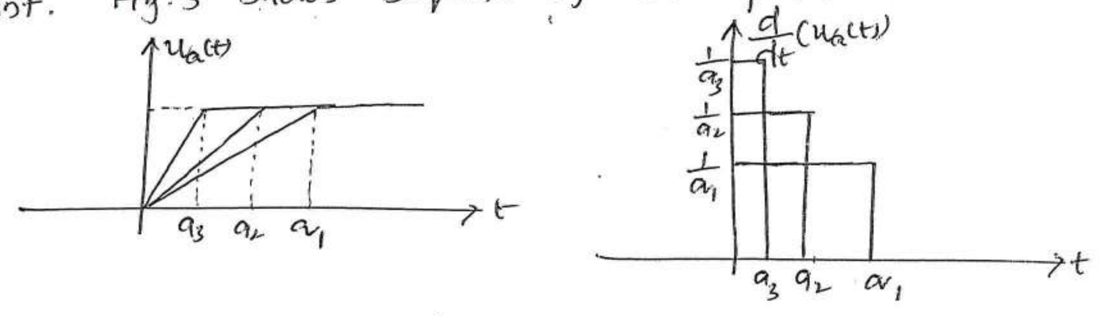


fig.3.

In the Limit when 'a' goes to Zero, the height Of the pulse goes to infinity and the width Of the pulse is Zero. The area Of the pulse however remains unity. Thus we define, the unit impulse function as the derivative Of a unit Step function. The unit impulse function is denoted by $\delta(t)$, i.e.

$$\delta(t) = \underset{a \to 0}{Lt} \frac{d}{dt}\left(u_a(t)\right) \rightarrow ②$$

Since $\frac{d}{dt}[u_a(t)]$ is a rectangular pulse Of height $(1/a)$ and width 'a' it can be described as,

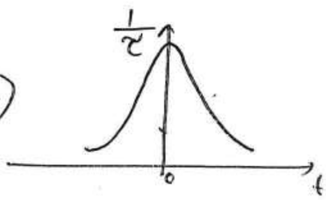
$$\delta(t) = \underset{a \to 0}{Lt} \frac{1}{a}\left(u(t) - u(t-a)\right)$$

$$\therefore \quad \left.\begin{aligned} \delta(t) &= \infty, \quad t = 0 \\ &\quad 0, \quad t \neq 0, \quad \&\ \int_{-\infty}^{\infty} \delta(t)\,dt = 1. \end{aligned}\right\} \rightarrow ③$$

<u>Note</u>: → It is not necessary to define the impulse function as a Limiting form Of a rectangular pulse. One can define $\delta(t)$ as a Limiting form Of a gaussian pulse, a triangular pulse, an exponential pulse or many other pulse forms.

→ The Sequences Of Some Of the pulse forms which Satisfy Equ ③ are given as,

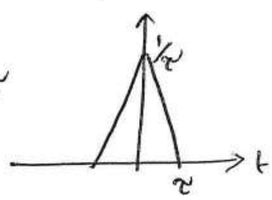
↦ Gaussian pulse: $\delta(t) = \underset{\tau \to 0}{Lt} \frac{1}{\tau} \cdot e^{(-\pi t^2/\tau^2)}$



→ Triangular pulse:

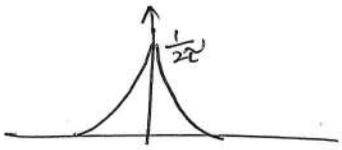
$$\delta(t) = \underset{\tau \to 0}{Lt} \frac{1}{\tau}\left(1 - \frac{|t|}{\tau}\right) \quad .. \ |t| < \tau$$

$$0 \qquad |t| > \tau$$

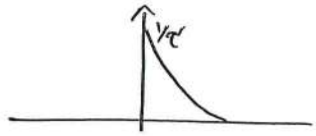


→ Exponential pulse:

$$\delta(t) = \underset{\tau \to 0}{Lt} \frac{1}{2\tau} e^{(-|t|/\tau)}$$

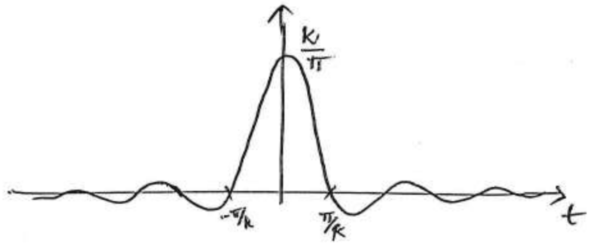


$$\delta(t) = \underset{\tau \to 0}{Lt} \frac{1}{\tau} e^{-t/\tau} \cdot u(t)$$



→ Sampling function:

$$\int_{-\infty}^{\infty} \frac{K}{\pi} Sa(Kt)\, dt = 1$$



as $K \to \infty$ the amplitudes of the function $\frac{K}{\pi} Sa(Kt)$ becomes Larger and the function Oscillates faster and decays Very rapidly away from the origin. In the Limit as $K \to \infty$, the function exists only at the origin and the net area under this curve is unity. Hence the Sampling function $K/\pi \, Sa(Kt)$ becomes an impulse function in the Limit as $K \to \infty$.

$$\therefore \quad \delta(t) = \underset{k \to \infty}{Lt} \left(\frac{K}{\pi} Sa(Kt)\right).$$

→ The impulse function and its derivatives are known as generalized functions and are justified by a relatively new discipline known as the "theory of distribution."

→ The Step function $u(t)$, the impulse function $\delta(t)$, and its higher derivatives are all known as "Singularity functions"

Derivative of Discontinuous functions:

If a function has a jump discontinuity at a point $t = t_0$ as shown in fig. then strictly speaking the function $f(t)$ does not possess a derivative at $t = t_0$. Therefore, a unit step function which has a jump discontinuity at $t = 0$ does not possess a derivative at the origin in a strict mathematical sense. We overcome this difficulty by considering the unit step function as a limit of the sequence of functions which were continuous. The limit of the derivative of this sequence was found to be an impulse function of unit strength. From this discussion it follows that, a function possess a derivative at a point of jump discontinuity and this derivative is an impulse function of strength equal to the amount of discontinuity. For a unit step function, the amount of discontinuity is unity and the derivative of $u(t)$ at $t = 0$, is an impulse function of unit strength

i.e $\quad \dfrac{du}{dt} = \delta(t) \Rightarrow \displaystyle\int_{-\infty}^{t} \delta(t)\,dt = u(t).$

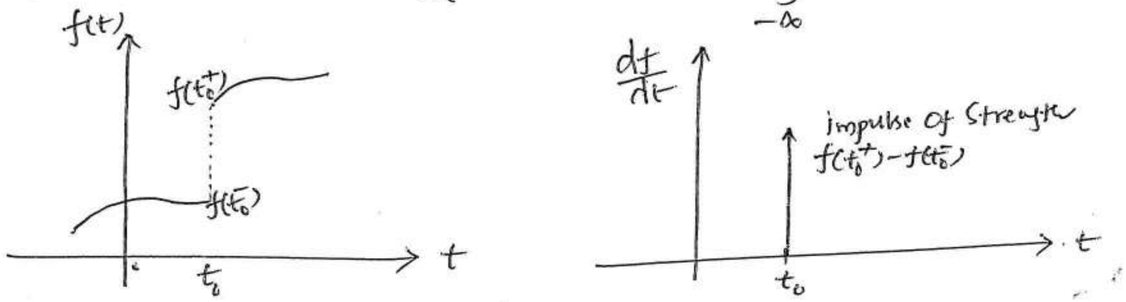


fig.

In the above figure, the function $f(t)$ has a jump discontinuity at $t = t_0$ of amount $(f(t_0^+) - f(t_0^-))$ and therefore the derivative of $f(t)$ at $t = t_0$ is given by

$$\left.\frac{df}{dt}\right|_{t=t_0} = \left[f(t_0^+) - f(t_0^-)\right] \delta(t - t_0).$$

Sampling Property of the impulse function:

Multiplication of any function $f(t)$ by an impulse $\delta(t - t_0)$ also yields an impulse of strength $f(t_0)$ at $t = t_0$.

$$f(t) . \delta(t - t_0) = f(t_0) . \delta(t - t_0).$$

$$\therefore \int_{-\infty}^{\infty} f(t).\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} f(t_0).\,\delta(t-t_0)\,dt$$

i.e

$$\int_{-\infty}^{\infty} f(t).\,\delta(t-t_0)\,dt = f(t_0). \rightarrow ①.$$

Equation ① Expresses the Sampling or Shifting property of an impulse function. Since $\delta(t-t_0)$ is Concentrated at $t=t_0$, and is zero everywhere else, it follows from Equ ① that,

$$\int_{t_0^-}^{t_0^+} f(t).\,\delta(t-t_0).\,dt = f(t_0) \text{ and}$$

$$\int_{0^-}^{0^+} f(t).\,\delta(t)\,dt = f(0).$$

→ Fourier Transform Of a Constant :

Constant is defined as $f(t) = A. \rightarrow ①$

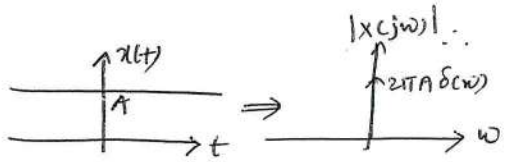
Since the above function does not Satisfy the Condition of absolute integrability, its Fourier transform does not exist. But it has transform in the Limit.

Consider the Fourier transform of a gate function of height 'A' and width 'τ' Sec. In the Limit as $\tau \rightarrow \infty$, the gate function tends to be a Constant function 'A'. The Fourier transform of a Constant 'A' is therefore the fourier transform of a gate function $G_\tau(t)$ as $\tau \rightarrow \infty$.

The Fourier Transform of $G_\tau(t)$ was found to be $A\tau\, Sa(\omega\tau/_2)$. Hence,

$$\mathcal{F}(A) = \underset{\tau\rightarrow\infty}{Lt}\; A\tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

$$= 2\pi A \underset{\tau\rightarrow\infty}{Lt}\; \frac{\tau}{2\pi}\, Sa\left(\frac{\omega\tau}{2}\right)$$



$$\mathcal{F}(A) = 2\pi A \cdot \delta(\omega)$$

$$\because \; \underset{T\to\infty}{Lt} \frac{T}{\pi} Sa(Tt) = \delta(t)$$

$$\therefore \quad \mathcal{F}(1) = 2\pi\, \delta(\omega).$$

Thus the fourier transform of a Constant Contains only a frequency component of $\omega = 0$. This is the logical result, Since a Constant function is a d.c Signal ($\omega = 0$) and does not have any other frequency Components.

⇒ Fourier Transform of $\cos\omega_0 t$ and $\sin\omega_0 t$:-

The functions $\cos\omega_0 t$ and $\sin\omega_0 t$ also do not Satisfy the Condition of absolute integrability. and fourier transform exist in the limit only. we shall first consider the function to exist only over the interval $-\tau/2$ to $\tau/2$ and zero outside this interval. In the limit 'τ' will be mark infinity, Therefore,

$$\mathcal{F}(\cos\omega_0 t) = \underset{\tau\to\infty}{Lt} \int_{-\tau/2}^{\tau/2} \cos\omega_0 t \, e^{-j\omega t} dt$$

$$= \underset{\tau\to\infty}{Lt} \int_{-\tau/2}^{\tau/2} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t} dt$$

$$= \underset{\tau\to\infty}{Lt} \int_{-\tau/2}^{\tau/2} \left(\frac{e^{-j(\omega-\omega_0)t}}{2} + \frac{e^{-j(\omega+\omega_0)t}}{2} \right) dt$$

$$= \frac{1}{2} \cdot \underset{\tau\to\infty}{Lt} \left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)} \right]_{-\tau/2}^{\tau/2}$$

$$= \underset{\tau\to\infty}{Lt} \frac{1}{2} \cdot \left[\frac{e^{-j(\omega-\omega_0)\tau/2}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)\tau/2}}{-j(\omega+\omega_0)} + \frac{e^{+j(\omega-\omega_0)\tau/2}}{j(\omega-\omega_0)} + \frac{e^{j(\omega+\omega_0)\tau/2}}{j(\omega+\omega_0)} \right]$$

$$= \underset{\tau \to \infty}{Lt} \frac{\tau}{2} \left[\frac{e^{j(w-w_0)\tau/2} - e^{-j(w-w_0)\tau/2}}{2j(w-w_0)\tau/2} + \frac{e^{j(w+w_0)\tau/2} - e^{-j(w+w_0)\tau/2}}{2j(w+w_0)\cdot\tau/2} \right]$$

$$= \underset{\tau \to \infty}{Lt} \frac{\tau}{2} \left[\frac{\sin(w-w_0)\tau/2}{(w-w_0)\tau/2} + \frac{\sin(w+w_0)\tau/2}{(w+w_0)\tau/2} \right]$$

$$= \underset{\tau \to \infty}{Lt} \pi\left(\frac{\tau}{\pi 2}\right)\left(Sa(w-w_0)\tau/2 + Sa(w+w_0)\tau/2 \right) \longrightarrow (1)$$

$$\mathcal{F}(\cos w_0 t) = \pi\left(\delta(w-w_0) + \delta(w+w_0)\right) \longrightarrow (2)$$

$$\therefore \underset{\tau \to \infty}{Lt} \frac{\tau}{\pi} Sa(\tau t) = \delta(t).$$

Similarly it can be shown that,

$$\mathcal{F}(\sin w_0 t) = j\pi\left(\delta(w+w_0) - \delta(w-w_0)\right) \longrightarrow (3)$$

Therefore, the fourier Spectrum for the above functions consists of two impulses at w_0 and $-w_0$ respectively. as shown below.

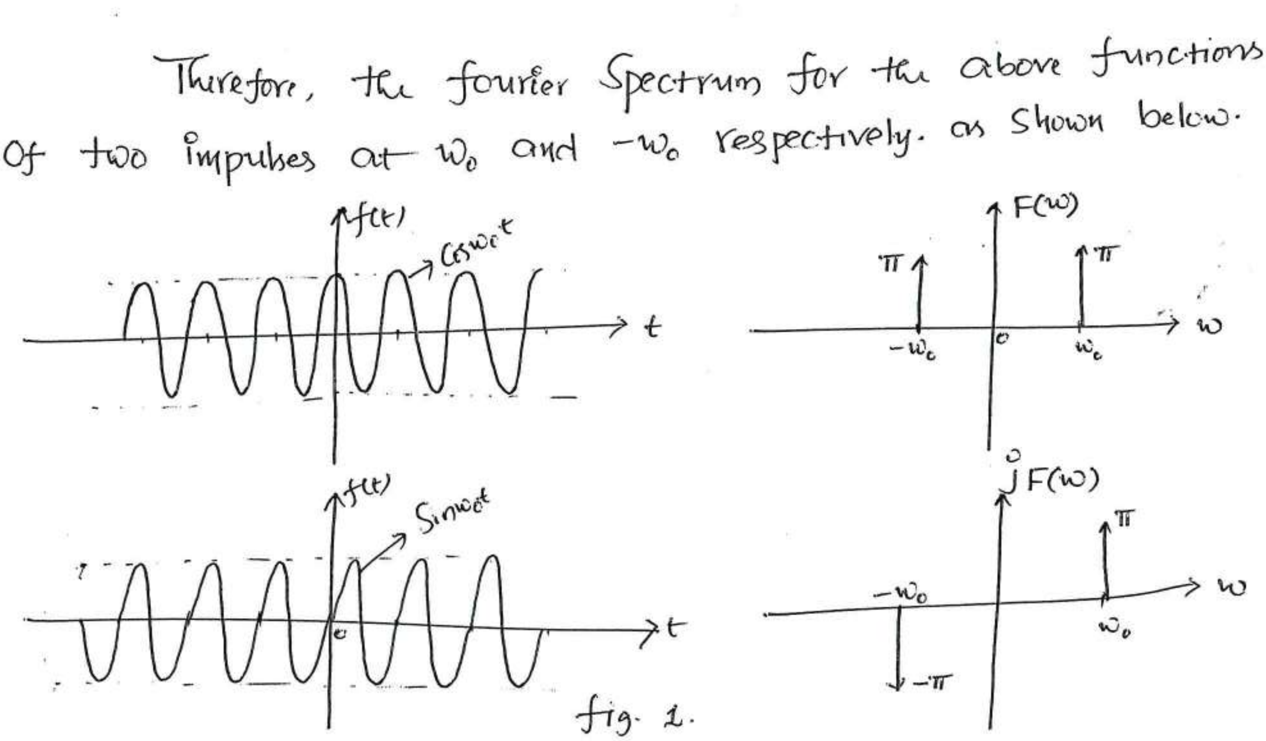


fig. 1.

For a finite 'τ' the Spectral density function of $\cos w_0 t$ is given by Equ (1). For $\tau = \frac{16\pi}{w_0}$, the Spectral density function is plotted as shown. ($\tau = \frac{16\pi}{w_0} = 8T$, means the function $\cos w_0 t$ is truncated by 8 cycles.

$$\therefore \quad f(t) = \begin{cases} \cos w_0 t , & |t| < \frac{16\pi}{w_0} \\ 0 , & |t| > \frac{16\pi}{w_0} . \end{cases}$$

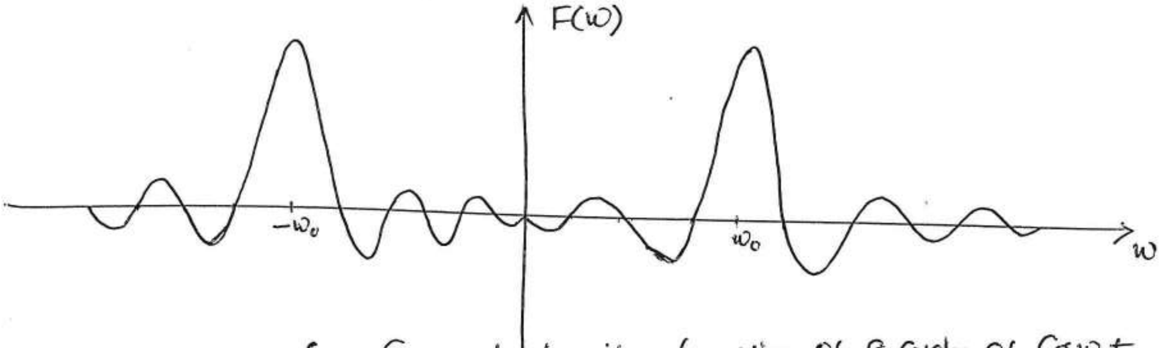


fig 2: Spectral density function of 8 cycles of $\cos w_0 t$.

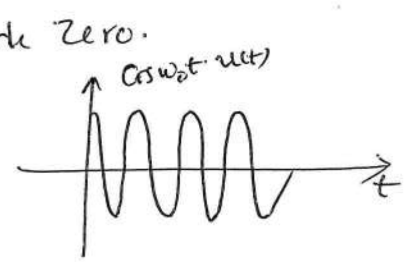
Note that in the Above figure there is a Large Concentration of Energy at frequencies near $\pm w_0$. As we increase the interval 'τ', the Spectral density Concentrates more and more around frequencies $\pm w_0$. In the Limit as $\tau \to \infty$, the Spectral density is zero everywhere except at frequencies $\pm w_0$, where it is infinite in Such a way that the area under the curve at each of these frequencies is 'π'. Therefore in the Limit, the distribution becomes two impulses of Strength 'π' units each, Located at frequencies $\pm w_0$ as Shown in fig. 2.

It is evident that, the Spectral density functions for $\cos w_0 t$ and $\sin w_0 t$ exists only at $w = w_0$. i.e these functions donot Contain Components of frequencies other than w_0.

⇒ Fourier Transform of a Unit Step function :

Since the unit Step function does not Satisfy the Condition of absolute integrability, its Fourier transform exist in the Limit only. Let us Consider the function $\cos w_0 t \, u(t)$. This function tends to $u(t)$ as w_0 is made zero.

$$\therefore \quad u(t) = \lim_{w_0 \to 0} \cos w_0 t \cdot u(t).$$



By definition,

$$\mathcal{F}(u(t)) = \lim_{w_0 \to 0} \mathcal{F}\left(\cos w_0 t \, u(t) \right)$$

$$\mathcal{F}(u(t)) = \underset{w_0 \to 0}{Lt} \left[\int_{-\infty}^{\infty} \cos w_0 t \cdot u(t) \cdot e^{-jwt} \, dt \right]$$

$$= \underset{w_0 \to 0}{Lt} \left[\int_{-\infty}^{\infty} \frac{e^{jw_0 t} \cdot u(t) \cdot e^{-jwt}}{2} \, dt + \int_{-\infty}^{\infty} \frac{e^{-jw_0 t} u(t) \cdot e^{-jwt}}{2} \, dt \right]$$

We know that, $\mathcal{F}(\cancel{u(t)} \cos w_0 t \cdot u(t)) = \frac{\pi}{2}\left(\delta(w-w_0) + \delta(w+w_0)\right) + \frac{jw}{w_0^2 - w^2}$

$$\therefore \mathcal{F}(u(t)) = \underset{w_0 \to 0}{Lt} \left[\frac{\pi}{2}\left[\delta(w-w_0) + \delta(w+w_0)\right] + \frac{jw}{w_0^2 - w^2} \right]$$

$$= \frac{\pi}{2}\left(\delta(w) + \delta(w) \right) + \frac{1}{jw}$$

$$\mathcal{F}(u(t)) = \pi \delta(w) + \frac{1}{jw} .$$

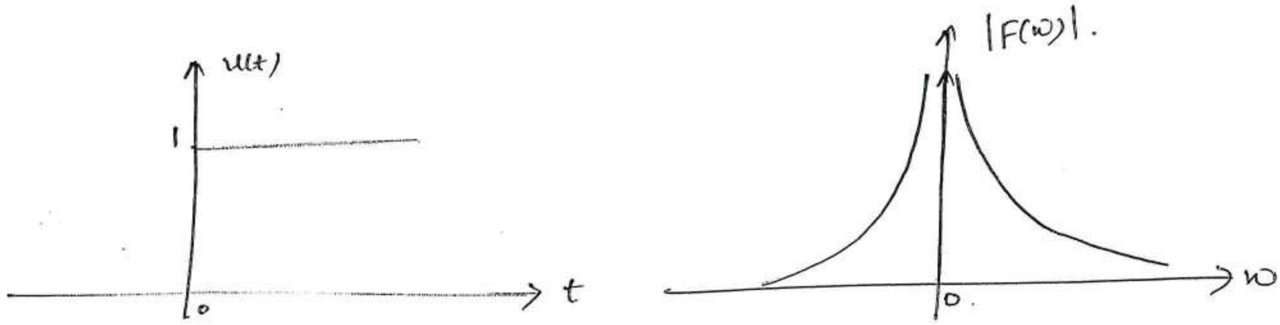
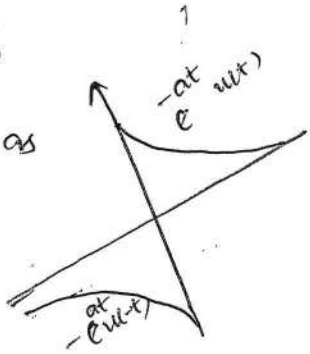


fig: The Unit Step function and its Spectral density function.

The Spectral density function of u(t) Contains an impulse at $w=0$. Thus the function u(t) Contains a Large D.C. Component. In addition it also has other frequency Components.

The function u(t) is not a Constant Since it is zero for $t<0$ and there is an abrupt discontinuity at $t=0$, giving rise to other frequency Components.

⇒ Fourier Transform of Signum function:



The Signum function is defined as

$$x(t) = Sgn(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

The Signum function can be expressed as a Sum of two one sided exponential Signals in the Limit $a \to 0$ as,

$$Sgn(t) = \underset{a \to 0}{Lt} \left(e^{-at} u(t) - e^{at} u(-t) \right)$$

$$\therefore \quad x(t) = Sgn(t) = \underset{a \to 0}{Lt} \left(e^{-at} u(t) - e^{at} u(-t) \right)$$

By definition,

$$X(j\omega) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} \left[\underset{a \to 0}{Lt}\, e^{-at} u(t) - e^{+at} u(-t) \right] e^{-j\omega t}\, dt$$

$$X(j\omega) = \underset{a \to 0}{Lt} \left(\int_0^{\infty} e^{-at} e^{-j\omega t}\, dt - \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt \right)$$

$$= \underset{a \to 0}{Lt} \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Bigg|_0^{\infty} - \frac{e^{(a-j\omega)t}}{(a-j\omega)} \Bigg|_{-\infty}^{\infty} \right]$$

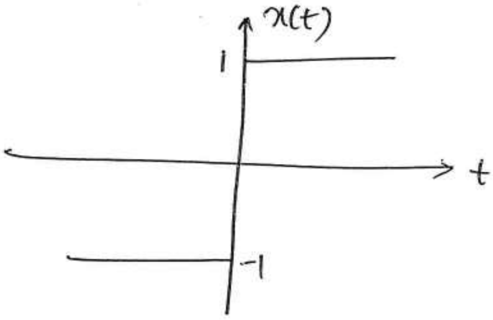
$$= \underset{a \to 0}{Lt} \left(0 - \frac{-1}{a+j\omega} - \left(\frac{1}{a-j\omega} \right) \right)$$

$$= \underset{a \to 0}{Lt} \left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right) = \frac{1}{j\omega} + \frac{1}{j\omega}$$

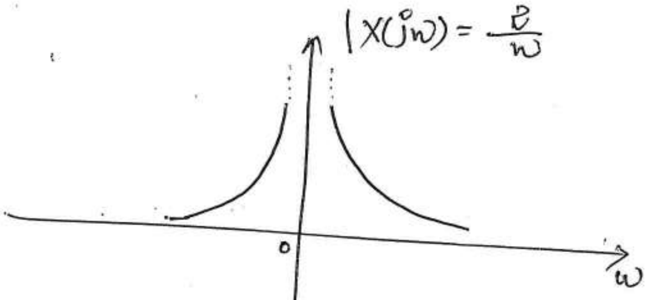
$$= \frac{2}{j\omega}.$$

$$X(j\omega) = \frac{2}{j\omega}$$

The plot of Signum function and its magnitude Spectrum are as Shown



Time domain Representation



Frequency Domain Representation.

⟹ Fourier Transform of $e^{-a|t|} \cdot \text{Sgn}(t)$:

The Signum function is defined as $\text{Sgn}(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases}$

$$e^{-a|t|} = \begin{cases} e^{-at}, & t>0 \\ e^{at}, & t<0 \end{cases}$$

∴ By definition

$$\mathcal{F}\left(e^{-a|t|} \cdot \text{Sgn}(t)\right) = \int_{-\infty}^{\infty} e^{-a|t|} \cdot \text{Sgn}(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{0^-} e^{+at} \cdot 1 \cdot e^{-j\omega t} dt + \int_{0^+}^{\infty} e^{-at} \cdot (-1) \cdot e^{-j\omega t} \cdot dt$$

$$= \frac{e^{(a-j\omega)t}}{(a-j\omega)} \Bigg|_{-\infty}^{0^-} + \frac{e^{-(a+j\omega)t}}{(a+j\omega)} \Bigg|_{0^+}^{\infty}$$

$$= \frac{+1}{a-j\omega} - \frac{1}{a+j\omega}$$

$$= \frac{a+j\omega - a + j\omega}{a^2+\omega^2}$$

$$\mathcal{F}\left(e^{-a|t|} \cdot \text{Sgn}(t)\right) = \frac{2j\omega}{a^2+\omega^2}$$

Properties of Continuous Time Fourier Transform:

→ **Linearity:** if $\mathcal{F}(x_1(t)) = X_1(j\omega)$ and

$\mathcal{F}(x_2(t)) = X_2(j\omega)$ then

$$\mathcal{F}(a x_1(t) + b x_2(t)) = a X_1(j\omega) + b X_2(j\omega).$$

proof. By definition of fourier Transform, we have,

$$X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t) \cdot e^{-j\omega t} \cdot dt \rightarrow ①, \quad X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t} dt \rightarrow ②$$

let $f(t) = a x_1(t) + b x_2(t)$

$$\therefore \mathcal{F}(f(t)) = F(j\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} [a \cdot x_1(t) + b x_2(t)] \cdot e^{-j\omega t} dt$$

$$= a \cdot \int_{-\infty}^{\infty} x_1(t) \cdot e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t} dt$$

$\because$ from ① & ②

$$F(j\omega) = a \cdot X_1(j\omega) + b \cdot X_2(j\omega)$$

→ **Time Shifting:** The time shifting property of Fourier Transform Says that, if $\mathcal{F}(f(t)) = F(j\omega)$, then,

$$\mathcal{F}(f(t-t_0)) = e^{-j\omega t_0} \cdot F[j\omega].$$

proof. By definition of fourier transform,

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt$$

$$\mathcal{F}(f(t-t_0)) = \int_{-\infty}^{\infty} f(t-t_0) \cdot e^{-j\omega t} dt$$

let $t - t_0 = \tau$
$t = t_0 + \tau$
$dt = d\tau$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot e^{-j\omega(t_0+\tau)} d\tau.$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot e^{-j\omega t_0} \cdot e^{-j\omega \tau} \cdot d\tau.$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt$$

$\because$ 'τ' is a dummy Variable.

$$\mathcal{F}(f(t-t_0)) = e^{-j\omega t_0} \cdot F(j\omega)$$

$\Rightarrow$ Time Scaling: The time Scaling property of fourier Transform Says that, if $\mathcal{F}(x(t)) = X(j\omega)$ then

$$\mathcal{F}(x(at)) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

proof. By definition,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$\therefore \mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t} \cdot dt$$

let $at = \tau$

$t = \frac{\tau}{a}$

$dt = \frac{d\tau}{a}$.

$$\therefore \mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\left(\frac{\tau}{a}\right)} \cdot \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t) \cdot e^{-j\left(\frac{\omega}{a}\right)t} \cdot dt$$

$\because$ 'τ' is a dummy Variable.

$$\mathcal{F}(x(at)) = \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

The above Transform is applicable for positive Values of 'a'.

if 'a' is $-$Ve then $\mathcal{F}(x(at)) = -\frac{1}{a} X\left(\frac{j\omega}{a}\right)$

$\therefore$ In general,

$$\mathcal{F}(x(at)) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$ for both positive and negative Values of 'a'.

$\Rightarrow$ Time Reversal: The time reversal property of Fourier Transform Says that, if $\mathcal{F}(x(t)) = X(j\omega)$ then

$$\mathcal{F}(x(-t)) = X(-j\omega).$$

proof: From Time Scaling property, we know that,

$$\mathcal{F}(x(at)) = \frac{1}{|a|} \cdot X\left(\frac{j\omega}{a}\right)$$

let $a = -1$.

$$\mathcal{F}(x(-t)) = X(-j\omega).$$

⇒ Conjugation: The Conjugation property of Fourier Transform Says that, if $\mathcal{F}(x(t)) = X(j\omega)$, then

$$\mathcal{F}(x^*(t)) = X^*(-j\omega).$$

proof: By definition of fourier Transform,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$\mathcal{F}(x^*(t)) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \left[\int_{-\infty}^{\infty} x(t) \cdot e^{+j\omega t}\right]^*$$

$$= \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j(-\omega) \cdot t} dt\right]^*$$

$$\mathcal{F}(x^*(t)) = X^*(-j\omega)$$

⇒ Frequency Shifting: The frequency Shifting property of fourier transform Says that, if

$$\mathcal{F}(x(t)) = X(j\omega), \text{ then}$$

$$\mathcal{F}(e^{j\omega_0 t} \cdot x(t)) = X(j(\omega - \omega_0))$$

proof: By definition of fourier transform,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$\mathcal{F}(e^{j\omega_0 t} \cdot x(t)) = \int_{-\infty}^{\infty} e^{+j\omega_0 t} \cdot x(t) \cdot e^{-j\omega t} dt$$

$$\mathcal{F}(e^{j\omega_0 t}\cdot x(t)) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j(\omega-\omega_0)t}\cdot dt$$

$$\mathcal{F}(e^{j\omega_0 t}\cdot x(t)) = X(j(\omega-\omega_0))$$

Similarly it can be Shown that,

$$\mathcal{F}(e^{-j\omega_0 t}\cdot x(t)) = X(j(\omega+\omega_0)).$$

⟹ Time Differentiation:

The time differentiation property of Fourier transform Says that, if $\mathcal{F}(x(t)) = X(j\omega)$, then

$$\mathcal{F}\left(\frac{d}{dt}x(t)\right) = j\omega X(j\omega).$$

proof By the definition of fourier Transform,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt.$$

$$\therefore \mathcal{F}\left(\frac{d}{dt}x(t)\right) = \int_{-\infty}^{\infty} \frac{d}{dt}x(t)\cdot e^{-j\omega t}\cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\cdot\left(\frac{d}{dt}x(t)\right)\cdot dt$$

$$\because \int u\cdot v = u\int v - \int[du\int v]$$

$$= e^{-j\omega t}\cdot x(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-j\omega)\cdot e^{-j\omega t}\cdot x(t)\cdot dt$$

$$= 0 + j\omega\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$\mathcal{F}\left(\frac{d}{dt}x(t)\right) = j\omega\cdot X(j\omega)$$

⟹ Time Integration: The integration property of fourier transform Says that, if $\mathcal{F}(x(t)) = X(j\omega)$, then,

$$\mathcal{F}\left(\int_{-\infty}^{t} x(\tau)\cdot d\tau\right) = \frac{1}{j\omega}\cdot X(j\omega).$$

~~proof~~ Let $x(t)$ be a Continuous time Signal and $X(j\omega)$ be Fourier Transform of $x(t)$. Since differentiation and integration are inverse Operations, $x(t)$ can be expressed as follows.

$$\frac{d}{dt}\left(\int_{-\infty}^{t} x(\tau)\cdot d\tau\right) = x(t).$$

On taking fourier Transform of the Above Equation, we get.

$$\mathcal{F}\left(\frac{d}{dt}\left(\int_{-\infty}^{t} x(\tau)\cdot d\tau\right)\right) = \mathcal{F}(x(t))$$

$$j\omega\cdot \mathcal{F}\left(\int_{-\infty}^{t} x(\tau)\cdot d\tau\right) = \mathcal{F}(x(t))$$

$$\therefore \quad \mathcal{F}\left(\int_{-\infty}^{t} x(\tau)\cdot d\tau\right) = \frac{1}{j\omega}\cdot X(j\omega).$$

⇒ **Frequency Differentiation:** (Multiplication by 't').

The frequency differentiation property of Fourier transform Says that,

$$\text{if} \quad \mathcal{F}(x(t)) = X(j\omega), \text{ then}$$

$$\mathcal{F}(t\cdot x(t)) = j\frac{d}{d\omega}X(j\omega).$$

~~proof:~~ By definition of Fourier Transform,

$$X(j\omega) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

Differentiate Above Equation w.r.t. 'ω' on both Sides,

$$\frac{d}{d\omega}(X(j\omega)) = \frac{d}{d\omega}\left(\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt\right)$$

$$= \int_{-\infty}^{\infty} x(t)\cdot\left(\frac{d}{d\omega}e^{-j\omega t}\right)dt.$$

$$= \int_{-\infty}^{\infty} x(t)\cdot(-jt)\cdot e^{-j\omega t}\cdot dt.$$

$$\frac{d}{dw}(X(jw)) = \frac{1}{j}\int_{-\infty}^{\infty}(t \cdot x(t)) \cdot e^{-jwt} \cdot dt$$

$$= \frac{1}{j} \cdot \mathcal{F}(t \cdot x(t))$$

$$\therefore \quad \mathcal{F}(t \cdot x(t)) = j \cdot \frac{d}{dw}(X(jw)).$$

⇒ Convolution Theorem : The Convolution of fourier Transform Says that, Fourier Transform Of Convolution of two Signals is given by the product Of the Fourier Transforms of the individual Signals. i.e.

$$\text{if} \quad \mathcal{F}(x_1(t)) = X_1(jw) \;\&\; \mathcal{F}(x_2(t)) = X_2(jw), \text{ then}$$

$$\mathcal{F}(x_1(t) * x_2(t)) = X_1(jw) \cdot X_2(jw). \rightarrow ①$$

Equation① is also known as Convolution property of fourier transform.

Convolution of two Signals $x_1(t)$ & $x_2(t)$ is defined as

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \cdot d\tau. \rightarrow ②.$$

$\because \tau \rightarrow$ dummy Variable for integration.

proof: let $x_1(t)$ & $x_2(t)$ be two Continuous time Signals, So

$$X_1(jw) = \mathcal{F}(x_1(t)) = \int_{-\infty}^{\infty} x_1(t) \cdot e^{-jwt} \cdot dt \rightarrow Ⓐ$$

$$X_2(jw) = \mathcal{F}(x_2(t)) = \int_{-\infty}^{\infty} x_2(t) \cdot e^{-jwt} \cdot dt \rightarrow Ⓑ$$

Using the ~~four~~ Definition Of Fourier Transform, we can write,

$$\mathcal{F}(x_1(t) * x_2(t)) = \int_{-\infty}^{\infty}(x_1(t) * x_2(t)) \cdot e^{-jwt} \cdot dt.$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \cdot d\tau\right) \cdot e^{-jwt} \cdot dt$$

$$\text{let } e^{-jwt} = e^{jw\tau} \cdot e^{-jw\tau} \cdot e^{-jwt}$$

$$= e^{-jw\tau} \cdot e^{-jw(t-\tau)}$$

let $M = t - \tau \Rightarrow dM = dt$

$$\therefore \quad e^{-j\omega t} = e^{-j\omega\tau} \cdot e^{-j\omega M}$$

$$\therefore \mathcal{F}(x_1(t) * x_2(t)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(M) \cdot e^{-j\omega\tau} \cdot e^{-j\omega M} d\tau \cdot dM$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} \cdot d\tau \cdot \int_{-\infty}^{\infty} x_2(M) \cdot e^{-j\omega M} \cdot dM.$$

Since τ & M are dummy Variables used for integration. we Can write,

$$\mathcal{F}(x_1(t) * x_2(t)) = \int_{-\infty}^{\infty} x_1(t) \cdot e^{-j\omega t} \cdot dt \cdot \int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t} \cdot dt$$

$$\mathcal{F}(x_1(t) * x_2(t)) = X_1(j\omega) \cdot X_2(j\omega).$$

⟹ **Frequency Convolution**: The frequency Convolution property of fourier Transform Says that,

if $\mathcal{F}(x_1(t)) = X_1(j\omega)$ and $\mathcal{F}(x_2(t)) = X_2(j\omega)$, then

$$\mathcal{F}(x_1(t) \cdot x_2(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda) \cdot X_2(j(\omega - \lambda)) \cdot d\lambda.$$

proof By the definition of fourier Transform,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt.$$

$$\therefore \quad \mathcal{F}(x_1(t) \cdot x_2(t)) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t) \cdot e^{-j\omega t} \cdot dt. \longrightarrow (1)$$

By the definition of inverse fourier transform, we get,

$$x_1(t) = \mathcal{F}^{-1}(X_1(j\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$x_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda) \cdot e^{j\lambda t} \cdot d\lambda. \longrightarrow (2)$$

On Substituting Equ (2) in Equ (1), we get.

$$\mathcal{F}(x_1(t)\cdot x_2(t)) = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(j\lambda)\cdot e^{j\lambda t}\cdot d\lambda\right)\cdot x_2(t)\cdot e^{-j\omega t}\cdot dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(j\lambda)\left(\int_{-\infty}^{\infty}x_2(t)\cdot e^{-j\omega t}\cdot e^{j\lambda t}\cdot dt\right)d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(j\lambda)\left(\int_{-\infty}^{\infty}x_2(t)\cdot e^{-j(\omega-\lambda)t}\,dt\right)d\lambda$$

$$\mathcal{F}(x_1(t)\cdot x_2(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(j\lambda)\cdot X_2(j(\omega-\lambda))\cdot d\lambda$$

⇒ **Parsevals Relation:** The parsevals Relation Says that,

if $\mathcal{F}(x(t)) = X(j\omega)$, then,

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega.$$

proof: Let $x(t)$ be a Continuous time Signal and $x^*(t)$ be Conjugate of $x(t)$. Now, $|x(t)|^2 = x(t)\cdot x^*(t)$.

On integrating Above Equation w.r.t. 't' we get

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}x(t)\cdot x^*(t)\cdot dt \longrightarrow ①$$

By the definition of inverse fourier transform, we Can write,

$$x(t) = \mathcal{F}^{-1}(X_1(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\cdot e^{j\omega t}\cdot d\omega.$$

$$\therefore\ x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)\cdot e^{-j\omega t}\cdot \cancel{dt}\,d\omega. \longrightarrow ②$$

from ① § ②,

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}x(t)\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)\cdot e^{-j\omega t}d\omega\right)dt$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega).\left(\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t} dt\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega).X(j\omega).d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Note: The term $|X(j\omega)|^2$ represents the distribution of Energy as a function of ω and so it is called Energy density Spectrum or Energy Spectral density of the Signal $x(t)$.

⟹ **Duality** : if $\mathcal{F}(x_1(t)) = X_1(j\omega)$ and $\mathcal{F}(x_2(t)) = X_2(j\omega)$.

and if $x_2(t) = X_1(j\omega)$ then

$\mathcal{F}(x_2(t)) = X_2(j\omega) = 2\pi\, x_1(-j\omega)$ i.e

if $x_2(t) \Longleftrightarrow X_1(j\omega)$ then

$X_2(j\omega) \Longleftrightarrow 2\pi\, x_1(-j\omega)$.

proof: let $\mathcal{F}(x_1(t)) = X_1(j\omega)$ and $\mathcal{F}(x_2(t)) = X_2(j\omega)$

let $x_2(t)$ and $X_1(j\omega)$ are in Similar form,

$$\therefore \quad x_2(t) = X_1(j\omega)\Big|_{j\omega = t}.$$

By definition of inverse fourier Transform,

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega).\, e^{j\omega t} d\omega$$

$$x_1(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega).\, e^{-j\omega t} d\omega$$

$$x_1(-t)\Big|_{t=j\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)\Big|_{t=j\omega} e^{-j\omega t} d\omega$$

$$x_1(-jw) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_2(t)\, e^{-jwt}\, dw$$

$$\therefore \int_{-\infty}^{\infty} x_2(t)\cdot e^{-jwt}\, dw = 2\pi\, x_1(-jw)$$

$$\therefore \int_{-\infty}^{\infty} \quad X_2(jw) = 2\pi\, x_1(-jw)$$

for Even function $x_1(-jw) = x_1(jw)$

$$\therefore \quad X_2(jw) = 2\pi\, x_1(jw).$$

⇒ Area Under a time Domain Signal :

if $\mathcal{F}(x(t)) = X(jw)$ then

$$\int_{-\infty}^{\infty} x(t)\cdot dt = X(0), \quad \text{where } X(0) = \underset{jw\to 0}{Lt}\, X(jw).$$

proof:

By definition,

$$X(jw) = \int_{-\infty}^{\infty} x(t)\cdot e^{-jwt}\, dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\cdot e^{-0}\cdot dt = \int_{-\infty}^{\infty} x(t)\cdot dt.$$

⇒ Area Under a frequency domain Signal :

if $\mathcal{F}(x(t)) = X(jw)$ then

$$\int_{-\infty}^{\infty} X(jw)\cdot dw = 2\pi\cdot x(0), \quad \text{where } x(0) = \underset{t\to 0}{Lt}\, x(t).$$

proof:

By the Definition of inverse fourier Transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\cdot e^{jwt}\cdot dw$$

$$x(0) = \underset{x\to 0}{Lt}\, x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\cdot 1\cdot dw$$

$$\therefore \int_{-\infty}^{\infty} X(jw)\cdot dw = 2\pi\cdot x(0).$$

→ Fourier Transform Of Complex Exponential Signal:

The Complex Exponential Signal is defined as

$$x(t) = A \cdot e^{j\omega_0 t}$$

$$\therefore \quad \mathcal{F}(x(t)) = X(j\omega) = \mathcal{F}\left(e^{j\omega_0 t} \cdot A\right)$$

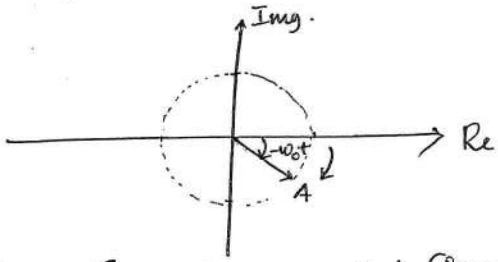
$$= 2\pi A\, \delta(\omega)\Big|_{\omega = \omega - \omega_0} \qquad \because \mathcal{F}\left(e^{j\omega_0 t} x(t)\right) = X(j(\omega - \omega_0))$$

$$\mathcal{F}\left(A e^{j\omega_0 t}\right) = 2\pi A\, \delta(\omega - \omega_0)$$

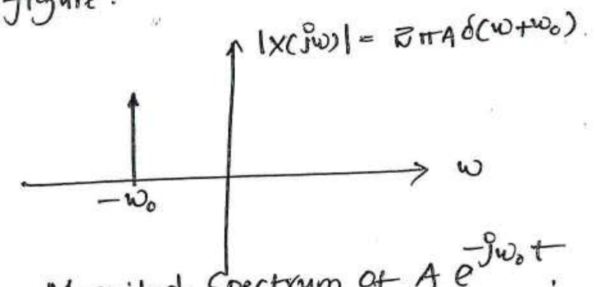
Similarly it can be shown that,

$$\mathcal{F}\left(A e^{-j\omega_0 t}\right) = 2\pi A\, \delta(\omega + \omega_0).$$

The Signal $A e^{-j\omega_0 t}$ can be represented by a rotating Vector of magnitude 'A' in clockwise direction in a Complex plane with an angular Speed of $\omega_0 t$ as Shown in below figure.

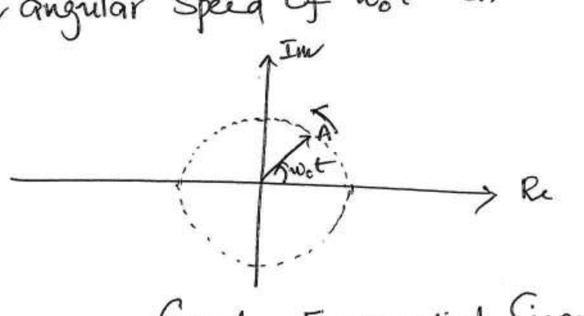


Complex Exponential Signal

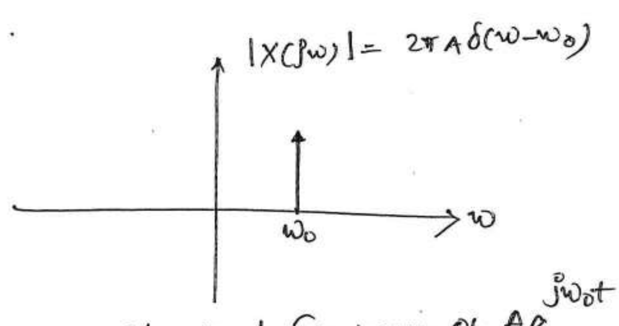


Magnitude Spectrum of $A e^{-j\omega_0 t}$.

Similarly the Signal $A e^{j\omega_0 t}$ can be represented by a rotating Vector of magnitude 'A' in anticlockwise direction in a Complex plane with an angular Speed of $\omega_0 t$ as Shown.



Complex Exponential Signal



Magnitude Spectrum of $A e^{j\omega_0 t}$.

→ Fourier Transform of a periodic Signal :

Let $x(t) \longrightarrow$ Continuous time periodic Signal.

$X(j\omega) = \mathcal{F}(x(t)) \rightarrow$ Fourier Transform of $x(t)$

The Exponential form of fourier Series Representation of $x(t)$ is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t}$$

$$\therefore \quad \mathcal{F}(x(t)) = \mathcal{F}\left(\sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot \mathcal{F}\left(e^{jn\omega_0 t}\right)$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot 2\pi \, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} F_n \cdot \delta(\omega - n\omega_0)$$

$$= \cdots + 2\pi C_{-2} F_{-2} \, \delta(\omega + 2\omega_0) + 2\pi F_{-1} \, \delta(\omega + \omega_0)$$
$$+ 2\pi C_0 \, \delta(\omega) + 2\pi C_1 \, \delta(\omega - \omega_0) + 2\pi C_2 \, \delta(\omega - 2\omega_0) + \cdots$$

$\therefore$ The magnitude of each term in Above Equation represents an impulse Located at its Harmonic frequency in the magnitude Spectrum. Hence we can say that the Fourier Transform of a periodic Continuous time Signal Consists of impulses Located at Harmonic frequencies of the Signal. The magnitude of each impulse is 2π times the magnitude of Fourier Coefficient. i.e the magnitude of n^{th} impulse is $2\pi |F_n|$.

Fourier Transform of Sinusoidal Signal:

The Sinusoidal Signal is defined as,

$$x(t) = A \sin \omega_0 t = \frac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

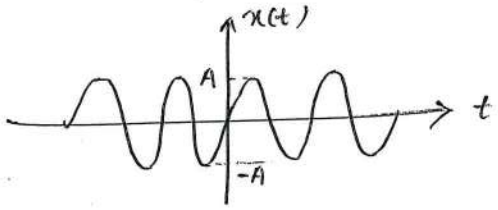
$$\therefore \mathcal{F}(x(t)) = X(j\omega) = \mathcal{F}\left(\frac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right)$$

$$= \frac{A}{2j}\left(\mathcal{F}(e^{j\omega_0 t}) - \mathcal{F}(e^{-j\omega_0 t})\right)$$

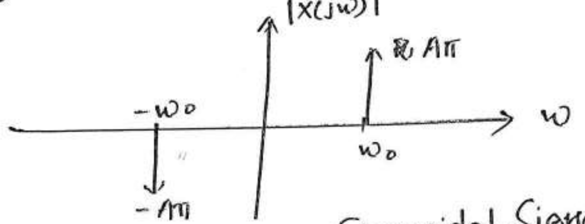
$$= \frac{A}{\cancel{2}j}\left(\cancel{2}\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right)$$

$$\mathcal{F}(A \sin \omega_0 t) = \frac{A\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

The plot of Sinusoidal Signal and its Spectrum are as follows.



Sinusoidal Signal



Spectrum of Sinusoidal Signal.

Fourier Transform of Cosinusoidal Signal:

The Cosinusoidal Signal is defined as,

$$x(t) = A \cos \omega_0 t = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

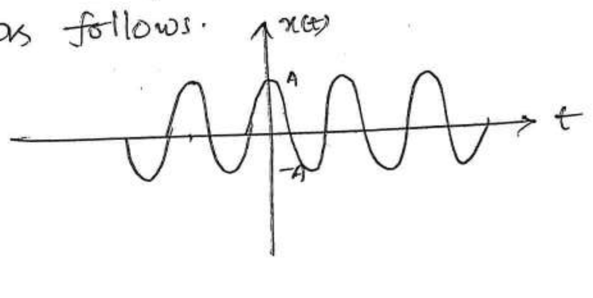
$$\therefore \mathcal{F}(x(t)) = X(j\omega) = \mathcal{F}\left(\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right)$$

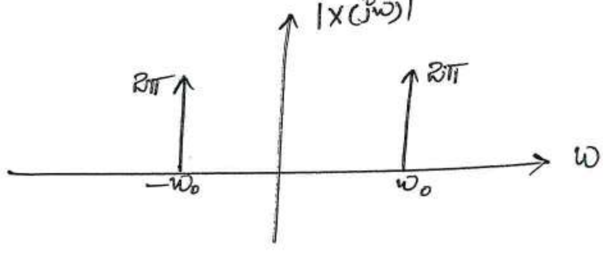
$$= \frac{A}{2}\left(\mathcal{F}(e^{j\omega_0 t}) + \mathcal{F}(e^{-j\omega_0 t})\right)$$

$$= \frac{A}{\cancel{2}}\left(\cancel{2}\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)\right)$$

$$\therefore \mathcal{F}(A \cos \omega_0 t) = A\pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

The plot of Cosinusoidal Signal and its magnitude Spectrum are as follows.

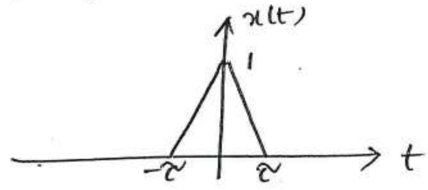




→ Fourier Transform of a Symmetrical Triangular pulse :

A Symmetrical Triangular pulse is defined as,

$$x(t) = \begin{cases} 1 - \dfrac{|t|}{\tau} & |t| < \tau \\ 0 & |t| > \tau \end{cases}$$



By the definition of Fourier Transform,

$$\mathcal{F}(x(t)) = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\tau}^{\tau} \left(1 - \frac{|t|}{\tau}\right) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\tau}^{0} \left(1 + \frac{t}{\tau}\right) \cdot e^{-j\omega t} \cdot dt + \int_{0}^{\tau} \left(1 - \frac{t}{\tau}\right) \cdot e^{-j\omega t} \cdot dt$$

$$= \left(1 + \frac{t}{\tau}\right) \cdot \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\tau}^{0} - \int_{-\tau}^{0} \frac{1}{\tau} \cdot \frac{e^{-j\omega t}}{-j\omega} dt + \left(1 - \frac{t}{\tau}\right) \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{0}^{\tau}$$

$$- \int_{0}^{\tau} \left(-\frac{1}{\tau}\right) \cdot \frac{e^{-j\omega t}}{-j\omega} \cdot dt \cdot$$

$$= \cancel{\frac{-1}{j\omega}} + \frac{1}{\tau j\omega} \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\tau}^{0} + \cancel{\frac{1}{j\omega}} - \frac{1}{\tau \cdot j\omega} \cdot \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{0}^{\tau}$$

$$= \frac{1}{\tau \cdot \omega^2}\left(1 - e^{+j\omega\tau} - \left(e^{-j\omega\tau} - 1\right)\right)$$

$$= \frac{1}{\tau \omega^2}\left(2 - e^{j\omega\tau} - e^{-j\omega\tau}\right)$$

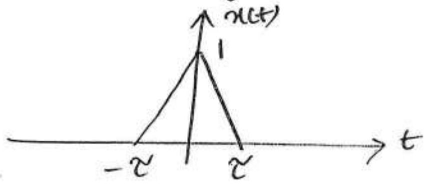
$$= \frac{2}{\tau \omega^2}\left(1 - \cos\omega\tau\right) = \frac{2}{\tau\omega^2}\left(\cancel{1} - \left(\cancel{1} - 2\sin^2\frac{\omega\tau}{2}\right)\right)$$

$$= \frac{4 \sin^2\left(\frac{\omega\tau}{2}\right)}{\tau\omega^2}$$

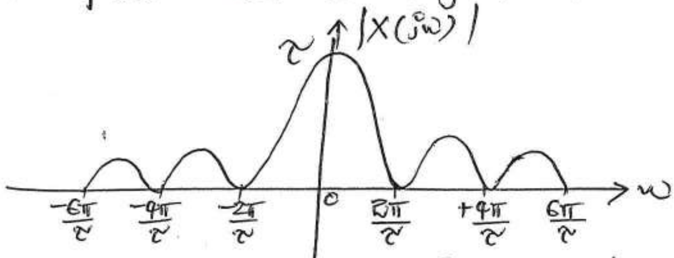
$$= \tau \cdot \left[\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}\right]^2$$

$$\boxed{X(j\omega) = \tau \cdot \left[Sa\left(\frac{\omega\tau}{2}\right)\right]^2}$$

The Symmetrical triangular pulse and its magnitude Spectrum are as follows.



Time Domain Representation



Frequency Domain Representation.
(Magnitude Spectrum)

→ Fourier Transform of $\cos w_0 t \cdot u(t)$:

$$x(t) = \cos w_0 t \, u(t).$$

$$= \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} \cdot u(t).$$

$$\mathcal{F}(x(t)) = X(jw) = \int_{-\infty}^{\infty} \cos w_0 t \cdot u(t) \cdot e^{-jwt} \cdot dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} \cdot e^{-jwt} \cdot u(t) \, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{-j(w-w_0) \cdot t} + e^{-j(w+w_0) t}}{2 .} \cdot u(t) \cdot dt \cdot$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-j(w-w_0) t} \cdot u(t) \cdot dt + \int_{-\infty}^{\infty} e^{-j(w+w_0)} \cdot u(t) dt \right]$$

We know that $\mathcal{F}(u(t)) = \pi \delta(w) + \frac{1}{jw}$.

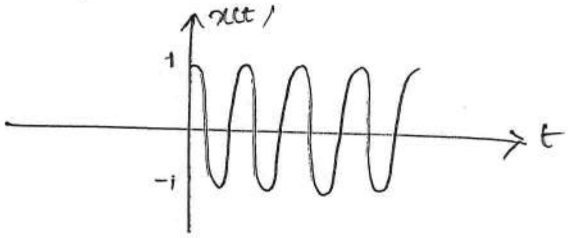
By using frequency Translation property, we can write

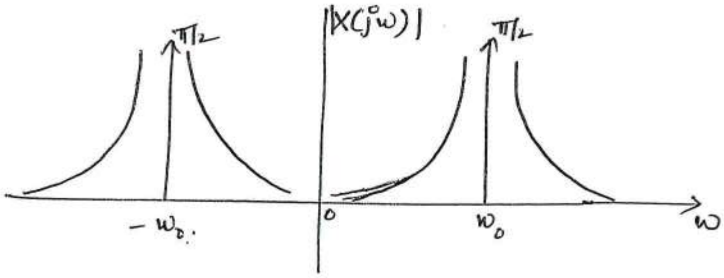
$$\mathcal{F}(\cos w_0 t \cdot u(t)) = \frac{1}{2} \left(\pi \delta(w-w_0) + \frac{1}{j(w-w_0)} + \pi \delta(w+w_0) + \frac{1}{j(w+w_0)} \right)$$

$$= \frac{\pi}{2} \Big(\delta(w-w_0) + \delta(w+w_0) \Big) + \frac{1}{\cancel{2}} \left(\frac{\cancel{2} jw}{\cancel{j^2} (w^2 - w_0^2)} \right)$$

$$\mathcal{F}(\cos w_0 t \cdot u(t)) = \frac{\pi}{2} \Big(\delta(w-w_0) + \delta(w+w_0) \Big) + \frac{jw}{w_0^2 - w^2} .$$

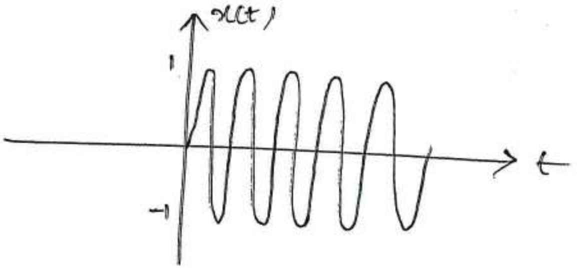
The time Domain And frequency Domain Representations of $x(t)$ are as follows.

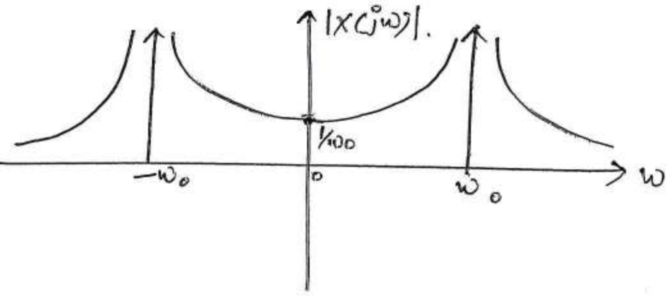




Similarly we can Show that,

$$\mathcal{F}(\sin w_0 t \cdot u(t)) = \frac{\pi}{2j}\left(\delta(w-w_0)-\delta(w+w_0)\right) + \frac{w_0}{w_0^2-w^2}$$

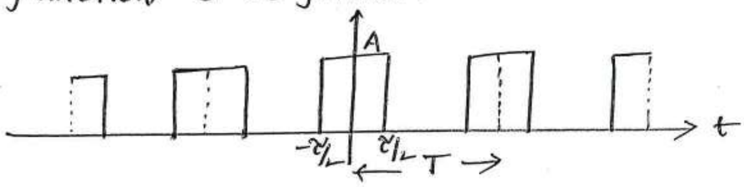




<u>Note</u>: The functions $\cos w_0 t\, u(t)$ and $\sin w_0 t\, u(t)$ are not eternal Sinusoidal Signals. These functions are zero for Values of $t<0$ and exists only for positive Values of 't'. Hence in addition to w_0 they also Contain other frequency Components.

prob: Find the Fourier Transform of a periodic gate function (rectangular pulse of width 'τ' Sec and repeating every 'T' Sec).

Soln: A periodic Gate function is as follows.



The fourier Transform of a periodic function $f(t)$ is given by

$$\mathcal{F}(f(t)) = 2\pi \sum_{n=-\infty}^{\infty} F_n \cdot \delta(w-nw_0).$$

Where F_n is the Exponential Fourier Series Coefficients of $f(t)$.

Therefore, The Spectral density function or the fourier Transform of a periodic Signal Consists of impulses Located at the Harmonic frequencies of the Signal and that the Strength of each impulse is Same as 2π times the Value of the Corresponding Coefficient in the Exponential Fourier Series.

We know that

$$F_n = \frac{1}{T}\int_T f(t)\cdot e^{-jn\omega_0 t}\cdot dt$$

$$f(t) = \begin{cases} A, & -\tau/2 < t < \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < t < T-\frac{\tau}{2} \end{cases}$$

$$\therefore \quad F_n = \frac{1}{T}\int_{-\tau/2}^{T-\tau/2} f(t)\cdot e^{-jn\omega_0 t}\cdot dt$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\cdot e^{-jn\omega_0 t}\cdot dt$$

$$= \frac{-A}{jn\omega_0 T}\cdot e^{-jn\omega_0 t}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{-A\cdot 2}{jn\omega_0 T}\cdot\left(\frac{e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}}{2}\right)$$

$$= \frac{A\tau}{T}\cdot\frac{\sin(n\omega_0\tau/2)}{\left(n\omega_0\frac{\tau}{2}\right)}$$

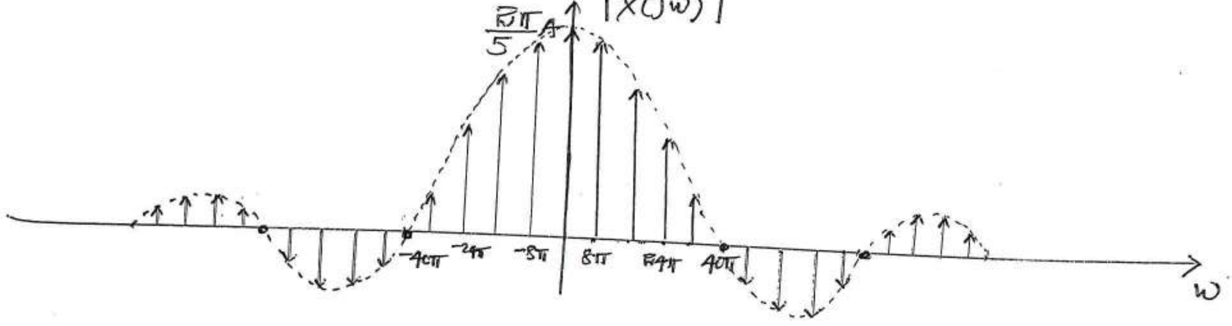
$$= \frac{A\tau}{T}\cdot Sa(n\omega_0\tau/2).$$

$$F_n = \frac{A\tau}{T}\cdot Sa\left(\frac{n\pi\tau}{T}\right) \qquad \because \omega_0 = \frac{2\pi}{T}.$$

$$\therefore \quad \mathcal{F}(f(t)) = 2\pi\frac{A\tau}{T}\sum_{n=-\infty}^{\infty} Sa\left(\frac{n\pi\tau}{T}\right)\cdot\delta(\omega-n\omega_0).$$

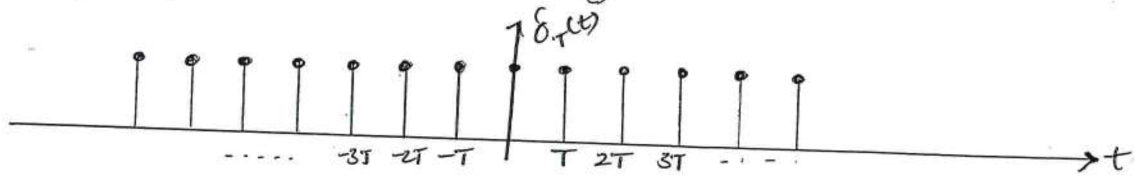
Therefore, the transform of $f(t)$ therefore Consists of impulses Located at, $\omega = 0, \pm\omega_0, \pm 2\omega_0 \cdots \pm n\omega_0 \cdots$ etc. The magnitude of impulse Located at $\omega = n\omega_0$ is given by, $2\pi \cdot \frac{A\tau}{T} Sa\left(\frac{n\pi\tau}{T}\right)$.

Let $\tau = \frac{1}{20}$ and $T = \frac{1}{4}$, $\therefore \omega_0 = 8\pi$. and the Spectrum for these parameters is as follows.



The Spectral Density function of a periodic Gate function.

prob: Find the Fourier Transform of a Sequence of equidistant impulses of unit Strength and Seperated by 'T' Seconds (periodic Train of Unit impulses) as Shown in below figure.



Soln: The periodic Train of impulses denoted by $\delta_T(t)$ has played a vital role in Sampling Theory. It is denoted as

$$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \cdots + \delta(t-nT) + \cdots$$
$$+ \delta(t+T) + \delta(t+2T) + \cdots + \delta(t+nT) + \cdots$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT). \longrightarrow ①.$$

Equation ① is obviously a periodic function with period 'T'. The Exponential Fourier Series of $\delta_T(t)$ is given by,

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t} \rightarrow ②$$

where,
$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\cdot e^{-jn\omega_0 t}\cdot dt$$

$$\delta_T(t) = \delta(t), \quad -T/2 < t < T/2$$

$$\therefore \quad F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\cdot e^{-jn\omega_0 t}\cdot dt$$

From the Sampling property of impulse function, one can write above Equation as,

$$F_n = \frac{1}{T}\, e^{-jn\omega_0 t}\Big|_{t=0} = \frac{1}{T}. \rightarrow ③$$

Therefore, the impulse train function of period 'T' contains Components of frequencies $\omega = 0, \pm\omega_0, \pm 2\omega_0 \cdots \pm n\omega_0$ etc, $\omega_0 = \frac{2\pi}{T}$.

$$\therefore \quad \delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}. \rightarrow ④$$

$\therefore$ The Fourier Transform of $\delta_T(t)$ is then given by,

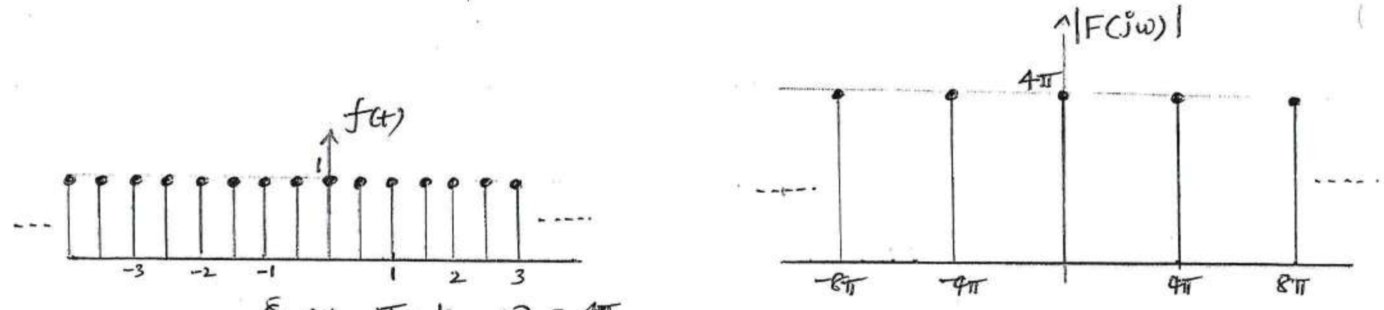
$$\mathcal{F}(\delta_T(t)) = 2\pi \sum_{n=-\infty}^{\infty} F_n.\ \delta(\omega - n\omega_0). \rightarrow ⑤$$

$$= \frac{2\pi}{T}.\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$\mathcal{F}(\delta_T(t)) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0). = \omega_0\, \delta_{\omega_0}(\omega) \rightarrow ⑥.$$

Above Equation States that, the Fourier Transform of a unit impulse train of period 'T' is also a train of impulses of Strength ω_0 and Seperated by 'ω_0' $(\frac{2\pi}{T})$ radians. Therefore the the impulse train function is its own transform.

The Sequence of impulses with periods $T = 1/2$ and $T = 1$ Sec. and their respective transforms are as shown in below. It is evident that as the periods of the impulse increases, the frequency Spectrum becomes dense.



$\delta_T(t)$, $T = 1/2$, $w_0 = 4\pi$

$\delta_T(t)$, $T = 1.$, $w_0 = 2\pi$

fig: Periodic Impulse functions And their Fourier Transforms.

⟹ Energy Density Spectrum :

We know that for a periodic function the power can be associated with the power contained in each discrete frequency component. This result can be extended to non periodic functions. For non periodic signals the Energy of the signal over the Entire interval $(-\infty,\infty)$ is usually finite and the avg. power tends to zero. A more useful concept for a non periodic signal is Energy 'E' defined as,

$$E = \int_{-\infty}^{\infty} |f^2(t)|\,dt \longrightarrow ①$$

if $F(jw)$ is fourier transform of $f(t)$ then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\cdot e^{jwt}\cdot dw$$

$$E = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} f(t)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\cdot e^{jwt}dw\right]\cdot dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\cdot\left[\int_{-\infty}^{\infty} f(t)\cdot e^{jwt}\cdot dt\right]\cdot dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\cdot\ F(-w)\cdot dw$$

We know that if $f(t)$ is real $F(w) = F^*(-w)$

$$F^*(w) = F(-w).$$

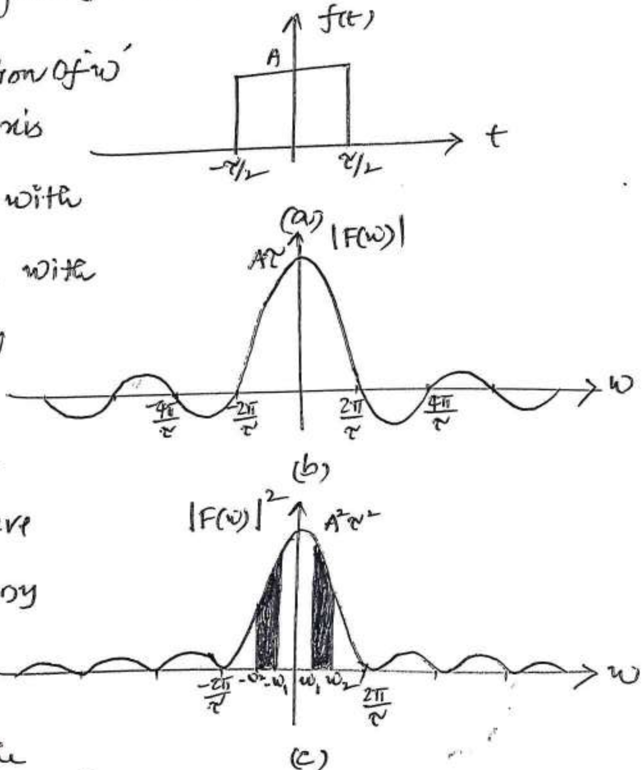
$$\therefore \quad E = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot F^*(\omega)\,d\omega. \qquad \because F(\omega)\cdot F(-\omega) = |F(\omega)|^2 \rightarrow ③$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \rightarrow ②$$

It is evident from Equ③ that, $|F(\omega)|^2$ is a real even function of 'ω'. Equ② States that the Energy of a Signal is given by $\frac{1}{2\pi}$ times the area under the $|F(\omega)|^2$ Curve.

Equ② is known as parsevals Theorem or Rayleigh's Theorem. The functions $F(\omega)$ & $|F(\omega)|^2$ for a gate function are as Shown.

Since $|F(\omega)|^2$ is an even function of ω it is Symmetrical about the Vertical axis passing through the origin. By analogy with periodic functions, we associate Energy with Each frequency Component. The Energy Spectrum However is Contionous and the Energy associated with any one particular frequency is Zero. But there is a finite Energy associated with any finite Band of frequencies.



$\therefore$ The Energy Contained in the frequency Components with in a band of frequencies (ω_1, ω_2) is given by $\frac{1}{2\pi}$ times the area of $|F(\omega)|^2$ under the band (ω_1, ω_2).

fig: a) Gate function $f(t)$
b) Spectral Density function
c) Energy Density Spectrum.

However there is also a band of negative frequencies $(-\omega_1, \omega_2)$ which also has Exactly the Same amount of Energy as that in (ω_1, ω_2). Therefore, the Energy in both of these bands Contributes to the Energy of the Components Lying in the frequency band (ω_1, ω_2). Hence it follows that the Energy Contained in the frequency band (ω_1, ω_2) is given by

$$E = \frac{1}{2\pi}2\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega \rightarrow ④$$

Note that in Equation ⊕ the quantity $\frac{|F(\omega)|^2}{\pi}$ represents Energy per Unit bandwidth and hence represents Energy density.

We define Energy density 'E' $S(\omega)$ as

$$S(\omega) = \frac{1}{\pi} |F(\omega)|^2$$

$\therefore$ If ΔE is the Energy associated with Components of frequencies Lying in the interval (ω_1, ω_2), then

$$\Delta E = \int_{\omega_1}^{\omega_2} S(\omega) \cdot d\omega \quad \text{and}$$

$$E = \int_{0}^{\infty} S(\omega) \cdot d\omega.$$

UNIT-III

SUMMARY AND PREVIOUS QUESTIONS

→ Fourier Transform is a tool that resolves a given function as a Continuous (integral) Sum of Exponential Signals.

→ Fourier Transform is particularly Suitable for Aperiodic (Nonperiodic) Signals.

→ Fourier Transform is a Limiting Case ($T \to \infty$) of Fourier Series.

→ An arbitrary function $f(t)$ can be represented by a Continuous Sum of Exponential functions over the entire interval by using Fourier Transform as,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega \cdot \longrightarrow ①$$

$$\text{Where } X(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt \cdot \longrightarrow ②$$

→ The function $X(\omega)$ is Called as Direct fourier Transform of $f(t)$, also known as Spectral density function or Analysis Equation. and is denoted by $\mathcal{F}(f(t))$

$$\therefore \mathcal{F}(f(t)) = F(\omega) = F(j\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt.$$

→ The function $f(t)$ given in Eqn ① is known as inverse fourier Transform of $X(\omega)$, also called as Synthesis Equation and is denoted by $\mathcal{F}^{-1}(X(\omega))$

$$\therefore \mathcal{F}^{-1}(X(\omega)) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

→ Equations ① and ② are usually referred to as Fourier Transform pair.

$$f(t) \xleftrightarrow{F.T} F(j\omega).$$

→ Since, the frequency domain Representation of $f(t)$ i.e $X(j\omega)$ is in general Complex, it needs two plots for it's representations.

i.e $F(j\omega) = |F(j\omega)| \cdot e^{j\theta(\omega)}$. Thus $F(j\omega)$ Can be represented by magnitude plot $|F(j\omega)|$ and a phase plot $\theta(\omega)$.

(Magnitude Spectrum) (phase Spectrum)

frequency Spectrum.

→ For a real function $f(t)$, the magnitude Spectrum is an even function (Symmetrical about the Vertical axis passing through the origin) of 'ω' and the phase Spectrum is an odd function (Asymmetrical about the Vertical axis passing through the origin) of 'ω'. i.e

$$|F(j\omega)| = |F(-j\omega)|$$

$$\angle F(j\omega) = -\angle F(-j\omega).$$

→ The fourier Transform of $f(t)$ exists if it Satisfies the following Dirchlet Conditions.

→ $\int_{-\infty}^{\infty} f(t)\, dt < \infty$

→ $f(t)$ Should have finite no. of Max & Min. over $(0,T)$.

→ $f(t)$ Should have finite no. of discontinuities over $(0,T)$.

→ The Absolute integrability of $f(t)$ is a Sufficient Condition but not a necessary Condition for the existence of the fourier transform because there are Such functions which are not absolutely integrable but have fourier Transforms.

→ $\mathcal{F}(\delta(t)) = 1$. i.e $\delta(t)$ has uniform Spectral density over the entire frequency interval. that is it Contains all frequency Components with the Same relative amplitudes.

→ An impulse function Can be treated as a Limiting Case of Triangular pulse, rectangular pulse, Gaussian pulse, an exponential pulse and a Sampling function. i.e.

$$\delta(t) = \underset{\tau\to 0}{Lt}\ \text{rect}\left(\frac{t}{\tau}\right),\ |t|<\tau/2, \quad \delta(t) = \underset{\tau\to 0}{Lt}\ \frac{1}{\tau}\cdot e^{-\frac{\pi t^2}{\tau^2}}, \quad \delta(t) = \underset{\tau\to 0}{Lt}\ \frac{1}{\tau}\left(1-\frac{|t|}{\tau}\right), |t|<\tau$$

$$\delta(t) = \underset{\tau\to 0}{Lt}\ \frac{1}{2\tau}\left(e^{-|t|/\tau}\right) = \underset{\tau\to 0}{Lt}\ \frac{1}{\tau}\cdot e^{-t/\tau}\cdot u(t).$$

$$\delta(t) = \underset{k\to\infty}{Lt}\ \frac{K}{\pi}\cdot Sa(kt), \quad \int_{-\infty}^{\infty} \frac{K}{\pi}\, Sa(kt)\, dt = 1.$$

→ The Step function, impulse function and its higher order derivatives are all known as Singularity functions.

⇒ If a function $f(t)$ has a jump discontinuity at $t=t_0$ of amount $(f(t_0^+)-f(t_0^-))$ then the derivative of $f(t)$ at $t=t_0$ is given by

$$\left.\frac{df(t)}{dt}\right|_{t=t_0} = \left[f(t_0^+)-f(t_0^-)\right]\delta(t-t_0).$$

⇒ Multiplication of any function $f(t)$ by an impulse $\delta(t-t_0)$ also yields an impulse of strength $f(t_0)$ at $t=t_0$. i.e.

$$f(t)\cdot\delta(t-t_0) = f(t_0)\cdot\delta(t-t_0).$$ This is known as Sampling property of impulse function.

$$\therefore \int_{-\infty}^{\infty} f(t)\cdot\delta(t-t_0)\cdot dt = f(t_0).$$

Since $\delta(t-t_0)$ is zero except at $t=t_0$, above Equation can be written as

$$\int_{t_0^-}^{t_0^+} f(t)\cdot\delta(t-t_0)\,dt = f(t_0) \text{ and}$$

$$\int_{0^-}^{0^+} f(t)\cdot\delta(t)\cdot dt = f(0).$$

⇒ The Summary of properties of Fourier Transforms are given in below table:

Let $\mathcal{F}(x(t)) = X(j\omega)$, $\mathcal{F}(x_1(t)) = X_1(j\omega)$; $\mathcal{F}(x_2(t)) = X_2(j\omega)$.

PROPERTY	TIME DOMAIN SIGNAL	FREQUENCY DOMAIN SIGNAL
→ Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\omega) + a_2 X_2(j\omega)$
→ Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}\cdot X(j\omega)$.
→ Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\cdot X\left(\frac{j\omega}{a}\right)$
→ Time Reversal	$x(-t)$	$X(-j\omega)$
→ Conjugation	$x^*(t)$	$X^*(-j\omega)$
→ Frequency Shifting	$e^{j\omega_0 t}\cdot x(t)$	$X(j(\omega-\omega_0))$
	$e^{-j\omega_0 t}\cdot x(t)$	$X(j(\omega+\omega_0))$.

→ Time Differentiation, $\frac{d}{dt} x(t)$. $\quad j\omega \cdot X(j\omega)$.

$\frac{d^n x(t)}{dt^n}$ $\quad (j\omega)^n X(j\omega)$.

→ Time Integration $\int_{-\infty}^{t} x(\tau) \cdot d\tau$ $\quad \frac{1}{j\omega} \cdot X(j\omega)$

→ Frequency Differentiation $t \cdot f(t)$ $\quad j \cdot \frac{d}{d\omega} X(j\omega)$

$t^n f(t)$ $\quad (j)^n \frac{d^n}{d\omega^n}(X(j\omega))$

$(-jt)^n f(t)$ $\quad \frac{d^n}{d\omega^n}[X(j\omega)]$.

⇒ Time Convolution (Frequency Multiplication) $\quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \cdot d\tau$ $\quad X_1(j\omega) \cdot X_2(j\omega)$.

⇒ Frequency Convolution (Time multiplication) $\quad x_1(t) \cdot x_2(t)$. $\quad \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X_1(j\lambda) \cdot X_2(j(\omega-\lambda)) d\lambda$

$= \frac{1}{2\pi} [X_1(j\omega) * X_2(j\omega)]$

*** The Convolution of two functions in the time domain is equivalent to multiplication of their Spectra in the frequency domain and that the multiplication of two functions in time domain is equivalent to $\frac{1}{2\pi}$ times Convolution of their Spectra in frequency domain.

⇒ Symmetry of Real Signals. $\quad$ if $x(t)$ is Real. $\quad X(j\omega) = X^*(j\omega)$

$|X(j\omega)| = |X(-j\omega)|$, $\angle X(j\omega) = -\angle X(j\omega)$

$Re(X(j\omega)) = Re(X(-j\omega))$

$Im(X(j\omega)) = -Im(X(-j\omega))$

⇒ Real And Even $\quad$ if $x(t)$ is Real & Even $\quad$ $X(j\omega)$ is also Real & Even

⇒ Real And odd $\quad$ if $x(t)$ is Real & odd $\quad$ $X(j\omega)$ is ~~also~~ Imaginary and odd.

→ **Duality** if $x_2(t) = X_1(j\omega)$, i.e $x_2(t)$ & $X_1(j\omega)$ are Same, then

$X_2(j\omega) = 2\pi x_1(-j\omega)$ i.e $X_2(j\omega)$ & $2\pi x_1(-j\omega)$ are Same.

→ Area Under a time Domain Signal. $\int_{-\infty}^{\infty} x(t).dt = X(0).$

→ Area Under a frequency domain Signal. $\int_{-\infty}^{\infty} X(j\omega).d\omega = 2\pi x(0).$

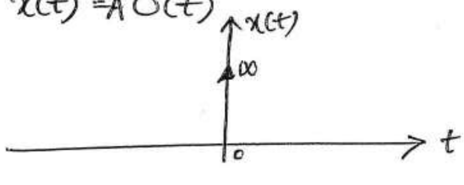
→ Parsevals Relation: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$

$$\Delta E = \int_{\omega_1}^{\omega_2} S(\omega).d\omega, \text{ where } S(\omega) = \frac{1}{\pi}|X(j\omega)|^2$$ is called as Energy density i.e 'E' is given by Area under the Energy density function.

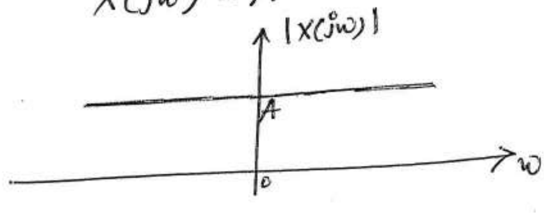
⟹ The Summary of fourier Transforms of Standard Signals and their Magnitude Spectrum are as follows:

$x(t)$	$X(j\omega)$, And Magnitude Spectrum

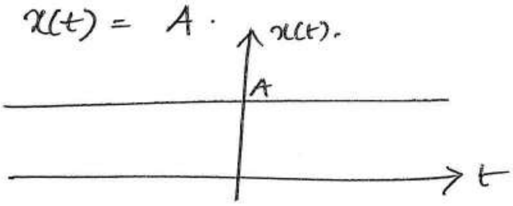
1) $x(t) = A\delta(t)$



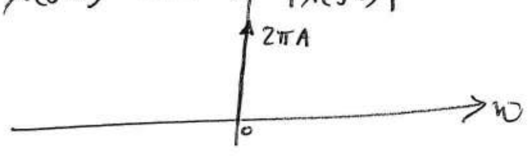
$X(j\omega) = A$



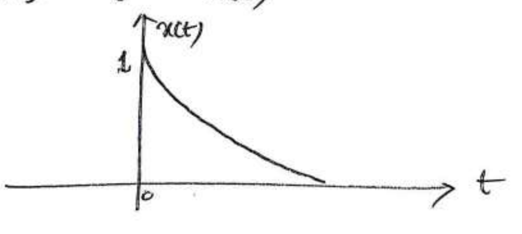
2) $x(t) = A.$



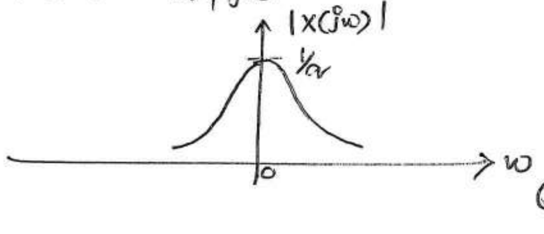
$X(j\omega) = 2\pi A\delta(\omega)$



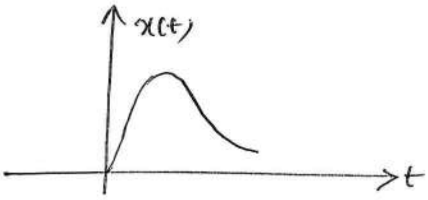
3) $x(t) = e^{-at} u(t)$



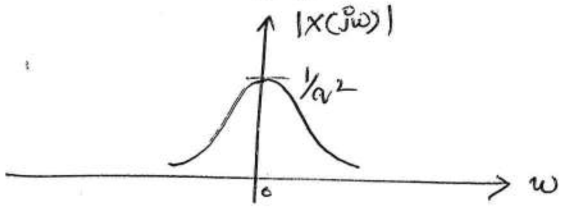
$X(j\omega) = \frac{1}{a+j\omega}$



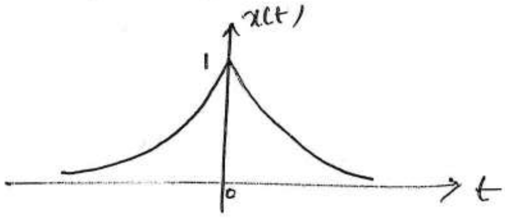
4) $x(t) = te^{-at}u(t)$



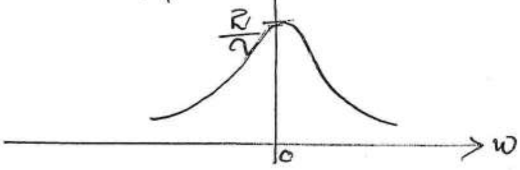
$X(j\omega) = \dfrac{1}{(a+j\omega)^2}$



5) $x(t) = e^{-a|t|}$



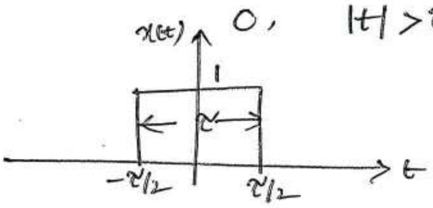
$X(j\omega) = \dfrac{2a}{a^2+\omega^2}$



6) $x(t) = rect\left(\frac{t}{\tau}\right)$

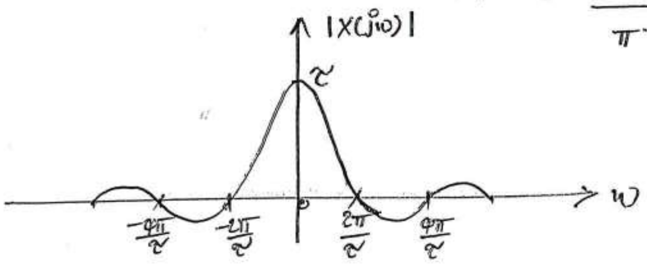
$= 1, \quad |t| < \tau/2$

$ 0, \quad |t| > \tau/2$

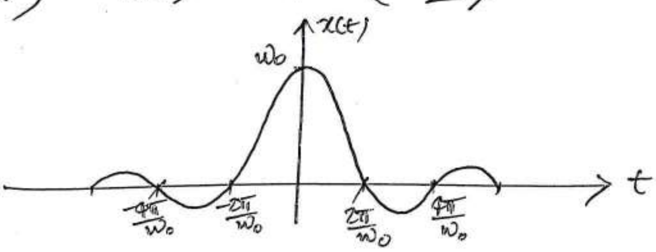


$X(j\omega) = \tau \cdot Sa\left(\frac{\omega\tau}{2}\right) = \tau \cdot \dfrac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$

$= \tau \cdot Sinc\left(\frac{\omega\tau}{2\pi}\right) = \tau \cdot \dfrac{\sin\frac{\pi\omega\tau}{2\pi}}{\pi\cdot\frac{\omega\tau}{2\pi}}$



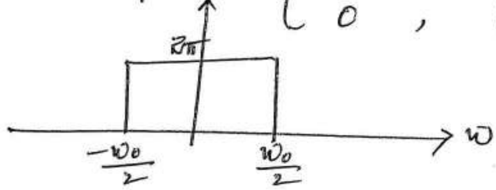
7) $x(t) = \omega_0\, Sa\left(\frac{\omega_0 t}{2}\right)$



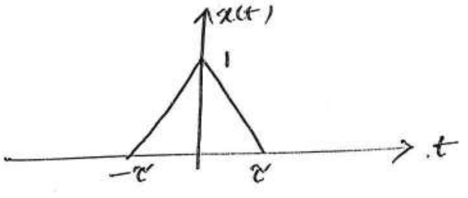
$X(j\omega) = 2\pi\, rect\left(-\frac{\omega}{\omega_0}\right)$

$= 2\pi\, rect\left(\frac{t}{\omega_0}\right)$

$|X(j\omega)| = \begin{cases} 2\pi, & |\omega| < \frac{\omega_0}{2} \\ 0, & |\omega| > \frac{\omega_0}{2} \end{cases}$

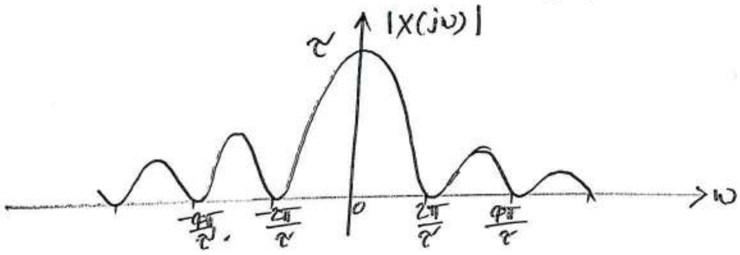


8) $x(t) = 1 - \frac{|t|}{\tau}$



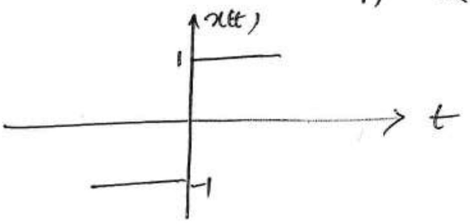
$X(j\omega) = \tau \cdot \left[Sa\left(\frac{\omega\tau}{2}\right)\right]^2$

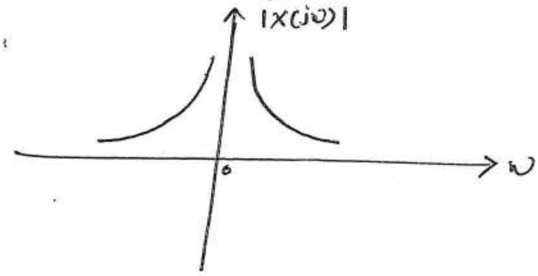
$= \tau \cdot \left[Sinc\left(\frac{\omega\tau}{2\pi}\right)\right]^2$



9) $x(t) = \mathrm{Sgn}(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases}$

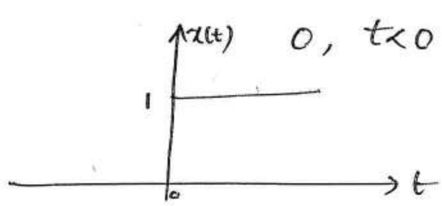
$X(j\omega) = \dfrac{2}{j\omega}$

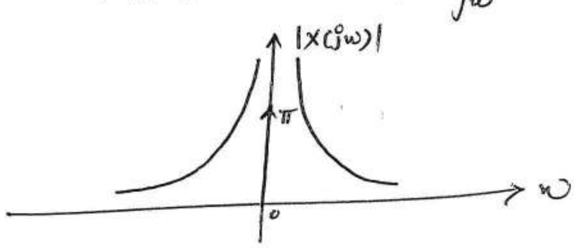




10) $x(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$

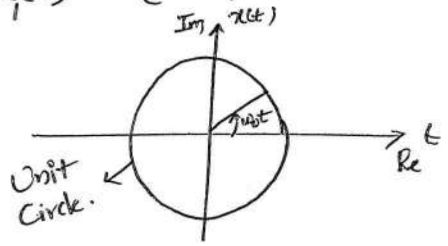
$X(j\omega) = \pi\delta(\omega) + \dfrac{1}{j\omega}$

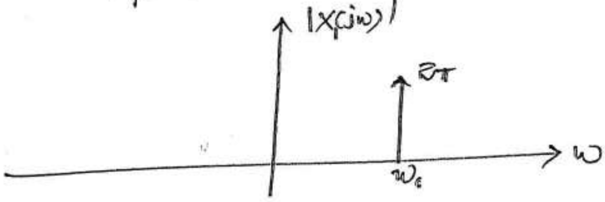




11) $x_1(t) = e^{j\omega_0 t}$

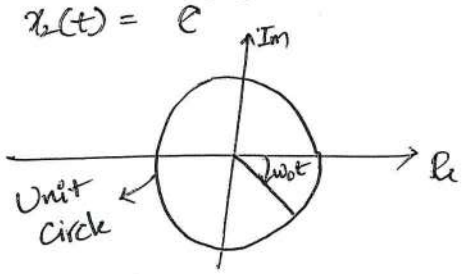
$X_1(j\omega) = 2\pi\delta(\omega-\omega_0)$

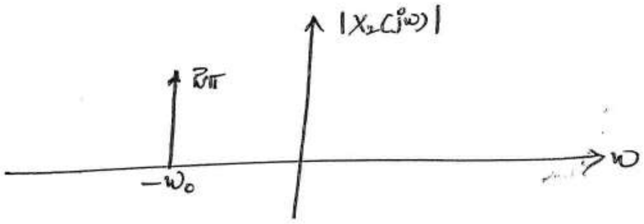




$x_2(t) = e^{-j\omega_0 t}$

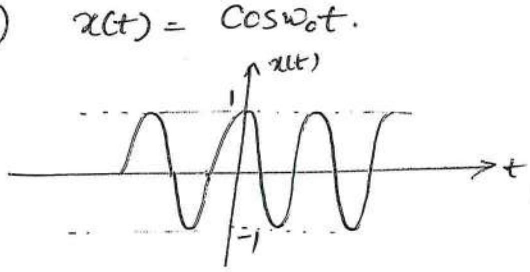
$X_2(j\omega) = 2\pi\delta(\omega+\omega_0)$

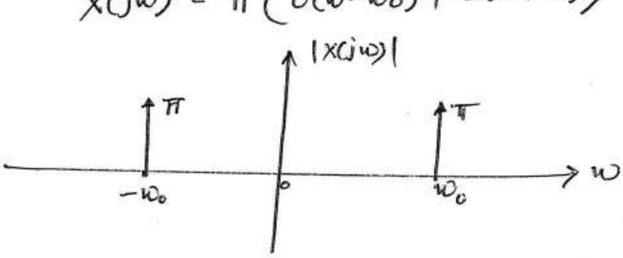




12) $x(t) = \cos\omega_0 t.$

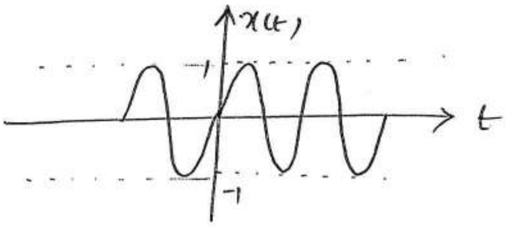
$X(j\omega) = \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$

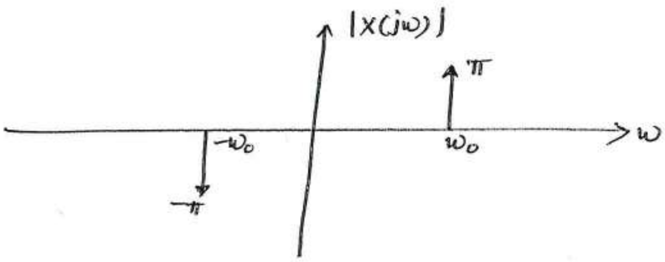




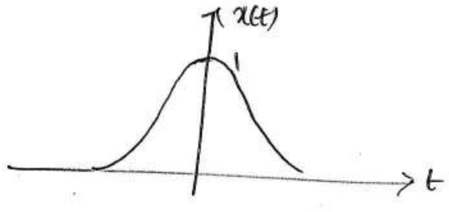
13) $x(t) = \sin\omega_0 t$

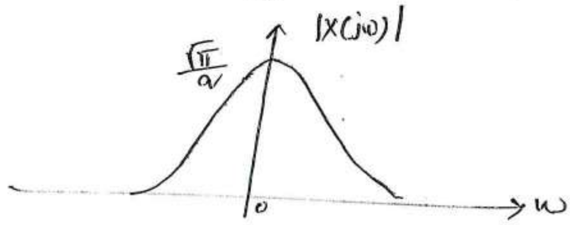
$X(j\omega) = \dfrac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$



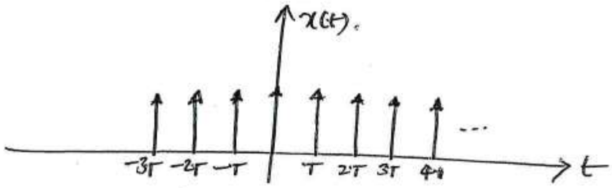


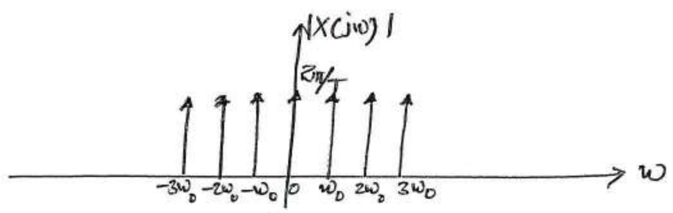
14) $x(t) = e^{-a^2t^2}$ $\qquad$ $X(j\omega) = \frac{\sqrt{\pi}}{a} \cdot e^{-\left(\frac{\omega}{2a}\right)^2}$





15) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ $\qquad$ $X(j\omega) = 2\pi \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \omega_0 = \frac{2\pi}{T}$





→ From the Above Summary, the following Observations Can made as follows.

→ The fourier Transform of a Gaussian pulse will be another gaussian pulse

→ The fourier Transform of an impulse train will be another impulse train.

→ The fourier transform of a rectangular pulse will be a Sinc pulse and Vice-Versa

→ The fourier Transform of a triangular pulse will be a Squared Sinc pulse

→ The fourier transform of a Constant will be an impulse and Vice-Versa.

⇒ The fourier Transform of a periodic Signal $x(t)$, with period 'T' is given by

$$\mathcal{F}(x(t)) = X(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_n \cdot \delta(\omega - n\omega_0).$$

where F_n → Exponential Fourier Series Coefficients given by

$$F_n = \frac{1}{T}\int_T x(t) \cdot e^{-jn\omega_0 t} \cdot dt.$$

Thus the fourier Transform of a periodic Continuous time Signal Consits of impulses Located at the Harmonic frequencies of the Signal and the Strength of each impulse is 2π times the magnitude of Fourier Coefficient.

$\Rightarrow$ The table of Standard Fourier Transform pairs is as follows.

$x(t)$	$X(j\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
A.	$2\pi A\,\delta(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$Sgn(t)$	$\frac{2}{j\omega}$
$t.u(t)$.	$\frac{1}{(j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ $n=1,2,3\ldots$	$\frac{1}{(j\omega)^n}$
$t^n u(t)$	$\frac{1}{(j\omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$
$Ae^{-a\|t\|}$	$\frac{2aA}{a^2+\omega^2}$
$Ae^{j\omega_0 t}$	$2\pi A\,\delta(\omega-\omega_0)$
$Sin\omega_0 t$	$\frac{\pi}{j}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$
$Cos\omega_0 t$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$.

Previous Questions:

1. a) Obtain the fourier Transform of the following functions.
   (i) impulse function $\delta(t)$
   (ii) D.C Signal (Constant = A).
   (iii) Unit Step function

   b) State And prove the time Differentiation property of CTFT.

Ans: Refer Notes or Material.

2 a) What is the duality property of CTFT? Explain.

__Ans:__ Refer Material. Write Answer by taking Example.

b) Obtain the fourier transform for,

(i) $x(t) = 6\sin(200\pi t)$

(ii) $x(t) = \frac{1}{2}\left(\delta(t+1) + \delta(t+1/2) + \delta(t+1/2) + \delta(t-1) + 5\right)$

__Soln:__ (i). Given Signal is

$$x(t) = 6\sin 200\pi t$$

Taking fourier transform on both sides,

$$\mathcal{F}(x(t)) = X(j\omega) = \mathcal{F}(6\sin 200\pi t)$$

$$= \int_{-\infty}^{\infty} 6\sin 200\pi t \cdot e^{-j\omega t} \cdot dt.$$

$$= \int_{-\infty}^{\infty} \cancel{6}^{3} \cdot \frac{e^{j200\pi t} - e^{-j200\pi t}}{\cancel{2}j} \cdot e^{-j\omega t} \cdot dt.$$

$$= \frac{3}{j}\left[\int_{-\infty}^{\infty} e^{j200\pi t} \cdot e^{-j\omega t} \cdot dt - \int_{-\infty}^{\infty} e^{-j200\pi t} \cdot e^{-j\omega t} dt\right]$$

we know that $\mathcal{F}\left(e^{j\omega_0 t}\right) = 2\pi\,\delta(\omega - \omega_0)$

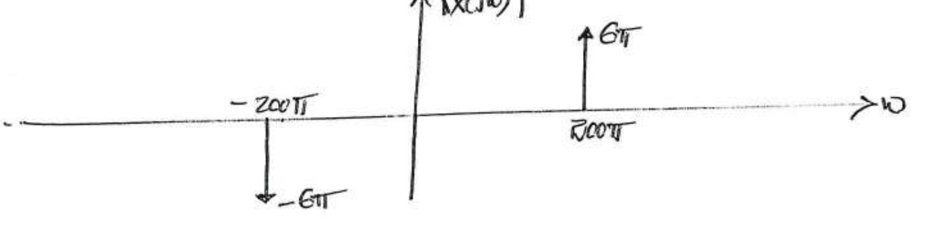
$$\mathcal{F}\left(e^{-j\omega_0 t}\right) = 2\pi\,\delta(\omega + \omega_0)$$

$$\therefore \mathcal{F}(x(t)) = \frac{3}{j}\left(\mathcal{F}\left(e^{j200\pi t}\right) - \mathcal{F}\left(e^{-j200\pi t}\right)\right)$$

$$= \frac{3}{j}\left(2\pi(\delta(\omega - 200\pi)) - 2\pi\,\delta(\omega + 200\pi)\right)$$

$$X(j\omega) = \frac{6\pi}{j}\left(\delta(\omega - 200\pi) - \delta(\omega + 200\pi)\right)$$

The Magnitude Spectrum $|X(j\omega)|$ is as follows.



(ii) The given Signal is

$$x(t) = \frac{1}{2}\left(\delta(t+1) + \delta(t+1/2) + \delta(t+1/2) + \delta(t-1) + 5 \right)$$

Taking Fourier Transform on both Sides.

$$\mathcal{F}(x(t)) = X(j\omega) = \mathcal{F}\left(\frac{1}{2}\left(\delta(t+1) + 2\,\delta(t+1/2) + \delta(t-1) + 5\right)\right)$$

By using Linearity property of CTFT,

$$X(j\omega) = \frac{1}{2}\,\mathcal{F}(\delta(t+1)) + \mathcal{F}(\delta(t+1/2)) + \mathcal{F}(\delta(t-1)) + \mathcal{F}(5).$$

we know that,

$$\mathcal{F}(\delta(t-t_0)) = e^{-j\omega t_0}$$

$$\mathcal{F}(\delta(t+t_0)) = e^{j\omega t_0}$$

$$\mathcal{F}(A) = 2\pi A \cdot \delta(\omega).$$

$$\therefore \quad X(j\omega) = \frac{1}{2}\, e^{j\omega} + e^{j\omega/2} + e^{-j\omega} + 2\pi(5)\cdot\delta(\omega).$$

$$X(j\omega) = \cos\omega + 10\pi\,\delta(\omega) + e^{j\omega/2} + 1/2\, e^{-j\omega}.$$

③ a) prove that the normalized power is given by $P = \sum_{n=-\infty}^{\infty} |C_n|^2$ where. $|C_n|$ are Complex Fourier Coefficients for the periodic wave form.

<u>Ans:</u> Write parsevals Theorem Applicable for Fourier Series.

b) If the waveform $V(t)$ has the Fourier transform $V(f)$, then Show that the transform of the integral of $V(t)$ is given by

$$\mathcal{F}\left(\int_{-\infty}^{t} V(\lambda)\,d\lambda\right) = \frac{V(f)}{j2\pi f}.$$

<u>Ans:</u> Time integration property of CTFT. write proof, $\omega = 2\pi f$.

④ State And prove the frequency Shifting property of CTFT.

<u>Ans:</u> See Material,

⑤ a) Determine the Fourier transform of a two Sided Exponential pulse $x(t) = e^{-|t|}$.

Soln: See Material , $a = 1$.

b) Find the Fourier Transforms of an even function $x_e(t)$ and an odd function $x_o(t)$ of $x(t)$.

Soln: Let $x(t)$ be a Continuous time Signal, then the Even and odd parts of $x(t)$ are defined as,

$$x_e(t) = \frac{x(t) + x(-t)}{2} \longrightarrow ①$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \longrightarrow ②.$$

Taking Fourier Transform on both Sides of Eqn①, we get.

$$\mathcal{F}(x_e(t)) = X_e(j\omega) = \mathcal{F}\left(\frac{x(t) + x(-t)}{2}\right)$$

By using Linearity property

$$X_e(j\omega) = \frac{1}{2}\left[\mathcal{F}(x(t)) + \mathcal{F}(x(-t))\right].$$

We know that if $\mathcal{F}(x(t)) = X(j\omega)$, then

$$\mathcal{F}(x(-t)) = X(-j\omega)$$

$$\therefore \quad X_e(j\omega) = \frac{1}{2}\left(X(j\omega) + X(-j\omega)\right) \longrightarrow ③.$$

Similarly, Taking Fourier transform on both Sides of Eqn②, we get

$$\mathcal{F}(x_o(t)) = X_o(j\omega) = \mathcal{F}\left(\frac{x(t) - x(-t)}{2}\right)$$

$$= \frac{1}{2}\left(\mathcal{F}(x(t)) - \mathcal{F}(x(-t))\right)$$

$$X_o(j\omega) = \frac{1}{2}\left(X(j\omega) - X(-j\omega)\right) \longrightarrow ④.$$

Thus Equations ③ & ④ represents the CTFT of even and odd parts of $x(t)$.

⑥ a) An AM Signal is given by,

$$f(t) = 15\sin(2\pi 10^6 t) + [5\cos 2\pi 10^3 t + 3\sin 2\pi 10^2 t]\sin 2\pi 10^6 t$$

Find the Fourier transform and Draw its Spectrum.

Soln:

An Amplitude modulated Signal is given by

$$f(t) = 15\sin(2\pi 10^6 t) + (5\cos 2\pi 10^3 t + 3\sin 2\pi 10^2 t)\sin 2\pi 10^6 t$$

Taking Fourier Transform on both Sides, we get,

$$\mathcal{F}(f(t)) = F(j\omega) = \mathcal{F}\left(15\sin 2\pi 10^6 t + (5\cos 2\pi 10^3 t + 3\sin 2\pi 10^2 t)\sin 2\pi 10^6 t\right)$$

By using Linearity property, we can write,

$$F(j\omega) = \mathcal{F}(15\sin 2\pi 10^6 t) + \mathcal{F}\left((5\cos 2\pi 10^3 t + 3\sin 2\pi 10^2 t)\sin 2\pi 10^6 t\right)$$

$$F(j\omega) = \mathcal{F}(15\sin 2\pi 10^6 t) + \mathcal{F}(5\cos 2\pi 10^3 t\,\sin 2\pi 10^6 t)$$
$$+ \mathcal{F}(3\sin 2\pi 10^2 t\,\sin 2\pi 10^6 t) \rightarrow ①$$

We know that,

$$\mathcal{F}(A\sin\omega_0 t) = \frac{A\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

$$\mathcal{F}(f(t).\sin\omega_0 t) = \frac{1}{2j}\left(F(j\omega-\omega_0) - F(\omega+\omega_0)\right)$$

$$\mathcal{F}(A\cos\omega_0 t) = A\pi\left(\delta(\omega-\omega_0) \pm \delta(\omega+\omega_0)\right)$$

$$\therefore\ \cancel{F(j\omega) = \frac{15\pi}{j}\left(\delta(\omega-2\pi 10^6) - \delta(\omega+2\pi 10^6)\right)} + \cancel{\left[5\pi\left(\delta(\omega-2\pi 10^3) + \delta(\omega+2\pi 10^3)\right)\right]}\times$$

$$\cancel{\boxed{2j}}\Big|_{\omega=}$$

$$\mathcal{F}(15\sin 2\pi 10^6 t) = \frac{15\pi}{j}\left(\delta(\omega-2\pi 10^6 t) - \delta(\omega+2\pi 10^6 t)\right)$$

$$\mathcal{F}(5\cos 2\pi 10^3 t) = 5\pi\left(\delta(\omega-2\pi 10^3) + \delta(\omega+2\pi 10^3)\right) \rightarrow ②$$

$$\mathcal{F}(5\cos 2\pi 10^3 t \cdot \sin 2\pi 10^6 t) = \frac{5\pi}{2j}\Big[\delta(\omega-2\pi 10^6-2\pi 10^3) + \delta(\omega-2\pi 10^6+2\pi 10^3)$$
$$- \delta(\omega+2\pi 10^6-2\pi 10^3) - \delta(\omega+2\pi 10^6+2\pi 10^3)\Big]$$
$$\rightarrow ③$$

$$\mathcal{F}(3\sin 2\pi 10^2 t) = \frac{3\pi}{j}\left(\delta(\omega-2\pi 10^2) - \delta(\omega+2\pi 10^2)\right)$$

$$\mathcal{F}(3\sin 2\pi 10^2 t\cdot \sin 2\pi 10^6 t) = \frac{3\pi}{j}\,\frac{1}{2j}\Big(\delta(\omega-2\pi10^2-2\pi10^6) - \delta(\omega+2\pi10^2-2\pi10^6) - \delta(\omega-2\pi10^2+2\pi10^6) + \delta(\omega+2\pi10^2+2\pi10^6)\Big)$$

$$= \frac{-3\pi}{2}\Big(\delta\big(\omega-(2\pi10^2+2\pi10^6)\big) - \delta\big(\omega-(2\pi10^6-2\pi10^2)\big) - \delta\big(\omega+(2\pi10^6-2\pi10^2)\big) + \delta\big(\omega+(2\pi10^6+2\pi10^2)\big)\Big] \longrightarrow (4)$$

By Substituting Equations (2), (3) & (4) in Equ (1), we get,

$$F(j\omega) = \frac{15\pi}{j}\Big(\delta(\omega-2\pi10^6) - \delta(\omega+2\pi10^6)\Big) + \frac{5\pi}{2j}\Big[\delta\big(\omega-(2\pi10^6+2\pi10^3)\big) + \delta\big(\omega-(2\pi10^6-2\pi10^3)\big) - \delta\big(\omega+(2\pi10^6-2\pi10^3)\big) - \delta\big(\omega+(2\pi10^6+2\pi10^3)\big)\Big] +$$

$$\frac{-3\pi}{2}\Big(\delta\big(\omega-(2\pi10^6+2\pi10^2)\big) - \delta\big(\omega-(2\pi10^6-2\pi10^2)\big) - \delta\big(\omega+(2\pi10^6-2\pi10^2)\big) + \delta\big(\omega+(2\pi10^6+2\pi10^2)\big)\Big]$$

The Spectrum of a two tone Amplitude modulated Signal is given as,

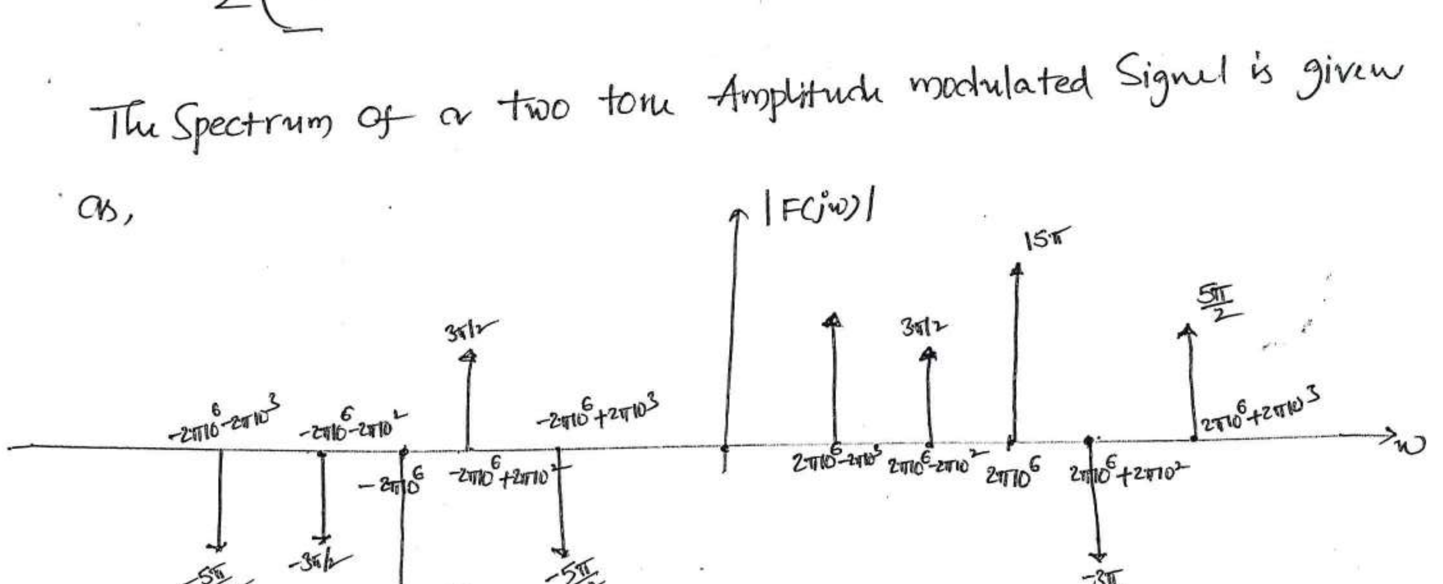
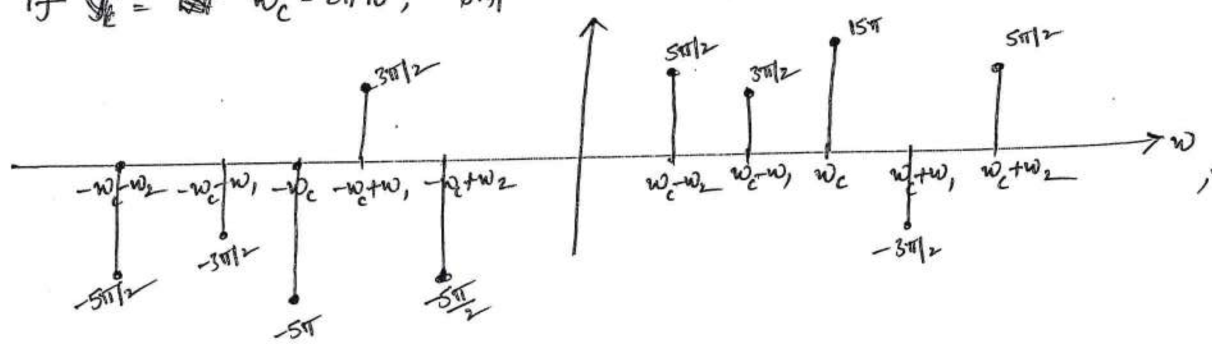


fig: Magnitude Spectrum of AM Signal $f(t)$.

if $\omega_c = 2\pi 10^6$, $\omega_{m_1} = 2\pi 10^2$, $\omega_{m_2} = 2\pi 10^3$, then



$, \omega_2 > \omega_1$

b) A Signal $x(t)$ has fourier Transform $X(f) = \dfrac{j2\pi f}{3+\left(\frac{j}{10}\right)}$

(i) what is the total net area Under the Signal $x(t)$.

(ii) let $y(t) = \int_{-\infty}^{t} x(\lambda)\cdot d\lambda$. what is the total net area under $y(t)$.

Soln: Given data:

$$\mathcal{F}(x(t)) = X(f) = \frac{j2\pi f}{3+\left(\frac{j}{10}\right)} \longrightarrow (1).$$

(i) Total net area under the Signal $x(t)$ is given by

$$\int_{-\infty}^{\infty} x(t)\cdot dt = X(0) = \frac{j2\pi(0)}{3+\left(\frac{j}{10}\right)} = 0.$$

(ii)
$$y(t) = \int_{-\infty}^{t} x(\lambda)\cdot d\lambda.$$

$$\mathcal{F}\left(y(t)\right) = \mathcal{F}\left(\int_{-\infty}^{t} x(\lambda)\, d\lambda\right) = \frac{1}{j\omega}\cdot X(f)$$

$$= \frac{1}{\cancel{j2\pi f}}\left(\frac{\cancel{j2\pi f}}{3+\left(\frac{j}{10}\right)}\right)$$

$$Y(f) = \frac{1}{3+\left(\frac{j}{10}\right)}$$

Total net area under $y(t)$ is given by

$$\int_{-\infty}^{\infty} y(t)\cdot dt = Y(0) = \frac{1}{3+\left(\frac{j}{10}\right)}.$$

(7) Find the Fourier Transform Of the following functions.

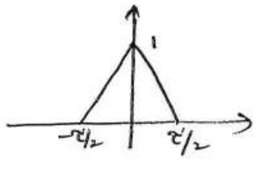
a) A Single Symmetrical triangular pulse,

b) A Single Symmetrical Gate pulse

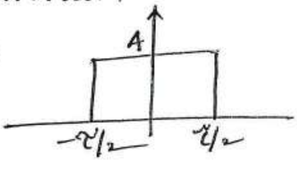
c) A Single Cosine wave at $t=0$.

Soln for a) & b) Refer Material.

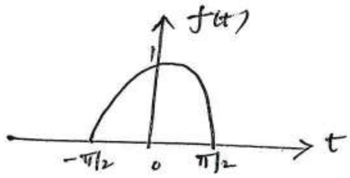
(a)



(b)



c) A Single Cosine wave at $t=0$ is defined as



$$f(t) = A \cos t.$$

$$F(jw) = \int_{-\pi/2}^{\pi/2} f(t) \cdot e^{-jwt}\, dt$$

$$= \int_{-\pi/2}^{\pi/2} A \cos t \; e^{-jwt} \cdot dt.$$

$$= \frac{A}{2} \int_{-\pi/2}^{\pi/2} \left(e^{jt} \cdot e^{-jwt} + e^{-jt} \cdot e^{-jwt} \right) dt$$

$$= \frac{A}{2} \left[\int_{-\pi/2}^{\pi/2} e^{-j(w-1)t}\, dt + \int_{-\pi/2}^{\pi/2} e^{-j(w+1)t}\, dt \right]$$

$$= \frac{A}{2} \left[\frac{e^{-j(w-1)t}}{-j(w-1)} \Bigg|_{-\pi/2}^{\pi/2} + \frac{e^{-j(w+1)t}}{-j(w+1)} \Bigg|_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{A}{2} \left[\frac{e^{-j(w-1)\pi/2}}{-j(w-1)} + \frac{e^{j(w-1)\pi/2}}{j(w-1)} - \frac{e^{-j(w+1)t}}{j(w+1)} + \frac{e^{j(w+1)\pi/2}}{j(w+1)} \right]$$

$$= \frac{A}{\cancel{2}} \left[\frac{\cancel{2}\cancel{j} \sin(w-1)\pi/2}{\cancel{j}(w-1)} + \frac{\cancel{2}\cancel{j} \sin(w+1)\pi/2}{\cancel{j}(w+1)} \right]$$

$$= \frac{\pi}{2} A \left(\frac{\sin(w-1)\pi/2}{(w-1)\pi/2} + \frac{\sin(w+1)\pi/2}{(w+1)\pi/2} \right)$$

$$F(jw) = \frac{A\pi}{2} \left(Sa(w-1)\pi/2 + Sa(w+1)\pi/2 \right)$$

(8). State And prove the following properties of CTFT.

a) Multiplication in time domain

b) Convolution in time domain.

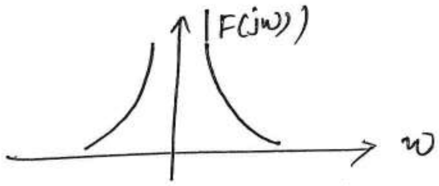
<u>Ans:</u> Refer Material

(9) Find the Fourier transform of Signum function and plot its Amplitude And phase Spectra.

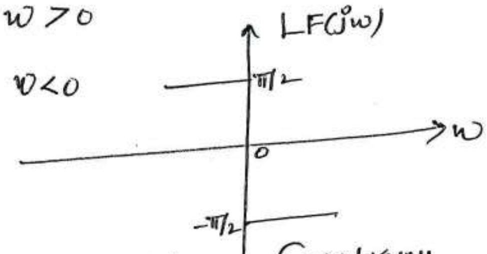
<u>Ans:</u> Refer Material for $\mathcal{F}(Sgn(t)) = \frac{1}{j\pi f} = \frac{2}{j\omega} = \frac{-2j}{\omega}$

$$|F(j\omega)| = \frac{2}{\omega}$$

$$\angle F(j\omega) = \begin{cases} -\pi/2, & \omega > 0 \\ \pi/2, & \omega < 0 \end{cases}$$



Amplitude Spectrum



phase Spectrum.

(9) If the waveform $V(t)$ has the fourier transform $V(f)$ then Show that the waveform delayed by time t_d i.e $V(t-t_d)$ has the transform $V(f)\cdot e^{-j\omega t_d}$.

<u>Ans:</u> Refer Material (Time Shifting property of CTFT).

(10) a) Explain How fourier transform is developed from Fourier Series.

b) Power Signals will have Fourier Transforms and Energy Signals will have fourier Series in the frequency domain. Justify this Statement.

<u>Ans:</u> a) Refer Material.

b) power Signals will have fourier Transforms:

→ A Continuous time Signal $x(t)$ is Said to be power Signal if it is having finite average power. The periodic Signals are always power Signals.

→ Fourier Transform is particularly Suitable for representation of aperiodic Signals. It is also possible to represent periodic (power) Signals by using fourier transform.

→ The fourier Transform of a periodic Signal $x(t)$ is given by,

$$\mathcal{F}(x(t)) = X(j\omega) = 2\pi \cdot \sum_{n=-\infty}^{\infty} F_n \cdot \delta(\omega - n\omega_0).$$

where $F_n = \frac{1}{T}\int_T x(t) \cdot e^{-jn\omega_0 t} dt$ is the Exponential Fourier Series Coefficients of $x(t)$.

Thus the fourier transform of a periodic (power) Signal Consists of train of impulses Located at, $\omega = 0, \pm\omega_0, \pm 2\omega_0, \cdots \pm n\omega_0$ whose Strength is 2π times F_n. Therefore the Strength of the impulse Located at $\omega = n\omega_0$ is $2\pi \cdot F_n$.

→ Energy Signals will have fourier Series:

→ A Continuous time Signal $x(t)$ is Said to be Energy Signal if it is having finite Energy. All finite duration (Aperiodic Signals) Signals are Energy Signals.

→ Fourier Series representation ~~of~~ is particularly Suitable for periodic (power) Signals. It is also possible to represent an aperiodic (Energy) Signal by using Fourier Series. over a Certain interval.

→ Consider a function as Shown.

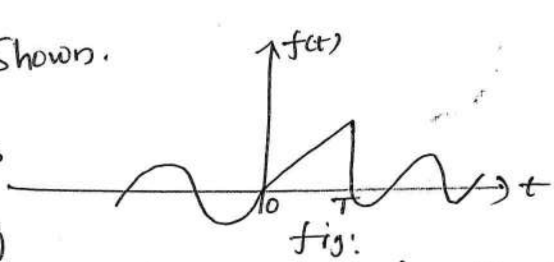


fig:

Here $f(t)$ is aperiodic Signal. It is possible to represent $f(t)$ by using fourier Series over $(0,T)$. But the Approximated function $f_a(t)$ is equal to $f(t)$ over $(0,T)$ but not over the entire interval $(-\infty, \infty)$. i.e $f_a(t) = f(t) \quad 0 < t < T$

Therefore from the Above discussion it is Concluded that power Signals will have Fourier transforms and Energy Signals will have fourier Series Representation.

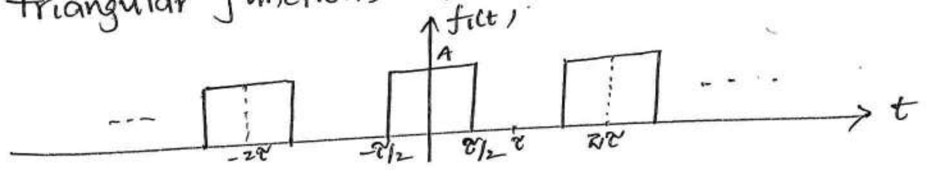
(11) State and prove the frequency Translation property of CTFT. From this Show that

Modulation Theorem. $\leftarrow$
$$\begin{cases} \mathcal{F}\left(f(t)\cdot \cos w_0 t\right) = \frac{1}{2}\left(F(w-w_0) + F(w+w_0)\right) \\ \mathcal{F}\left(f(t)\,\sin w_0 t\right) = \frac{1}{2j}\left(F(w-w_0) - F(w+w_0)\right) \\ \text{where } F(w) = \mathcal{F}(f(t)). \end{cases}$$

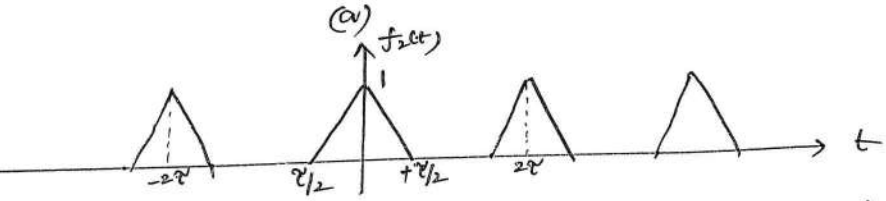
Ans: See Class notes.

(12) Show that the fourier transform of a periodic train of impulses $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is also a periodic train of impulses. Also Sketch the magnitude Spectrum for $T=\frac{1}{2}$, $T=1$. Make a Comment on what happens if T tends to infinity.

(13) Determine the fourier transform of a periodic rectangular and periodic triangular functions as shown.



Ans: Refer Material (or) class notes



(14) Show that for a real function $f(t)$, the Spectral density function (Magnitude Spectrum) is Symmetrical About the Vertical axis passing through the origin.

(15) prove that $\mathrm{Sinc}(0) = 1$. and plot the Sinc function.

Ans:
$$\mathrm{Sinc}(x) = \frac{\sin \pi x}{\pi x}$$

$$\sin \pi x = \frac{e^{j\pi x} - e^{-j\pi x}}{2j},$$

we know that, $e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

$$\therefore\ e^{j\pi x} = 1 + j\pi x + \frac{(j\pi x)^2}{2!} + \frac{(j\pi x)^3}{3!} + \cdots$$

$$e^{-j\pi x} = 1 - j\pi x + \frac{(j\pi x)^2}{2!} - \frac{(j\pi x)^3}{3!} + \cdots$$

$$\therefore\ e^{j\pi x} - e^{-j\pi x} = 2j\pi x + \frac{2}{3!}(j\pi x)^3 + \frac{2}{5!}(j\pi x)^5 + \cdots$$

$$\sin\pi x = \frac{e^{j\pi x} - e^{-j\pi x}}{2j} = \pi x + \frac{1}{3!}\frac{(j\pi x)^3}{j} + \frac{1}{5!}\frac{(j\pi x)^5}{j} + \cdots$$

88, 89, 90, 91, 92, 93, 95, A2, A4, A5, B1, B3, B5, C0, C2, B6, 131, 09, 02, 01

$$\frac{\sin\pi x}{\pi x} = 1 + \frac{1}{3!}(j\pi x)^2 + \frac{1}{5!}(j\pi x)^4 + \cdots$$

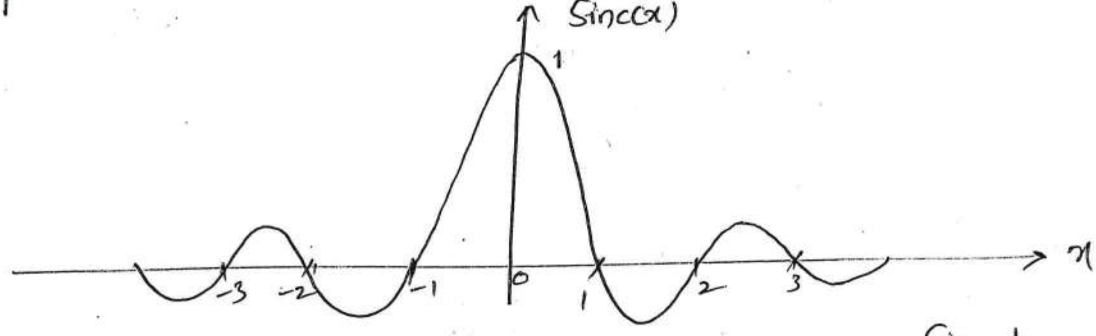
$$\mathrm{Sinc}(x) = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \cdots$$

putting $x=0$ in the Above equation,

$$\boxed{\mathrm{Sinc}(0) = 1.}$$

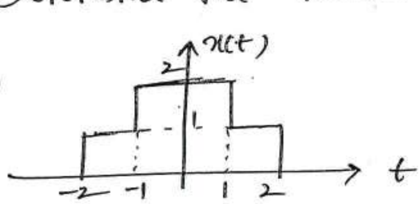
$\mathrm{Sinc}(x) = 0$ at $x = \pm1, \pm2, \pm3 \ldots$ Since $\sin\pi x = 0$.

Other values of Sinc(x) Can be Obtained by putting 'x' at different points. Below fig. Shows the plot of Sinc function.

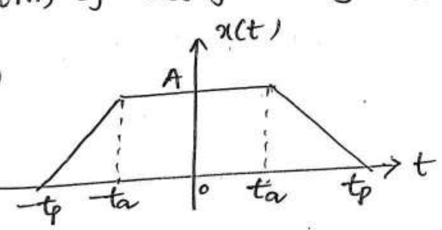


(16) Determine the Fourier Transform of the following Signals.

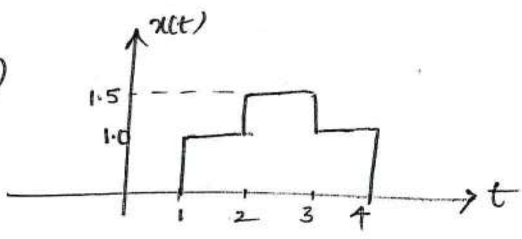
a)



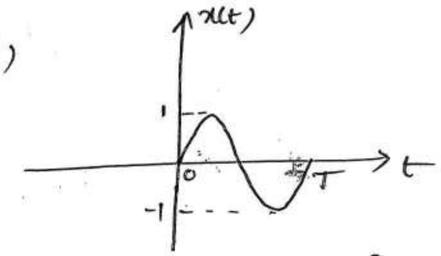
(b)



(c)



(d)



(One Cycle of Sine wave)

UNIT - V

SAMPLING

INTRODUCTION :- Sampling process is an operation by which an analog Signal is Converted into a discrete time Signal. To enable Digital Transmission of Data Sampling is Very useful in Digital Communication. An analog Signal can be Converted into a Corresponding Sequence of Samples that may be Spaced Uniformely or non Uniformely in time. Usually the Samples are Spaced Uniformely and the Corresponding Sampling process is Called Uniform Sampling. Below fig. Shows the Continuous time Signal and its Sampled (Discrete time) Signal. Observe that the Continuous time Signal is Sampled at $t = 0, T_s, 2T_s \cdots$ and So on.

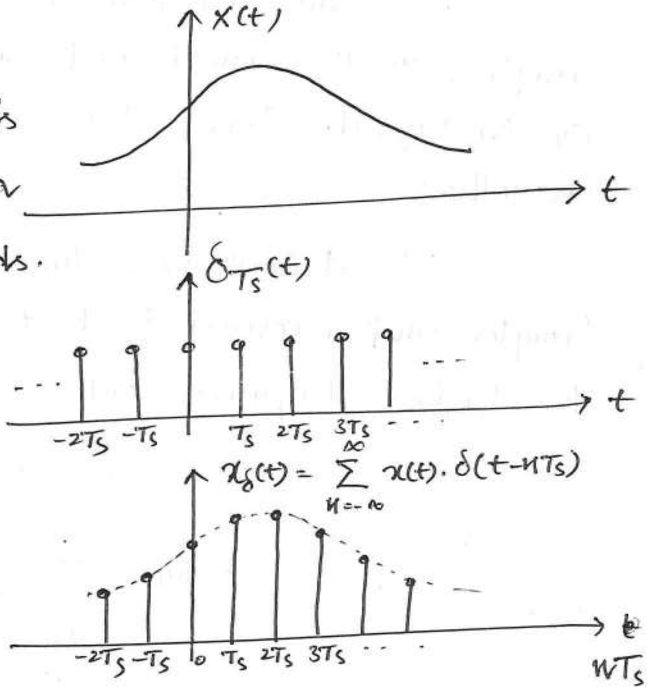


fig: Continuous time and its Sampled Signal.

Here the CT Signal $x(t)$ is Sampled instantaneously and at a Uniform Rate once every T_s Seconds. Thus we Obtain an infinite Sequence of Samples Spaced 'T_s' Seconds apart as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT_s)$$

where 'n' takes all possible integer Values.

We refer 'T_s' as Sampling period and $f_s = \frac{1}{T_s}$ as the Sampling rate. The instantaneous form of Sampling is Called as Uniform Sampling.

Sampling Theorem Gives the Criteria for Spacing 'T_s' between two Successive Samples. The Sampling period 'T_s' is Chosen Such that the Samples of $x_\delta(t)$ must represent all the information Contained in $x(t)$.

Sampling Theorem for Band Limited Signals (Lowpass Signals):

Statement:

"A band limited Signal of finite Energy, which has no frequency Components higher than 'W' Hz Can be Completely described by Specifying the Values of the Signal at instants of time Seperated by $1/2W$ Seconds" and

"A band limited Signal of finite Energy, which has no frequency Components higher than 'W' Hz, may be Completely recovered from the knowledge of its Sample Values taken at the rate of '2W' Samples per Second."

The first part of the Above Statement tells about Sampling of the Signal and Second part tells about reconstruction of the Signal. Above Statement Can be Combined and Stated alternatively as follows

"A Continuous time Signal can be Completely represented in its Samples and recovered back if the Sampling frequency is twice of the highest frequency Content of the Signal. i.e

$$f_s \geq 2W \text{ Samples/Sec.}$$

where $f_s \rightarrow$ Sampling frequency

$W \rightarrow$ Higher frequency Content of the Signal.

$$T_s \leq \frac{1}{2W} \text{ Sec.}$$

The minimum Sampling frequency $f_{s(min)} = 2W$ Samples/Sec is known as Nyquist Rate Of Sampling. and

The maximum Sampling period $T_{s(max)} = \frac{1}{2W}$ Sec is known as Nyquist interval of Sampling.

Nyquist Rate plays an important Role in the reconstruction Of the Signal from the Sampled Version.

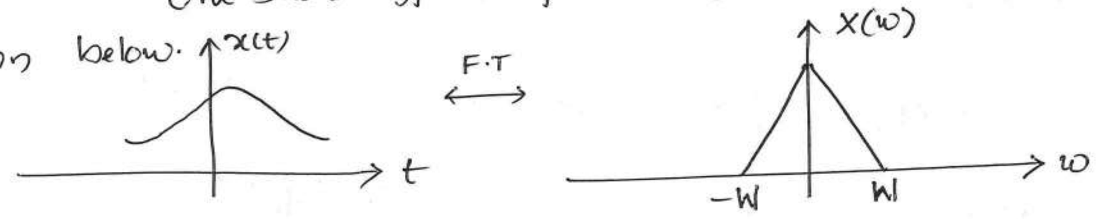
proof:

Representation of $x(t)$ interms of its Samples

let us Consider a Continuous time Signal $x(t)$ whose Spectrum is band limited to 'W' Hz. This means that the Signal $x(t)$ has no frequency Components beyond 'W' Hz. Then the Corresponding Fourier Transform of the CT Signal $x(t)$ is given by

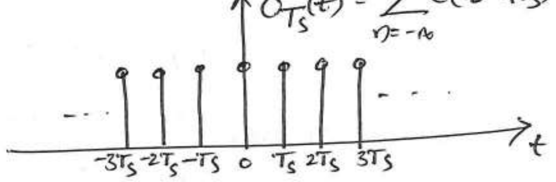
$$X(w) = 0 \quad , \quad |w| > W$$

One Such typical Spectrum of a band limited Signal is Shown below.



The Sampled Signal of $x(t)$ Can be Obtained by multiplying $x(t)$ by the periodic impulse train, whose period is $T_s = \frac{1}{f_s}$.

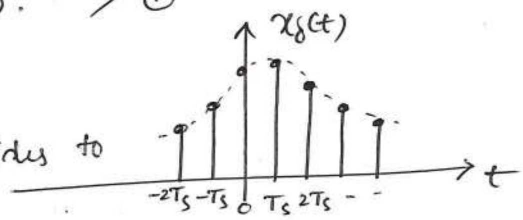
$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$\therefore \quad x_\delta(t) = x(t) * \delta_{T_s}(t)$$

$$\therefore \quad x_\delta(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t) . \delta(t - nT_s) . \longrightarrow ①$$



→ By taking Fourier Transform on Both Sides to Equ ① we get.

$$X_\delta(f) = F.T \left\{ \sum_{n=-\infty}^{\infty} x(t) . \delta(t - nT_s) \right\}$$

$$x(t) \longleftrightarrow X(f)$$

$$\delta(t - nT_s) \longleftrightarrow \frac{1}{T_s} \delta(f - nf_s)$$

$$\therefore \quad X_\delta(f) = X(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

Since Convolution is a Linear Operation

$$X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f) . * \delta(f - nf_s).$$

$$\therefore\ X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \rightarrow ②$$

$$X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right) \rightarrow ③$$

From Equ③ it can be seen that $X_\delta(f)$ represents a continuous spectrum which is periodic with a period equal to $\frac{1}{T_s}$. This means that the process of uniformly sampling a signal in the time domain results in a periodic spectrum in the frequency domain with a period equal to sampling rate.

$$③ \Rightarrow X_\delta(f) = \cdots \frac{1}{T_s}X\left(f-\frac{2}{T_s}\right) + \frac{1}{T_s}X\left(f-\frac{1}{T_s}\right) + \frac{1}{T_s}X(f) + \frac{1}{T_s}X\left(f+\frac{1}{T_s}\right) + \frac{1}{T_s}X\left(f+\frac{2}{T_s}\right) \cdots$$

$$X_\delta(f) = \cdots f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f+f_s) + f_s X(f+2f_s) \cdots$$

From the above equation it is evident that the spectrum $X_\delta(f)$ consisting of $X(f)$, which is repeating with period $f_s = \frac{1}{T_s}$.

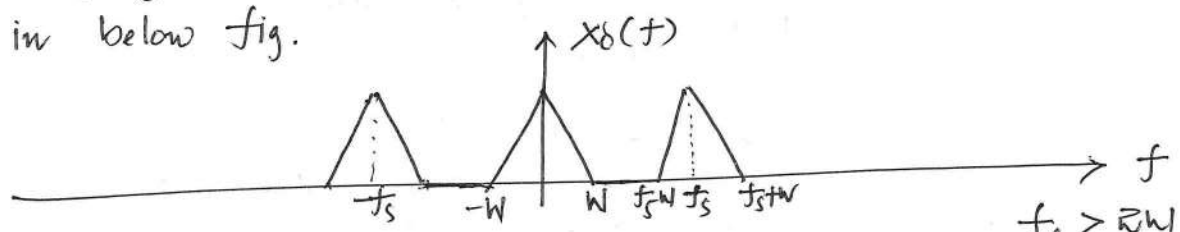
Now if we have to reconstruct $x(t)$ from $x_\delta(t)$, we must recover the signal frequency spectrum $X(f)$ from $X_\delta(f)$, which is possible if there is no overlaping between successive cycles of $X_\delta(f)$. This requires

$$f_s \geq 2W$$

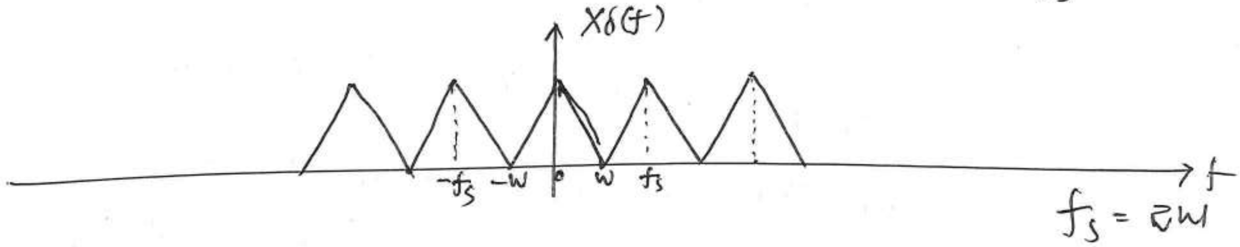
(or)

$$T_s \leq 2W.$$

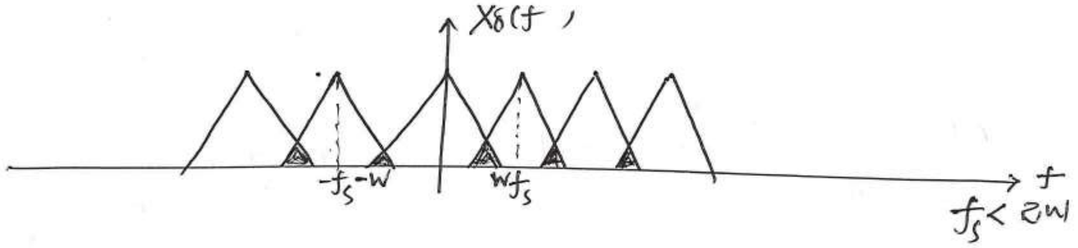
The resulting frequency spectrum obtained by the sampling theorem with different conditions is as shown in below fig.



$f_s > 2W$



$f_s = 2W$



→ if $f_s = 2W$ then we have

$$X_\delta(f) = f_s \cdot X(f) \qquad -W \le f \le W$$

$$= \frac{\text{[illegible]}}{2W} X(f) \longrightarrow (1)$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s)$$

Take F.T on Both Sides

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot e^{-j2\pi f n T_s}$$

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{f_s}\right) \cdot e^{-j2\pi n \frac{f}{f_s}}$$

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot e^{-\frac{j\pi n f}{W}} \quad , \longrightarrow (2)$$

$$\therefore (1) \Rightarrow \quad X(f) = \frac{1}{2W} X_\delta(f)$$

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-\frac{j\pi n f}{W}} , \quad -W \le f \le W$$

$$x(t) = \mathcal{F}^{-1}\left(\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot e^{-j\cancel{2}\pi \frac{n f}{W}} \right)$$

→ Reconstruction Of $x(t)$ from its Samples :

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi f t} \, df$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot e^{-\frac{j\pi n f}{W}} \right] e^{j2\pi f t} \, dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f \left(t - \frac{n}{2W}\right)} \, df$$

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \frac{1}{2W} \left. \frac{e^{j2\pi f\left(t-\frac{n}{2W}\right)}}{j2\pi\left(t-\frac{n}{2W}\right)} \right|_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \frac{1}{2W} \frac{1}{j2\pi\left(t-\frac{n}{2W}\right)} \left[e^{j2\pi W\left(t-\frac{n}{2W}\right)} - e^{-j2\pi W\left(t-\frac{n}{2W}\right)} \right]$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \frac{1}{\cancel{2W}} \left(\cancel{2W} \frac{\sin \cancel{2}\pi \cancel{W}\left(\frac{2Wt-n}{\cancel{2W}}\right)}{\pi\left(2Wt-n\right)} \right)$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \text{Sinc}\left(2Wt-n\right)$$

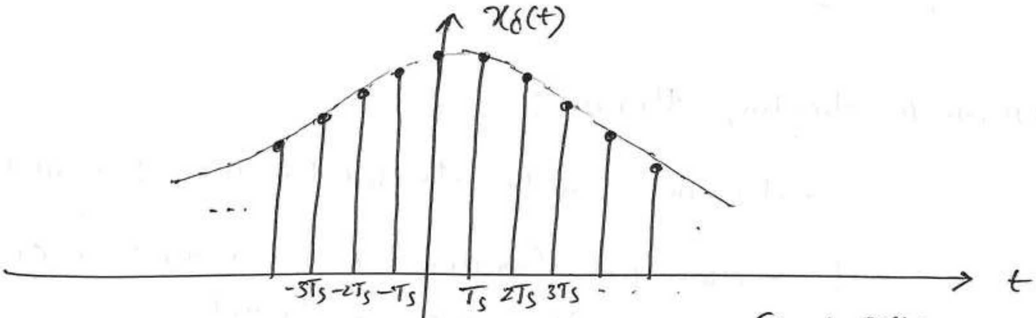
$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\ \text{Sinc}(2Wt-n), \quad -\infty < t < \infty \quad \rightarrow ③$$

$$x(t) = \cdots x(-2T_s) \cdot \text{Sinc}(2Wt+2) + x(-T_s) \cdot \text{Sinc}(2Wt+1) + x(0)\ \text{Sinc}(2Wt) + x(T_s) \cdot \text{Sinc}(2Wt-1) + \cdots$$

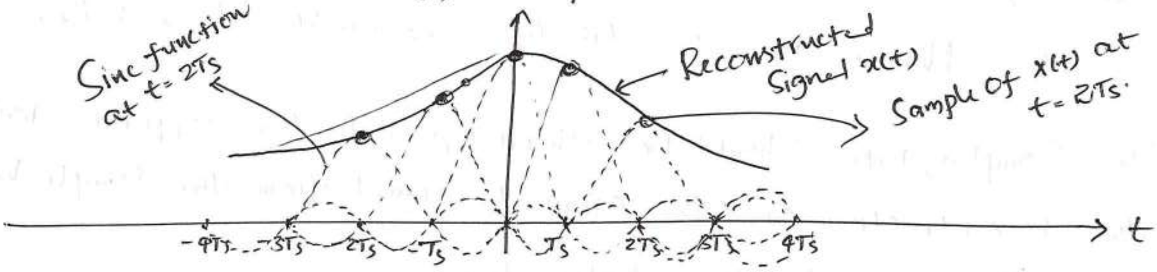
Equation ③ is known as interpolation formula for the reconstruction of the original Signal $x(t)$ from the Sequence of the Sample Values $x(nT_s)$. The Sinc function Sinc $(2Wt)$ is called the interpolation function because each Sample multiplied by a delayed Version of this function adds up to produce the original Signal $x(t)$.

Equation ③ also represents the response of an ideal Low pass filter of bandwidth 'W' and zero transmission delay when the input to the filter consists of the Sequence of Samples $x\left(\frac{n}{2W}\right)$, for $-\infty \leq n \leq \infty$.

When the interpolated Signal (Eqn ③) is passed through the Low pass filter of band width $-W \leq f \leq W$, the reconstructed waveform is (Shown in below fig) is obtained. The individual Sinc functions are interpolated to get Smooth $x(t)$.



(a) Sampled Version of Signal $x(t)$



(b) Reconstruction of $x(t)$ from its Samples.

Effects of Under Sampling (Aliasing):

Nyquist rate plays an important role in the reconstruction of the Signal from its Sampled Version. For Example when the Sampling rate f_s is less than $2W$, the frequency Shifted replicas of the Original Spectrum $X(f)$ are Overlapped to produce the Spectrum of $X_\delta(f)$.

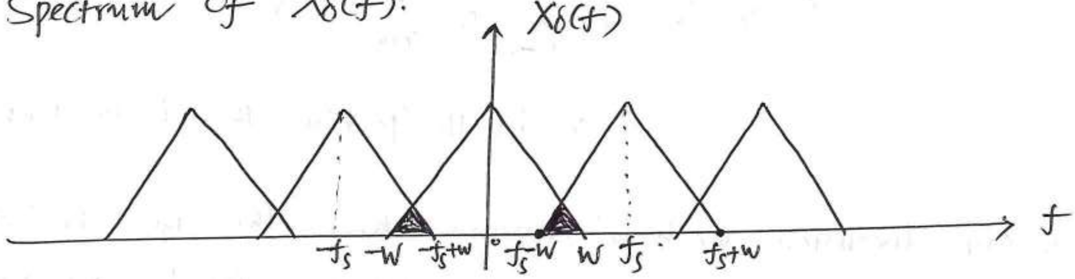


fig: Effects of Under Sampling (Aliasing).

Def: When $f_s < 2W$ then the high frequency Components are getting folded over onto the Low frequency Components. This phenomena is known as Aliasing or folded over Effect.

Effects of Aliasing:

→ Since high frequency & Low frequency Components are interfere with each other, distortion is generated.

→ Data is lost and recovery is not possible.

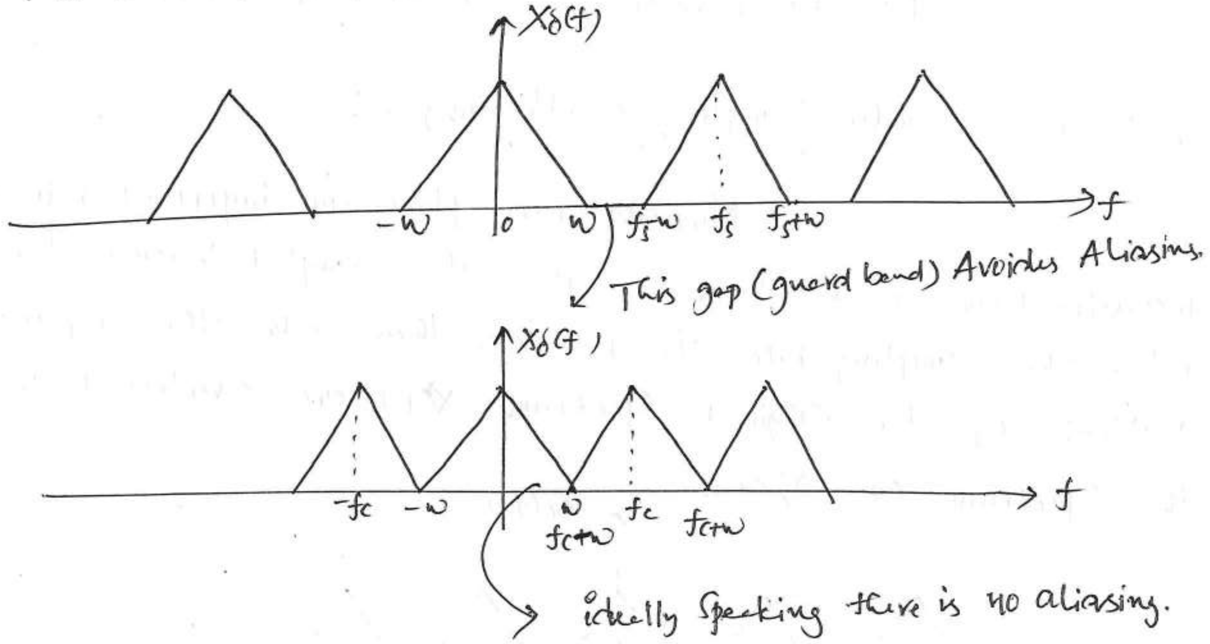
→ Methods to Avoiding Aliasing:

Aliasing can be Avoided by using two methods.

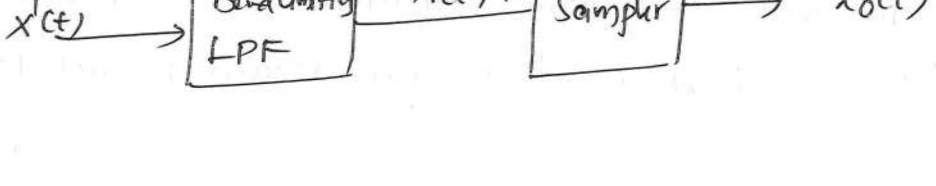
→ $f_s \geq 2W$ i.e Sampling rate is greater than or equal to twice the highest frequency of the message Signal.

→ Message Signal is Strictly band Limited to W Hz.

* The Sampling rate Should be atleast equal to the Nyquist rate for the reconstruction of the original Signal from the Sample Values. This is illustrated in the below fig.



* In our discussion we have assumed that the Signal is Strictly band Limited. But practically the message Signals are not generally Strictly band Limited. Hence distortion may arises in such cases from the application of Sampling theorem to these Signals. Thus in order to band Limiting the Signals a Low pass filter is used before Sampling the Signals as Shown below. Thus the output of Low pass filter is Strictly band Limited and there are no frequency Components higher than W Hz and there will be no aliasing.



→ <u>Nyquist Rate And Nyquist interval</u>:

The minimum Sampling rate $f_{s(min)} = 2W$ Samples/Sec for a given Band width of W Hz is known as nyquist rate.

$$\text{Nyquist rate} = f_{s(min)} = 2W \text{ Hz or Samples/Sec.}$$

Nyquist interval is the time interval between any two adjacent Samples when Sampling rate is Nyquist rate.

$$\therefore \quad \text{Nyquist interval} = \frac{1}{2W} \text{ Sec.}$$

Ex: Find the Nyquist rate And Nyquist interval for the following Signals.

(i) $m(t) = \frac{1}{2\pi} \cos 4000\pi t \cos 1000\pi t$

(ii) $m(t) = \frac{\sin 500\pi t}{\pi t}$

Soln:

(i) $m(t) = \frac{1}{2\pi} \cos 4000\pi t \cos 1000\pi t$

$$= \frac{1}{4\pi}\left(\cos 5000\pi t + \cos 3000\pi t\right)$$

$$= \frac{1}{4\pi}\left(\cos 2\pi f_1 t + \cos 2\pi f_2 t\right)$$

$\therefore$ On Comparing, we get

$$f_1 = 2500\text{Hz}, \quad f_2 = 1500\text{ Hz}$$

$\therefore$ Highest frequency $W = f_2 = 2500$Hz.

$$\text{Nyquist rate} = f_s = 2W = 5000 \text{ Hz}$$

$$\text{Nyquist interval} \quad T_s = \frac{1}{2W} = \frac{1}{5000} = 0.2 \text{ mSec.}$$

(ii) $$m_2(t) = \frac{\sin 500\pi t}{\pi t} = \frac{\sin 2\pi f t}{\pi t}$$

On Comparing we get $f = W = 250Hz$

$\therefore$ Nyquist rate $= 2W = 500Hz$

Nyquist interval $= \frac{1}{2W} = \frac{1}{500} = 2 m\,Sec.$

prob: Determine the Nyquist rate for the following Signals.

(i) $x(t) = \text{rect } 300t$

(ii) $x(t) = -10 \sin 40\pi t \cos 300\pi t$

Soln:

(i) $x(t) = \text{rect}(300t)$

$$X(f) = \frac{1}{300} \text{Sinc}\left(\frac{f}{300}\right)$$

The Sinc function never goes to zero and stays there at a finite frequency. Therefore its highest frequency is infinite. Hence the nyquist rate is also infinite.

(ii) $x(t) = -10 \sin 40\pi t \cos 300\pi t$

$$= -5\left(\sin 340\pi t + \sin(-2\cancel{60}\pi t)\right)$$

$$= 5\left(\sin 260\pi t - \sin 340\pi t\right)$$

Highest frequency $f_2 = 170Hz$

Minimum frequency $f_1 = 130Hz$

$\therefore$ The Max. frequency present in the Signal $x(t)$ is given by $W = f_2 = 170Hz$.

$\therefore$ Nyquist Rate $= 2W = 340Hz$.

prob: Determine the Nyquist rate Corresponding to each of the following Signals.

(i) $X(t) = 1 + \cos 2000\pi t + \sin 4000\pi t.$

(ii) $X(t) = \dfrac{\sin 4000\pi t}{\pi t}$

Soln: (i) Given

$$x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$$

The frequencies present in the Above Signal are

$$f_1 = 1000 Hz$$
$$f_2 = 2000 Hz.$$

Therefore, the maximum frequency present in the Signal $x(t)$ is given by $f_2 = W = 2000 Hz$

$\therefore$ Nyquist rate $f_s = 2W = 4000 Hz$

Nyquist interval $T_s = \dfrac{1}{2W} = \dfrac{1}{4000} = 0.25 m\,Sec.$

(ii) Given $x(t) = \dfrac{\sin 4000\pi t}{\pi t}$

The frequency of the Signal $f = W = 2000 Hz$

$\therefore$ Nyquist rate $= 2W = 4000 Hz$

Nyquist interval $= \dfrac{1}{2W} = \dfrac{1}{4000} = 0.25\ Sec.$

prob: Determine the Nyquist Sampling rate and Nyquist Sampling interval for the Signals,

a) $\text{Sinc}(100\pi t)$ b) $\sin(100\pi t)$

c) $\text{Sinc}(100\pi t) + \text{Sinc}(50\pi t)$ d) $\text{Sinc}(100\pi t) + 3\,\text{Sinc}^2(60\pi t)$

Soln.

prob: Consider the Signal $X(t) = \left[\frac{\sin 50\pi t}{\pi t}\right]^2$ which is to be Sampled with a Sampling frequency of $w_s = 150\pi$ to obtain a Signal $x(t)$ with fourier Transform $G(jw)$. Determine the maximum Value of w_0 for which it is guaranteed that $G(jw) = 75\,X(jw)$ for $|w| \leq w_0$ where $X(jw)$ is the F.T. of $x(t)$

Solw: Given Data,

$$x(t) = \left(\frac{\sin 50\pi t}{\pi t}\right)^2$$

Sampling frequency $f_s = 75Hz \therefore w_s = 150\pi$

$$G(jw) = 75 \cdot X(jw), \quad |w| \leq w_0$$

$$w_0 = ?$$

According to Fourier Transform

$$X(jw) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt}\,dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} \left(\frac{\sin 50\pi t}{\pi t} \right)^2 e^{-j\omega t} dt$$

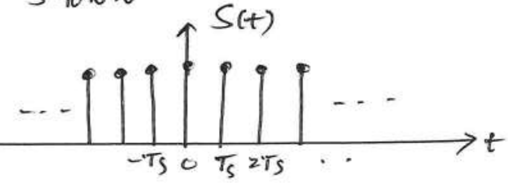
Types of Sampling :

① Ideal Sampling (or) Instantaneous Sampling (or) Impulse Sampling.

② Natural Sampling

③ flat top Sampling.

① **Ideal (Impulse) Sampling :** In impulse Sampling the Sampling Signal is a true impulse Sequence. as Shown.

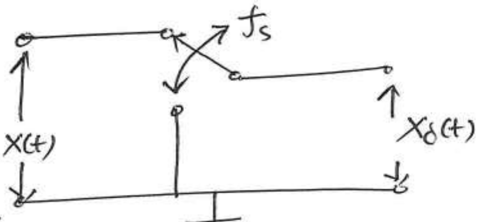


∴ The Sampled Signal $x_\delta(t)$ is thus given by

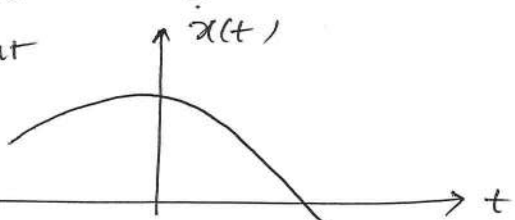
$$x_\delta(t) = x(t) * S(t)$$

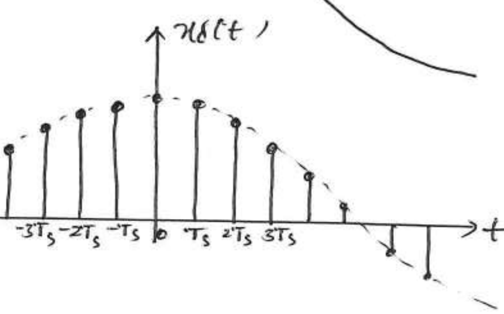
$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \rightarrow ①$$

Below fig. Shows the Switching Sampler. If closing time 't' of the Switch Approaches zero, the output $x_\delta(t)$ gives only the instantaneous Value.



Since the width of the pulse approaches zero, the instantaneous Sampling gives train of impulses in $x_\delta(t)$. The area of each impulse in the Sampled Version is equal to instantaneous Value of the input Signal $x(t)$.





From Eqn ①, the Fourier Transform of the ideally Sampled Signal is given by, i.e

Spectrum of Ideally Sampled Signal is

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \longrightarrow ②$$

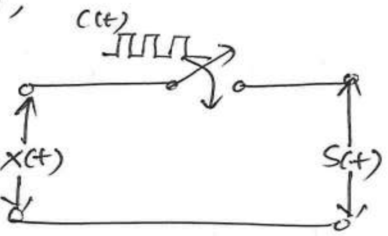
From Equ ② it is clear that, the Spectrum of ideally Sampled Signal is periodic in 'f_s' and weighted by 'f_s'. Instantaneous Sampling is possible in theory, because it is not possible to have a pulse whose width approaches zero.

Natural (Chopper) Sampling:

In natural Sampling, the Sampling Signal is a periodic train of pulses of finite width 'τ'. Natural Sampling is Some times Called Chopper Sampling because the waveform of the Sampled Signal appears to be Chopped off from the original Signal.

let us Consider an analog Continuous time Signal $x(t)$ to be Sampled at the rate of f_s Hz and 'f_s' is higher than the nyquist rate Such that Sampling theorem is Satisfied. A Sampled Signal [$S(t)$] is Obtained by multiplication of a Sampling function [$C(t)$] and the Signal $x(t)$. Sampling function $C(t)$ is a train of periodic pulses of width 'τ' and frequency equal to 'f_s'.

Below fig. Shows a functional Diagram of natural Sampler. When $C(t)$ goes high, Switch 's' is Closed Therefore,



$$S(t) = x(t), \quad \text{when } C(t) = A$$
$$\quad\;\; 0, \quad \text{when } C(t) = 0$$

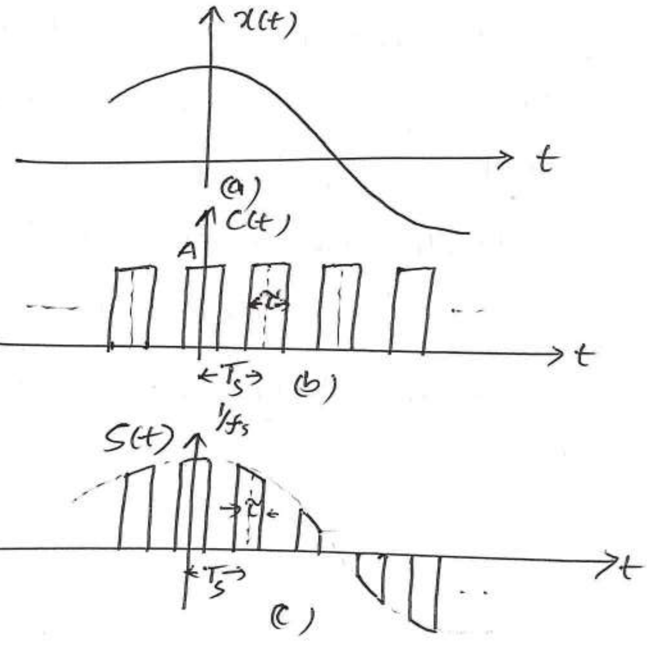
Where 'A' is Amplitude of $C(t)$.

The waveforms of $x(t)$, $c(t)$ and $S(t)$ are as Shown.

fig (a) → Continuous time Signal

fig (b) → Sampling function Waveform (periodic pulse train)

fig (c) → Naturally Sampled Signal $S(t)$.



Mathematically $S(t)$ Can be defined as

$$S(t) = x(t) \cdot C(t) \longrightarrow ①$$

Where, $C(t)$ is a periodic train of pulses of width 'τ' & frequency 'f_s'.

The Exponential fourier Series for $C(t)$ is given by

$$C(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{j2\pi n f_s t}$$

where $F_n = \frac{\tau A}{T_s} \operatorname{Sinc}(f_n \tau)$.

$$\therefore \quad S(t) = C(t) \cdot x(t)$$

$$S(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{Sinc}(f_n \tau)\, e^{j2\pi f_s n t} \cdot x(t).$$

Above Equation represents naturally Sampled Signals.

$$\therefore \quad S(f) = \mathcal{F}(S(t))$$

$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{Sinc}(f_n \tau)\, \mathcal{F}\left(e^{j2\pi f_s n t} \cdot x(t) \right)$$

Spectrum of Naturally Sampled Signal $S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{Sinc}(n f_s \tau)\, X(f - n f_s)$.

Flat top Sampling (Rectangular pulse Sampling):

Flat top Sampling is a practical Sampling method.

In Natural Sampling, the top of the pulse follows the Envelope of the Signal. where as in flat top Sampling, the pulse top is prevented from following the Envelope of the message Signal. The top of the Samples remains Constant and equal to instantaneous Value of base band Signal $x(t)$ at the Start of Sampling. The duration of each Sample is 'τ' and the Sampling rate is equal to $f_s = \frac{1}{T_s}$. Below fig. Shows the functional block diagram of Sample & hold Circuit generating flat top Samples.

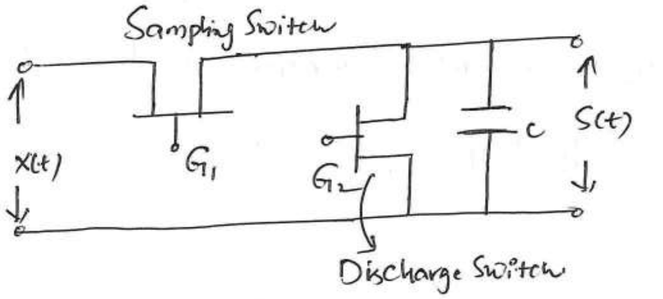
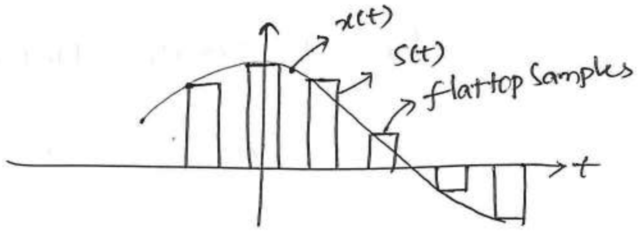


fig.(a) Sample & Hold Circuit.



fig(b) Waveforms of flat top Sampling.

From fig.(b) it is clear that, only Starting Edge of the pulse represents instantaneous Value of the base band Signal. The flat top pulse of S(t) is mathematically equivalent to the Convolution of instantaneous Sample and pulse h(t) as shown below.

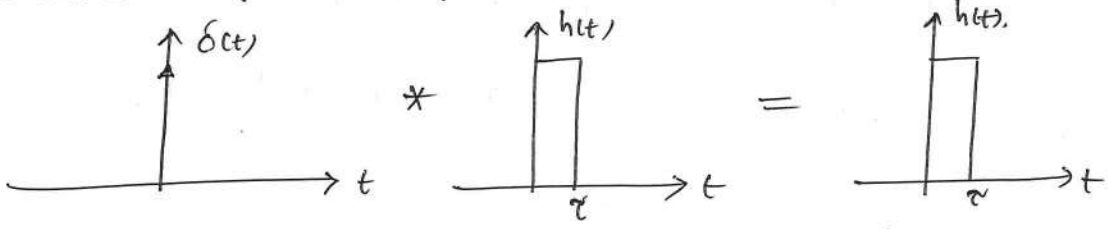
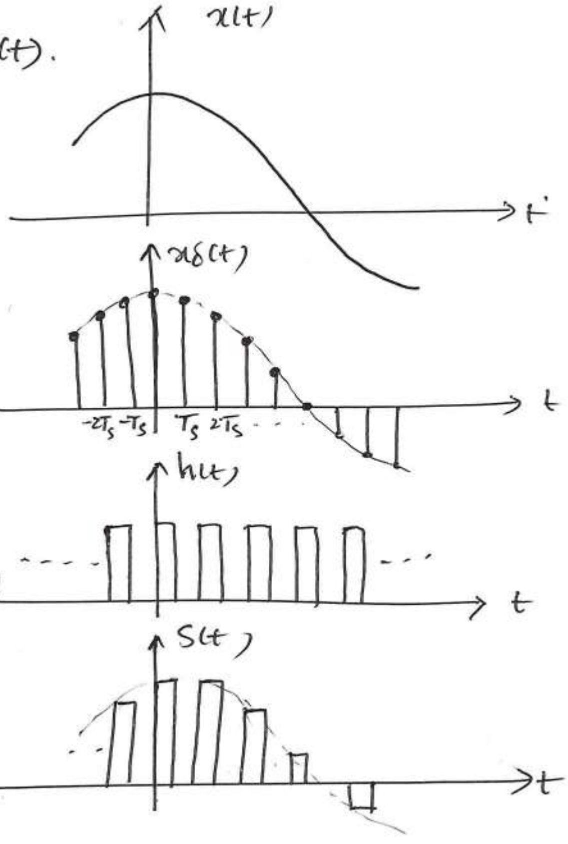


fig: Convolution of Any function with delta function is equal to that function only.

The width of the pulse in S(t) is determined by width of h(t) and Sampling instant is determined by delta function.



$$\therefore \quad S(t) = x_\delta(t) * h(t).$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s).$$

$$\therefore \quad S(t) = x_\delta(t) * h(t),$$

$$= \int_{-\infty}^{\infty} x_\delta(u).\, h(t-u).du$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(u-nT_s).\, h(t-u)du$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s).\int_{-\infty}^{\infty} \delta(u-nT_s).\, h(t-u)du$$

From the Sampling property of impulse function, we have,

$$S(t) = \sum_{n=-\infty}^{\infty} x(nT_s) . h(t-nT_s).$$

From Convolution Theorem, we have,

$$S(f) = X_\delta(f) . H(f).$$

We know that,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

∴ Spectrum of flat top Sampled Signal becomes,

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) . H(f).$$

UNIT - IV

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

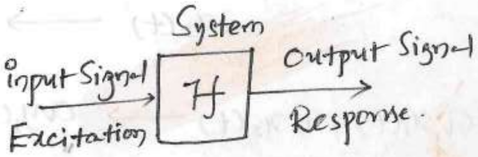
Definition Of System : A System refers to a physical device which produces an output Signal in response to an input Signal. The input Signal is referred to as excitation and the output as response.

(or)

Any process that exhibits Cause and Effect relation Can be called a System. A System will have an input Signal and an output Signal. The Output Signal will be a transformed Version of the input Signal.

A System is denoted by a letter H. The diagramatic representation of a System is as follows.

$$\text{Output} = H(\text{input})$$



where H denotes the System Operation. or System operator.

fig: Representation of a System.

Classification Of Systems :-

→ Depending on the type of Energy used to Operate the Systems the Systems can be classified as follows.

- Electrical Systems
- Mechanical Systems
- Thermal Systems
- Hydraulic Systems.

→ Depending on the type of input and output Signals the Systems can be classified as follows.

- Continuous time System
- Discrete time System.

→ A System which can process Continuous time Signals is Called as Continuous time System.

→ A System which can process Discrete time Signals is Called as Discrete time System.

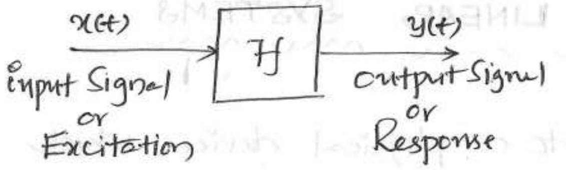


fig: Representation Of Continuous time System.

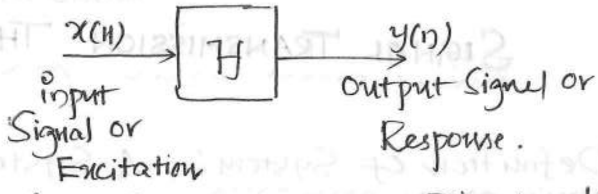


fig: Representation of Discrete time System.

Linear / Non Linear Systems :

A System which Satisfies both Homogenity and Superposition principle is known as Linear System. i.e

if $x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$

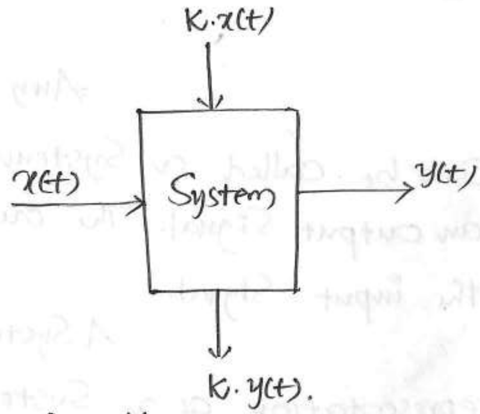


fig: Homogenity principle.

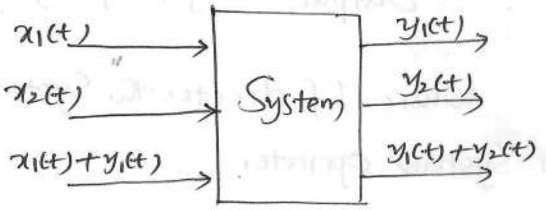


fig: Super position principle.

A System that donot Satisfy the above Condition is called as Non Linear Systems.

→ A Linear System Can be Characterized by Linear algebraic Equations, difference Equation or differential Equations.

→ There are no Streight forward methods for the Analysis Of non-Linear Systems.

Classification Of Linear Systems :

Linear Systems may be Classified as

→ Lumped And Distributed Systems

→ Time invariant and Time Variant Systems

→ Causal and Non Causal Systems.

→ Static (Memoryless) and Dynamic Systems

→ Stable And Unstable Systems.

→ **Lumped And Distributed Systems:**

A System is a Collection of individual Elements interconnected in a particular fashion.

A Lumped System Consists of Lumped elements. In a Lumped System, the Energy in the System is Considered to be Stored or Dissipated in Distinct isolated elements (R, L, C, etc). Lumped Systems are described by using ordinary differential Equations.

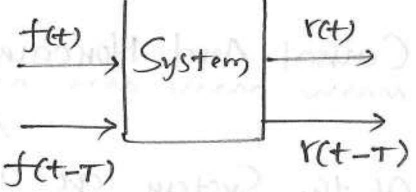
In Contrast to Lumped Systems, we have distributed Systems Such as transmission Lines, waveguides, Antennas, Semiconductor devices etc where it is not posible to describe a System by Lumped parameters. The distributed Systems are described by using partial differential Equations.

→ **Time Variant and Time invariant Systems:**

The System ~~parameters~~ whose parameters donot change with time are Called Constant parameter or time invariant Systems. Most of the physically realizable Systems are belongs to this Category. Circuits using passive elements are an example of time invariant Systems.

The Systems whose parameters Change with time are Called as Variable parameter or time Variant or time dependent Systems.

Example: $\frac{d^2r}{dt^2} + a\frac{dr}{dt} + b\cdot r = f(t)$ $\longrightarrow$ LTI System



$\frac{d^2r}{dt^2} + \frac{dr}{dt} + (2t+1)r = f(t)$ $\longrightarrow$ Time Variant

It is evident that for a time invariant System if a driving function f(t) yields a response r(t) then the Same driving function delayed by 'T' yield the Same response r(t) but delayed by time 'T' i.e.

$$\left.\begin{array}{l} \text{if } f(t) \longrightarrow r(t), \text{ then} \\ f(t-T) \longrightarrow r(t-T). \end{array}\right\} \rightarrow (1)$$

The time Variant Systems however donot Satisfy Eqn (1).

Linear time invariant (LTI) / Linear Time Variant (LTV) Systems:

A System which Satisfies both Linearity and time invariance properties is known as LTI System.

A System that Satisfies Linearity but not time invariance properties is known as LTV Systems.

Most of the physically Realizable Systems are Linear time invariant Systems.

The input output Relation of an LTI Continuous time System is represented by Constant Coefficient D.E as shown.

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_{N-1} \frac{d}{dt} y(t) + a_N y(t) = b_0 \frac{d^M}{dt^M} x(t)$$
$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_{M-1} \frac{d}{dt} x(t) + b_M x(t)$$

where $N \rightarrow$ Order of the System, $M \leq N$, $a_0 = 1$.

The Solution of the Above D.E is the response $y(t)$ of the Continuous time System for the input $x(t)$.

⇒ Causal And Noncausal Systems:

A System is Said to be Causal if the output of the System at any time 't' depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the System output at any time 't' depends on future inputs or outputs. then the System is called a noncausal System. The Causality refers to a System that is realizable in real time.

⇒ Static And Dynamic Systems :

A Continuous time System is called Static or memoryless if its output at any instant of time 't' depends at most on the input Signal at the Same time but not on the future or past input. In any other case the System is Said to be dynamic or to have memory.

Stable And Unstable Systems : An arbitrary relaxed System is Said to be BIBO (Bounded input Bounded o/p) Stable if and only if every bounded input produces a bounded output.

The term bounded input refers to finite Value of the input Signal $x(t)$ for any Value of 't'. Hence if input $x(t)$ is bounded then there exists a Constant M_x Such that $|x(t)| < M_x$ and $M_x < \infty$ for all 't'.

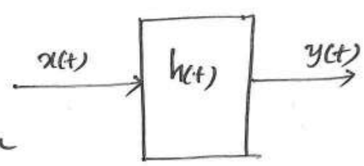
The term bounded output refers to finite and predictable output for any Value of 't'. Hence if output $y(t)$ is bounded then there exists a constant M_y Such that $|y(t)| < M_y$ and $M_y < \infty$ for all 't'.

Examples of bounded i/p Signals: Step, decaying Exponential, impulse.

Examples of unbounded Signals: Ramp, rising Exponential etc.

Condition for Stability of an LTI System : For BIBO Stability of an LTI Continuous time System, the integral of impulse response Should be finite.

$$\int_{-\infty}^{\infty} |h(t)| < \infty, \text{ for Stability of a System.}$$



proof: The response of a System $y(t)$ for any input $x(t)$ is given by Convolution of the input and the impulse response. i.e

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) \cdot d\tau \longrightarrow ①$$

$$① \Rightarrow \quad |y(t)| = \int_{-\infty}^{\infty} |h(\tau).x(t-\tau)|\, d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|.|x(t-\tau)|\, d\tau \longrightarrow ②$$

if $x(t)$ is bounded then there exists a constant M_x such that

$|x(t-\tau)| \leq M_x < \infty$.

$$\therefore ② \Rightarrow \quad |y(t)| = \int_{-\infty}^{\infty} |h(\tau)|.M_x.\, d\tau$$

$$= M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau. \rightarrow ③.$$

From Equation ③ it is clear that, the output $y(t)$ is bounded if the impulse response satisfies the condition

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Since 'τ' is a dummy variable above Equation can be written as

$$\boxed{\int_{-\infty}^{\infty} |h(t)|\, dt < \infty}$$

Convolution of Continuous time Signals:

The Convolution of two continuous time Signals $x_1(t)$ and $x_2(t)$ is defined as,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda).x_2(t-\lambda).\, d\lambda.$$

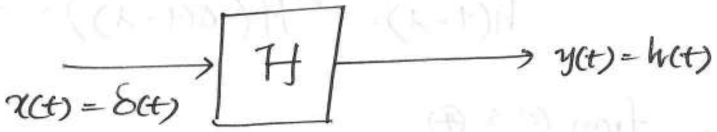
λ : dummy Variable used for integration.

$*$: Convolution Operation.

Impulse Response :

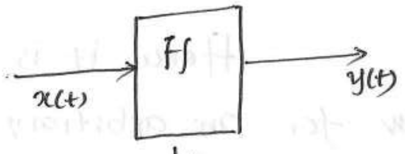
When the input to a Continuous time System is an Unit impulse Signal $\delta(t)$, then the response (O/P) $y(t)$ is called an impulse response of the System and it is denoted by $h(t)$,

$$\text{impulse response}, \quad h(t) = H(\delta(t))$$



Continuous time System.

Response of a Linear System :



In an LTI System the response $y(t)$ for an arbitrary input $x(t)$ is given by Convolution of input $x(t)$ and the impulse response $h(t)$ of the System.

$$\therefore \quad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(t-\lambda) \cdot d\lambda.$$

where the Symbol '$*$' represents Convolution Operation.

In an LTI System if the input $x(t)$ is a unit Step Signal, then the response is Called a unit Step response.

proof : Let $y(t)$ be response Of the System 'H' for an input $x(t)$

$$\therefore \quad y(t) = H(x(t)). \longrightarrow ①$$

We know that any Signal $x(t)$ Can be expressed as an integral of impulses. $x(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot \delta(t-\lambda) \cdot d\lambda. \rightarrow ②$

from ① & ②

$$y(t) = H\left(\int_{-\infty}^{\infty} x(\lambda) \cdot \delta(t-\lambda) \cdot d\lambda \right)$$

$$= \int_{-\infty}^{\infty} \left(H(x(\lambda) \cdot \delta(t-\lambda)) \cdot d\lambda \, . \right)$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot H(\delta(t-\lambda)) \cdot d\lambda \rightarrow ③$$

let the impulse response of the LTI System is

$$h(t) = H(\delta(t))$$

By using Time invariance property

$$h(t-\lambda) = H(\delta(t-\lambda)) = \rightarrow ④$$

$\therefore$ from ③ & ④

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(t-\lambda) \cdot d\lambda$$

Hence it is Concluded that the response of an LTI System for an arbitrary input $x(t)$ is given by Convolution of input $x(t)$ with impulse response $h(t)$ of the System.

Unit Step response of an LTI System using Convolution:

The response $y(t)$ of a System is given by Convolution of the input $x(t)$ and impulse response $h(t)$ of the System.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(t-\lambda) \cdot d\lambda$$

Let the input $x(t)$ be unit Step Signal $u(t)$ and the Corresponding response be $S(t)$. Now the Unit Step response is given by

$$S(t) = x(t) * h(t) \quad , \quad x(t) = u(t)$$

$$= u(t) * h(t)$$

$$= \int_{\lambda=-\infty}^{\lambda=\infty} h(\lambda) \cdot u(t-\lambda)$$

In the Above Convolution Operation,

$$u(\lambda) = 1, \ \lambda > 0$$

$$u(-\lambda) = 1, \ \lambda < 0$$

$$u(t-\lambda) , 1, \ \lambda < t, \quad u(t-\lambda) = 0, \ \lambda > t.$$

Therefore the Unit Step response $S(t)$ is given by

$$S(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda) \cdot d\lambda .$$

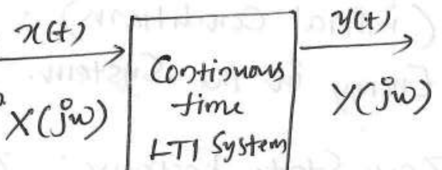
Analysis Of LTI Continuous Time Systems Using Fourier Transform:

Transfer function Of LTI Continuous time System:

The Ratio Of Fourier Transform Of output and the fourier transform of the input is called as Transfer function Of LTI Continuous time System in frequency domain.

let $x(t) \rightarrow$ input to a Contionous time System

$y(t) \rightarrow$ output to a Continuous time System



$X(j\omega) \rightarrow$ fourier transform Of $x(t)$

$Y(j\omega) \rightarrow$ fourier transform Of $y(t)$

Now, $\text{Transfer function} = \dfrac{Y(j\omega)}{X(j\omega)}$

Response Of LTI Continuous time System in Time Domain:

An LTI System is usually represented by using a Constant Coefficient differential Equation as

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_{N-1} \frac{d}{dt} y(t) + a_N y(t) = b_0 \frac{d^M}{dt^{M-1}} x(t) + b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \longrightarrow ①$$

where $M \leq N$, $a_0 = 1$.

$N \rightarrow$ Order of the System.

The Solution Of Equ ① is the response $y(t)$ Of the LTI System, which consists of two parts. In mathematics two parts of the Solution $y(t)$ are, homogeneous Solution $y_h(t)$ and particular Solution $y_p(t)$

$$\therefore \text{Response } y(t) = y_h(t) + y_p(t) \longrightarrow ②$$

The Homogeneous Solution is the response of the System when there is no input. The particular Solution is the Solution of the N^{th} order differential equation governing the System for Specific input Signal $x(t)$, for $t \geq 0$.

In Signals and Systems two parts of the Solution $y(t)$ are called Zero input response $y_{zi}(t)$ and zero State response $y_{zs}(t)$.

$$\text{Response } y(t) = y_{zi}(t) + y_{zs}(t).$$

Zero input Response (free response or Natural response):

Zero input response is given by the homogeneous Solution with constants evaluated using initial Values of output (initial Conditions). This response is mainly due to the initial Stored Energy in the System.

Zero State Response: Zero State response is given by the Sum of homogeneous Solution and particular Solution with zero initial Conditions. The Zero State response is the response of the System due to input Signal and with zero initial output Condition. Hence the zero State response is also Called forced Response.

$$\text{Zero input response} = y_{zi}(t) = y_h(t) \Big| \text{ constants Evaluated using initial output Conditions.}$$

$$\text{Zero-State Response} = y_{zs}(t) = \left[y_h(t) + y_p(t)\right] \Big| \text{ Constants Evaluated using zero initial output Conditions.}$$

Note:

input	$y_p(t)$	
$x(t) = A$	K	
$x(t) = A\,u(t)$	$K \cdot u(t)$	
$x(t) = A e^{\alpha t}$ $(\alpha \neq \lambda_i)$	$K e^{\alpha t}$	λ_i is one of the root of Characteristic polynomial.
$= A e^{\alpha t}$ $(\alpha = \lambda_i)$	$K t\, e^{\alpha t}$	
$x(t) = A \cos w_0 t$ $A \sin w_0 t$	$K_1 \cos w_0 t + K_2 \sin w_0 t.$	

Prob: Determine the natural response of the System described by the equation

$$\frac{d^2y(t)}{dt^2} + 6\cdot\frac{dy}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t), \quad y(0) = 1, \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = -2.$$

Soln: The natural response is the response of the System due to initial Conditions and so is given by Zero input response.

$$\text{Zero input (Natural) response } y_{zi}(t) = y_h(t) \Big| \text{ with Constants evaluated using initial Conditions.}$$

where $y_h(t) \rightarrow$ homogeneous Solution.

Given System Equation is

$$\frac{d^2y(t)}{dt^2} + 6\cdot\frac{dy}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t) \rightarrow (1)$$

Homogeneous Solution: It is the response of the System when $x(t)=0$

$$\therefore (1) \Rightarrow \frac{d^2y(t)}{dt^2} + 6\cdot\frac{dy}{dt} + 5\cdot y(t) = 0$$

The Characteristic polynomial is

$$D^2 + 6D + 5 = 0$$

$$(D+1)(D+5) = 0$$

$$D = -1, -5. \quad (\text{Distinct Roots})$$

$\therefore$ homogeneous Solution is $y_h(t) = C_1 e^{-t} + C_2 e^{-5t} \rightarrow (2)$

Zero input Response (Natural response):

$$y_{zi}(t) = y_h(t) \Big| \text{ Constants Evaluated using initial Conditions.}$$

$$= C_1 e^{-t} + C_2 e^{-5t} \Big|_{C_1, C_2 \text{ evaluated using initial Conditions}}$$

$$\frac{dy_{zi}(t)}{dt} = -C_1 e^{-t} - 5C_2 e^{-5t}$$

At $t=0$, $y_{zi}(0) = C_1 + C_2 \Rightarrow C_1 + C_2 = 1 \rightarrow (3) \quad \because y(0) = 1,$

$$\left.\frac{dy_{zi}(t)}{dt}\right|_{t=0} = -C_1 - 5C_2 = -2$$

$$C_1 + 5C_2 = 2 \rightarrow (4)$$

After Solving (3) & (4) we get, $C_2 = 1/4$, $C_1 = 3/4$.

Therefore, the natural response is given by

$$y_{zi}(t) = \frac{3}{4}e^{-t} + \frac{1}{4}e^{-5t}, \quad t \geq 0$$

$$y_{zi}(t) = \left(\frac{3}{4}e^{-t} + \frac{1}{4}e^{-5t}\right) u(t).$$

prob: Determine the forced response (Zero State response) of the System described by the below equation. $5\frac{dy}{dt} + 10\,y(t) = 2x(t)$, $x(t) = 2u(t)$.

Soln:

The forced response is the response of the System due to input Signal with Zero initial conditions and so it is given by Zero-State response. as

$$y_{zs}(t) = y_h(t) + y_p(t) \Big| \text{ with Constants Evaluated using initial Conditions}$$

where $y_h(t) \longrightarrow$ Homogeneous Solution

$y_p(t) \longrightarrow$ particular Solution.

Given System Equation is

$$5 \cdot \frac{dy(t)}{dt} + 10\,y(t) = 2 \cdot x(t) \longrightarrow (1)$$

where $x(t) = 2 \cdot u(t)$.

Homogeneous Solution:

Homogeneous Solution is the Solution of the System equation when $x(t) = 0$,

$$(1) \Rightarrow \quad 5\frac{dy(t)}{dt} + 10\,y(t) = 0$$

The Characteristic polynomial is

$$5D + 10 = 0$$

$$D = -2.$$

$$\therefore \quad y_h(t) = Ce^{-2t} \longrightarrow (2).$$

Particular Solution:

Particular Solution is the Solution of the System equation for Specific input.

Here input $x(t) = 2 \cdot u(t)$.

Let the particular Solution is of the form,

$$y_p(t) = K \cdot x(t)$$
$$= 2K\,u(t) \rightarrow ③$$

$$\frac{dy_p(t)}{dt} = 2K\,\delta(t) \rightarrow ④$$

Substitute ③ & ④ in System Equation, we get,

$$10K \cdot \delta(t) + 20K\,u(t) = 4u(t)$$

at $t=1$, $\quad 10K\,\delta(1) + 20K\,u(1) = 4u(1)$

$$0 + 20K = 4 \Rightarrow K = 1/5.$$

$\therefore$ particular Solution $y_p(t) = 2 \cdot 1/5\, u(t)$

forced Response (Zero-State response):

$y_{zs}(t) = y_h(t) + y_p(t)$ | Constants Evaluated using zero initial Conditions.

$$y_{zs}(t) = Ce^{-2t} + \frac{2}{5}u(t)$$

At $t=0$, $\quad y_{zs}(0) = C + \frac{2}{5} = 0 \Rightarrow C = -2/5.$

$\therefore$ Forced response $\quad y_{zs}(t) = -\frac{2}{5}e^{-2t} + \frac{2}{5}u(t), \; t \geq 0$

$$y_{zs}(t) = \frac{2}{5}(1 - e^{-2t})\,u(t)$$

Note: Complete response: $\quad y(t) = y_h(t) + y_p(t) \rightarrow$ Method-①

(or)

$$y(t) = y_{zi}(t) + y_{zs}(t) \rightarrow \text{Method-②}$$

Transfer function of a LTI System in the frequency domain:

Transfer function of a Linear Time invariant System is defined as the Ratio of fourier transform of output to the fourier transform of input Signal i.e

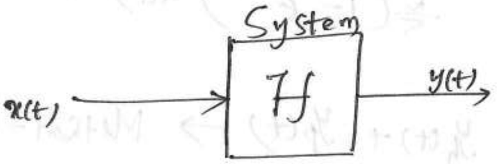
$$\text{Transfer function} = \frac{Y(j\omega)}{X(j\omega)}.$$

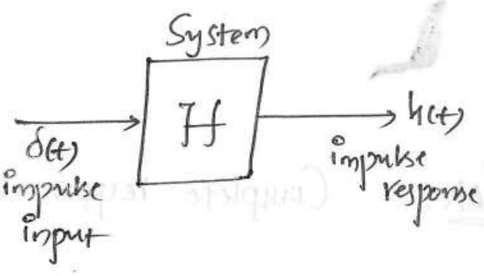
The transfer function of LTI Continuous time System in frequency domain can be obtained from the differential equation governing the input-output relation of an LTI Continuous time System. that is, if we take the fourier transform of the below equation and rearranging the resultant equation as a ratio of $Y(j\omega)$ and $X(j\omega)$, the transfer function of the System in frequency domain is obtained.

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_{N-1}\frac{d}{dt}y(t) + a_N y(t) = b_0 \frac{d^M}{dt^M}x(t) + b_1 \frac{d^{M-1}}{dt^{M-1}}x(t) + \cdots + b_{M-1}\frac{dx(t)}{dt} + b_M x(t).$$

Impulse response & Transfer function:

Consider an LTI Continuous time System 'H' as shown in the below fig. and let $x(t)$ and $y(t)$ be the input and output of the System respectively.





For a Continuous time System 'H', if the input is impulse Signal $\delta(t)$ as shown in fig. then the output is called impulse response which is denoted by $\delta(t)$.

$$\therefore \quad H(j\omega) = \frac{\cancel{Y}(j\omega)}{X(j\omega)}$$

$$\text{if } x(t) = \delta(t) \Rightarrow X(j\omega) = 1 \Rightarrow H(j\omega) = Y(j\omega)$$

$$h(t) = y(t).$$

Therefore, transfer function can also be defined as the impulse response of the system.

The importance of impulse response is that the response for any input to LTI system is given by convolution of input and impulse response. i.e.

$$x(t) * h(t) = y(t)$$

Mathematically convolution operation is defined as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau).d\tau. \longrightarrow ①$$

where 'τ' is dummy variable for integration.

let $H(j\omega) \longrightarrow$ Fourier transform of $h(t)$.

$X(j\omega) \longrightarrow$ Fourier Transform of $x(t)$.

$Y(j\omega) \longrightarrow$ Fourier Transform of $y(t)$.

By taking Fourier transform on both sides of Eqn ① and from the convolution property of CTFT, we can write,

$$\mathcal{F}(y(t)) = \mathcal{F}(x(t) * h(t))$$

$$Y(j\omega) = X(j\omega).H(j\omega)$$

$$\therefore \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

From the above analysis it is evident that, the transfer function in the frequency domain is given by fourier transform of the impulse response.

$$\therefore \quad \boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

Response of LTI System using Fourier Transform:

The transfer function of an LTI System $H(j\omega)$ is given by

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \rightarrow ①$$

① $\Rightarrow$ response in frequency domain $Y(j\omega) = H(j\omega).X(j\omega) \rightarrow ②$

Since Equ ② will be a rational function of $j\omega$, and So $Y(j\omega)$ can be expressed as a ratio of two factorized polynomial in $j\omega$ as Shown below.

$$Y(j\omega) = \frac{(j\omega+z_1)(j\omega+z_2)(j\omega+z_3)\cdots}{(j\omega+P_1)(j\omega+P_2)(j\omega+P_3)\cdots} \rightarrow ③$$

By partial fraction expansion technique Equ③ Can be expressed as Shown below.

$$Y(j\omega) = \frac{K_1}{j\omega+P_1} + \frac{K_2}{j\omega+P_2} + \frac{K_3}{j\omega+P_3} + \cdots \rightarrow ④$$

where, K_1, K_2, K_3 are residues.

Now, the time domain response can be obtained by taking inverse fourier transform of above equation. The inverse fourier transform of each term in Equ④ Can be obtained by Comparing the terms with Standard fourier transform pairs.

$$Eg: \mathcal{F}\left(e^{-at}u(t)\right) = \frac{1}{a+j\omega}.$$

$\therefore$ ④ $\Rightarrow$

$$y(t) = K_1 e^{-P_1 t}u(t) + K_2 e^{-P_2 t}u(t) + K_3 e^{-P_3 t}u(t) + \cdots \rightarrow ⑤$$

Since the transfer function is defined with zero initial Conditions, the response obtained by using Equ⑤ is the time domain Steady State (or forced) response of the LTI System.

__Note:__ Zero State (or) Steady State or forced response only Can be obtained Via frequency domain.

Frequency Response of LTI Continuous time System:

The output $y(t)$ of an LTI Continuous time System is given by Convolution of $h(t)$ and $x(t)$.

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) . x(t-\tau) . d\tau \rightarrow ①$$

Let the input $x(t) = Ae^{j\omega t} = A(\cos\omega t + j\sin\omega t)$,

where, $A \rightarrow$ amplitude

$\omega \rightarrow$ Angular frequency (rad/sec)

$$\therefore \ x(t-\tau) = Ae^{j\omega(t-\tau)} \rightarrow ②$$

from ① & ②, we get

$$y(t) = \int_{-\infty}^{\infty} h(\tau) . A . e^{j\omega(t-\tau)} d\tau$$

$$= Ae^{j\omega t} \int_{-\infty}^{\infty} h(\tau) . e^{-j\omega\tau} . d\tau .$$

$$y(t) = x(t) . H(j\omega) \rightarrow ③ \qquad \because \int_{-\infty}^{\infty} h(\tau) . e^{-j\omega\tau} d\tau = H(j\omega)$$

Equation ③ Shows that, the output $y(t)$ Contains the Same Signal as input $Ae^{j\omega t}$ multiplied by $H(j\omega)$. This $H(j\omega)$ is Called as frequency response of the LTI Continuous time System.

By the Convolution property of fourier transform Equ① Can be written as $Y(j\omega) = X(j\omega) . H(j\omega)$

(or)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \rightarrow ④$$

from ③ & ④ we can say that, the frequency response $H(j\omega)$ of an LTI-CT System is Same as transfer function in frequency domain and so, the frequency response is also given by the ratio of fourier transform of output to Fourier transform of input.

$$\text{Frequency Response } H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}.$$

Advantages of frequency response Analysis:

→ Testing of practical Systems can be easily carried out with available Sinusoidal Signal generators.

→ Transfer functions of Complicated Systems can be determined experimentally.

→ The Effects of noise disturbance and parameter Variations are relatively easy to Visualize inorder to determine Corrective measures.

→ Frequency response Analysis can be extended to Certain nonLinear Systems also.

prob: The impulse response of a Continuous time System is given as

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Determine the frequency response and plot the magnitude and phase plots.

Soln: The fourier transform of impulse response is given by

$$\mathcal{F}(h(t)) = H(j\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} u(t) \cdot e^{-j\omega t} \cdot dt$$

$$H(j\omega) = \frac{1}{RC}\int_0^{\infty} e^{-t/RC} \cdot e^{-j\omega t} \cdot dt$$

$$= \frac{1}{RC}\int_0^{\infty} e^{-(j\omega+\frac{1}{RC})t} \cdot dt$$

$$= \frac{1}{RC} \cdot \left. \frac{e^{-(j\omega+\frac{1}{RC})t}}{-(j\omega+\frac{1}{RC})} \right|_0^{\infty}$$

$$= \frac{1}{RC}\left(0 + \frac{1}{j\omega+\frac{1}{RC}} \right)$$

Freq. Response, $H(j\omega) = \dfrac{1}{1+j\omega RC} \longrightarrow ①$

Since $H(j\omega)$ is Complex it needs two plots for its representation, magnitude response and phase response.

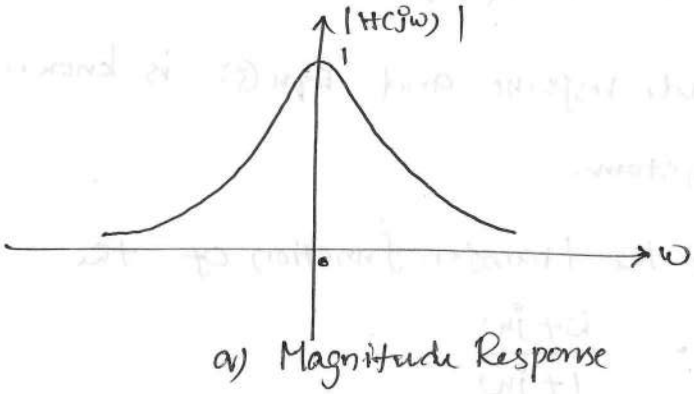
The magnitude of Equ ① is

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \longrightarrow ②$$

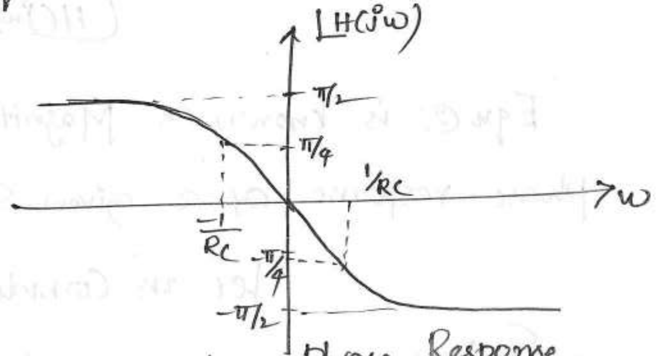
The plot of Equ ② as a function of 'w' gives the magnitude response of the System.

$$\angle H(j\omega) = -\tan^{-1}(\omega RC) \longrightarrow ③$$

The plot of Equ ③ as a function of 'w' gives the phase response of the System. Below fig. Shows the magnitude and phase response given by Equations ② & ③



a) Magnitude Response



(b) Phase Response.

From the Above figures it is clear that the magnitude response is Symmetric but phase response is antisymmetric. And the magnitude response is monotonically decreasing. Hence this is a Lowpass filter.

prob: The System produces an output $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the System.

Soln:

$$\text{Given } y(t) = e^{-t}u(t)$$

$$x(t) = e^{-2t}u(t).$$

$$\text{we know that } e^{-at}u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$$

$$\therefore \quad \mathcal{F}(y(t)) = Y(j\omega) = \frac{1}{1+j\omega}$$

$$\mathcal{F}(x(t)) = X(j\omega) = \frac{1}{2+j\omega}$$

$$\therefore \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1+j\omega} \cdot \frac{2+j\omega}{1}$$

$$H(j\omega) = \frac{2+j\omega}{1+j\omega} \longrightarrow ①$$

The magnitude of Eqn ① $|H(j\omega)|$ is given as

$$|H(j\omega)| = \sqrt{\frac{4+\omega^2}{1+\omega^2}} \longrightarrow ②$$

The phase of Eqn ① $\angle H(j\omega)$ is given by

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}(\omega) \rightarrow ③$$

Eqn ② is known as Magnitude response and Eqn ③ is known as phase response of a given System.

Let us consider the transfer function of the System

$$H(j\omega) = \frac{2+j\omega}{1+j\omega}$$

Let us rearrange the above equation as

$$H(j\omega) = 1 + \frac{1}{1+j\omega} \longrightarrow (4)$$

Therefore inverse fourier transform of the above equation gives the impulse response of the System.

$$\mathcal{F}^{-1}(H(j\omega)) = \mathcal{F}^{-1}\left(1 + \frac{1}{j\omega+1}\right)$$

$$= \mathcal{F}^{-1}(1) + \mathcal{F}^{-1}\left(\frac{1}{1+j\omega}\right)$$

$$\boxed{h(t) = \delta(t) + e^{-t}u(t)}$$

Differential Equations: An LTI-CT System is represented by a Constant Coefficient differential equation as,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \longrightarrow (1)$$

$$, \quad a_0 = 1 \text{ \& } M \leq N.$$

'N' $\rightarrow$ The order of the System.

Apply time differentiation property of fourier Transform to Eqn (1), we get,

$$\mathcal{F}\left(\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t)\right) = \mathcal{F}\left(\sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)\right)$$

$$\sum_{k=0}^{N} a_k . (j\omega)^k \, Y(j\omega) = \sum_{k=0}^{M} b_k . (j\omega)^k \, X(j\omega)$$

$\therefore$ The System Transfer function $H(j\omega)$ is given by

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \longrightarrow (2)$$

The impulse response and frequency response can be obtained from Eqn (2) prob: The differential equation of the System is given by

$$\frac{d^2y(t)}{dt^2} + 5.\frac{dy(t)}{dt} + 6.y(t) = -\frac{dx(t)}{dt}$$

Determine the frequency response and impulse response of this System.

Soln: The Differential equation of the System is given by

$$\frac{d^2y(t)}{dt^2} + 5.\frac{dy(t)}{dt} + 6.y(t) = -\frac{dx(t)}{dt}$$

By taking F.T of the above equation and by using time differentiation property we get

$$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6.Y(j\omega) = -j\omega X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 5.j\omega + 6\right] = -j\omega X(j\omega)$$

$$\therefore H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega}{(j\omega)^2+5(j\omega)+6} \rightarrow ①$$

By using Equ ① one can easily determine the magnitude and phase response (frequency response).

Expand Equ ① By using partial fractions, we get

$$H(j\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)}$$

$$H(j\omega) = \frac{2}{2+j\omega} - \frac{3}{3+j\omega}$$

By taking inverse fourier Transform of the above equation, we get,

$$h(t) = \mathcal{F}^{-1}\left(\frac{2}{2+j\omega} - \frac{3}{3+j\omega}\right)$$

$\therefore$ The impulse response is given by

$$h(t) = 2e^{-2t}u(t) - 3e^{-3t}u(t)$$

$$h(t) = (2e^{-2t} - 3e^{-3t})u(t)$$

prob: The input Voltage to the R.C circuit is given as $x(t) = te^{-t/RC}u(t)$ and the impulse response of this circuit is given by $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$. Determine the output.

Soln: Given Data,

$$x(t) = te^{-t/RC}u(t)$$

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t).$$

$$y(t) = ?$$

we know that $y(t) = x(t) * h(t)$.

(or)

$$Y(j\omega) = X(j\omega) \cdot H(j\omega). \rightarrow (1)$$

$$H(j\omega) = \mathcal{F}(h(t)).$$

$$= \mathcal{F}\left[\frac{1}{RC}e^{-t/RC}u(t)\right]$$

$$= \frac{1}{RC} \cdot \frac{1}{j\omega + \frac{1}{RC}} \rightarrow (2)$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$X(j\omega) = \mathcal{F}(x(t))$$

$$= \mathcal{F}\left(te^{-t/RC}u(t)\right)$$

$$= \frac{1}{\left(\frac{1}{RC} + j\omega\right)^2} = \frac{R^2C^2}{[1 + j\omega RC]^2} \rightarrow (3)$$

$$\therefore Y(j\omega) = \frac{R^2C^2}{(1 + j\omega RC)^3}$$

$$Y(j\omega) = \frac{1}{RC} \cdot \frac{1}{\left[\frac{1}{RC} + j\omega\right]^3}$$

$$\therefore \quad y(t) = \mathcal{F}^{-1}\left(Y(j\omega) \right)$$

$$= \mathcal{F}^{-1}\left(\frac{1}{RC} \cdot \frac{1}{\left[\frac{1}{RC} + j\omega\right]^3} \right)$$

$$\underline{\underline{y(t) = \frac{1}{RC} \cdot \frac{t^2}{2!} e^{-t/RC} u(t).}}$$

prob: Consider a LTI-CT System described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$ Using Fourier Transform find the output $y(t)$ to each of the following input Signals.

(i) $x(t) = e^{-t} u(t)$

(ii) $x(t) = u(t)$

Solu: Given data;

An LTI-CT System is described by

$$\frac{dy(t)}{dt} + 2 \cdot y(t) = x(t)$$

Take fourier Transform on Both Sides, we get,

$$j\omega Y(j\omega) + 2Y(j\omega) = X(j\omega)$$

$$Y(j\omega) = \frac{1}{j\omega + 2} \cdot X(j\omega) \longrightarrow (1).$$

(i) The input to the System is $x(t) = e^{-t} u(t)$

$$\therefore \quad X(j\omega) = \frac{1}{1+j\omega} \longrightarrow (2)$$

from (1) & (2)

$$Y(j\omega) = \frac{1}{j\omega + 2} \cdot \frac{1}{1+j\omega}$$

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{j\omega + 2}$$

Taking inverse fourier transform on Both Sides

$$y(t) = \mathcal{F}^{-1}\left(\frac{1}{1+j\omega}\right) - \mathcal{F}^{-1}\left(\frac{1}{2+j\omega}\right)$$

$$= e^{-t} u(t) - e^{-2t} u(t)$$

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

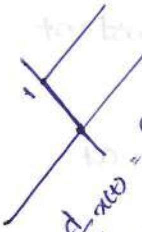
(ii) The input to the System is $x(t) = u(t)$

$$\therefore \quad X(j\omega) = \mathcal{F}(u(t)) = \frac{1}{j\omega} \rightarrow ③$$

from ① & ③ we get,

$$Y(j\omega) = \frac{1}{2+j\omega} \cdot \frac{1}{j\omega}$$

$$Y(j\omega) = \frac{1/2}{j\omega} - \frac{1/2}{j\omega+2}$$



$\frac{d}{dt}x(t) = (1-0)\,\delta(t)$
$j\omega X(j\omega) = 1$
$X(j\omega) = \frac{1}{j\omega}$

By Taking inverse fourier Transform on both Sides

$$y(t) = \mathcal{F}^{-1}\left(\frac{1/2}{j\omega}\right) - \mathcal{F}^{-1}\left(\frac{1/2}{2+j\omega}\right)$$

$$= \tfrac{1}{2}u(t) - \tfrac{1}{2}e^{-2t}u(t)$$

$$y(t) = \tfrac{1}{2}(1-e^{-2t})\,u(t)$$

⇒ Distortionless Transmission through System:

Distortionless Transmission means that the output of the System is an exact replica of the input Signal. This means that the difference between input and output of the System is Such a way that,

→ The amplitude of the output Signal may increase or decrease by Some factor with respect to input and,

→ The output Signal may be delayed in time with respect to input Signal because of System delay.

Therefore the output Signal $y(t)$ Can be Written interms of $x(t)$ as,

$$y(t) = K \cdot x(t-t_0) \longrightarrow ①$$

where $K \rightarrow$ Constant represents Change in amplitude

$t_0 \rightarrow$ delay in transmission of Signal through a System.

By taking Fourier Transform of Eqn ① on both Sides

$$Y(j\omega) = K \cdot e^{-j\omega t_0} X(j\omega).$$

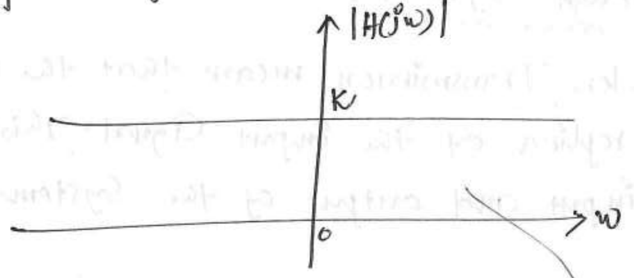
Therefore, the transfer function $H(j\omega)$ is thus given by

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = K \cdot e^{-j\omega t_0} \rightarrow ②$$

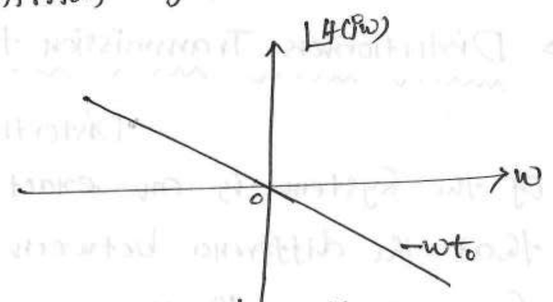
$$|H(j\omega)| = K \quad \text{and} \quad \angle H(j\omega) = -\omega t_0 \rightarrow ③$$

Equ② gives the transfer function for a distortionless Transmission System. It is clear from Equ③ that the Amplitude response is independent of frequency i.e the transfer function has constant amplitude at all frequencies. The phase shift is linearly proportional to frequency 'ω'. Here the phase shift is linear at all frequencies.

Below fig. shows the Amplitude response and phase response of a Distortionless Transmission System.



a) Magnitude Response

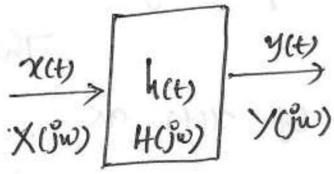


(b) phase response.

Filter Characteristics Of Linear Systems :

If a signal $x(t)$ is applied to a system having impulse response $h(t)$.

Then the output or the response is given by



$$y(t) = x(t) * h(t)$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) \rightarrow ①$$

Therefore the system modifies the spectral density function of the input signal according to Equ①. This means that the system acts like a filter to various frequency components. Some frequency components are boosted in strength, some are attenuated and some may remein uneffected. Similarly each frequency Component undergoes a certain amount of phase shift in the process of transmission.

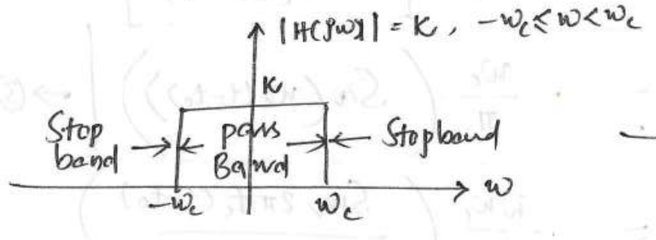
The System modifies the Spectral density function Of the input according to its filter characteristics $H(j\omega)$. The resultant response has a Spectral density function $X(j\omega) \cdot H(j\omega)$. Therefore, $H(j\omega)$ acts as a weighting function to different frequencies.

Ideal Low pass Filters: An ideal Low pass filter transmitts all Of the Signals below a Certain frequency "ω_c" without any distortion. The range of frequencies from 0 to ω_c Hz is called pass band of the Low pass filter. The range of frequencies outside this pass band is called as Stop band. It rejects all the Signals which lie outside of the pass band. The frequency ω_c is called as Cut-off frequency Of the ideal Low pass filter.

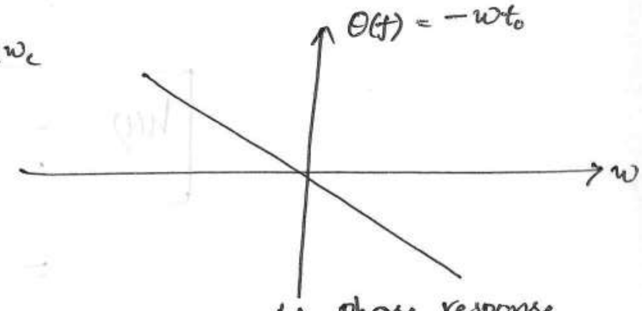
Since the filter is ideal and distortionless the transfer function of an ideal Low pass filter can be written as,

$$H(j\omega) = \left.\begin{matrix} K e^{-j\omega t_0}, & -\omega_c \leq \omega < \omega_c \\ 0, & |\omega| > \omega_c \end{matrix}\right\} \rightarrow (1)$$

Below fig. Shows the magnitude and phase responses for an ideal Low pass filter.



(a) Magnitude response.



(b) phase response

let $K=1$, in Equation (1), the transfer function will be

$$H(j\omega) = \left.\begin{matrix} e^{-j\omega t_0} & -\omega_c \leq \omega < \omega_c \\ 0 & |\omega| > \omega_c \end{matrix}\right\} \rightarrow (2)$$

Therefore, the impulse response Of a ideal Low pass filter Can be

Obtained by taking inverse fourier transform of Equation ②,

$$\therefore \quad h(t) = \mathcal{F}^{-1}(H(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\cdot e^{j\omega t}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega t_0}\, e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{+j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\cdot \left.\frac{e^{j\omega(t-t_0)}}{j\omega(t-t_0)}\right|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi j(t-t_0)}\left[e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{2j}\right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\sin \omega_c(t-t_0)\right]$$

$$= \frac{\omega_c}{\pi}\left[\frac{\sin \omega_c(t-t_0)}{\omega_c(t-t_0)}\right]$$

$$\boxed{h(t) = \frac{\omega_c}{\pi}\left(Sa(\omega_c(t-t_0))\right)} \rightarrow ③$$

$$= \frac{2\omega_c}{2\pi}\left(\frac{\sin 2\pi f_c(t-t_0)}{2\pi f_c(t-t_0)}\right)$$

$$= \frac{\omega_c}{\pi}\left(\text{Sinc}(2f_c(t-t_0))\right)$$

$$\boxed{h(t) = 2f_c\ \text{Sinc}(2f_c(t-t_0))} \rightarrow ④$$

Equations ③ & ④ gives the impulse response of an ideal Low pass filter.

Below fig. Shows the impulse response of an ideal Low pass filter.

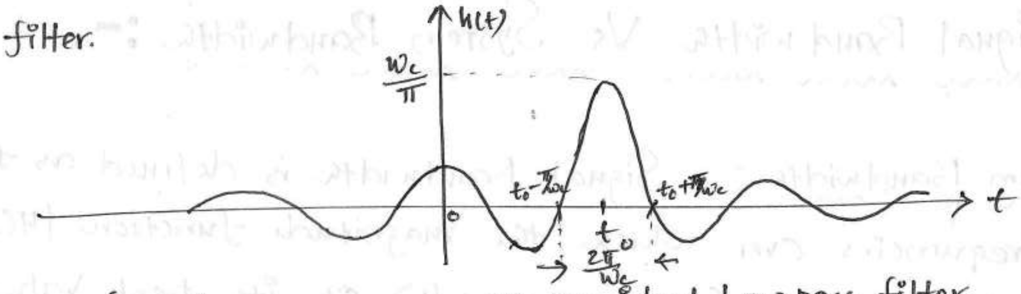


fig: impulse response of an ideal Low pass filter.

Above fig. Shows that, the impulse response exist for negative Values of 't'. But actually unit impulse is applied at $t=0$ always. Thus the response appears before the unit impulse is applied. practically it is impossible to implement Such a System. Because $h(t)=0, t<0$ for a Causal System, an ideal Low pass filter is not a Causal System.

Therefore it is clear that, although ideal Low pass filter is Very desirable it cannot be physically realizable. Thus the impulse response of a causal filter Start atmost at $t\geq 0$ and not at $t<0$ i.e negative Values of 't'.

Ideal High pass filter And Ideal Band pass filters:

An ideal Low pass filter is anticipatory (noncausal) because its response begins before the input is applied. Hence it is not physically realizable.

The Magnitude response of ideal High pass filters and ideal Band pass filters is as Shown where these filters have Sharp transition at Cutoff frequencies. This Sharp transition in frequency response results in noncausal impulse response. This means that all ideal filters are physically not realizable Since their impulse response is non causal.

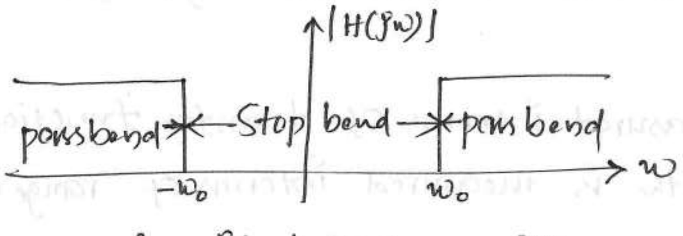


fig: ideal High pass filter

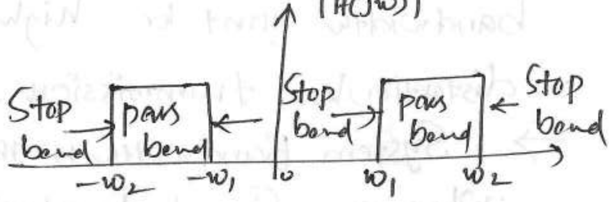
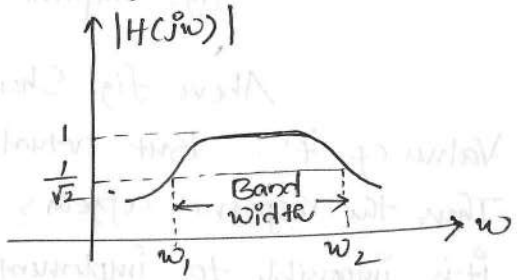


fig: ideal Band pass filter.

Signal Bandwidth Vs System Bandwidth :-

System Bandwidth : Signal Bandwidth is defined as the interval of frequencies over which the magnitude function $|H(j\omega)|$ remains within $\frac{1}{\sqrt{2}}$ times (i.e within 3db) of its peak Value.



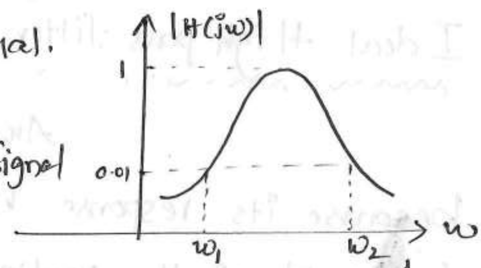
from fig. it is observed that the System Band width $= \omega_2 - \omega_1$.

$\therefore |H(j\omega)|$ was the magnitude less than $\frac{1}{\sqrt{2}}$ (i.e 3db) for $\omega > \omega_2$ & $\omega < \omega_1$.

Signal Band width : It is the range of Significant Signal frequencies which are present in the Signal.

(or)

Signal Bandwidth is defined as the maximum frequency Component present in the Signal.



From fig, it is observed that the Signal $x(t)$ was Significant frequencies from ω_1 to ω_2. Normally all the physically obtained Signals have Limited Band width. $\therefore$ $B.W = \omega_2 - \omega_1$.

Note : → The transmission through the System will be distortionless if it has infinite Band width.

→ When the Signal was finite bandwidth then the System bandwidth must be high to accommodate the Signal for distortionless transmission.

→ System Bandwidth is measured interms of transfer function where as Signal band width is measured interms of range of frequencies.

Causality And, Paley-wiener Criterian for physical Realization:

Paley-wiener Criterion gives the Condition for Causality in frequency domain. Causality relates to physical realizability of a System.

The necessary and Sufficient Condition for magnitude function $|H(j\omega)|$ to be physically realizable is given as

$$\int_{-\infty}^{\infty} \frac{|\ln|H(j\omega)||}{1+\omega^2}\, d\omega < \infty \rightarrow (1)$$

For Eqn (1) to be Valid, $|H(j\omega)|$ must be Square integrable. i.e

$$\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega < \infty.$$

Eqn (1) is usually known as paley-wiener Criterion.

The System which Violates Paley wiener Criterion is anticipatory. Such Systems produces response before excitation is applied. Such Systems are physically not realizable.

The System which Satisfies paley-wiener Criterion has Causal unit Sample response. i.e

$$h(t) = 0, \quad t \leq 0$$

Interpretations Of paley-Wiener Criterion:

→ For a physically realizable System, $|H(j\omega)|$ may be zero at Certain discrete frequencies. But it can not be zero over a finite band of frequencies. Otherwise integral of Eqn (1) will become infinite.

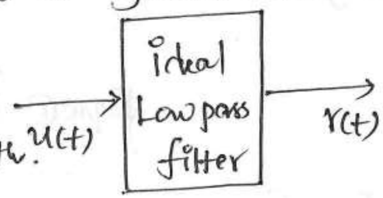
→ For a physically realizable System, $|H(j\omega)|$ can not decay faster than a function of Exponential order. For Example $H(j\omega) = e^{-\alpha\omega}$ represents realizable System, but $H(j\omega) = e^{-\alpha\omega^2}$ is not realizable.

Relationship between Bandwidth And Rise time:

Rise time (t_r) is defined as the time taken by the output (response) to reach its final value from initial value.

A Sharp change in the Signal amplitude gives rise to high frequency components in the Signal. A Signal which is Smooth Contains Low frequency Components.

Let a unit Step function be applied to an ideal Low pass filter. The response r(t) will Show a gradual rise. The rise time will depends upon the Cutoff frequency of the filter i.e System Bandwidth.

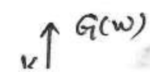


→ The transfer function of an ideal Low pass filter is given by

$$H(\omega) = |H(j\omega)| \cdot e^{j\theta(\omega)}$$

$$= G(\omega) \cdot e^{-j\omega t_0} \longrightarrow (1)$$

Here, $G(\omega)$ is a rectangular function as shown



$$\therefore \quad R(w) = \pi\delta(w) + \frac{1}{jw}H(jw). \rightarrow ④$$

→ The time response can be obtained by taking inverse fourier transform of above equation i.e

$$r(t) = \mathcal{F}^{-1}\Big(R(w)\Big) = \mathcal{F}^{-1}\left(\pi\delta(w) + \frac{1}{jw}H(jw)\right)$$

$$= \frac{1}{2}\mathcal{F}^{-1}\Big(2\pi\delta(w)\Big) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{jw}G(w).e^{-jwt_0}e^{jwt}dw$$

$$\therefore \quad r(t) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{jw}.G(w).e^{+jw(t-t_0)}.dw$$

$$\because \quad G(w) = 1, \; -w_c < w < w_c,$$

$$r(t) = \frac{1}{2} + \frac{1}{2\pi}\int_{-w_c}^{w_c}\frac{e^{jw(t-t_0)}}{jw}.dw$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{-w_c}^{w_c}\frac{\cos w(t-t_0) + j\sin w(t-t_0)}{jw}.dw$$

$$r(t) = \frac{1}{2} + \frac{1}{2\pi}\left[\int_{-w_c}^{w_c}\frac{\cos w(t-t_0)}{jw}dw + \int_{-w_c}^{w_c}\frac{\sin w(t-t_0)}{w}dw\right]$$

The first integral in the above equation is zero because the term in the integral is zero. And the term in the second integral is an even function we can write,

$$r(t) = \frac{1}{2} + \frac{1}{\cancel{2}\pi}\left(0 + \cancel{2}\int_{0}^{w_c}\frac{\sin w(t-t_0)}{w}dw\right)$$

$$r(t) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{w_c}\frac{\sin w(t-t_0)}{w}dw.$$

$$= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{w_c(t-t_0)}\frac{\sin x}{x}dx$$

$$r(t) = \frac{1}{2} + \frac{1}{\pi} Si[w_c(t - t_0)]$$

Above equation gives the response of the System interms of Bandwidth w_c. The rise time is given as

$$t_r = \frac{2\pi}{w_c} = \frac{1}{f_c}$$

Below fig. Shows the plot of $r(t)$. As w_c increases $r(t)$ increases Slowly. Thus rise time and Bandwidth are inversely proportional to each other.

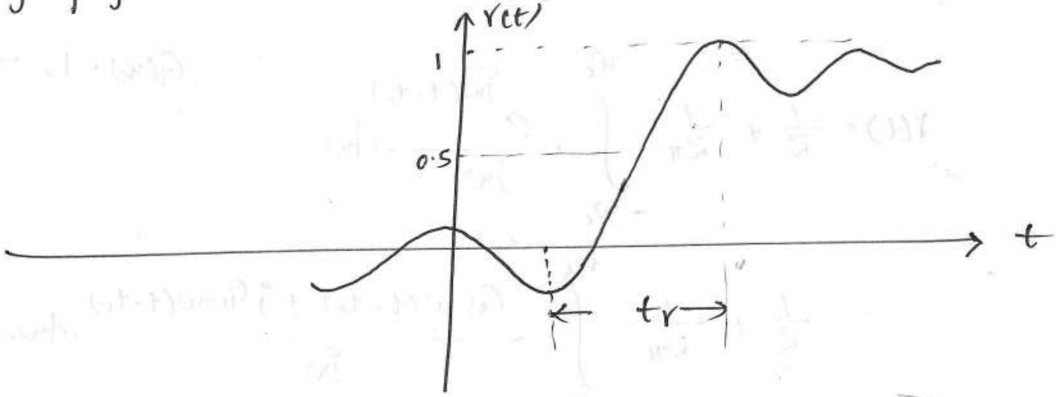
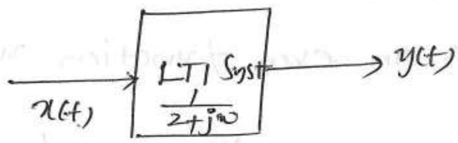


fig: plot of $r(t)$

PREVIOUS PROBLEMS

① Let the System function of a LTI System be $\frac{1}{jw+2}$. What is the output of the System for an input $(0.8)^t u(t)$.

Solw: Given Data,



Transfer function

$$H(jw) = \frac{1}{2+jw}$$

Input $x(t) = 0.8^t u(t)$.

According to Convolution Theorem,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) \cdot d\tau$$

Impulse response of the Linear time invariant System is

$$h(t) = \mathcal{F}^{-1}(H(jw))$$

$$h(t) = \mathcal{F}^{-1}\left(\frac{1}{j\omega+2}\right)$$

$$h(t) = e^{-2t}u(t) \qquad \because \mathcal{F}^{-1}\left(\frac{1}{a+j\omega}\right) = e^{-at}u(t)$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\cdot(0.8)^{t-\tau}\cdot u(\tau)\cdot d\tau$$

$$= \int_{0}^{\infty} e^{-2\tau}\cdot(0.8)^{t-\tau}\cdot d\tau$$

$$= (0.8)^t \int_0^{\infty} (0.8e^2)^{-\tau}\, d\tau$$

$$\text{let } (0.8e^2)^{-1} = a$$

$$y(t) = (0.8)^t \int_0^{\infty} a^{\tau}\, d\tau \qquad \because \int a^x\cdot dx = \frac{a^x}{\log a}$$

$$= (0.8)^t \left.\frac{a^{\tau}}{\log a}\right|_0^{\infty}$$

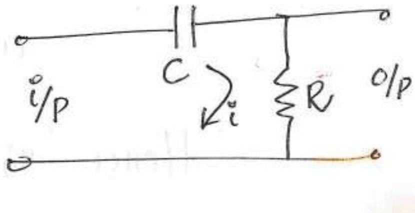
$$= (0.8)^t \cdot \left.\frac{(0.8e^2)^{-\tau}}{\log(0.8e^2)^{-1}}\right|_0^{\infty}$$

$$= (0.8)^t \cdot \frac{1}{\log(0.8e^2)^{-1}}(0-1)$$

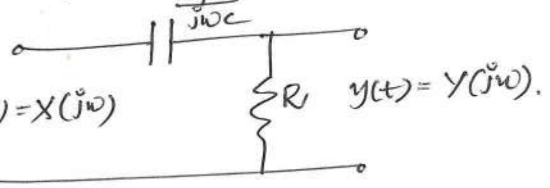
$$= \frac{\cancel{-}0.8^t}{\cancel{-}\log(0.8e^2)} = \frac{0.8^t}{\log 0.8 + 2\log e}$$

$$y(t) = \frac{0.8^t}{2+\log 0.8}$$

(2) Find the impulse response of the System Shown in fig. Find the Transfer function. Sketch its frequency response.



Soln: Transform the given network into frequency domain, we have,



$$\therefore \quad Y(j\omega) = \frac{X(j\omega) \cdot R}{R + \frac{1}{j\omega C}}$$

$$= \frac{j\omega RC \cdot X(j\omega)}{1 + j\omega RC}$$

$\therefore$ Transfer function (or) the impulse response is given by

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega RC}{1 + j\omega RC} \longrightarrow (1)$$

$$= \frac{1}{1 + \frac{1}{j\omega RC}}$$

$$\text{let } \omega' = \frac{1}{RC}$$

$$\therefore \quad H(j\omega) = \frac{1}{1 + \frac{\omega_1}{j\omega}} = \frac{1}{1 - j\frac{\omega_1}{\omega}}$$

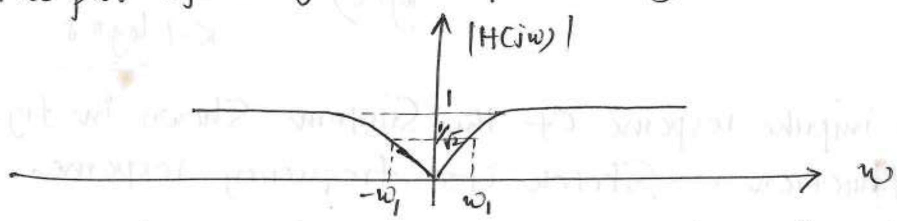
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega_1}{\omega}\right)^2}} \longrightarrow (2)$$

$$H(j\omega) = 1 - \frac{1}{1 + j\omega RC} = 1 - \frac{1}{RC}\left(\frac{1}{\frac{1}{RC} + j\omega}\right)$$

$$\therefore \quad h(t) = \mathcal{F}^{-1}(H(j\omega)) = \mathcal{F}^{-1}\left(1 - \frac{1}{RC}\left(\frac{1}{\frac{1}{RC} + j\omega}\right)\right)$$

$$h(t) = \delta(t) - \frac{1}{RC} \cdot e^{-t/RC} u(t) \longrightarrow (3).$$

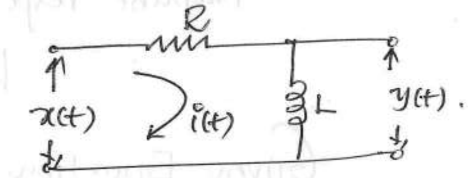
The plot of Magnitude response is given by



at $\omega = \omega_1$, $|H(j\omega)| = \frac{1}{\sqrt{2}}$ (or) -3db

Hence 'ω_1' is called 3db frequency.

③ Find the impulse response of the R-L filter as shown.



Soln: From the figure it is clear that

$$x(t) = R\,i(t) + L\frac{di(t)}{dt} \rightarrow ①$$

$$y(t) = L\frac{di(t)}{dt} \longrightarrow ②$$

$$① \Rightarrow \quad X(j\omega) = R\cdot I(j\omega) + L\cdot(j\omega)\cdot I(j\omega) \rightarrow ③$$

$$② \Rightarrow \quad Y(j\omega) = L\cdot(j\omega)\cdot I(j\omega).$$

$$I(j\omega) = \frac{-j}{\omega L}\,Y(j\omega).$$

$$③ \Rightarrow \therefore X(j\omega) = (R + j\omega L)\cdot\frac{1}{j\omega L}\cdot Y(j\omega).$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \frac{R}{j\omega L}}$$

The impulse response $h(t) = \mathcal{F}^{-1}(H(j\omega))$

$$= \mathcal{F}^{-1}\left(\frac{j\omega L}{R + j\omega L}\right)$$

$$= \mathcal{F}^{-1}\left(\frac{j\omega}{\frac{R}{L} + j\omega}\right)$$

$$= \mathcal{F}^{-1}\left(1 - \frac{R/L}{R/L + j\omega}\right)$$

$$h(t) = \delta(t) - \frac{R}{L}e^{-R/L\,t}\,u(t)$$

④ Consider an LTI System with the input and output related through,

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}\,x(\tau-2)\cdot d\tau.$$

what is the impulse response of the System.

Soln: Given,

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}\cdot x(\tau-2)\cdot d\tau.$$

impulse response of the system is given by

$$h(t) = T(\delta(t))$$

Given Equation can be viewed as convolution of $x(t-2)$ and e^{-t}.

$$\text{if} \quad x(t) = \delta(t) \rightarrow y(t) = h(t)$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \cdot x(\tau-2)\, d\tau$$

$$\therefore \quad h(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \cdot \delta(\tau-2)\, d\tau$$

$$= e^{-(t-\tau)} \Big|_{\tau=2},$$

$$= e^{-(t-2)}, \quad t>0$$

$$h(t) = e^{-(t-2)} u(t)$$

LAPLACE TRANSFORM

Definition: Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Provided the integral exists; is called Laplace Transform of $f(t)$.

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \cdot dt}$$

Sufficient Condition- For Existence of Laplace Transform.

1)- $f(t)$ should be Continuous, or piecewise continuous on the sub-interval of $(0, \infty)$

2) $f(t)$ should be exponential order.

i.e. $|f(t)| \leq Me^{\alpha t}$

where, $\alpha > 0$ is known as exponential Order

i.e $\lim_{t \to \infty} [F(t)\, e^{-\alpha t}] \xrightarrow{\text{finite}}$ for $t > \alpha$

and $f(t)$ is Continuous then Laplace Transform of $f(t)$ $\int_0^\infty e^{-st} f(t)\, dt$ exist.

Note:- Above Condition are Sufficient not necessary.

Important Example:.

(1.) $L\{1\} = \frac{1}{s}$

(2.) $L\{e^{at}\} = \frac{1}{s-a}$: $(s>a)$

(3.) $L\{t^n\} = \frac{n!}{s^{n+1}}$ $n \in \mathbb{N} \cup \{0\}$

(4) $L\{\sin at\} = \frac{a}{s^2+a^2}$; $(s>0)$

(5) $L\{\cos at\} = \frac{s}{s^2+a^2}$; $(s>0)$

(6) $L\{\sinh at\} = \frac{a}{s^2+a^2}$ $(s>0)$

(7) $L\{\cosh at\} = \frac{s}{s^2-a^2}$ $(s>0)$

Proof of Some.

Solⁿ.:

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

or $$= \int_0^\infty e^{-st} \cdot t^n\, dt$$

Now put

$st = x$.

$s\,dt = dx$

$t=0 \Rightarrow x=0$

$t=\infty \Rightarrow x=\infty$

or $$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

or $$= \int_0^\infty \frac{e^{-x} \cdot x^n}{s^n} \cdot \frac{dx}{s}$$

or $$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

Gamma function -

$$\because \Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}$$

or, $$= \frac{1}{s^{n+1}} \cdot \Gamma{n+1}$$

and, $\Gamma{n+1} = n!$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

2). $L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at}\, dt$ $\qquad$ $s > a$

Put

$(s-a).t = x$

$(s-a)\, dt = dx$

$t = 0 \Rightarrow x = 0$

$t = \infty \Rightarrow x = \infty$

$= \int_0^\infty e^{-(s-a)t}\, dt$

$= \int_0^\infty e^{-x} \frac{dx}{(s-a)}$

$= \frac{1}{s-a} \left[\frac{e^{-x}}{-1} \right]_0^\infty$

$e^{-\infty} \rightarrow 0$

if $s-a > 0$

$= \frac{1}{s-a} [0 + 1]$

a. $\boxed{L\{e^{at}\} = \frac{1}{s-a}}$ $\qquad$ $s > a$

if $s < a$

then, $e^{-(s-a)t} \rightarrow e^{+(n)t}$

as $t \rightarrow \infty$, $e^{-(s-a)t} \rightarrow \infty$

then Laplace doesn't exist.

Similarly, we can find.

$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$ $\qquad$ $\cosh(at) = \frac{e^{at} + e^{-at}}{2}$

$\sin(at) = \frac{e^{iat} - e^{-iat}}{2i}$ $\qquad$ $\cos(at) = \frac{e^{iat} + e^{-iat}}{2}$

$i^2 = -1$

change in above formulae and

use $L\{e^{at}\} = \frac{1}{s-a}$, we can find the Result.

Do Yourself.

Exp:

Find Laplace Transform of-

$$f(t) = \begin{cases} t & 1 < t < 2 \\ 4-t & 2 < t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

sol^n: From the definition of Laplace-

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}.f(t)\,dt$$

or $$F(s) = \int_0^1 e^{-st}.f(t)\,dt + \int_1^2 e^{-st}.f(t)\,dt + \int_2^3 e^{-st}.f(t)\,dt + \int_3^\infty e^{-st}.f(t)\,dt$$

or $$F(s) = 0 + \int_1^2 e^{-st}.t\,dt + \int_2^3 e^{-st}.(4-t)\,dt + 0$$

or $$F(s) = \int_1^2 e^{-st}.t\,dt + \int_2^3 (4-t).e^{-st}\,dt$$

By Using Integral By-Part.

or $$F(s) = \left[\frac{t.e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\,dt\right]_1^2 + \left[\frac{(4-t)e^{-st}}{-s} - \int \frac{-1.e^{-st}}{-s}\,dt\right]_2^3$$

or $$F(s) = \left[\frac{te^{-st}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\right]_1^2 + \left[\frac{(4-t)e^{-st}}{-s} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\right]_2^3$$

$$= \left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - \frac{1.e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right]$$

$$\Rightarrow F(s) = \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2 + \left[\frac{(t-1)e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_2^3$$

$$\Rightarrow F(s) = \left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right] + \left[\frac{e^{-3s}}{-s} + \frac{e^{-3s}}{s^2} + \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right]$$

$$F(s) = \frac{-2e^{-2s}}{s^2} + \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} + \left(\frac{1}{s^2} - \frac{1}{s}\right)e^{-3s}$$

$$\boxed{F(s) = \frac{(s+1)}{s^2} \cdot e^{-s} - \frac{2}{s^2} e^{-2s} + \frac{(1-s)}{s^2} e^{-3s}}$$

Exercise:

1). $f(t) = \begin{cases} \cos 3t & 0 < t < 2 \\ 0 & t > 2 \end{cases}$

2) $f(t) = \begin{cases} 4 & 0 < t < 2 \\ 2t & t > 2 \end{cases}$

Properties of Laplace Transform.

(1) $L\{a f_1(t) \pm b f_2(t)\} = a L\{f_1(t)\} \pm b L\{f_2(t)\}$

$$\Rightarrow \int_0^\infty e^{-st}[a f_1(t) \pm b f_2(t)]\,dt = a\int_0^\infty e^{-st} f_1(t)\,dt \pm b\int_0^\infty e^{-st} f_2(t)\,dt$$

↳ Linear Property.

(2) Change of Scale Property

If $L\{f(t)\} = F(s)$ then $\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

$$\Rightarrow L\{f(at)\} = \int_0^\infty e^{-st} \cdot f(at)\,dt$$

$at = u$

$a\,dt = du$

$t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$$= \int_0^\infty e^{-\frac{su}{a}} \cdot f(u)\,\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-Su} \cdot F(u)\,du \qquad S = \frac{s}{a}$$

$$= \frac{1}{a}F(S) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

so, $\boxed{L\{f(at\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

Exp:- $L\{(6t)^4\} = ?$

as, $L\{t^4\} = \frac{4!}{s^5}$ then, $L\{(6t)^4\} = \frac{1}{6} \cdot \frac{4!}{\left(\frac{s}{6}\right)^5} = \frac{6^5 \cdot 4!}{6 \cdot s^5} = \frac{6^4 \cdot 4!}{s^5}$

(3) If $f(t)$ is defined for $t>0$ and $f(t)$ is sectionally continuous and of exponetial order then $F(s) \longrightarrow 0$ as $s \rightarrow \infty$

(4) First shifting property:-

if $L\{f(t)\} = F(s)$

then $L\{e^{at} f(t)\} = F(s-a)$

$$\Rightarrow \quad L\{e^{at} f(t)\} = \int_0^\infty e^{-st} . e^{at} . f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} . f(t)\, dt$$

$$= F(s-a).$$

Exp:- $L\{e^{2t} . \sin 4t\}$.

as, $L\{\sin 4t\} = \dfrac{4}{s^2+(4)^2} = \dfrac{4}{s^2+16}$

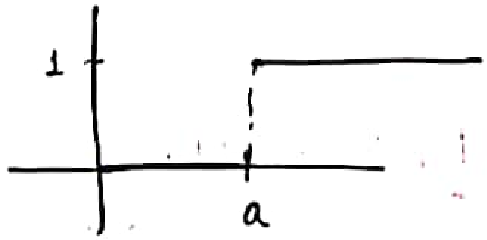
so, by shifting property -

$$L\{e^{2t} \sin 4t\} = \frac{4}{(s-2)^2+16}.$$

5 Unit Step function (Heaviside's).

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$U(t-a) = \begin{cases} 0 & 0 < t < a \\ 1 & t \geq a \end{cases}$$



Now, $\boxed{L\{U(t-a)\} = \dfrac{e^{-sa}}{s}}$

Sol

$$L\{U(t-a)\} = \int_0^\infty e^{-st} \cdot U(t-a) \cdot dt$$

$$= \int_0^a e^{-st} \times 0 \times dt + \int_a^\infty e^{-st} \cdot 1 \, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \left[0 + \frac{e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

* 6.

If $L\{f(t)\} = F(s)$ (Second shifting Property)

and $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & 0 < t < a \end{cases}$

then, $\boxed{L\{g(t)\} = e^{-as} \cdot F(s)}$

Initial and Final Value Theorem -

(a) Initial Value Theorem :-

If $L\{f(t)\} = F(s)$

then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} [sF(s)]$ Provided the limit exist

Proof: As we known that -

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

$$\lim_{s\to\infty} \int_0^\infty e^{-st}.f'(t)\,dt = \lim_{s\to\infty} [sF(s) - f(0)]$$

or $$\int_0^\infty \left(\lim_{s\to\infty} e^{-st}\right) f'(t)\,dt = \lim_{s\to\infty} [sF(s)] - f(0)$$

$$0 = \lim_{s\to\infty} [sF(s)] - f(0)$$

or, $$\lim_{s\to\infty} [sF(s)] = f(0) = \lim_{t\to 0} f(t).$$

$$\Rightarrow \boxed{\lim_{s\to\infty} [sF(s)] = \lim_{t\to 0} f(t).}$$

(10)

2) Final Value Theorem:-

$L\{f(t)\} = F(s)$

$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$ Provided the limit exists

So $L\{f'(t)\} = sF(s) - f(0)$

$\therefore \int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$

$\lim_{s\to 0}\int_0^\infty e^{-st}\cdot f'(t)\,dt = \lim_{s\to 0}[sF(s) - f(0)]$

or $\int_0^\infty \left(\lim_{s\to 0} e^{-st}\right)\cdot f'(t)\,dt = \lim_{s\to 0}[sF(s)] - f(0)$

or $\int_0^\infty f'(t)\,dt = \lim_{s\to 0}[sF(s)] - f(0)$

$\Rightarrow [f(t)]_0^\infty = \lim_{s\to 0}[sF(s)] - f(0)$

$\Rightarrow \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0}[sF(s)] - f(0)$

$\Rightarrow \boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]}$.

Laplace Transform of Error function

Error function is defined as –

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx.$$

then,

$$\boxed{L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}} = F(s)$$

Now,

$$L\{\operatorname{erf} a\sqrt{t}\} = L\{\operatorname{erf}\sqrt{a^2 t}\} = \frac{1}{a^2}F\left(\frac{s}{a^2}\right)$$

Exp. Find $L\{\operatorname{erf} 2\sqrt{t}\}$

Since,

$$L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}} = F(s)$$

$$= L\{\operatorname{erf}\sqrt{4t}\}$$

$$= \frac{1}{4}\cdot\frac{1}{\left(\frac{s}{4}\right)\sqrt{\frac{s}{4}+1}}$$ by change of Scale Property.

$$= \frac{1}{4}\times\frac{4\times 2}{s\sqrt{s+4}} = \frac{2}{s\sqrt{s+4}}$$

Laplace Transform of Bessel function $J_0(x)$ and $J_1(x)$

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$ and, $$L\{J_0(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{\frac{s^2}{a^2}+1}}$$

$$= \frac{1}{a}\cdot\frac{a}{\sqrt{s^2+a^2}}$$

so, $$L\{J_0(at)\} = \frac{1}{\sqrt{s^2+a^2}}$$

$$L\{J_1(x)\} = -L\{J_0'(x)\} = -\left[sL\{J_0(x)\} - J_0(0)\right]$$

$$L\{J_1(x)\} = -\left[\frac{s}{\sqrt{s^2+1}} - 1\right]$$

$$\boxed{L\{J_1(x)\} = 1 - \frac{s}{\sqrt{s^2+1}}}$$

$$L\{J_1(ax)\} = \left(1 - \frac{s/a}{\sqrt{\frac{s^2}{a^2}+1}}\right)\cdot\frac{1}{a}$$

$$= \frac{1}{a}\left(1 - \frac{s}{\sqrt{s^2+a^2}}\right)$$

Convolution Theorem:

If $L\{f_1(t)\} = F_1(s)$

and $L\{f_2(t)\} = F_2(s)$

then, $L\left\{\int_0^t f_1(x) \cdot f_2(t-x) \cdot dx\right. = F_1(s) \cdot F_2(s)$.

Mostly used to find Laplace Inverse. Transformation.

for, Example:

Find Laplace of $\int_0^t e^x \sin(t-x)\, dx$.

Let by Convolution theorem-

$$f_1(x) = e^x$$

$$f_2(x) = \sin x$$

then $L\left\{\int_0^t f_1(x) \cdot f_2(t-x)\, dx\right\} = F_1(s) \cdot F_2(s)$

$\therefore$ $L\left\{\int_0^t e^x \cdot \sin(t-x)\, dx\right\} = L\{e^t\} \cdot L\{\sin x\}$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2-1}$$

$$= \frac{1}{(s-1)^2(s+1)}$$

Laplace Transform of Derivatives and Integrals.

Periodic function, Impulse function.

Laplace Transform of the derivative of f(t):

Let $L\{f(t)\} = F(s)$

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

Proof:-

$$L\{f'(t)\} = \int_0^\infty e^{-st}.\, f'(t)\, dt$$

by integration by Part

or $$L\{f'(t)\} = \left[e^{-st}.\, f(t)\right]_0^\infty - \int_0^\infty (-se^{-st}).\, f(t)\, dt$$

or $$L\{f'(t)\} = \left[0 - f(0)\right] + s\int_0^\infty e^{-st}.\, f(t)\, dt$$

$$\therefore \boxed{L\{f'(t)\} = s\,L\{f(t)\} - f(0)}$$

In General form:-

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - f^{n-1}(0)$$

i.e. For, $n=3$.

$$\boxed{L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f(0).}$$

Laplace Transform of Integral of $f(t)$. (15).

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s) \qquad \text{where, } L\{f(t)\} = F(s)$$

Proof: Let $\phi(t) = \int_0^t f(t)\,dt$ and, $\phi(0) = 0$.

$\phi'(t) = f(t)$

Now,

$$L\{\phi'(t)\} = sL\{\phi(t)\} - \phi(0)$$

or, $$L\{f(t)\} = sL\{\phi(t)\}.$$

or, $$L\{\phi(t)\} = \frac{1}{s}L\{f(t)\}$$

or, $$\boxed{L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s).}$$

Laplace transform of $t^n f(t)$.

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)].$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt.$$

$\therefore$ Differenitiating w.r.to 's'

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$= \int_0^\infty \left(\frac{\partial}{\partial s} e^{-st}\right)\cdot f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty t\cdot e^{-st} f(t)\,dt$$

$$= -L\{t\cdot f(t)\}.$$

$$\Rightarrow \quad L\{t\cdot f(t)\} = (-1)\frac{d}{ds}F(s)$$

Similarly -

$$L\{t^2 f(t)\} = (-1)^2\frac{d^2}{ds^2}F(s)$$

$$L\{t^3 f(t)\} = (-1)^3\frac{d^3}{ds^3}F(s)$$

$$\boxed{L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)}$$

Laplace Transform of $\frac{1}{t} f(t)$.

$L\{f(t)\} = F(s)$ then- $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(s)\, ds$.

Proof:-

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

on integrating with respect to 's'

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt \right] ds$$

or $$= \int_0^\infty \left[\int_s^\infty e^{-st} ds \right] \cdot f(t)\, dt$$

or $$= \int_0^\infty \left[\frac{e^{-st}}{-t} \right]_s^\infty \cdot f(t)\, dt$$

or $$= \int_0^\infty \frac{-1}{t} \left[e^{-\infty} - e^{-st} \right] f(t)\, dt$$

or, $$= \int_0^\infty \frac{1}{t} \cdot e^{-st} \cdot f(t)\, dt = L\left\{\frac{1}{t} f(t)\right\}$$

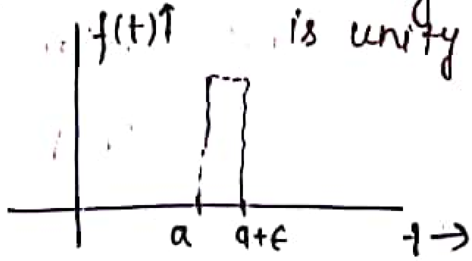
so, $$\boxed{L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(s)\, ds.}$$

Impulse Function:-

When a large force acts for a short time, then the product of the force and the time is called impulse in applied Mechanics. The unit impulse function is the limiting function - as $\epsilon \to 0$

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} & a < t < a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

• area of rectangle is unity.



Laplace Transform of Unit Impulse function.

$$\int_a^b f(t)\,dt = (b-a)\cdot f(\eta)$$

$, a < \eta < a+\epsilon$
(Mean Value Theorem)

$$\int_0^\infty f(t)\cdot\delta(t-a)\,dt = \int_a^{a+\epsilon} f(t)\cdot\frac{1}{\epsilon}\,dt$$

$$= (a+\epsilon-a)\cdot\frac{1}{\epsilon} f(\eta) = f(\eta).$$

Now, Property =

$$\boxed{\int_0^\infty f(t)\cdot\delta(t-a)\,dt = f(a).}$$

as - $\epsilon \to 0$
then, $\eta \to a$.

So, $L\{\delta(t-a)\} = \int_0^\infty e^{-st}\cdot\delta(t-a)\,dt = e^{-as}.$

$f(t) = e^{-st}.$

② Exp:. $L\{t^3 \delta(t-2)\}$

$$= \int_0^\infty e^{-st}\cdot t^3 \cdot \delta(t-2)\,dt$$

here, $f(t) = e^{-st} \cdot t^3$

so, $\int_0^\infty e^{-st} \cdot t^3 \, \delta(t-2) = f(2) = e^{-2s} \cdot 2^3 = 8e^{-2s}$.

Thus, $\boxed{L\{t^3 \delta(t-2)\} = 8e^{-2s}}$.

<u>Property (2)</u>

$$\boxed{\int_{-\infty}^{\infty} f(t)\cdot \delta'(t-a)\,dt = -f'(a)}.$$

The Unit Impulse Function is defined as -

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}.$$

thus, $\boxed{\int_0^\infty \delta(t-a)\,dt = 1}$.

Laplace Transform of Periodic function :

Let $f(t)$ be periodic function with period T, then -

$$\frac{\int_0^T e^{-st} \cdot f(t)\, dt}{1 - e^{-sT}} = L\{f(t)\} = F(s)$$

Exp: Find Laplace Transform.

$$f(t) = 2t \quad ; \quad \text{Period of } t = 3,$$

Sol:

Thus, $T = 3$.

so,
$$L\{f(t)\} = \frac{1}{1-e^{sT}} \cdot \int_0^T e^{-st} \cdot f(t)\, dt$$

or
$$L\{f(t)\} = \frac{1}{1-e^{3s}} \cdot \int_0^3 e^{-st} \cdot 2t\, dt$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}} \left[\frac{t e^{-st}}{-s} - \int \frac{e^{-st}}{-s} \right]_0^3$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}} \left[\frac{t e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_0^3 .$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}} \left[\frac{3 \cdot e^{-3s}}{-s} + \frac{e^{-3s}}{s^2} - 0 - \frac{1}{s^2} \right]$$

Inverse Laplace Transform

If $F(s)$ is the Laplace Transform of a function $f(t)$. then $f(t)$ is known as Laplace inverse or Inverse Laplace transform

If $L\{f(t)\} = F(s)$

then, $L^{-1}\{F(s)\} = f(t)$.

Where, L^{-1} is called the inverse Laplace transform operator.

Important formulae :-

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

3) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

4) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

5) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

6) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

7) $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

8) $L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}$
$= e^{at}.f(t)$.

9) $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(t)\,dt$.

10) $L^{-1}\{1\} = \delta(t)$.

11) $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -tf(t)$

(12) $L^{-1}\{e^{-as}F(s)\} = f(t-a)\cdot U(t-a)$. (Second shifting property)

(13) $L^{-1}\{e^{-as}\} = \delta(t-a)$

14) $L^{-1}\{sF(s)\} = \frac{d}{dt}f(t) + f(0)\cdot\delta(t)$.

Derivative.

(15) $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\,f(t)$

or, $L^{-1}\{F(s)\} = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}F(s)\right\}$

Integral.

(16) $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

$\Rightarrow L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$

Inverse Laplace Transform by Convolution.

Since $L\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1(s)\cdot F_2(s)$

or, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(x)\cdot f_2(t-x)\,dx$.

(1) Find Laplace Inverse of —

$$F(s) = \frac{7}{s^2+4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{7}{s^2+4}\right\}$$

or, $f(t) = 7\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$

$\because L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

or, $L\{\sin at\} = \frac{a}{s^2+a^2}$

or, $f(t) = \frac{7}{2}\,L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

or, $\boxed{f(t) = \frac{7}{2}\sin(2t)}$

(2) $F(s) = \frac{5}{s-4} + \frac{7s}{s^2+16}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{5}{s-4} + \frac{7s}{s^2+16}\right\}$

$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

or $f(t) = 5\,L^{-1}\left\{\frac{1}{s-4}\right\} + 7\,L^{-1}\left\{\frac{s}{s^2+16}\right\}$

or $\boxed{f(t) = 5e^{4t} + 7\cos(4t)}$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

(3)

$$L^{-1}\left\{\frac{s+1}{s^2+4s-5}\right\}$$

$$= L^{-1}\left\{\frac{(s+2)}{(s+2)^2-9} - \frac{1}{(s+2)^2-9}\right\}$$

$$= L^{-1}\left\{\frac{(s+2)}{(s+2)^2-3^2}\right\} - L^{-1}\left\{\frac{1}{(s+2)^2-9}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{s}{s^2-3^2}\right\} - e^{-2t} L^{-1}\left\{\frac{1}{s^2-9}\right\}$$

$$= e^{-2t}\cosh(3t) - \frac{e^{-2t}}{3} L^{-1}\left\{\frac{3}{s^2-9}\right\}$$

$$= e^{-2t}\cosh(3t) - \frac{e^{-2t}}{3}\sinh(3t)$$

$$= \frac{e^{-2t}}{3}\left[3\cosh(3t) - \sinh(3t)\right]$$

so,

$$\boxed{L^{-1}\left\{\frac{s+1}{s^2+4s-5}\right\} = \frac{e^{-2t}}{3}\left[3\cosh(3t) - \sinh(3t)\right]}$$

Partial fraction.

$$\frac{s+1}{s^2+4s-5} = \frac{(s+1)}{(s^2+4s+4-9)}$$

$$= \frac{(s+1)}{(s+2)^2-9}$$

$$= \frac{(s+2)-1}{(s+2)^2-9}$$

By shifting Property -

If

$L\{f(t)\} = F(s)$

$L\{e^{at}f(t)\} = F(s-a)$

i.e.

$L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}$

$= e^{at}f(t)$.

and,

$L\{\cosh(at)\} = \frac{s}{s^2-a^2}$

$L\{\sinh(at)\} = \frac{a}{s^2-a^2}$.

(4) $F(s) = \dfrac{s}{(s-4)^5}$

or, $F(s) = \dfrac{(s-4+4)}{(s-4)^5}$

or $F(s) = \dfrac{(s-4)}{(s-4)^5} + \dfrac{4}{(s-4)^5}$

or $F(s) = \dfrac{1}{(s-4)^4} + \dfrac{4}{(s-4)^5}$

$\because$ By shifting Property :

If $L^{-1}\{F(s)\} = f(t)$

then, $L^{-1}\{F(s-a)\} = e^{at}.f(t)$.

Taking Laplace inverse -

$L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{(s-4)^4}\right\} + 4L^{-1}\left\{\dfrac{1}{(s-4)^5}\right\}$

or $F(t) = e^{4t} L^{-1}\left\{\dfrac{1}{s^4}\right\} + 4e^{4t} L^{-1}\left\{\dfrac{1}{s^5}\right\}$

$L^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}$

or $F(t) = e^{4t}.\dfrac{t^3}{3!} + 4e^{4t}.\dfrac{t^4}{4!}$

or, $F(t) = \dfrac{e^{4t}}{6}[t^3+t^4]$

$$\boxed{\text{or, } F(t) = \dfrac{t^3 e^{4t}}{6}(t+1)}$$

(5) $F(s) = \dfrac{e^{-3s} - e^{-6s}}{s^8}$

or, $F(s) = \dfrac{e^{-3s}}{s^8} - \dfrac{e^{-6s}}{s^8}$

Taking Laplace Inverse-

$L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{e^{-3s}}{s^8}\right\} - L^{-1}\left\{\dfrac{e^{-6s}}{s^8}\right\}$.

Now,

$L^{-1}\left\{\dfrac{1}{s^8}\right\} = \dfrac{t^7}{7!}$

Thus,

$L^{-1}\left\{\dfrac{e^{-3s}}{s^8}\right\} = \dfrac{(t-3)^7}{7!}\, U(t-3)$

and,

$L^{-1}\left\{\dfrac{e^{-6s}}{s^8}\right\} = \dfrac{(t-6)^7}{7!}\, U(t-6)$

$L\{f(t)\} = F(s)$

$L\{f(t)\cdot U(t-a)\}$

$= F(s-a)\cdot e^{-as}$.

So, inverse-

$L^{-1}\{F(s)\} = f(t)$

$L^{-1}\{e^{-as} F(s)\} = f(t-a)\cdot U(t-a)$

$$\boxed{U(t-a) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a. \end{cases}}$$

Unit step function.

and,

$L^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$

so,

$$\boxed{f(t) = L^{-1}\{F(s)\} = \frac{1}{7!}\left[(t-3)^7\cdot U(t-3) - (t-6)^7\, U(t-6)\right]}$$

6) $F(s) = -\dfrac{e^{-4s}(s+7)}{s^2+25}$

Let $f(s) = \dfrac{s+7}{s^2+25} = \dfrac{s}{s^2+25} + \dfrac{7}{s^2+25}$

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{7}{5}L^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$L^{-1}\{f(s)\} = \cos(5t) + \frac{7}{5}\sin(5t)$$

Now, using Property

Unit step function:

$$U(t-a) = \begin{cases} 1 & 0<t<a \\ 0 & t>a \end{cases}$$

if $L^{-1}\{f(s)\} = f(t)$

then, $L^{-1}\{e^{-as}f(s)\} = f(t-a).U(t-a).$

Since, $L^{-1}\{f(s)\} = \cos(5t) + \frac{7}{5}\sin(5t)$

then, $L^{-1}\{e^{-4s}f(s)\} = \left[\cos[5(t-4)] + \frac{7}{5}\sin[5(t-4)]\right] U(t-4)$

Now, Given that

$$F(s) = -e^{-4s}f(s) = -e^{-4s}\frac{(s+7)}{s^2+25}$$

so, $\boxed{L^{-1}\{F(s)\} = -\left[\cos(5t-20) + \frac{7}{5}\sin(5t-20)\right] U(t-4)}$

Find Laplace Inverse.

(28)

$$L^{-1}\left\{\ln\left(\frac{s+2}{s+4}\right)\right\} = f(t)$$

$$\Rightarrow L^{-1}\{\ln(s+2) - \ln(s+4)\} = f(t)$$

$$\Rightarrow L^{-1}\{\ln(s+2) - \ln(s+4)\} = f(t)$$

From Differential theorem -

$$\Rightarrow \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}[\ln(s+2) - \ln(s+4)]\right\} = f(t)$$

$$\Rightarrow F(t) = \frac{-1}{t} L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\}$$

$$F(t) = \frac{-1}{t} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{t} L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$F(t) = \frac{-1}{t} e^{-2t} + \frac{1}{t} e^{-4t}$$

$$F(t) = \frac{1}{t}\left[e^{-4t} - e^{-2t}\right]$$

$$\Rightarrow \boxed{L^{-1}\left\{\ln\left(\frac{s+2}{s+4}\right)\right\} = \frac{1}{t}\left[e^{-4t} - e^{-2t}\right]}$$

If
$$L\{f(t)\} = F(s)$$
then
$$L\{t \cdot f(t)\} = \frac{-d}{ds} F(s)$$
$$\Downarrow$$

$$L^{-1}\{F(s)\} = f(t)$$

$$L^{-1}\left\{\frac{-d}{ds} f(s)\right\} = t \cdot f(t)$$

$$f(t) = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds} F(s)\right\}$$

$$f(t) = 5 + 3t^2 + 4\int_0^t f(u)\sin(4(t-u))\,du \quad \text{——(1)}$$

Using Convolution Theorem

using Laplace Transform.

Since, Convolution Theorem -

$$\int_0^t f(t-u)\cdot g(u)\,du = F(s)\cdot G(s)$$

Let $L\{f(t)\} = F(s)$.

Apply Laplace in -

$$f(t) = 5 + 3t^2 + 4\int_0^t f(u)\,\sin(4(t-u))\,du$$

$$\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

or $$f(t) = 5 + 3t^2 + 4\int_0^t f(t-u)\sin 4u \cdot du$$

Apply Laplace -

$$L\{f(t)\} = L\{5\} + 3L\{t^2\} + 4L\left\{\int_0^t f(t-u)\cdot\sin 4u \cdot du\right\}$$

or $$F(s) = L\{5\} + 3L\{t^2\} + 4\cdot L\{f(t)\} * L\{\sin 4t\}$$

or $$F(s) = \frac{5}{s} + 3\cdot\frac{2!}{s^3} + 4\,F(s)\cdot\frac{4}{s^2+4^2}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

or, $$F(s) = \frac{5}{s} + \frac{6}{s^3} + \frac{16}{s^2+16}\cdot F(s)$$

or $$F(s)\left[1-\frac{16}{s^2+16}\right] = \frac{5s^2+6}{s^3}$$

or, $$F(s)\left[\frac{s^2+16-16}{s^2+16}\right] = \frac{5s^2+6}{s^3}$$

or, $$F(s) = \frac{(5s^2+6)(s^2+16)}{s^5}$$

or, $$F(s) = \frac{(5s^4+80s^2+6s^2+96)}{s^5}$$

or, $$F(s) = \frac{5}{s}+86\cdot\frac{1}{s^3}+96\cdot\frac{1}{s^5}$$

Taking Laplace inverse -

$$L^{-1}\{F(s)\} = 5L^{-1}\left\{\frac{1}{s}\right\}+86L^{-1}\left\{\frac{1}{s^3}\right\}+96\left\{\frac{1}{s^5}\right\}$$

or $$f(t) = 5+86\cdot\frac{1}{2}\cdot t^2+96\cdot\frac{1}{4!}t^4$$

or, $$\boxed{f(t) = 5+43t^2+4t^4}$$

$\because$ $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}t^n$

Scanned with CamScanner

Application to solve Simple linear and Simultaneous differential equation.

Ordinary differential equation with Constant coefficient can be easily solved by Laplace transform Method.

In this Method we use –

$$L\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$$

$$L\{y'(t)\} = s\,Y(s) - y(0).$$

$$L\{y(t)\} = Y(s).$$

where, $y(0)$ and $y'(0)$ is Given.

* Same use in Simultaneous Differential equation

$$L\{y^n(t)\} = s^n y(s) - s^{n-1} y(0) - s^{n-2} y'(0) \;--\; - y(0).$$

For this, we solve some example:

1) Use the Laplace Transform to slove the following IVP

$$y' + y = \sin(t) \qquad y(0) = 0$$

Sol^n:

Taking Laplace -

Let $L\{y(t)\} = y(s)$

$$L\{y'(t) + y(t)\} = L\{\sin t\}$$

or $L\{y'(t)\} + L\{y(t)\} = L\{\sin t\}$

$L\{y'(t)\} = s\,y(s) - y(0)$

or, $s\,y(s) - y(0) + y(s) = \dfrac{1}{s^2+1}$

or $(s+1)\,y(s) - 0 = \dfrac{1}{s^2+1}$

or, $y(s) = \dfrac{1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{(s^2+1)}$

Decompose by Partial Fraction.

or $y(s) = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+1)}$$

or $$1 = A(s^2+1) + (Bs+C)(s+1)$$

or $$1 = A(s^2+1) + Bs^2 + Bs + Cs + C$$

or $$1 = (A+B)s^2 + (B+C)s + (A+C)$$

$\Rightarrow$ $A+B = 0 \Rightarrow B = -A$

$B+C = 0 \Rightarrow B = -C$

$A+C = -1 \Rightarrow -B-B = -1 \Rightarrow 2B = 1 \Rightarrow \boxed{B = 1/2}$

so, $\boxed{A = -B = -1/2}$, $\boxed{C = -B = -1/2}$

so,

$$y(s) = \frac{-1/2}{(s+1)} + \frac{-\frac{s}{2} + \frac{1}{2}}{(s^2+1)}$$

or, $$y(s) = -\frac{1}{2} \cdot \frac{1}{(s+1)} - \frac{1}{2} \frac{(s-1)}{(s^2+1)}$$

or, $$y(s) = \frac{-1}{2} \cdot \frac{1}{(s+1)} - \frac{1}{2} \frac{s}{(s^2+1)} + \frac{1}{2} \frac{1}{(s^2+1)}$$

Now, Taking Laplace Inverse:

$$L^{-1}\{y(s)\} = \frac{-1}{2} L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2} L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or, $y(t) = \frac{-1}{2} e^{-t} - \frac{1}{2} \cos t + \frac{1}{2} \sin t$

or, $\boxed{y(t) = \frac{1}{2}\left[\sin t - \cos t - e^{-t}\right]}$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

Solving using Laplace Transform -

(2)

$$y'' + 2y' + 4y = e^{2t} \qquad y(0) = 1 \qquad y'(0) = 0$$

Taking Laplace -

$$\mathcal{L}\{y'' + 2y' + 4y\} = \mathcal{L}\{e^{2t}\}$$

or, $\left[s^2 y(s) - s y(0) - y'(0)\right] + 2\left[s y(s) - y(0)\right] + 4y(s) = \frac{1}{(s-2)}$

or $(s^2 + 2s + 4)\, y(s) - s - 0 + 2(-1) = \frac{1}{(s-2)}$

or $(s^2 + 2s + 4)\, y(s) = \frac{1}{(s-2)} + s + 2$

or $(s^2 + 2s + 4)\, y(s) = \frac{1 + (s+2)(s-2)}{(s-2)}$

or $y(s) = \frac{1 + s^2 - 4}{(s-2)(s^2+2s+4)} = \frac{s^2 - 3}{(s-2)(s^2+2s+4)}$

Partial fraction decomposition -

$$\frac{s^2 - 3}{(s-2)(s^2+2s+4)} = \frac{A}{(s-2)} + \frac{Bs + C}{(s^2+2s+4)}$$

or, $s^2 - 3 = A(s^2 + 2s + 4) + (Bs - C)(s-2)$

or $s^2 - 3 = A(s^2 + 2s + 4) + (Bs^2 - 2Bs - Cs + 2C)$

or $s^2 - 3 = (A+B)s^2 + (2A - 2B - C)s + (4A + 2C)$

$$A+B = 1 \Rightarrow 4A+4B = 4 \quad \text{—(1)}$$

$$2A-2B-C=0 \Rightarrow \underline{C = 2A-2B} \quad \text{Put in}$$

$$4A+2C=-3$$

$$4A+2(2A-2B)=-3$$

$$8A-4B=-3 \quad \text{—(2)}$$

on adding (1) & (2)

$$12A = 1 \Rightarrow \boxed{A = 1/12}$$

$$B = 1-A = 1-\frac{1}{12} = \frac{11}{12} \Rightarrow \boxed{B = 11/12}$$

or $$C = 2(A-B) = 2\left(\frac{11}{12}-\frac{1}{12}\right) = \frac{2\times10}{12} = \frac{20}{12}$$

$$y(s) = \frac{1/12}{(s-2)} + \frac{\left(\frac{11}{12}s+\frac{20}{12}\right)}{(s^2+2s+4)}$$

$$y(s) = \frac{1}{12}\cdot\frac{1}{(s-2)} + \frac{1}{12}\cdot\frac{(11s+20)}{(s^2+2s+4)}$$

or $$y(s) = \frac{1}{12}\cdot\frac{1}{(s-2)} + \frac{11}{12}\cdot\frac{s}{(s^2+2s+4)} + \frac{20}{12}\cdot\frac{1}{(s^2+2s+4)}$$

or $$y(s) = \frac{1}{12}\left[\frac{1}{(s-2)} + \frac{11s}{(s+1)^2+(\sqrt{3})^2} + \frac{20}{(s+1)^2+(\sqrt{3})^2}\right]$$

Taking laplace inverse-

$$L^{-1}\{y(s)\} = \frac{1}{12}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{11}{12}\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{20}{12}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

$$= \frac{1}{12}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{11}{12}\mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{20}{12}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

$$= \frac{1}{12}e^{2t} + \frac{11}{12}\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{(20-11)}{12}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

or, $$y(t) = \frac{1}{12}e^{2t} + \frac{11}{12}e^{-t}\cos(\sqrt{3}t) + \frac{9}{12\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$$

or, $$\boxed{y(t) = \frac{1}{12}\left[e^{2t} + 11e^{-t}\cos(\sqrt{3}t) + 3\sqrt{3}\,e^{-t}\sin(\sqrt{3}t)\right]}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

(3) Use Laplace to Solve-

$$\frac{dx}{dt} = -x + y \qquad x(0)=0, \quad y(0)=7$$

$$\frac{dy}{dt} = 2x$$

Taking Laplace

$$\mathcal{L}\{x'(t)\} = \mathcal{L}^{-1}\{-x(t)+y(t)\}$$
$$\mathcal{L}\{y'(t)\} = \mathcal{L}^{-1}\{2x(t)\}$$

or,
$$sX(s) - X(0) = -X(s) + Y(s)$$
$$sY(s) - Y(0) = 2X(s)$$

or,
$$sX(s) + X(s) - Y(s) = 0$$
$$sY(s) - 7 - 2X(s) = 0$$

or
$$\left.\begin{array}{l} (s+1)X(s) - Y(s) = 0 \quad \longrightarrow \text{multiply by } s \\ -2X(s) + sY(s) = 7 \end{array}\right\} \text{——(1)}$$

we get
$$s(s+1)X(s) - sY(s) = 0$$
$$-2X(s) + sY(s) = 7$$

on adding

$$(s(s+1)-2)\,X(s) = 7$$

$$X(s) = \frac{7}{s^2+s-2} = \frac{7}{s^2+2s-s-2}$$

or $$X(s) = \frac{7}{(s+2)(s-1)}$$

Now, Partial Fraction -

$$\frac{7}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

or $$7 = (A+B)s + (2B-A)$$

$$\left.\begin{aligned} A+B &= 0 \\ -A+2B &= 7 \end{aligned}\right\} \Rightarrow 3B = 7 \Rightarrow B = \frac{7}{3}, \quad A = \frac{-7}{3}.$$

$$X(s) = \frac{-7/3}{s+2} + \frac{7/3}{s-1}$$

Taking Laplace Inverse -

$$L^{-1}\{X(s)\} = \frac{-7}{3} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{7}{3} L^{-1}\left\{\frac{1}{s-1}\right\}$$

or, $$x(t) = \frac{-7}{3} e^{-2t} + \frac{7}{3} e^{t}$$

or, $$\boxed{x(t) = \frac{7}{3}\left[e^{t} - e^{-2t}\right]}$$

Now From (1),

$$Y(s) = (s+1)X(s) = \frac{(s+1)\cdot 7}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

Now, Partial Fraction–

$$7s + 7 = (A+B)s + (2B-A)$$

$$\left.\begin{aligned} A+B&=7 \\ -A+2B&=7 \end{aligned}\right\} \Rightarrow 3B = 14 \Rightarrow B = \frac{14}{3}$$

$$A = 7 - B = 7 - \frac{14}{3} = \frac{21-14}{3} = \frac{7}{3}$$

so $$Y(s) = \frac{7/3}{(s+2)} + \frac{14/3}{(s-1)}$$

Taking Laplace inverse -

$$L^{-1}\{Y(s)\} = \frac{7}{3} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{14}{3} L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$y(t) = \frac{7}{3}e^{-2t} + \frac{14}{3}e^{t} \qquad \because L\{e^{at}\} = \frac{1}{s-a}$$

or $$\boxed{y(t) = \frac{7}{3}\left[e^{-2t} + 2e^{t}\right]}$$

Use the Laplace. Solve, Simultaneous. Eqn.

$$\frac{dx}{dt} = -y + t \quad \Rightarrow \quad x'(t) + y(t) = t$$

$$\frac{dy}{dt} = -4x \quad \Rightarrow \quad y'(t) + 4x(t) = 0$$

$$x(0) = 1, \quad y(0) = 2$$

Taking Laplace both side-

$$L\{x'(t) + y(t)\} = L\{t\}$$

$$L\{y'(t) + 4x(t)\} = 0$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $L\{x(t)\} = X(s)$

$L\{y(t)\} = Y(s)$

or $sX(s) - X(0) + Y(s) = \frac{1}{s^2}$

$sY(s) - Y(0) + X(s) = 0$

or $sX(s) - 1 + Y(s) = \frac{1}{s^2}$

$X(s) - 2 + sY(s) = 0$

or $sX(s) + Y(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$ $\longrightarrow \times 1$

$X(s) + sY(s) = 2$ $\longrightarrow$ multiply by s

$\Big\}$ – Eqn (1)

we get -

$$sX(s) + Y(s) = \frac{s^2+1}{s^2}$$

$$sX(s) + s^2Y(s) = 2s$$

On subtracting

$$Y(s)\left[1-s^2\right] = \frac{s^2+1}{s^2} - 2s$$

or, $$(s^2-1)\,Y(s) = 2s - \frac{s^2+1}{s^2} = \frac{2s^3-s^2-1}{s^2}$$

or, $$Y(s) = \frac{2s^3-s^2-1}{s^2(s^2-1)}$$

Now, $$\frac{2s^3-s^2-1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s+1)}$$

or, $$2s^3-s^2-1 = As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1)$$

or, $$2s^3-s^2-1 = A(s^3-s) + B(s^2-1) + C(s^3+s^2) + D(s^3-s^2)$$

or, $$2s^3-s^2-1 = (A+C+D)s^3 + (B+C-D)s^2 - As - B$$

on comparing.

$$\left.\begin{array}{lll} A+C+D=2 & \Rightarrow & C+D=2 \\ B+C-D=-1 & \Rightarrow & C-D=-2 \end{array}\right\} \Rightarrow \begin{array}{l} C=0 \\ D=2 \end{array}$$

$$-A=0 \Rightarrow A=0$$

$$-B=-1 \Rightarrow B=1$$

so, $$Y(s) = \frac{1}{s^2} + \frac{2}{(s+1)}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + 2L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = t + 2e^{-t}}$$

Now,

$$X(s) = 2 - s\,Y(s) \longrightarrow \underline{\text{from eq}^n (1)}$$

or $$X(s) = 2 - s\left(\frac{2s^3 - s^2 - 1}{s^2(s-1)(s+1)}\right)$$

or $$X(s) = \frac{2s(s^2-1) - (2s^3 - s^2 - 1)}{s(s-1)(s+1)}$$

or, $$X(s) = \frac{2s^3 - 2s - 2s^3 + s^2 + 1}{s(s-1)(s+1)} = \frac{s^2 - 2s + 1}{s(s-1)(s+1)}$$

or, $$\frac{s^2 - 2s + 1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

or $$s^2 - 2s + 1 = A(s^2-1) + B(s^2+s) + C(s^2-s)$$

or $$s^2 - 2s + 1 = (A+B+C)s^2 + (B-C)s - A$$

$$\left.\begin{aligned} A+B+C &= 1 \\ B-C &= -2 \\ \underline{A} &\underline{= -1} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} B+C &= 2 \\ B-C &= -2 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= 0 \\ C &= +2 \end{aligned}$$

$$X(s) = \frac{-1}{s} + \frac{2}{s+1}$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{-1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{x(t) = -1 + 2e^{-t}}$$

So, solution is-

$$\left.\begin{aligned} x(t) &= 2e^{-t} - 1 \\ y(t) &= 2e^{-t} + t \end{aligned}\right\}$$

Unit-5
Z-transforms

- Z transform is an extended part of discrete time fourier transform.
- discrete time fourier transform canbe applied only to Stable System but Z-transforms canbe applied or they canbe use to calculate the functioning of unstable Systems as well.
- Z transform is discrete counter part of Laplace transform.
- the primary use of Z transform is to study the System characteristic and derivation of Computational structures for implementation of discrete time systems on Computers.

* **Defination of Z-transforms:-**

Z-transform of $x(n)$ is denoted by $X(z)$. It is defined as.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \quad -①$$

where z is a complex variable given by $z = re^{j\Omega}$

where r is the magnitude, Ω = angle of z.

Substituting z in equation ① we have.

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\Omega})^{-n}$$

$$X(re^{j}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\Omega n} \quad -②$$

from the equation (2) we see that

$X(re^{j\Omega})$ is the fourier transform of the sequence $x(n).r^{-n}$ i.e $X(re^{j\Omega}) = F\{x(n).r^{-n}\}$.

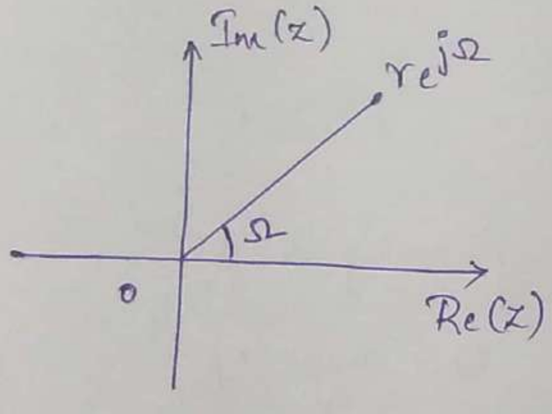
- the exponential weighting factor r^{-n} may be decaying or growing with increasing n, depending on whether 'r' is greater than or less than unity.

when $r = 1 \Rightarrow |z| = 1$.

$$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} = F\{x(n)\}$$

$$\Rightarrow X(e^{j\Omega}) = X(z)\Big|_{z = e^{j\Omega}}.$$

ploting in Zplane. $z = re^{j\Omega}$ is located at a distance 'r' from origin and angle Ω relative to real axis.



$\therefore$ the relationship b/w $x(n)$ and $X[z]$ can be represented as

$$x(n) \xleftrightarrow{ZT} X[z]$$

* Types of z-transform :-

1. Bilateral Z- transform: Z transform defines both +ve & -ve sides of the Zplane they are called as Bilateral or both sided Z-transform.

2. Unilateral or one Sided z-transform:-

It is defined as $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$.

here Summation starts from $n=0$ to ∞ ∴ only +ve axis of zplane.

* Region of Convergence (ROC):-

- ROC is the region where z transform Converges.

- from the defination of Z transform it is clear that Z trasform is an infinite power series.

- this series is not convergent for all values of z.

1. Finite Duration Sequence:-

a. Right-Sided Sequence:-

- A right-sided sequence is one for which $x(n)=0$ for all $n \leq n_0$, where n_0 is +ve or -ve, but finite.

- If $n_0 \geq 0$, the resulting sequence $x(n)$ is a positive time sequence.

- For a Casual finite Sequence ROC is entire zplane except for $z=0$

Eg $x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$

By defination $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$ (+ve Sequence)

$$= \sum_{n=0}^{3} x(n) z^{-n}.$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

$= 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{1}{z^3}$

∴ from above expression $x(z)$ is finite for all the values of z except $z=0$. Hence ROC is entire z plane except $z=0$ ∴ mathematically it can be writen as.

$$\boxed{ROC : |z| > 0}$$

b. Left sided Sequence :-

- Sequence $x(n)$ is one for which $x(n)=0$ for all $n > n_0$ where n_0 is positive or negative, but finite.
- If $n_0 \leq 0$, the resulting sequence $x(n)$ is called anti causal or negative sequence.
- for such kind of sequence ROC is entire z-plane except for $z=\infty$.

Eg $x(n) = (1, 1, 2, \underset{\uparrow}{2})$

By defination $x(z) = \sum_{n=-\infty}^{0} x(n) z^{-n}$.

$$\Rightarrow x(z) = \sum_{n=-3}^{0} x(n) . z^{-n}$$

$$= x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0)$$

$$= z^3 + z^2 + 2z + 2.$$

the expression $x(z)$ becomes infinite at $z=\infty$ hence Roc is entire z plane except for $z=\infty$

$$\boxed{Roc : |z| < \infty}$$

C. Double-Sided Sequence:-

- A signal that has finite duration in both the +ve & -ve side is known as double-sided sequence.
- In this case, ROC is the entire z-plane except for points at $z=0$ and $z=\infty$.

Eg 1. $x(n) = (2, 1, \underset{\uparrow}{1}, 2)$

By defination $X(z) = \sum_{n=\infty}^{-\infty} x(n) z^{-n}$.

$$\Rightarrow X(z) = \sum_{n=-2}^{1} x(n) z^{-n}.$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1}$$

$$= 2z^{2} + z^{1} + 1 + 2z^{-1}$$

$\therefore$ ROC : $0 < |z| < \infty$.

$X(z)$ becomes infinity at $z=0$ and $z=\infty$

$\Rightarrow$ ROC is the entire z-plane except for $z=0$ & $z=\infty$.

2. ROC of infinite Duration Sequence:-

a. Positive time exponential Sequence:-

A positive time exponential sequence is defined by.

$$x(n) = a^{n} u(n).$$

$\therefore$ By z transfor defination

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{n} u(n) z^{-n}$$

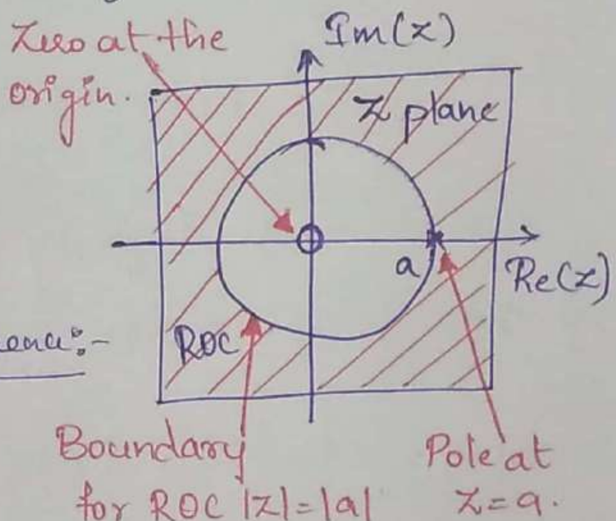
$$\Rightarrow X(z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad [u(n) = 0 \text{ for } n < 0]$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n.$$

we know that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$; $|\alpha| < 1$.

$$\Rightarrow X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}. \quad \text{Roc}: |z| > |a|$$

the above result converges if $|a z^{-1}| < 1$ which is equal to $|a| < |z|$. Values of z for which $X(z) = 0$ are called. Zeros of $X(z)$, while the value of z for which $X(z) \to \infty$ are called the poles of $X(z)$



- poles are indicated as x
Zeros are indicated by 0.

b. Negative time exponential Sequence:-

Sequence is defined by.

$$x(n) = -b^n u(-n-1)$$

we know that $u(-n-1) = \begin{cases} 1, & n \leq -1 \\ 0, & n > -1. \end{cases}$

By z transformation defination

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$x(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (b/z)^n$$

Substituting $n = -m$ in the above summation, we get

$$x(z) = -\sum_{m=\infty}^{1} (z/b)^m$$

$$= -\left[\sum_{m=1}^{\infty} (z/b)^m + 1 - 1\right]$$

$$= -\left[\sum_{m=1}^{\infty} (z/b)^m + (z/b)^0 - 1\right]$$

$$= -\left[\sum_{m=0}^{\infty} (z/b)^m - 1\right]$$

$$= 1 - \sum_{m=0}^{\infty} (z/b)^m$$

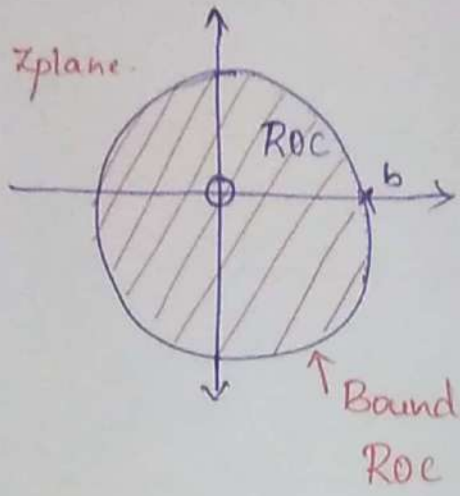
We know that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

$$\therefore \; x(z) = 1 - \frac{1}{1-(z/b)}$$

$\therefore$ Result Converges at $|z/b| < 1 = |z| < |b|$

ROC : $|z| < |b|$

ROC and pole-Zero plot for negative time Sequence is (8)



※ Note :-

1. positive time Sequence has z transform with ROC exterior to the circle $|z| = |a|$ $(|z| > |a|)$

2. negative time Sequence has z transform with ROC interior to the circle $|z| = |b|$ $(|z| < |b|)$

* Double Sided exponential Sequence :-

double Sided Sequence is the Sum of positive and negative exponential Sequence.

i.e $x(n) = a^n u(n) - b^n u(-n-1)$

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

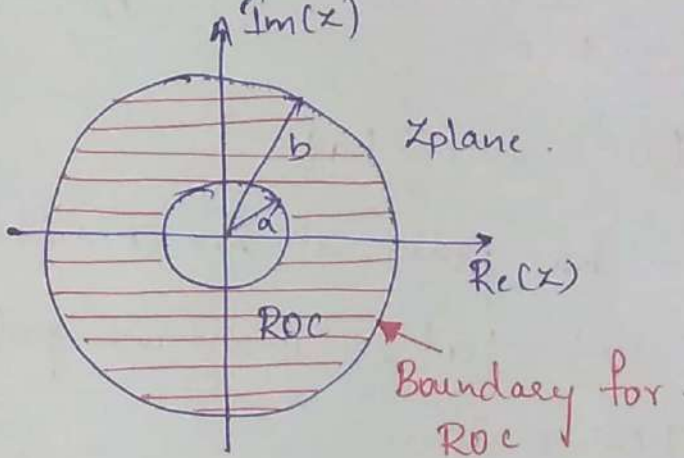
$$\therefore X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \left[\frac{1}{1-az^{-1}}\right] + \left[1 - \frac{1}{1-(z/b)}\right]$$

$$= \frac{z}{z-a} + \frac{z}{z-b} \quad ROC : |z| > |a| \cap |z| < |b|$$

- the first power series converges if $|az^{-1}| < 1$. or $|z| > |a|$
- the second power series converges if $|(z/b)| < 1$ or $|z| < |b|$
- where as $x(z)$ is the sum of positive and negative time sequence with ROC equal to the intersection of the two region of convergence.



* Properties of ROC :-

We assume $x(z)$ is a rational function of z.

Property 1 : "ROC for a finite duration sequence includes entire Z-plane, except $|z|=0$ and $|z|=\infty$"

Proof : $x(n) = \{1, 2, \underset{\uparrow}{1}, 2\}$

By defination of z transform

$$x(z) = 1z^{2} + 2z^{1} + 1 + 2z^{-1}$$

$$= z^{2} + 2z^{1} + 1 + \frac{2}{z}.$$

$x(z) = \infty$ for $z = 0$, & $z = \infty$.

Hence proved.

Property 2: ROC doesnot Contain any poles.

Proof: We have calculate z transform of $a^n u(n)$

$$X(Z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad [\text{By defination}]$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{Z}{Z-a}.$$

ROC: $|Z| > |a|$

this function has poles at $Z = a$. Note that ROC is $|Z| > |a|$ ∴ this means poles donot lie inside ROC region.

Property 3: "ROC is the ring in the Z-plane Centered about Origin"

Proof: Consider. $a^n u(n) \xleftrightarrow{ZT} \frac{Z}{Z-a}$, ROC $|Z| > |a|$

or $-b^n u(-n-1) \xleftrightarrow{ZT} \frac{Z}{Z-b}$; ROC: $|Z| < |b|$

Here we can observe that when $Z = 0$ the value obtained is 0 ∴ $|Z|$ is always a circular region centered around origin.

Property 4: ROC of a casual Sequence (Right hand side Sequence) is of the form $|Z| > r$.

Proof: Right hand Side Sequence = $a^n u(n)$

we know that is ROC: $|Z| > |a|$

thus ROC of right hand sided sequence is of the form $|z| > r$. where r is the radius of the circle.

Property 5: ROC of left sided sequence is of the form $|z| < r$.

Proof: left sided sequence $= -b^n u(-n-1)$.

we know its ROC is $|z| < |b|$

this is of the form $|z| < r$ where r is the radius of the circle.

Property 6: ROC of two sided sequence is the concentric ring in Z-plane.

Proof:- Double sided sequence $= a^n u(n) - b^n u(-n-1)$

we know its ROC is $|a| < |z| < |b|$

$\therefore$ ROC is the intersection of two regions of convergence hence it is concentric rings.

Property 7: If $x(n)$ is a finite casual sequence, then its ROC is entire Z-plane except $z=0$.

Proof: $x(n) = \{1, 2, 3\}$ Z-transform will be

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{2} x(n) z^{-n} = x(0) + x(1) z^{-1} + x(2) z^{-2}$$

$$= 1 + 2z^{-1} + 3z^{-2}.$$

hence the sequence converges in entire zplane except for $z=0$.

Property 8: the ROC of a stable LTI system contains unit circle in the z-plane.

Property 9: The ROC is a connected region.

Proof: the convergence of the sequence exists over certain area, rather than discrete points.

Hence ROC is connected Region.

1. Determine the z-transform of following sequence.

a. $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

By defination of z transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{6} x(n) z^{-n}.$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5} + x(6) z^{-6}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 0.z^{-5} + 7z^{-6}$$

$$\therefore X_1(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{7}{z^6}.$$

$X_1(z) = \infty$ if $z = 0$

$\therefore X_1(z)$ is convergent for all values of z. except for $z = 0$.

$\Rightarrow$ ROC: Entire z plane except for $z = 0$.

Scanned with CamScanner

b. $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

By Defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$X_2(z) = \sum_{n=-3}^{3} x(n) z^{-n}.$$

$$\Rightarrow x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}.$$

$$X_2(z) = 1z^3 + 2z^2 + 3z + 4 + 5z^{-1} + z^{-2} \cdot 0 + 7z^{-3}$$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

$X_2(z) = \infty$ for $z = 0$, $z = \infty$.

$\therefore$ ROC is entire z plane except for $z=0$, $z=\infty$.

c. $x(n) = \{5, 2, \underset{\uparrow}{-2}, 1, 1, -3\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=-2}^{3} x(n) z^{-n}.$$

$$= x(-2) z^{+2} + x(-1) z^{+1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^2 + 2z - 2 + 1z^{-1} + 1z^{-2} + (-3) z^{-3}$$

$$= 5z^2 + 2z - 2 + \frac{1}{z} + \frac{1}{z^2} - \frac{3}{z^3}$$

$\therefore$ each term in $X(z)$ is finite and Consequently $X(z)$ will Converge for entire z plane except for $z=0$, $z=\infty$.

2. find the z transform of $\delta(n)$

$$\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0. \end{cases}$$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} \delta(n) z^{-n}.$$

$$= 1 \cdot z^{-0} = 1.$$

this is a fixed value for any z. Hence ROC is entire z plane.

3. find the z transform of unit step sequence $u(n)$.

$$\text{Unit step } u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} u(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}.$$

$$\boxed{\sum_{n=0}^{\infty} \alpha^{n} = \frac{1}{1-\alpha}}$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

$|z| > 1 \rightarrow$ ROC of $x(z)$

$|z|$ represents a circle in z plane where radius $= 1$.
"a unit circle" in zplane.

(4) For the Signal $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$, find the z-transform and ROC. (15)

given: $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$.

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$X(z) = \sum_{n=-\infty}^{\infty} \left[7(1/3)^n u(n) - 6(1/2)^n u(n)\right] z^{-n}.$$

$$= \sum_{n=0}^{\infty} 7(1/3)^n z^{-n} - \sum_{n=0}^{\infty} 6(1/2)^n z^{-n}.$$

$$= 7\sum_{n=0}^{\infty} (1/3\, z^{-1})^n - 6\sum_{n=0}^{\infty} (1/2\, z^{-1})^n.$$

$$= 7\cdot\frac{1}{1-1/3\, z^{-1}} - 6\cdot\frac{1}{1-1/2\, z^{-1}}$$

$$\boxed{\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.}$$

$$= \frac{7(1-1/2\, z^{-1}) - 6(1-1/3\, z^{-1})}{(1-1/3\, z^{-1})(1-1/2\, z^{-1})}$$

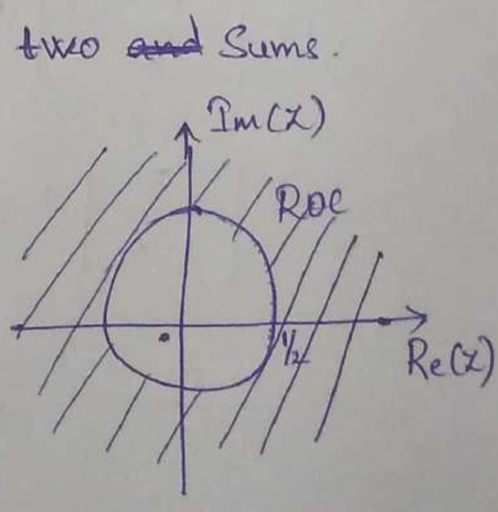
$$= \frac{1 - 3/2\, z^{-1}}{(1-1/3\, z^{-1})(1-1/2\, z^{-1})}$$

the expression of $X(z)$ contains two ~~and~~ Sums.

$\therefore |1/3\, z^{-1}| < 1$ & $|1/2\, z^{-1}| < 1$

$|1/3| < |z|$ & $|1/2| < |z|$

ROC: $|z| > |1/2|$



5. Determine the z-transform of $x(n) = -u[-n-1] + (1/2)^n u(n)$ (16) find the ROC and pole-zero location of $X(z)$ in the z-plane.

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} [-u(-n-1) + (1/2)^n u(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [-u(-n-1)] z^{-n} + \sum_{n=-\infty}^{\infty} (1/2)^n u(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-1) z^{-n} + \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= (-1) \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} (1/2 z^{-1})^n$$

$$= (-1) \left[\sum_{n=1}^{\infty} z^n + \cancel{1}^{\,z^0} - 1 \right] + \sum_{n=0}^{\infty} (1/2 z^{-1})^n$$

$$= (-1) \left[\sum_{n=0}^{\infty} z^n - 1 \right] + \frac{1}{1 - 1/2 z^{-1}}$$

$$= (-1) \left[\frac{1}{1-z} - 1 \right] + \frac{1}{1 - 1/2 z^{-1}}$$

$$= (-1) \left[\frac{z}{1-z} \right] + \frac{1}{1 - 1/2 z^{-1}}$$

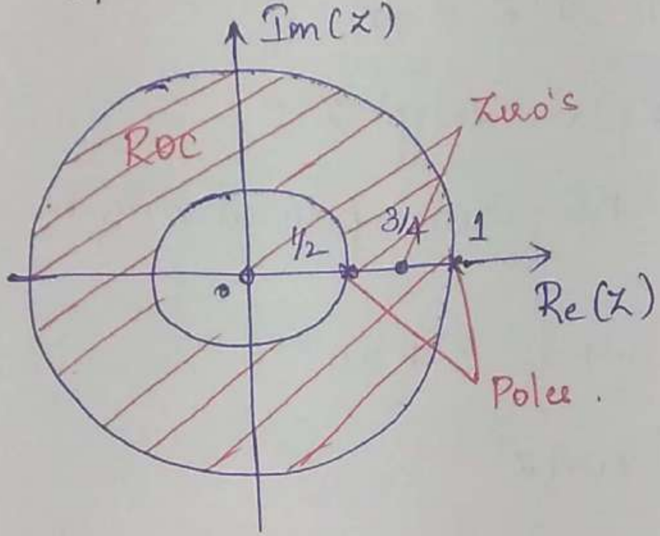
$$= \frac{-z}{1-z} + \frac{1}{1 - 1/2 z^{-1}}$$

$$= \frac{z}{(z-1)} + \frac{z}{(z-1/2)}$$

$$= \frac{2z^2 - 3/2\,z}{(z-1)(z-1/2)} = \frac{z(2z-3/2)}{(z-1)(z-1/2)}$$

$\therefore\ |z| < 1 \qquad |1/2\, z^{-1}| < 1. \Rightarrow |1/2| < |z|$

$\therefore\ ROC = |1/2| < |z| < 1.$



6. Determine the z transform and ROC for the following time signals. Sketch the ROC, poles and zero's in the z plane.

1. $x(n) = \delta(n)$

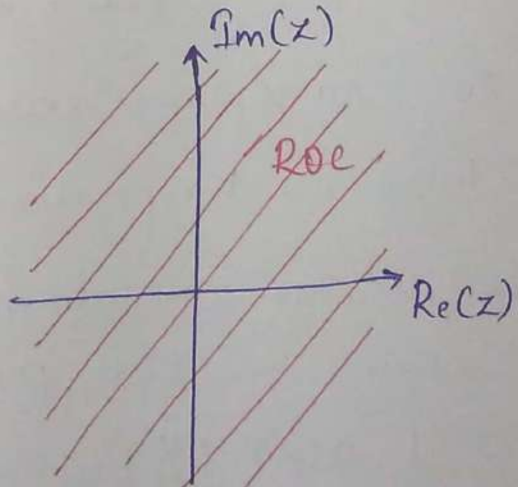
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$\delta(n)$ exist only when $n=0$

$X(z) = 1 \cdot z^0 = 1.$

ROC = entire z plane.

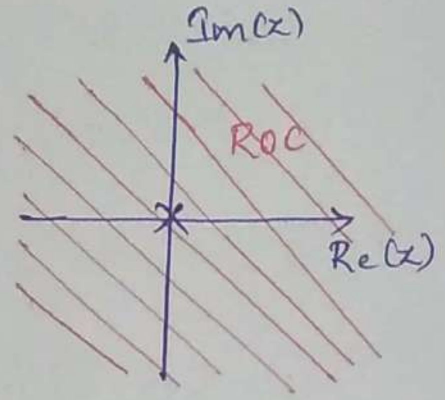
No zero's no poles



2. $x(n) = \delta(n-K)$; $K > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-K) z^{-n}$$



$\delta(n-k)$ exist only when $n=K$.

$\therefore X(z) = 1 \cdot z^{-K} = 1/z^{K}$

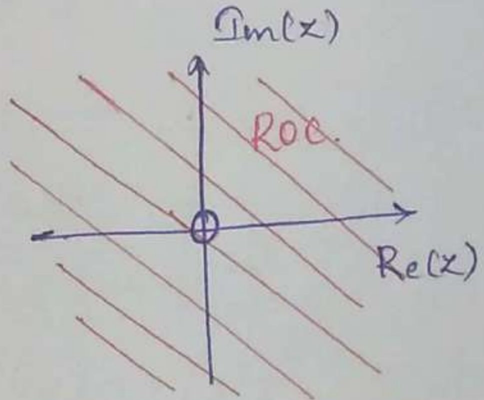
ROC : All z plane except $z=0$

$\therefore$ there are K repetative poles at origin.

3. $x(n) = \delta(n+K)$; $K > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n+K) z^{-n}$$



$\delta(n+k)$ exisit only when $n=-K$.

$\therefore X(z) = 1 \cdot z^{K} = z^{K}$.

ROC : All z plane exupt $z=0$.

there are K repetative zeroes at origin.

7. Determine z transform and ROC for the given signals. Sketch the ROC, poles and zeroes.

1. $x(n) = (1/2)^{n} u(n-2)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[(1/2)^n u(n-2) \right] z^{-n}$$

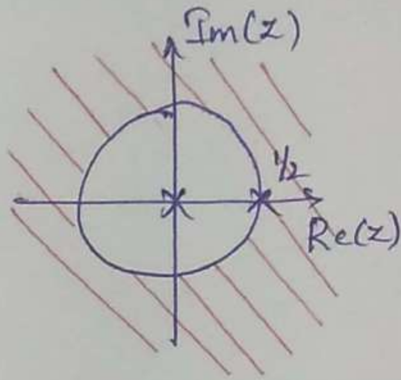
$$= \sum_{n=2}^{\infty} (1/2)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} (1/2\, z^{-1})^n$$

$$\boxed{\sum_{n=2}^{\infty} \alpha^n = \frac{\alpha^2}{1-\alpha}}$$

$$X(z) = \frac{(1/2\, z^{-1})^2}{1 - 1/2\, z^{-1}} = \frac{1}{4z^2(1 - 1/2\, z^{-1})}$$

$$= \frac{1}{4z(z - 1/2)}$$



ROC : $|1/2\, z^{-1}| < 1 \Rightarrow |1/2| < |z|$

2. $x(n) = 2^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

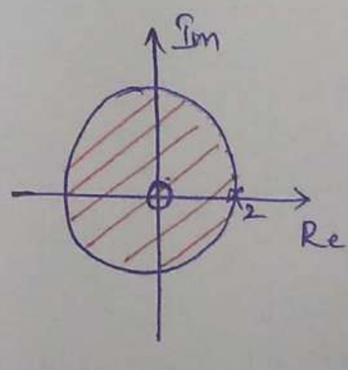
$$= \sum_{n=-\infty}^{\infty} 2^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 2^n \cdot z^{-n}$$

$$= \sum_{n=1}^{\infty} 2^{-n} \cdot z^{n} = \sum_{n=1}^{\infty} (2^{-1} z)^n$$

$$\boxed{\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}}$$

$$= \frac{2^{-1} z}{1 - 2^{-1} z} = \frac{-z}{z-2}$$



ROC : $|z| < 2$. hole at $z = 0$.

Pole at $z = 2$.